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THE LARGE LIMITS THAT ALL GOOD CATEGORIES ADMIT

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1. Introduction

A category $\mathcal{A}$ is called *complete* if every functor $F:\mathcal{K}\to\mathcal{A}$, with $\mathcal{K}$ small, has a limit. Here 'small' means 'of cardinal less than a chosen (strongly) inaccessible ∞'; ∞ might be ω (the case of finitely-complete $\mathcal{A}$), or an inaccessible in ZFC corresponding to a Grothendieck universe, or the cardinal of the universal class if we discuss categories in the language of GB.

A complete $\mathcal{A}$, unless it is merely a preorder, cannot admit a limit for all functors $F:\mathcal{K}\to\mathcal{A}$ with $\mathcal{K}$ large and discrete, as is well known. However there are certain large $\mathcal{K}$ such that each $F:\mathcal{K}\to\mathcal{A}$ with $\mathcal{A}$ complete *does* have a limit; it is trivially so, for instance, if $\mathcal{K}$ has an initial object. We might seek to characterize such $\mathcal{K}$.

The question becomes a more reasonable one, however, if we require of $\mathcal{A}$ a little more than completeness. If $\mathcal{A}$ is any of the common categories of mathematical structures, or the dual of such a category, or a category manufactured from these categories by forming product-categories and functor-categories, then not only is $\mathcal{A}$ complete: it also admits all intersections, large or small, of the subobjects of any object. Usually, indeed, because it is well-powered; but not necessarily: a non-wellpowered example is provided by the quasi-topological spaces of Spanier [2], and others are provided by functor-categories $[\mathcal{B},\mathcal{A}]$ with large $\mathcal{B}$. At the very least it is reasonable to suppose that a good $\mathcal{A}$ admits arbitrary intersections, if not of *all* monomorphisms, then at any rate of *regular* monomorphisms.

Call a set $\mathcal{L}$ of objects of $\mathcal{K}$ *ancestral* (Freyd's word is 'pre-initial') if for each $K\in\mathcal{K}$ there is a map $L\to K$ from some $L\in\mathcal{L}$. We prove:

Theorem 1. *Let a category $\mathcal{K}$ be given. Then every functor $F:\mathcal{K}\to\mathcal{A}$, where $\mathcal{A}$ is* *complete and admits arbitrary intersections of regular monomorphisms, admits a limit if $\mathscr{X}$ has a small ancestral set $\mathscr{L}$. The converse is true if $\infty = \omega$ (the finitely-complete case), or if* card(ob $\mathscr{X}$) $\leq \infty$ *(the* GB *case); and the converse is always true under the General Continuum Hypothesis.*

In fact a somewhat more general question suggests itself. The hypothesis that $\mathscr{A}$ admits arbitrary intersections of monomorphisms asserts the existence in $\mathscr{A}$ of certain large limits under the condition that the large diagram consists of *monomorphisms*. We prove:

Theorem 2. *Let a category $\mathscr{X}$ be given, and a set Σ of its maps. Then every functor $F : \mathscr{X} \to \mathscr{A}$, where $\mathscr{A}$ is complete and admits arbitrary intersections of monomorphisms, and where Fk is a monomorphism for each $k \in \Sigma$, admits a limit if the category of fractions $\mathscr{X}[\Sigma^{-1}]$ has a small ancestral set $\mathscr{L}$. The converse is true under the same conditions as in Theorem* 1.

Of course Theorem 2 subsumes Theorem 1 if we are content to deal with $\mathscr{A}$ admitting all intersections.

A closely-related question has been discussed in correspondence between Ross Street and P.J. Freyd: that of characterizing those $\mathscr{X}$ for which every functor $F : \mathscr{X} \to$ **Set** has a limit, where **Set** is the category of small sets. This question assumes some importance in the context of Street's 2-categorical studies of the foundations of category theory. They have:

Theorem 3 (Freyd–Street). *Every $F : \mathscr{X} \to$* **Set** *has a limit if, for every set $\mathscr{N}$ of objects of $\mathscr{X}$ with card $\mathscr{N} = \infty$, there is a small set $\mathscr{L}$ of objects of $\mathscr{X}$ which is ancestral for $\mathscr{N}$, in the sense that for each $N \in \mathscr{N}$ there is a map $L \to N$ with $L \in \mathscr{L}$. The converse is true if $\infty = \omega$.*

To this we can now add:

Proposition 4. *The converse of Theorem 3 is also true if* card(ob $\mathscr{X}$) $\leq \infty$, *or more generally if $\mathscr{X}$ has an ancestral set of cardinal $\leq \infty$.*

We have been privileged to see the correspondence between Street and Freyd, and some of our central arguments in proving the converse have been adapted from, or suggested by, theirs.

