Spectra of graphs attached to the space of melodies

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Abstract
For any integer \( n \) greater than or equal to two, two intimately related graphs on the vertices of the \( n \)-dimensional cube are introduced. All of their eigenvalues are found to be integers, and the largest and the smallest ones are also determined. As a byproduct, certain kind of generating function for their spectra is introduced and shown to be quite effective to compute the eigenvalues of some broader class of adjacency matrices of graphs.

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0. Introduction

In this paper we introduce two infinite families of graphs \( \{G_n\}, \{G'_n\} \) and investigate their spectra. These graphs arise naturally in our mathematical study of melodies [3]. We regard a melody as an element of \([0, k]^n = \{0, 1, \ldots, k\}^n\) for suitable integers \( k, n \), and attach a graph with vertices in \( \mathbb{Z}_2 \) to it by a simple rule. The rule is originally introduced in [2], where the authors study among other things the distribution of slopes which are obtained by the least-square method from associated graphs of various types of songs. Based on their study we introduce in [3] two distances on \([0, k]^n\), the set of melodies, and consider how two melodies resemble each other. When we try to find the pairs of melodies which have distance equal to one, we are naturally led to the respective unit spheres around the origin with respect to the two distances and hence to the definition of the graphs \( G_n, G'_n \) on \([0, 1]^n\). Thus it comes to one of the central problems to investigate the spectra of their incidence matrices \( A_n, A'_n \), which are the objects of our main concern in this paper. Thanks to the regularity of the graphs, which are acted upon by \( F_2^n \) transitively, our task is reduced to understanding the shape of the set of certain charactersums. We need, however, to know which character sum contributes to which eigenvalue. For this purpose we introduce certain types of generating functions which depict the overall structure of the spectra, and help us to discover various interesting patterns of the distributions of the eigenvalues. As a result we have found the largest and the smallest eigenvalues, and determined completely when \( A_n, A'_n \) are invertible. The latter result has an application to the theory of Hodge cycles on abelian varieties (see Remark 4.1). Furthermore the results in this paper strongly suggest that our argument can be generalized to one applicable to much broader classes of regular graphs (see Remark 5.1). Such generalizations, however, will be described in another article.

The plan of this paper is as follows. In Section 1 we introduce various objects of our main concern in this paper. They are defined through the so-called discrete Fréchet distance, whose definition is recalled there too. Section 2 is devoted to the computation of the sizes of the unit spheres with respect to the two distances defined in the previous section. Several subsets of the unit spheres are introduced too and they will turn out to play an important role throughout the paper. Section 3 describes several methods of computation of the eigenvalues of the incidence matrices. We see the significance of certain character sums and related generating functions, and in particular we derive a formula which enables one to compute the whole spectra of the incidence matrices of size \( 2^n \) quite effectively. In Section 4 we focus our attention on the invertibility of
the incidence matrices, and determine completely the set of $n$ such that $A_n$ is invertible. This result can be used to prove the Hodge conjecture for certain abelian varieties of CM-type. In Section 5 we determine the largest and the smallest eigenvalue of $A_n$, $A'_n$ as well as the second largest eigenvalue of $A'_n$. Actually we determine which character sums contribute to these eigenvalues and hence the multiplicities of those are completely determined too.

1. Main objects

Let $B^m_0$ denote the set $\{0, 1\}^n \subset \mathbb{R}^n$. For any pair of $n$-tuples $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in B^m_0$ of points in $B_2$, let $d_f(p, q)$ denote the discrete Fréchet distance between $p$ and $q$. We recall the definition for the convenience of the reader. (See [4] for details.) A coupling $L$ between $p$ and $q$ is a sequence

$$(p_{a_1}, q_{b_1}), (p_{a_2}, q_{b_2}), \ldots, (p_{a_m}, q_{b_m})$$

of distinct pairs from $\{p_1, \ldots, p_n\} \times \{q_1, \ldots, q_n\}$ such that $a_1 = b_1 = 1$, $a_m = b_m = n$, and for any $i = 1, \ldots, m - 1$, we have $a_i+1 = a_i$ or $a_i+1 = a_i + 1$, and $b_i+1 = b_i$ or $b_i+1 = b_i + 1$. The length $\|L\|$ of the coupling $L$ is defined by

$$\|L\| = \max_{i=1, \ldots, m} \{d(a_i, q_{b_i})\},$$

where $d(\ast, \ast)$ denotes the Euclidean distance on $\mathbb{R}^2$. The discrete Fréchet distance $d_f(p, q)$ between $p, q \in B^m_0$ is defined to be

$$d_f(p, q) = \min[\{\|L\| \mid L \text{ is a coupling between } p \text{ and } q\}].$$

Intuitively this can be defined as follows. A man is walking a dog on a leash. The man can move on the points in the sequence $p$, and the dog in the sequence $q$, but backtracking is not allowed. The discrete Fréchet distance $d_f(p, q)$ is the length of the shortest leash that is sufficient for traversing both sequences. By using the discrete Fréchet distance, we introduce two distances $d_n, d'_n$ on $B_n$, as follows. For any $a = (a_i) \in B_n$, let $p : B_n \rightarrow B^2_0$ and $p' : B_n \rightarrow B'^2_0$ be defined by

$$p(a) = ((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n), (a_n, a_1)),$$

$$p'(a) = ((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)).$$

For any $a, b \in B_n$, we put

$$d_n(a, b) = d_f(0, p(a + b)),$$

$$d'_n(a, b) = d_f(0, p'(a + b)),$$

where $0$ denotes the origin of $B_n$ (resp. $B_{n-1}$), and the addition “$a + b$” is done by identifying $B_n$ with $F^n_2$. By definition, both of $d_n, d'_n$ are equivalent under the natural action of $F^n_2$ on $B_n$.