Apart from guessing the 'right' conditions (in so far as they are the right ones), the only non-straightforward part is the proof of the converses. These reduce entirely to questions, not about general categories, but about preordered sets. We therefore turn first to these questions, at the same time changing the variance so that we discuss ordinals rather than their duals.

2. Preordered sets

2.1. Cofinality

We work in ZFC. An ordinal is the set of smaller ordinals, and a cardinal is an initial ordinal. The *regular* cardinals are 0, 1, and those infinite cardinals that are not the sum of a smaller number of smaller cardinals. An *inaccessible* cardinal is an infinite regular one for which the set of smaller cardinals is closed under exponentiation; one such is $\omega = \aleph_0$. When an inaccessible ∞ (which may be ω) is chosen once for all, sets of cardinal $< \infty$ are called *small.*

It is convenient, in view of some of the proofs below, to deal with preordered rather than ordered sets. If $(X, \leq)$ is such a preordered set, we write $x<y$ if $x \leq y$ and $y \not\leq x$. A subset of X, unless the contrary is clear, is taken with its induced preorder. A *totally ordered* set is a preordered one where exactly one of $x=y$, $x<y$, $y<x$ is true; thus its preorder is an order.

We extend the preorder $\leq$ to the set $\mathscr{P}X$ of subsets of X, writing $A \leq B$ for subsets A, B of X if for each $a \in A$ there is some $b \in B$ with $a \leq b$. If we identify $x \in X$ with the singleton $\{x\}$, this does extend the original preorder. Clearly $A \subset B$ implies $A \leq B$. Note that (by our definition of $<$) $A<x$ means that $a<x$ for all $a \in A$.

An element a of X is said to be *maximal* if there is no $x \in X$ with $a<x$. A subset A of X is called *invincible* if there is no $x \in X$ with $A<x$; such a subset need not be maximal in $\mathscr{P}X$.

A subset A of X is *cofinal* if $X \leq A$. In the language of the introduction, this is just to say that A is ancestral in X^{op}; for preordered sets we retain the older term. The *cofinal character* $\operatorname{cof} X$ of X is the least cardinal α such that X has a cofinal subset of cardinal α; of course $\operatorname{cof} X = 0$ only for empty X. It is immediate that a cardinal α is regular if and only if $\alpha = \operatorname{cof} \alpha$.

A map $f: X \to Y$ between preordered sets is an increasing function; that is, one for which $x \leq x'$ implies $f(x) \leq f(x')$. It is a monomorphism exactly when injective, and an epimorphism exactly when surjective; but it may of course be bijective without being an isomorphism. We shall for emphasis use a Greek letter such as $\phi: X \to Y$ in the few cases where we merely consider a map of the underlying sets, not necessarily increasing.

Proposition 5. *There is a cofinal $Z \subset X$ and a bijection $f: Z \to \operatorname{cof} X$ of preordered sets.*

Proof. Let $\operatorname{cof} X = \alpha$, let A be cofinal in X with $\operatorname{card} A = \alpha$, and let $\phi: A \to \alpha$ be a bijection of sets. Set $Z = \{a \in A \mid \text{for all } b \in A,\ a \leq b \text{ implies } \phi(a) \leq \phi(b)\}$.

Now given $a \in A$ define $z \in A$ by $\phi(z) = \min\{\phi(b) \mid b \in A \text{ and } a \leq b\}$. Since ϕ is a bijection, we have $a \leq z$. So for $b \in A$, $z \leq b$ gives $a \leq b$ and so $\phi(z) \leq \phi(b)$ by the definition of z. Hence $z \in Z$, showing that Z is cofinal in A and thus in X.

The restriction $g: Z \to \alpha$ of $\phi: A \to \alpha$ is now an increasing injection. Since Z is cofinal we have card $Z=\alpha$; thus $g(Z)\subset\alpha$ is in fact isomorphic to α, and we have an increasing bijection $f: Z\to\alpha$. □

Corollary 6. *If X is totally ordered, the cardinal* $\operatorname{cof} X$ *is regular.*

Proof. The bijection $f: Z\to\alpha=\operatorname{cof} X$ of Proposition 5 is now an isomorphism. Since clearly $\operatorname{cof} Z=\alpha$, we have $\operatorname{cof}\alpha=\alpha$, and α is regular. □

The corollary is of course a classical result.

Proposition 7. *If the map $f: X\to Y$ of preordered sets is an epimorphism, we have* $\operatorname{cof} Y\leq\operatorname{cof} X$.

Proof. If A is cofinal in X, $f(A)$ is cofinal in Y. □

2.2. The spectrum

By the *spectrum* $\operatorname{spec} X$ of a preordered set X we mean the set of those *regular cardinals* α for which there exists an epimorphism $X\to\alpha$ of preordered sets. Except for empty X, it always contains 1; moreover, since $\operatorname{cof}\alpha=\alpha$ for a regular cardinal α, Proposition 7 gives:

Proposition 8. $\operatorname{spec} X\leq\operatorname{cof} X$.

For a totally-ordered set the spectrum is trivial:

Proposition 9. *For a nonempty totally ordered X, we have* $\operatorname{spec} X=\{1,\operatorname{cof} X\}$.

Proof. Let $\operatorname{cof} X=\alpha$, which is regular by Corollary 6. Since the f of Proposition 5 is here an isomorphism, we may identify α with a cofinal subset of X. Define $g: X\to\alpha$ by $g(x)=\min\{\beta\in\alpha \mid x\leq\beta\}$; then g is clearly increasing, and it is surjective since $g(\beta)=\beta$ for $\beta\in\alpha$. Thus $\alpha\in\operatorname{spec} X$.

Now suppose $\beta\neq 1$ belongs to $\operatorname{spec} X$, and let $g: X\to\beta$ be an epimorphism. Considered just as a map of the underlying sets, g has a right inverse given by an injection $h:\beta\to X$; in fact h is itself a map of ordered sets, but we do not use this. The image $h(\beta)\subset X$ is cofinal in X; for given $x\in X$ we can choose $\gamma\in\beta$ with $\gamma>g(x)$, and then $h(\gamma)\geq x$: otherwise we should have $h(\gamma)<x$, giving the contradiction $\gamma=g(h(\gamma))\leq g(x)$. We conclude that $\operatorname{cof} X\leq\beta$. But $\beta\leq\operatorname{cof} X$ by Proposition 8, so that $\beta=\operatorname{cof} X$. □

It is immediate from the definition of spec that:

Proposition 10. *If $f: X\to Y$ is epimorphic,* $\operatorname{spec} Y\subset\operatorname{spec} X$.

Theorem 11. *For any* X *we have* $\operatorname{cof}(\operatorname{cof} X) \in \operatorname{spec} X$.

Proof. Choose Z and f as in Proposition 5. Define $g: X \to \operatorname{cof} X$ by $g(x) = \min\{f(z) \mid z \in Z \text{ and } x \leq z\}$. Clearly g is increasing; and for $x \in Z$ we have $g(x) = f(x)$, so that g is epimorphic. It follows from Proposition 10 that $\operatorname{spec}(\operatorname{cof} X) \subset \operatorname{spec} X$; but by Proposition 9 we have $\operatorname{cof}(\operatorname{cof} X) \in \operatorname{spec}(\operatorname{cof} X)$. □

Combining this with Proposition 8 gives:

Corollary 12. *If* $\operatorname{cof} X$ *is regular, then* $\operatorname{cof} X \in \operatorname{spec} X$, *and* $\operatorname{cof} X$ *is the greatest element of* $\operatorname{spec} X$.