**Remark 1.1.** If one defines a distance $d$ by $d(a, b) = d_f(p_n(a), p_n(b))$, then it is not equivariant. For example,

$$d((0, 0, 0, 1), (0, 1, 1, 0)) = 1,$$

because one can find a good coupling, but

$$d((0, 0, 0, 1) + (0, 0, 0, 1), (0, 0, 0, 1) + (0, 1, 1, 0)) = d((0, 0, 0, 0), (0, 1, 1, 0))$$

$$= \sqrt{2} \neq 1,$$

because the man must stay at the origin and the dog must go to $(1, 1)$ sometime.

Now we can introduce two graphs and two matrices of our main concern in this paper. For any $n \geq 2$, let $G_n$ denote the graph on the set of vertices $B_n$ with edge set

$$E_n = \{(a, b) \in B_n \times B_n \mid d_n(a, b) = 1\}.$$

Similarly let $G'_n$ denote the graph on the set of vertices $B_n$ with edge set

$$E'_n = \{(a, b) \in B_n \times B_n \mid d'_n(a, b) = 1\}.$$

Regarding the elements of $B_n$ as the 2-adic expansions of integers between zero and $2^n - 1$, we can introduce a natural total order on $B_n$. With this order understood, we denote by $A_n$ (resp. $A'_n$) the adjacency matrix of the graph $G_n$ (resp. $G'_n$). Let $B_0(0)$ (resp. $B'_0(0)$) denote the unit sphere around the origin with respect to the distance $d_n$ (resp. $d'_n$). When we refer to these spheres, we call $B_n(0)$ the unit sphere and $B'_n(0)$ the enlarged unit sphere. The following proposition actually proves that $B_n(0) \subset B'_n(0)$ and gives a characterization of their elements:

**Proposition 1.1.** (1) For any $a \in B_n - \{0\}$, we have $a \in B_n(0)$ if and only if $(a_i, a_{i+1}) \neq (1, 1)$ for any $i = 1, 2, \ldots, n$, where we set $a_{n+1} = a_1$.

(2) For any $a \in B_n - \{0\}$, we have $a \in B'_n(0)$ if and only if $(a_i, a_{i+1}) \neq (1, 1)$ for any $i = 1, 2, \ldots, n - 1$. 
Example 1.1. For small values of $n$, the respective unit spheres are specified as follows: (Here we represent a vector by the juxtaposition of its coordinates.)

\[
\begin{align*}
B_2(0) &= \{10, 01\},
B_3(0) &= \{100, 010, 001\}, \\
B_4(0) &= \{1000, 0100, 0010, 0001, 1010, 0101\}, \\
B_5(0) &= \{10000, 01000, 00100, 00010, 00001, 10100, 01100, 00101, 10010, 01001\}, \\
B_6(0) &= \{10, 01\}, \\
B_7(0) &= \{100, 010, 001\}, \\
B_8(0) &= \{1000, 0100, 0010, 0001, 1010, 0101, 1001\}, \\
B_9(0) &= \{10000, 01000, 00100, 00010, 00001, 10100, 01100, 00101, 10010, 01001, 10001, 10101\}.
\end{align*}
\]

Remark 2.2. By Proposition 1.1, both unit spheres are stable under the inversion $\iota : (a_1, \ldots, a_n) \mapsto (a_n, \ldots, a_1)$ of the order of coordinates. Furthermore the unit sphere $B_n(0)$ is stable under the cyclic transformation $(a_1, a_2, \ldots, a_n-1, a_n) \mapsto (a_2, a_3, \ldots, a_n, a_1)$.

2. Sizes of the unit spheres

In this section we compute the numbers of the elements in the unit spheres.

2.1. The enlarged unit sphere $B_n'(0)$

For any $a \in B_n$, let $|a| = \sum_{k=1}^n a_k$, and let

\[
\begin{align*}
B_n'(0)^e &= \{ a \in B_n'(0) : |a| \equiv 0 \text{ mod } 2 \}, \\
B_n'(0)^o &= \{ a \in B_n'(0) : |a| \equiv 1 \text{ mod } 2 \}.
\end{align*}
\]

Furthermore we put

\[
\begin{align*}
b_n' &= \#B_n'(0), \\
b_n'^e &= \#B_n'(0)^e, \\
b_n'^o &= \#B_n'(0)^o.
\end{align*}
\]

Example 2.1. Example 1.1 together with a bit of computation gives us the table of values of these terms. (The terms $b_n', b_n'^e, b_n'^o$ will be defined later.)

We will show that the sequences $\{b_n'\}$, $\{b_n'^e\}$ and $\{b_n'^o\}$ satisfy simple recurrence relations. For this purpose, we introduce certain maps. For any $n \geq 3$, let $jux_0, jux_{01} : B_{n-1} \rightarrow B_n$ denote the maps defined by

\[
\begin{align*}
jux_0(a_1, \ldots, a_{n-1}) &= (a_1, \ldots, a_{n-1}, 0), \\
jux_{01}(a_1, \ldots, a_{n-1}) &= (0, a_1, \ldots, a_{n-1}).
\end{align*}
\]

Furthermore, for any $n \geq 4$, let $jux_{01}, jux_{10} : B_{n-2} \rightarrow B_n$ denote the maps defined by

\[
\begin{align*}
jux_{01}(a_1, \ldots, a_{n-2}) &= (a_1, \ldots, a_{n-2}, 0, 1), \\
jux_{10}(a_1, \ldots, a_{n-2}) &= (1, 0, a_1, \ldots, a_{n-2}).
\end{align*}
\]

Proposition 2.1. For any $n \geq 4$, the equality $b_n' = b_{n-1}' + b_{n-2}' + 1$ holds.

Proof. One sees that $jux_0$ induces a bijection from $B_{n-1}'(0)$ onto the subset $B_0' = \{ a \in B_n'(0) : a_n = 0 \}$. Furthermore, since an element $a \in B_n'(0)$ with $a_n = 1$ must satisfy $a_{n-1} = 0$ by Proposition 1.1, $jux_{01}$ induces a bijection from $B_{n-2}'(0) \sqcup \{0\}$ onto the subset $B_1' = \{ a \in B_n'(0) : a_n = 1 \}$. Noting that $B_n'(0) = B_0' \sqcup B_1'$, we have the equality $b_n' = b_{n-1}' + b_{n-2}' + 1$. □

For later use, we record the disjoint sum decomposition established in this proposition as well as its inversion:

Proposition 2.2. For any $n \geq 4$, we have

\[
\begin{align*}
B_n'(0) &= jux_0(B_{n-1}'(0)) \sqcup jux_{01}(B_{n-2}'(0) \sqcup \{0\}), \\
B_n'(0) &= jux_0(B_{n-1}'(0)) \sqcup jux_{10}(B_{n-2}'(0) \sqcup \{0\}).
\end{align*}
\]
Table 2.1
Sizes of the unit spheres and related numbers.