2.3. Partially-proved conjectures

The strongest conjecture we shall contemplate is:

Conjecture 13. *If the regular cardinal* α *is not in* $\operatorname{spec} X$ *and if* $Z \subset X$ *has cardinal* α, *there is some* $A \subset X$ *with* $Z \leq A$ *and* $\operatorname{card} A < \alpha$.

For this we have no very strong evidence one way or the other. However, a special case is:

Conjecture 14. *If the inaccessible* ∞ *is not in* $\operatorname{spec} X$ *and if* $Z \subset X$ *has cardinal* ∞, *there is some* $A \subset X$ *with* $Z \leq A$ *and* $\operatorname{card} A < \infty$.

The conjecture of immediate concern to us, which we shall see in Theorem 23 to be implied by Conjecture 13, is:

Conjecture 15. *If* $\operatorname{spec} X < \infty$ *then* $\operatorname{cof} X < \infty$.

By Corollary 12, we have:

Theorem 16. *Conjecture* 15 *is true if* $\operatorname{cof} X$ *is regular; and hence it is true if* $\operatorname{cof} X \leq \infty$.

The following results will establish Conjecture 15 when $\infty = \omega$. They are in part adapted from arguments in the Freyd–Street correspondence. Recall from Section 2.1 the definition of an *invincible* $A \subset X$.

Proposition 17. *If* $A \subset X$ *is totally ordered and invincible, then* $\operatorname{cof} A \in \operatorname{spec} X$.

Proof. We may as well replace A by the regular cardinal $\alpha = \operatorname{cof} A$, treating this as a subset of X with the induced order. Define $f: X \to \alpha$ by $f(x) = \min\{\beta \in \alpha \mid \beta \not< x\}$. Clearly f is increasing; and it is epimorphic since $f(\beta) = \beta$ for $\beta \in \alpha$. □

Proposition 18. *If every totally ordered and invincible $A \subset X$ has* $\operatorname{cof} A < \infty$, *then for each $x \in X$ there is a small invincible A, isomorphic to a regular cardinal, with $x \le A$.*

Proof. By Zorn's lemma there is a maximal totally ordered subset B containing x; then B is invincible, and we take A to be cofinal in B and isomorphic to $\operatorname{cof} B$. □

Theorem 19. *Conjecture* 15 *is true if* $\infty = \omega$.

Proof. By Proposition 17 we have the hypothesis and hence the conclusion of Proposition 18; but in the case $\infty = \omega$ the regular cardinal $\operatorname{card} A$ in that conclusion must be 1, so that A is a singleton $\{a\}$, and a maximal element of X. Thus the set M of maximal elements is cofinal. If M were infinite there would be an epimorphism $g: M \to \omega$, which would extend to an epimorphism $f: X \to \omega$ on setting $f(x) = 0$ for $x \notin M$, giving $\omega \in \operatorname{spec} X$. Since this is false by hypothesis, M is finite. □

2.4. A transformation of the question

Given a subset Z of X, which is fixed for the time being, we define a new preorder $\le'$ on X by:

$$x \le' y \quad \text{if and only if} \quad \{z \in Z \mid z \le x\} \subset \{z \in Z \mid z \le y\},$$

and write X' for the set X with this new preorder; observe that X' has at most $\operatorname{card}(2^Z)$ non-isomorphic elements. We at once verify:

Proposition 20. *The identify map $X \to X'$ is a map of preordered sets, so that by Propositions* 7 *and* 10 *we have* $\operatorname{cof} X' \le \operatorname{cof} X$ *and* $\operatorname{spec} X' \subset \operatorname{spec} X$. *Moreover* $\operatorname{cof} X' \le \operatorname{card}(2^Z)$.