<table>
<thead>
<tr>
<th>n</th>
<th>b_n^e</th>
<th>b_n^o</th>
<th>b_n''</th>
<th>b_n'</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
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<td>5</td>
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<td>6</td>
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<tr>
<td>12</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Proof. The equality (1) is just proved. We obtain (2) from (1) by applying the inversion map \( \iota \) to both its sides. \( \Box \)

By Proposition 2.3 together with Table 2.1, one can show the following corollary by induction on \( n \).

Corollary 2.1. \( b'_n \equiv 0 \mod 2 \) if and only if \( n \equiv 0, 2 \mod 3 \).

By using the maps \( jux_0 \) and \( jux_00 \), we can also prove recurrence relations for \( b'_n \), \( b''_n \).

Proposition 2.3. For any \( n \geq 4 \), the equalities \( b'_n = b'_{n-1} + b''_{n-2} \) and \( b''_n = b'_{n-1} + b''_{n-2} + 1 \) hold.

Proof. For \( \epsilon \in \{0, 1\} \), let \( B'_n(0)^\epsilon = \{ a \in B'_n(0); a_n = \epsilon, |a| \equiv 0 \mod 2 \} \) and \( B''_n(0)^\epsilon = \{ a \in B''_n(0); a_n = \epsilon, |a| \equiv 1 \mod 2 \} \).

Then the map \( jux_0 \) induces a bijection from \( B'_{n-1}(0)^\epsilon \) onto \( B'_n(0)^\epsilon \), and \( jux_{01} \) induces a bijection from \( B'_{n-2}(0)^\epsilon \) onto \( B''_n(0)^\epsilon \). Since \( B''_n(0)^\epsilon = B''_n(0)^1 \cup B''_n(0)^0 \), we see that \( b''_n = b'_{n-1} + b''_{n-2} \). Similarly, \( jux_0 \) induces a bijection from \( B'_{n-1}(0)^\epsilon \) onto \( B'_n(0)^\epsilon \), and \( jux_{01} \) induces a bijection from \( B'_{n-2}(0)^\epsilon \cup \{1\} \) onto \( B''_n(0)^\epsilon \). Since \( B'_n(0)^2 = B'_n(0)^1 \cup B'_n(0)^0 \), we see that \( b'_n = b'_{n-1} + b''_{n-2} + 1 \). \( \Box \)

These relations will enable us to find recurrence relations for \( b_n \) in the next subsection.

2.2. The unit sphere \( B_n(0) \)

We compute the number of elements in the unit sphere. We introduce some notation similar to the above ones:

\[
B_n(0)^e = \{ a \in B_n(0); |a| \equiv 0 \mod 2 \},
\]

\[
B_n(0)^o = \{ a \in B_n(0); |a| \equiv 1 \mod 2 \}.
\]

Furthermore we put

\[
b_n = \#B_n(0),
\]

\[
b_n^e = \#B_n(0)^e,
\]

\[
b_n^o = \#B_n(0)^o.
\]

Proposition 2.4. For any \( n \geq 5 \), the equalities \( b_n^e = b_{n-1}^e + b_{n-3}^o \) and \( b_n^o = b_{n-1}^o + b_{n-3}^e + 1 \) hold.

Proof. As in the proof of Proposition 2.3, we put \( B_n(0)^e = \{ a \in B_n(0); a_n = \epsilon, |a| \equiv 0 \mod 2 \} \) and \( B_n(0)^o = \{ a \in B_n(0); a_n = \epsilon, |a| \equiv 1 \mod 2 \} \) for \( \epsilon \in \{0, 1\} \). The map \( jux_0 \) induces a bijection from \( B'_{n-1}(0)^\epsilon \) onto \( B'_n(0)^\epsilon \). As for the elements in \( B_n(0)^\epsilon \), note that if \( a \in B_n(0)^\epsilon \), then \( a_1 \) as well as \( a_{n-1} \) must be equal to zero. Therefore if we define \( jux_{001} : B_{n-3} \rightarrow B_n \) by \( jux_{001}(a_1, \ldots, a_{n-3}) = (0, a_1, \ldots, a_{n-3}, 0, 1) \), then it induces a bijection from \( B'_{n-3}(0)^\epsilon \) onto \( B'_n(0)^\epsilon \). Since \( B'_n(0)^e = B'_n(0)^1 \cup B'_n(0)^0 \), we see that \( b'_n = b'_{n-1} + b''_{n-3} \). By a similar argument, we see that \( b''_n = b'_{n-1} + b''_{n-3} + 1 \). \( \Box \)

Proposition 2.5. For any \( n \geq 4 \), the equality \( b_n = b_{n-1} + b_{n-2} + 1 \) holds.

Proof. When \( n = 4, 5 \) or \( 6 \), the equality holds by Table 2.1. When \( n \geq 7 \), by adding the equalities in Proposition 2.4 we see that

\[
(*) \ b_n = b'_{n-1} + b''_{n-3} + 1.
\]

Therefore it follows from Proposition 2.1 that

\[
b_n = b'_{n-1} + b''_{n-3} + 1
\]

\[
= (b'_{n-2} + b''_{n-3} + 1) + (b'_{n-4} + b''_{n-5} + 1) + 1
\]

\[
= (b'_{n-2} + b''_{n-4} + 1) + (b'_{n-3} + b''_{n-5} + 1) + 1
\]

\[
= b'_{n-1} + b''_{n-2} + 1,
\]

the last equality being the consequence of \((*)\) again. This completes the proof of Proposition 2.5. \( \Box \)
In a similar way we obtain the following:

**Corollary 2.2.** Let \( d_n = b^n_n - b^0_n \). Then for any \( n \geq 4 \), the equality \( d_n = d_{n-1} - d_{n-2} - 1 \) holds. In particular the value of \( d_n \) depends only on \( n \bmod 6 \).

### 3. Eigenvalues and spectral zetas

In this section we establish a simple method to compute the eigenvalues of the adjacency matrices \( A_n, A'_n \). Furthermore we introduce a generating function, called spectral zeta function of a matrix, and investigate how it enables us to find the overall structure of the eigenvalues.

An important fact we should keep in mind is that our graphs \( G_n \) and \( G'_n \) are acted upon transitively by the group \( F_2^n \). In particular, if we denote by \( d \) (resp. \( d' \)) the distance function of \( G_n \) (resp. \( G'_n \)), then we have

\[
d(a, b) = d(0, a + b),
\]

\[
d'(a, b) = d'(0, a + b),
\]

for any \( a, b \in F_2^n \). This enables us to find the eigenvalues of the adjacency matrices \( A_n, A'_n \) with the help of the notion of group matrix and its determinant \[ 1 \]. Let \( x_0, x_00, \ldots, x_1 \) be variables indexed by the elements of \( F_2^n \). The group matrix \( X_n \) is defined to be the matrix whose \((a, b)\)th element is \( x_{a+b} \). Therefore if we put \( x_a = 1 \) (resp. 0) for \( a \in B_n(0) \) (resp. \( a \notin B_n(0) \)), then \( X_n \) specializes to \( A_n \), and \( A'_n \) is obtained from \( X_n \) by a similar specialization. On the other hand, if we denote by \( X(F_2^n) \) the group of characters of \( F_2^n \), then it is known that

\[
\det X_n = \prod_{x \in X(F_2^n)} \left( \sum_{a \in F_2^n} \chi(a)x_a \right).
\]

In order to specify the characters, we use the following notation. For any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in F_2^n \), let

\[
\chi_{\alpha}(a) = (-1)^{a \cdot \alpha}, \quad (a \in F_2^n)
\]

where \( a \cdot \alpha = \sum_{i=1}^{n} \alpha_i a_i \). Then one knows that \( X(F_2^n) = \{ \chi_{\alpha}; \alpha \in F_2^n \} \). Therefore the formula above can be rewritten as

\[
\det X_n = \prod_{\alpha \in F_2^n} \left( \sum_{a \in F_2^n} \chi_{\alpha}(a)x_a \right).
\]

Thus we have

\[
\det (\lambda E_{2n} + A_n) = \prod_{\alpha \in F_2^n} \left( \sum_{a \in F_2^n} \chi_{\alpha}(a)e_a \right), \quad (5)
\]

where

\[
e_a = \begin{cases} 1, & \text{if } a \in B_n(0), \\ \lambda, & \text{if } a = 0, \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
\det (\lambda E_{2n} + A'_n) = \prod_{\alpha \in F_2^n} \left( \sum_{a \in F_2^n} \chi_{\alpha}(a)e'_a \right), \quad (6)
\]

where

\[
e'_a = \begin{cases} 1, & \text{if } a \in B'_n(0), \\ \lambda, & \text{if } a = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

**Example 3.1.** The adjacency matrix \( A_2 \) is given by

\[
A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.
\]
It follows from formula (5) that

$$\det(\lambda E_4 + A_2) = \prod_{a \in F_2^0} \left( \sum_{a' \in F_2^0} \chi_{\alpha}(a)e_a \right)$$

$$= (\chi(00)_{00} + \chi(01)_{00} + \chi(10)_{10} + \chi(11)_{11})$$

$$\times (\chi(00)_{00} + \chi(01)_{01} + \chi(10)_{10} + \chi(11)_{11})$$

$$\times (\chi(00)_{00} + \chi(10)_{01} + \chi(10)_{10} + \chi(10)_{11})$$

$$\times (\chi(10)_{00} + \chi(11)_{00} + \chi(11)_{10} + \chi(11)_{11})$$

$$= (\lambda + 1 + 1 + 0)(\lambda - 1 + 1 - 0)(\lambda + 1 - 1 - 0)(\lambda - 1 - 1 + 0)$$

$$= (\lambda + 2)\lambda^2(\lambda - 2).$$

The eigenvalues of $A_n$ and $A'_n$ for small values of $n$ are tabulated below. The expression $(a^n_1 \cdots a^n_k)$ means that $a_i$ is an eigenvalue that occurs with multiplicity $n_i$ for $1 \leq i \leq k$.

Next we introduce a certain type of characteristic function and see that it enables one to compute the eigenvalues more efficiently.

**Definition 3.1.** For any finite subset $S \in \mathbb{Z}^n$, let

$$f_S(x_1, \ldots, x_n) = \sum_{(a_1, \ldots, a_n) \in S} x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

and call it the generating function attached to $S$.

Regarding the set $\{0, 1\}$ as a subset of $\mathbb{Z}$, we can attach the generating function $f_{B_n(0)}, f'_{B_n(0)}$ to the unit spheres $B_n(0), B'_n(0)$. Here are some examples:

**Example 3.2.**

$$f_{B_2(0)} = x_1 + x_2$$

$$f_{B_3(0)} = x_1 + x_2 + x_3$$

$$f_{B_4(0)} = x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4$$

$$f_{B_5(0)} = x_1 + x_2 + x_3 + x_4 + x_5 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2$$

$$f_{B_6(0)} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_6 + x_5x_1 + x_6x_2 + x_1x_3x_5 + x_2x_4x_6.$$

Furthermore we introduce a counting function of eigenvalues of a matrix. Let $A$ be a square matrix all of whose eigenvalues are integers, and let $V_A(\lambda)$ denote the eigenspace of $A$ for an eigenvalue $\lambda$. We define the spectral zeta function $\zeta_A(q)$ for $A$ by the following rule:

$$\zeta_A(q) = \sum_{\lambda \in \mathbb{Z}} \dim V_A(\lambda)q^\lambda \in \mathbb{Z}[q, q^{-1}].$$