Proposition 21. *For $z \in Z$ we have $z \le' x$ if and only if $z \le x$. So if A is cofinal in X' we have $Z \le A$.*

If, therefore, there is no $A \subset X$ with $Z \le A$ and $\operatorname{card} A < \operatorname{card} Z$, we must have $\operatorname{cof} X' \ge \operatorname{card} Z$, as well as $\operatorname{cof} X' \le \operatorname{card}(2^Z)$. Then by Theorem 11 and Proposition 20 we have $\operatorname{cof}(\operatorname{cof} X') \in \operatorname{spec} X' \subset \operatorname{spec} X$. Thus:

Proposition 22. *For $Z \subset X$, either there is an $A \subset X$ with $Z \le A$ and* $\operatorname{card} A < \operatorname{card} Z$, *or there is a cardinal α with* $\operatorname{card} Z \le \alpha \le \operatorname{card}(2^Z)$ *and* $\operatorname{cof} \alpha \in \operatorname{spec} X$.

Theorem 23. *If we assume* GCH, *or if Conjecture* 13 *is true, we have* $\operatorname{cof} X = \sup \operatorname{spec} X$, *provided that* $\operatorname{cof} X$ *is infinite. Hence Conjecture* 15 *is then true.*

Proof. Let $\alpha = \sup \operatorname{spec} X$. Since $\operatorname{cof} X$ is infinite, α is infinite by Theorem 11. By

Proposition 8 we have to show that $\operatorname{cof} X \leq \alpha$. Suppose the contrary. Then there are subsets Z of X such that $Z \leq A$ for $A \subset X$ implies $\operatorname{card} A > \alpha$; for X itself is one such. Choose such a Z with smallest possible cardinal β. Then $\beta > \alpha$ since $Z \leq Z$; and in fact $Z \leq A$ implies $\operatorname{card} A \geq \beta$, for otherwise there would be some $B \geq A$ with $\operatorname{card} B \leq \alpha$, a contradiction since $Z \leq A \leq B$.

The infinite cardinal β must be regular. For otherwise $Z = \bigcup_{i \in \operatorname{cof}\beta} Z_i$, where $\operatorname{cof}\beta < \beta$ and each $\operatorname{card} Z_i < \beta$. Then we have $Z_i \leq A_i$ with $\operatorname{card} A_i \leq \alpha$, giving the contradiction $Z \leq A = \bigcup_i A_i$ with $\operatorname{card} A \leq \max(\operatorname{cof}\beta, \alpha) < \beta$.

But β cannot be regular if Conjecture 13 is true, since $\beta \notin \operatorname{spec} X$. Nor can β be regular if we assume GCH. For then, by Proposition 22, either $\operatorname{cof}\beta$ or $\operatorname{cof}(2^\beta)$ belongs to $\operatorname{spec} X$; but $\operatorname{cof}\beta = \beta$ and $\operatorname{cof}(2^\beta) = 2^\beta$, and $\beta > \sup \operatorname{spec} X$. □

We now turn to Conjecture 14. Once again let $Z \subset X$ and let the new preordered set $(X', \leq')$ be defined in terms of Z as at the beginning of this section.

Theorem 24. *Conjecture* 14 *is true if* $\operatorname{cof} X \leq \infty$.

Proof. By Proposition 20, $\operatorname{cof} X' \leq \infty$; and $\infty \notin \operatorname{spec} X'$ since $\infty \notin \operatorname{spec} X$. By Corollary 12, $\operatorname{cof} X' \neq \infty$; so $\operatorname{cof} X' < \infty$, which gives the desired result by Proposition 21. □

Proposition 25. *If* $A \subset X'$ *is totally ordered we have* $\operatorname{cof} A \leq \operatorname{card} Z$, *unless* Z *is empty.*

Proof. We may replace A by any cofinal subset, and hence may suppose that A is isomorphic to the regular cardinal $\operatorname{cof} A$. If this cardinal is 0 or 1, there is nothing to prove. So we may suppose that each $a \in A$ has a successor a^* in A. Let $W = \{z \in Z \mid z \leq A\}$, and define $g: W \to A$ by taking $g(w)$ to be the first $a \in A$ (in the sense of $\leq'$) for which $w \leq a^*$. Since $W \subset Z$, it suffices to prove that g is surjective. For $a \in A$ we have $a <' a^*$ and hence, by the definition of $\leq'$, there is some $z \in Z$ with $z \leq a^*$ and $z \nleq a$; and $z \in W$. Now $g(z) = a$. For by the definition of g we have $z \leq g(z)^*$, whence $z \leq' g(z)^*$ by Proposition 21; and if $g(z) <' a$ we have $g(z)^* \leq' a$, giving $z \leq' a$, and so by Proposition 21 the contradiction $z \leq a$. □

Theorem 26. *Conjecture* 14 *is true if* $\infty = \omega$.