Recall that all of the eigenvalues of $A_n, A'_n$ are integers by (5) and (6), hence the spectral zeta functions are well defined for them. We show that the generating functions $f_{B_n(0)}, f'_{B_n(0)}$ and the spectral zeta functions $\zeta_A(\lambda), \zeta_A'(\lambda)$ are related in an intimate way:

**Proposition 3.1.** (1) $\zeta_{A_n}(q) = \sum_{(e_1, \ldots, e_n) \in \{1, -1\}^n} q^{f_{B_n(0)}(e_1, \ldots, e_n)}.$

(2) $\zeta_{A'_n}(q) = \sum_{(e_1, \ldots, e_n) \in \{1, -1\}^n} q^{f'_{B_n(0)}(e_1, \ldots, e_n)}.$

**Proof.** (1) For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in F_2^n$, we have

$$\sum_{a \in F_2^n} \chi_{\alpha}(a)e_a = \lambda + \sum_{a \in B_n(0)} (-1)^{\alpha, a}$$

$$= \lambda + \sum_{a \in B_n(0)} (-1)^{\alpha_1}, \ldots, (-1)^{\alpha_n}$$

Therefore the eigenvalue corresponding to $\alpha$ coincides with the respective value $f_{B_n(0)}((-1)^{\alpha_1}, \ldots, (-1)^{\alpha_n})$, and hence assertion (1) follows. Assertion (2) can be proved in the same way. □
Table 3.1
Eigenvalues of $A_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Eigenvalues of $A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^10^2(-2)^1$</td>
</tr>
<tr>
<td>3</td>
<td>$3^11^1(-1)^1(-3)^1$</td>
</tr>
<tr>
<td>4</td>
<td>$6^12^2(-2)^2$</td>
</tr>
<tr>
<td>5</td>
<td>$10^24^20^4(-2)^2(10)^5$</td>
</tr>
<tr>
<td>6</td>
<td>$17^29^552^7(-1)^{18}(-3)^{18}(-5)^6$</td>
</tr>
<tr>
<td>7</td>
<td>$28^112^88^44^40^1(-4)^{42}(-8)^7$</td>
</tr>
</tbody>
</table>

Table 3.2
Eigenvalues of $A'_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Eigenvalues of $A'_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^10^2(-2)^1$</td>
</tr>
<tr>
<td>3</td>
<td>$4^12^0(-1)^3$</td>
</tr>
<tr>
<td>4</td>
<td>$7^32^1(-1)^3(-3)^3$</td>
</tr>
<tr>
<td>5</td>
<td>$12^26^24^20^5(-2)^{10}(-4)^5$</td>
</tr>
<tr>
<td>6</td>
<td>$20^110^25^24^22^08(-2)^{18}(-4)^6(-6)^6$</td>
</tr>
<tr>
<td>7</td>
<td>$33^117^215^113^211^29^27^31^2(-1)^{27}(-3)^{28}(-5)^{14}(-9)^7$</td>
</tr>
</tbody>
</table>

Table 3.3
Spectral zeta for $A_n$ and $A'_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Spectral zeta function for $A_n$</th>
<th>Its irreducible decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$q^{-2} + 2 + q^2$</td>
<td>$(1 + q)^2 / q^2$</td>
</tr>
<tr>
<td>3</td>
<td>$q^{-3} + 3q^{-1} + 3q + q^3$</td>
<td>$(1 + q)^3 / q^3$</td>
</tr>
<tr>
<td>4</td>
<td>$9q^{-7} + 6q^{-1} + q^7$</td>
<td>$(3 + q^2)^3 / q^7$</td>
</tr>
<tr>
<td>5</td>
<td>$5q^{-10} + 10q^{-1} + 6 + 5q^2 + 5q^4 + q^{-10}$</td>
<td>$(1 + q)^5 f_5(q) / q^4$</td>
</tr>
<tr>
<td>6</td>
<td>$6q^{-13} + 18q^{-1} + 18q^{-1} + 7q^2 + 14q^{-3} + q^{-17}$</td>
<td>$(1 + q)^6 f_6(q) / q^3$</td>
</tr>
<tr>
<td>7</td>
<td>$7q^{-16} + 42q^{-4} + 50 + 14q^{-8} + q^{-19}$</td>
<td>$(1 + q)^7 f_7(q) / q^2$</td>
</tr>
<tr>
<td>8</td>
<td>$8q^{-19} + 40q^{-12} + 40q^{-4} + q^{-12}$</td>
<td>$(1 + q)^8 f_8(q) / q^2$</td>
</tr>
<tr>
<td>9</td>
<td>$9q^{-22} + 54q^{-9} + 54q^{-7} + q^{-17}$</td>
<td>$(1 + q)^9 f_9(q) / q^3$</td>
</tr>
<tr>
<td>10</td>
<td>$10q^{-25} + 70q^{-14} + 70q^{-10} + q^{-17}$</td>
<td>$f_{10}(q) / q^{10}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>Spectral zeta function for $A'_n$</th>
<th>Its irreducible decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$q^{-2} + 2 + q^2$</td>
<td>$(1 + q)^2 / q^2$</td>
</tr>
<tr>
<td>3</td>
<td>$3q^{-2} + 3 + q^{-1} + q^4$</td>
<td>$(1 + q^3)(3 + q^4) / q^4$</td>
</tr>
<tr>
<td>4</td>
<td>$4q^{-3} + 5q^{-2} + 4q + 2q^3$</td>
<td>$(1 + q^4)g_4(q) / q^3$</td>
</tr>
<tr>
<td>5</td>
<td>$5q^{-4} + 10q^{-2} + 2q^4 + 2q^2 + q^{-12}$</td>
<td>$(1 + q^5)g_5(q) / q^4$</td>
</tr>
<tr>
<td>6</td>
<td>$6q^{-6} + 6q^{-4} + 18q^{-2} + q^{-20}$</td>
<td>$(1 + q^6)g_6(q) / q^6$</td>
</tr>
<tr>
<td>7</td>
<td>$7q^{-7} + 14q^{-5} + 28q^{-3} + q^{-11}$</td>
<td>$(1 + q^7)g_7(q) / q^5$</td>
</tr>
<tr>
<td>8</td>
<td>$8q^{-11} + 16q^{-8} + 16q^{-6} + q^{-14}$</td>
<td>$(1 + q^8)g_8(q) / q^3$</td>
</tr>
<tr>
<td>9</td>
<td>$9q^{-14} + 18q^{-12} + 18q^{-10} + q^{-17}$</td>
<td>$(1 + q^9)g_9(q) / q^4$</td>
</tr>
<tr>
<td>10</td>
<td>$10q^{-15} + 20q^{-13} + 10q^{-17} + q^{-19}$</td>
<td>$(1 + q^{10})g_{10}(q) / q^5$</td>
</tr>
</tbody>
</table>