Proof. Under the hypotheses of Conjecture 14 we have $\operatorname{card} Z = \infty$ and $\infty \notin \operatorname{spec} X$. By Proposition 20 we have $\infty \notin \operatorname{spec} X'$. By Propositions 17 and 25, we have the hypotheses and hence the conclusion of Proposition 18, applied to X' in place of X. The proof of Theorem 19 carries over, since $\infty = \omega \notin \operatorname{spec} X'$, to show that X' has a finite cofinal subset; whence the desired result by Proposition 21. □

3. Proofs of the results of the introduction

For each category $\mathscr{A}$ it is understood that mor $\mathscr{A}$ is a set; and $\mathscr{A}$ is *small* when this set is small.

By a *regular monomorphism* in $\mathscr{A}$ we mean a map i which is the joint equalizer of all the pairs f, g with $fi = gi$. Every regular monomorphism is a monomorphism; every equalizer of a single pair is a regular monomorphism; and every regular monomorphism is such an equalizer if $\mathscr{A}$ admits cokernel-pairs. The regular monomorphisms are closed under intersection; for all this see [1]. *We suppose until further notice that $\mathscr{A}$ is complete and admits all intersections of regular monomorphisms, and that $F : \mathscr{K} \to \mathscr{A}$ is a functor.*

The positive parts of Theorems 1 and 2 follow from a series of easy lemmas:

Lemma 27. *If $T : \mathscr{L} \to \mathscr{K}$ is surjective on objects, then $F : \mathscr{K} \to \mathscr{A}$ admits a limit if FT does, and* $\lim F \to \lim FT$ *is a regular monomorphism.*

Proof. If $(h_L : N \to FTL)$ is $\lim FT$, consider, for each object $\tau = (L, k : TL \to TL', L')$ of the comma-category T/T, the equalizer $j_\tau : B_\tau \to N$ of the pair $h_{L'}$, $Fk \cdot h_L : N \to FTL'$; and let $i : M \to N$ be the intersection of all the j_τ. Then $h_L i : M \to FTL$ depends only on $K = TL$; and if we call it f_K, clearly $(f_K : M \to FK)$ is $\lim F$. □

Corollary 28. *$F : \mathscr{K} \to \mathscr{A}$ has a limit if* ob $\mathscr{K}$ *is small.*

Call a family $(h_K : N \to FK)$, indexed by the objects of $\mathscr{K}$ but not necessarily a cone over F, a *quasi-limit* of F if every cone $(g_K : A \to FK)$ factorizes as $g_K = h_K t$ for a unique t.

Lemma 29. *If $(h_K : N \to FK)$ is a quasi-limit of F, there is a regular monomorphism $i : M \to N$ such that $(h_K i : M \to FK)$ is* $\lim F$.

Proof. For each $k : K \to K'$ in $\mathscr{K}$, let j_k be the equalizer of the pair $h_{K'}$, $Fk \cdot h_K : N \to FK'$, and let i be the intersection of all the j_k. □

Lemma 30. *If $\mathscr{L}$ denotes both an ancestral set of objects of $\mathscr{K}$, and the full subcategory they determine, and if $F|\mathscr{L}$ has a limit, so does F; and* $\lim F \to \lim F|\mathscr{L}$ *is a regular monomorphism.*

Proof. For each $K \in \mathscr{K}$ choose an $r_K : L(K) \to K$ with $L(K) \in \mathscr{L}$, taking care to choose $L(K) = K$ and $r_K = 1$ if $K \in \mathscr{L}$. If $(h_L : N \to FL)$ is the limit of $F|\mathscr{L}$, extend h to all objects K by setting $h_K = Fr_K \cdot h_{L(K)} : N \to FK$. Then $(h_K : N \to FK)$ is a quasi-limit for F, and we use Lemma 29. □

Proof of the positive part of Theorem 1: Corollary 28 and Lemma 30. □

Now suppose that $\mathscr{K}$ admits *all* intersections of monomorphisms.