Example 3.3. Tables 3.1–3.3 illustrate the spectral zeta functions for small values of $n$. (Here $f_5, \ldots, f_{10}$ and $g_4, \ldots, g_{10}$ represent some irreducible polynomials in $q$.) We observe that the spectral zeta functions are divisible by $1 + q^2$ quite often. More precisely we have the following:

**Proposition 3.2.** (1) For any integer $n \geq 2$, the spectral zeta function $\zeta_{A_n}(q)$ is always divisible by $1 + q^2$.
(2) For any integer $n \geq 2$, the spectral zeta function $\zeta_{A'_n}(q)$ is divisible by $1 + q^2$ if and only if $n \equiv 0, 2 \mod 3$.

We will prove this as a consequence of a more general result. For any family $\{A_a\}_{a \in \mathbb{F}_2^2}$ of integers, let $f(\{A_a\}, \mathbf{x}) = \sum_{a \in \mathbb{F}_2^2} A_a x_1^{a_1} \cdots x_n^{a_n}$, and let $\zeta_{\{A_a\}}(q) = \sum_{e_1, \ldots, e_n \in \{1, -1\}^n} q^{(\{A_a\}, (e_1, \ldots, e_n))}$. For example, when $A_n$ is the characteristic function of $B_n(0)$ in $\mathbb{F}_2^2$, we have $f(\{A_n\}, \mathbf{x}) = f_{2n}(0)$ and $\zeta_{\{A_n\}}(q) = \zeta_{A_n}(q)$. We can show the following:

**Proposition 3.3.** $\zeta_{\{A_a\}}(q)$ is divisible by $1 + q^2$ if and only if $\sum_{a} A_a \mathbf{a} \neq \mathbf{0}$ in $\mathbb{F}_2^2$.

**Proof.** For any $(e_1, \ldots, e_n) \in \{1, -1\}^n$, let $e_i = (-1)^{b_i} b_i$, $1 \leq i \leq n$, with $b_i \in \{0, 1\}$. Note that

$$
eq (-1)^{\sum_{i \leq n} a_i b_i} = \chi_b(\mathbf{a}).$$
where \( b = (b_1, \ldots, b_n) \). It follows that
\[
q^{e_1 + \cdots + e_n} = q^{\nu_b(a)}.
\]

Therefore we have
\[
q^{f((A_1, (1, \ldots, 1)))} = q^{\sum a(1 - \nu_b(a))} = \prod_a q^{a(1 - \nu_b(a))} = \prod_{a, \nu_b(a) = -1} q^{2a}.
\]

Setting \( q = \sqrt{-1} \), we see that
\[
(\sqrt{-1})^{f((A_1, (1, \ldots, 1)))} = \prod_{a, \nu_b(a) = -1} (\sqrt{-1})^{2a} = \prod_{a, \nu_b(a) = -1} (-1)^{a} = \prod_a (\chi_b(a))^a = \chi_b \left( \sum_a A_a a \right).
\]

For ease of description, we call the leftmost side \( r(e_1, \ldots, e_n) \), which belongs to \( \{ \pm 1 \} \) by the above computation. Then we have
\[
\sum_{(e_1, \ldots, e_n) \in \{1, -1\}^n} (\sqrt{-1})^{f((A_1, (1, \ldots, 1)))} = (\sqrt{-1})^{f((A_1, (1, \ldots, 1)))} \sum_{(e_1, \ldots, e_n) \in \{1, -1\}^n} r(e_1, \ldots, e_n) = (\sqrt{-1})^{f((A_1, (1, \ldots, 1)))} \sum_{b \in F_2^n} \chi_b \left( \sum_a A_a a \right).
\]

By the orthogonality relation of characters, the last sum \( = 0 \) if and only if \( \sum_a A_a a \neq 0 \) in \( F_2^n \). This completes the proof. \( \Box \)

Now we proceed to the proof of Proposition 3.2. First we deal with assertion (2) which concerns the matrix \( A_n' \). Identifying \( B_n \) with \( F_2^n \), we put
\[
\sigma_n' = \sum_{(a_1, \ldots, a_n) \in \mathbb{F}_2^n} a_1, \ldots, a_n \in \mathbb{F}_2^n.
\]

By Proposition 3.3, in order to prove that \( \zeta_n'(q) \) is always divisible by \( 1 + q^2 \), we need to show that \( \sigma_n' \neq 0 \). We can prove, however, more precise results as follows:

**Proposition 3.4.** For any \( n \geq 2 \), the sum \( \sigma_n' \) of the elements in \( B_n'(0) \) is given by
\[
\sigma_n' = \begin{cases} (010)^m, & \text{if } n = 3m, \\ (100)^m1, & \text{if } n = 3m + 1, \\ (110)^m11, & \text{if } n = 3m + 2. \\ \end{cases}
\]

In particular \( \sigma_n' \neq 0 \), and hence \( \zeta_n'(q) \) is always divisible by \( 1 + q^2 \) for any \( n \geq 2 \).

**Proof.** We employ the decomposition (1) and prove our assertion by induction on \( n \). As is seen in Example 1.1, the assertion holds for \( n = 2, 3 \). So we assume that \( n \geq 4 \) and that it holds for integers smaller than \( n \). We divide our argument into three cases according to the value of \( n \) modulo 3. In the proof we use Corollary 2.1 several times which specifies the parity of \( b_n' = \# B_n'(0) \).

**Case 1.** \( n = 3m \): In this case, it follows from the decomposition (1) that
\[
\sigma_n' = (\sigma_{3m-1}', 0) + (\sigma_{3m-2}', b_{3m-2}') + (0, \ldots, 0, 1)
= (\sigma_{3m-1}', 0) + (\sigma_{3m-2}', 0, 1) + (0, \ldots, 0, 1)
= (\sigma_{3m-1}', 0) + (\sigma_{3m-2}', 0, 0)
\]
For any Proposition 3.2 that 

Proposition 3.5.

Zero eigenvalue of $M$ is invariant under the action. Hence if we denote by $b$ then the number of elements of the set $B_1 = \{ a \in B_n(0) ; a_n = 1 \}$. Hence it is equal to $b_{n-3} + 1$. It follows from Corollary 2.1 that $b_{n-3} + 1 \equiv 1 \mod 2$ if and only if $n - 3 \equiv 0, 2 \mod 3$, namely $n \equiv 0, 2 \mod 3$. □

Since $B_n(0)$ is stable under the natural action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the coordinates, the generating function $f_{B_n(0)}$ is invariant under the action. Hence if we denote by $c_n$ the number of monomials in $F_{B_n(0)}$ which contains $x_n$, then we have

$$\sum_{a} A_a a = c_n(1, \ldots, 1),$$

which is $\neq 0$ in $F_2^n$ if and only if $n \equiv 0, 2 \mod 3$ by Proposition 3.5. Thus Proposition 3.2(1) follows from Proposition 3.3.

4. Zero eigenvalue of $A_n$

In this section, we determine when the set of the eigenvalues of $A_n$ contains 0. For any square matrix $M$, we denote the set of eigenvalues of $M$ by $\text{Spec } M$. In view of the formula (5) and (6), if we set

$$S(\alpha) = \sum_{a \in B_n(0)} \chi_a(a),$$

$$S'(\alpha) = \sum_{a \in B_n(0)} \chi_a(a),$$

for any $\alpha \in F_2^n$, then we have

$$\text{Spec } A_n = \{ S(\alpha) ; \alpha \in F_2^n \},$$

$$\text{Spec } A_n' = \{ S'(\alpha) ; \alpha \in F_2^n \}. \quad (7)$$

Thus we need to investigate which character sums vanish. We begin with the character $\chi(1^n)$. Recall that we set $b_n^+ - b_n^0 = d_n$ in Corollary 2.2.

Proposition 4.1. For any $n \geq 2$, the character sum $S(1^n)$ is equal to $d_n$. In particular it vanishes if and only if $n \equiv 1, 5 \mod 6$. 

Proof. By definition, we have $\chi_{(n-1)}(a) = (-1)^{a_1+\cdots+a_n}$, which is equal to 1 (resp. $-1$) if $\sum_i a_i$ is even (resp. odd). Therefore, using the notation introduced in (3) and (4), we have $S(1^n) = b_n^e - b_n^o = d_n$. The last assertion follows from Corollary 2.2 and Table 2.1. □

Next we examine the character $\chi_{(n-1)}$. For ease of description, we define an integer sequence $\{c_n\}$ by the rule

$$c_n = \begin{cases} -1, & \text{if } n \equiv 0, 3 \text{ mod } 6, \\ 0, & \text{if } n \equiv 1, 2 \text{ mod } 6, \\ -2, & \text{if } n \equiv 4, 5 \text{ mod } 6. \end{cases}$$

Proposition 4.2. For any $n \geq 2$, the character sum $S(1^{n-1})$ is equal to $c_n$.

Proof. By definition, we have $\chi_{(n-1)}(a) = (-1)^{a_1+\cdots+a_{n-1}}$, which is equal to 1 (resp. $-1$) if $\sum_{i=1}^{n-1} a_i$ is even (resp. odd). Therefore, by the argument in the proof of Proposition 2.4, we have $S(1^{n-1}) = #B(0)^o - #B(0)^e = b_{n-1}^o - b_{n-1}^e + (b_{n-3}^e + 1)$. By Proposition 2.3 and Table 2.1, we see that the value of the rightmost side depends only on $n \mod 6$ and is equal to $c_n$. This finishes the proof. □

Combining this with Proposition 4.1, we obtain the following:

Proposition 4.3. The adjacency matrix $A_n$ has the eigenvalue zero if $n \equiv 1, 2, 5 \text{ mod } 6$.

Thus we are reduced to considering the remaining cases $n \equiv 0, 3, 4 \text{ mod } 6$. The cases when $n \equiv 0, 3 \text{ mod } 6$ are rather easy to deal with as follows:

Proposition 4.4. For any integer $n$ with $n \equiv 0 \text{ mod } 3$, the spectrum of the adjacency matrix $A_n$ does not contain zero.

Proof. The value of the generating function $f_{B_n(0)}(e_1, \ldots, e_n)$ is congruent modulo 2 to $\#\{\text{terms in } f_{B_n(0)}\}$, $\#B_n(0) = b_n$. On the other hand it follows by induction from Proposition 2.5 and Table 2.1 that

$$b_n \mod 2 = \begin{cases} 1, & \text{if } n \equiv 0 \text{ mod } 3, \\ 0, & \text{if } n \equiv 1, 2 \text{ mod } 3. \end{cases}$$

Hence if $n \equiv 0 \text{ mod } 3$, then $f_{B_n(0)}(e_1, \ldots, e_n)$ is odd. Thus the assertion follows from Proposition 3.1(1). □

Lastly we consider the case $n \equiv 4 \text{ mod } 6$. The result is as follows:

Proposition 4.5. For any integer $n$ with $n \equiv 4 \text{ mod } 6$, the spectrum of the adjacency matrix $A_n$ does not contain zero.