Lemma 31. *If $\mathscr{L}$ is a full subcategory of $\mathscr{K}$ such that for each $K \in \mathscr{K}$ there is some $m_K: K \to L(K)$ with $L(K) \in \mathscr{L}$ and with Fm_K monomorphic, and if $F|\mathscr{L}$ has a limit, so does F; and* $\lim F \to \lim F|\mathscr{L}$ *is a monomorphism.*

Proof. Choose the m_K as above so that $L(K) = K$ and $m_K = 1$ if $K \in \mathscr{L}$. Let $(h_L: N \to FL)$ be the limit of $F|\mathscr{L}$. For each $K \in \mathscr{K}$ define the monomorphism j_K by the pullback

$$\begin{array}{ccc} M_K & \xrightarrow{\nu_K} & FK \\ {\scriptstyle j_K}\downarrow & & \downarrow{\scriptstyle Fm_K} \\ N & \xrightarrow[h_{L(K)}]{} & FL(K), \end{array}$$

let $i: M \to N$ be the intersection of all the j_K, and define $f_K: M \to FK$ as $\nu_K|M$. Then (f_K) is a quasi-limit for F, and we use Lemma 29. □

Lemma 32. *Let $\mathscr{K}$ be the union of an increasing sequence of full subcategories $\mathscr{K}_n$, indexed by the natural numbers. Let each restriction $F|\mathscr{K}_n$ have a limit M_n, and let all the comparison maps $M_{n+1} \to M_n$ be monomorphisms. Then F has a limit, namely the intersection of all the maps $M_n \to M_0$.* □

Proof of the positive part of Theorem 2. To say that the set $\mathscr{L}$ of objects of $\mathscr{K}$ is ancestral in $\mathscr{K}[\Sigma^{-1}]$ is to say that for each $K \in \mathscr{K}$ there is for some $n \geq 0$ a zig–zag

$$K \xleftarrow[r_1]{} P_1 \xrightarrow[m_1]{} Q_1 \xleftarrow[r_2]{} P_2 \xrightarrow[m_2]{} Q_2 \cdots Q_{n-1} \xleftarrow[r_n]{} P_n \xrightarrow[m_n]{} Q_n$$

with $Q_n \in \mathscr{L}$ and each $m_i \in \Sigma$. Let $\mathscr{K}_n$ be the full subcategory of $\mathscr{K}$ given by those K for which a zig–zag of length n suffices. The result now follows at once from Corollary 28 and Lemmas 30, 31, and 32. □

We now turn to the converses of Theorems 1 and 2. For an infinite regular cardinal α, write $\mathbf{Set}_\alpha$ for the category of all cardinals $< \alpha$ and all maps (of sets) between them; so that $\mathbf{Set} = \mathbf{Set}_\infty$. If $\alpha \geq \infty$, $\mathbf{Set}_\alpha$ is cocomplete, the colimit of $F: \mathscr{K} \to \mathbf{Set}_\alpha$ with $\mathscr{K}$ small being as usual the set $\pi(elF)$ of connected components of the category elF of elements $(K \in \mathscr{K}, x \in FK)$ of F, which has cardinal $< \alpha$ by the regularity of α. A slightly more subtle point is:

Proposition 33. *$F: \mathscr{K} \to \mathbf{Set}_\alpha$ has no colimit if $\pi(elF)$ has cardinal $\geq \alpha$.*

Proof. Set $Q = \pi(elF)$ and let $(q_K: FK \to Q)$ be the canonical cone; this is the colimit of F in $\mathbf{Set}_\beta$ for any regular $\beta > \operatorname{card} Q$. If F has a colimit $(r_K: FK \to R)$ in $\mathbf{Set}_\alpha$, we have a comparison map $t: Q \to R$ with $r_K = tq_K$. Since card $Q \geq \alpha$, t is not monomorphic. Thus there is a map $u: Q \to \{0,1\}$ that does not factorize through t. But (uq_K) is a cone over F in $\mathbf{Set}_\alpha$, so that $uq_K = vr_K$ for some v, giving the contradiction $u = vt$. □

Proof of the converses of Theorems 1 and 2. For the converse of Theorem 1, suppose $\mathscr{X}$ has no small ancestral $\mathscr{L}$. Let the quotient of $\mathscr{X}$, when we identify each pair of maps with the same domain and the same codomain, be the preordered set X^{op}, the projection being $T: \mathscr{X}^{\mathrm{op}} \to X$; then X has no small cofinal set, so that $\operatorname{cof} X \geq \infty$. Whenever Conjecture 15 is true, as it is by Theorems 16, 19, and 23, under the various hypotheses of Theorem 1 (and more generally if $\mathscr{X}$ has an ancestral $\mathscr{L}$ with card $\mathscr{L} < \infty$), there is a regular cardinal $\alpha \geq \infty$ in Spec X, and thus a functor $S: X \to \alpha$ that is surjective on objects. $\mathbf{Set}_\alpha^{\mathrm{op}}$ is complete, and admits all intersections of monomorphisms. Let $F: \alpha \to \mathbf{Set}_\alpha$ be the functor sending $\beta \in \alpha$ to β (seen as the set of ordinals $< \beta$) and sending $\beta \to \gamma$ to the inclusion of β in γ as an initial segment. Then $\pi(el\,F) = \alpha$; so that, by Proposition 33, F has no colimit. By Lemma 27, the functor $FST: \mathscr{X}^{\mathrm{op}} \to \mathbf{Set}_\alpha$ has no colimit either.

For the converse of Theorem 2 under the same hypotheses, we observe that, if $\mathscr{X}[\Sigma^{-1}]$ has no small ancestral set, the above argument gives a functor $G: \mathscr{X}[\Sigma^{-1}] \to \mathbf{Set}_\alpha$ with no colimit. If $P: \mathscr{X} \to \mathscr{X}[\Sigma^{-1}]$ is the projection, GP has no colimit either, by Lemma 27; and GP takes the maps in Σ to isomorphisms. □

Proof of Theorem 3 and Proposition 4. We include for completeness the proof of Theorem 3, essentially as given by Freyd and Street. For the positive part, let $F: \mathscr{X} \to \mathbf{Set}$, and let M be the set of projective cones over F with vertex 1; if M is small it is the desired limit of F.

Suppose then that card $M \geq \infty$, and choose a monomorphism $\phi: \infty \to M$, writing ϕ_α for $\phi(\alpha)$. Then for each pair $\alpha \neq \beta \in \infty$ there is some object $K_{\alpha\beta}$ of $\mathscr{X}$ for which the elements $\phi_\alpha(K_{\alpha\beta})$ and $\phi_\beta(K_{\alpha\beta})$ of $FK_{\alpha\beta}$ are different. By the hypothesis of Theorem 3 there is a small set $\mathscr{N}$ of objects of $\mathscr{X}$ such for each such (α,β) there is a map $N \to K_{\alpha\beta}$ with $N \in \mathscr{N}$. Since ϕ_α and ϕ_β are cones, this implies that $\phi_\alpha(N) = \phi_\beta(N)$. In other words the composite of $\phi: \infty \to M$ with $M \to \prod_{N \in \mathscr{N}} FN$ is monomorphic; which contradicts the smallness of $\mathscr{N}$ and the FN.

We have the converse whenever Conjecture 14 is true, as it is, by Theorems 24 and 26, under the hypotheses in Theorem 3 or Proposition 4. For if X^{op} is the preordered quotient of $\mathscr{X}$ used above in proving the converse of Theorem 1, and if $\mathscr{N} \subset \operatorname{ob} \mathscr{X}$ with card $\mathscr{N} = \infty$ and no small $\mathscr{L}$ ancestral for $\mathscr{N}$, Conjecture 14 gives $\infty \in \operatorname{spec} X$. We thus have a functor $\mathscr{X} \to \infty^{\mathrm{op}}$ that is surjective on objects, and it only remains by Lemma 27 to produce a functor $F: \infty^{\mathrm{op}} \to \mathbf{Set}$ with no limit. Such a functor is given by the contravariant power-set functor taking α to 2^α. □

References

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