Proof. Write $f_{B_n(0)}$ as $f_{B_n(0)} + x_{n-1}f_2(x_1, \ldots, x_{n-1})$, then the set of terms in $f_2(x_1, \ldots, x_{n-1})$ corresponds bijectively to $P_{n-3}(0) \cup \{0\}$, as is shown in the proof of Proposition 2.4. On the other hand the number of elements in $P_{n-3}(0) \cup \{0\}$ is equal to $b_{n-3}^e + 1$, which is even for any $n \equiv 4 \text{ mod } 6$ by Corollary 2.1. Therefore for any $(e_1, \ldots, e_{n-1}) \in \{\pm 1\}^{n-1}$, the value $f_{B_n(0)}(e_1, \ldots, e_{n-1})$ is even, and hence we have

$$f_{B_n(0)}(e_1, \ldots, e_{n-1}, 1) - f_{B_n(0)}(e_1, \ldots, e_{n-1}, -1) = (+1)f_2(e_1, \ldots, e_{n-1}) - (-1)f_2(e_1, \ldots, e_{n-1}) \equiv 0 \text{ mod } 4.$$

Furthermore, as we have already noticed, the natural action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{Z}[x_1, \ldots, x_n]$ keeps $f_{B_n(0)}$ invariant. Therefore if we change only the value of $x_k$ from 1 to $-1$ for any $k$ with $1 \leq k \leq n$, the value of $f_{B_n(0)}$ changes by a multiple of four. Therefore, since the hypercube graph is connected, we see that the congruence

$$f_{B_n(0)}(e_1, \ldots, e_n) \equiv f_{B_n(0)}(1, \ldots, 1) \mod 4$$

holds for any $(e_1, \ldots, e_n) \in \{\pm 1\}^n$. Since $f_{B_n(0)}(1, \ldots, 1) = b_n$ and $b_n$ is easily seen to be congruent to 2 mod 4 if $n \equiv 4 \text{ mod } 6$, we see that $f_{B_n(0)}(e_1, \ldots, e_n) \equiv 2 \text{ mod } 4$ for any $(e_1, \ldots, e_n) \in \{\pm 1\}^n$. Hence the spectrum of $A_n$ does not contain zero in this case. This finishes the proof. □

Combining Propositions 4.3–4.5, we obtain the following:

Theorem 4.1. For any $n \geq 2$, the adjacency matrix $A_n$ is invertible if and only if $n \equiv 0, 3, 4 \text{ mod } 6$.

Remark 4.1. By taking a CM-extension field $K$ of degree $2^{n+1}$ such that $\text{Gal}(K/Q) \cong (\mathbb{Z}/2\mathbb{Z})^{n+1}$ and attaching to it a CM-type constructed naturally from the unit sphere $B_n(0)$, we can define an abelian variety $X_n$ of dimension $2^n$. As a consequence of Theorem 4.1, we can show that $X_n$ satisfies the Hodge conjecture when $n \equiv 0, 3, 4 \text{ mod } 6$. 

We prove this by induction on $n$. By employing another decomposition $S'(0^n - 1^2)$ for any $n$, and for any $n$, Proposition 5.2 and Proposition 5.3, and that it holds for any integers smaller than $n$. When $\alpha \in F_2$ begins with $0$, it follows from (8) that

$$S'(\alpha) = S'(0, \alpha_2, \ldots, \alpha_n)$$

$$= S'(\alpha_2, \ldots, \alpha_n) + S'(\alpha_3, \ldots, \alpha_n) + 1.$$

By the induction hypothesis, the rightmost side is greater than or equal to $m'_n - 1 + m'_{n-2} + 1$, which is equal to $m'_n$ by Proposition 5.2. Hence the assertion follows in this case. When $\alpha \in F_2^n$ begins with $1$, we compute as follows using formula (9) this time:

$$S'(\alpha) = S'(1, \alpha_2, \ldots, \alpha_n)$$

$$= S'(\alpha_2, \ldots, \alpha_n) - S'(\alpha_3, \ldots, \alpha_n) - 1.$$

When $\alpha_2 = 0$, the last expression is equal to $S'(\alpha_4, \ldots, \alpha_n)$, which is greater than or equal to $m'_{n-3}$ by the induction hypothesis. Since the sequence $\{m'_n\}$ is decreasing, it follows that $S'(\alpha) > m'_n$. When $\alpha_2 = 1$, the last expression is

Table 5.1
Some character sums.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m'_n = S'(0^n - 1^2)$</th>
<th>$S'(0^n - 1^2)$</th>
<th>$S'(10^n - 1)$</th>
<th>$S'(010^n - 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-2</td>
<td>-2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>-4</td>
<td>-4</td>
<td>2</td>
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<td>-6</td>
<td>-6</td>
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<tr>
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</tr>
<tr>
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<td>-90</td>
<td>-90</td>
<td>88</td>
<td>88</td>
</tr>
</tbody>
</table>

5. The largest and smallest eigenvalues

We investigate the largest and smallest eigenvalues of $A_n$ and $A'_n$. The largest eigenvalues are rather easy to understand:

**Proposition 5.1.** For any $n \geq 2$, we have

$$\max_{\alpha \in F_2^n} \chi_{A_n}(\alpha) = b_n,$$

and the multiplicity of the eigenvalue $b_n$ (resp. $b'_n$) of $A_n$ (resp. $A'_n$) is equal to one.

**Proof.** By formula (7), the largest eigenvalue of $A_n$ is equal to the maximum among the character sums $S(\alpha)$ with $\alpha \in F_2^n$, and the sum $S(\alpha) = \sum_{\beta \in B_n} \chi_{A}\beta(\alpha)$ attains its maximum if and only if each summand $\chi_{A}\beta(\alpha)$ is equal to one. Hence only $S(0^n)$ gives rise to the maximum and its value is equal to $\#B_n(0) = b_n$. The assertion for $A'_n$ is proved in the same way.

As for the minimum of Spec $A'_n$, we will show that $S'(\alpha)$ attains its minimum at $\alpha = (0^n - 1^2)$ and also at $\alpha = (0^n - 1^3)$. Let $m''_n = S'(0^n - 1^2)$. In order to prove the minimality, we need a recurrence relation among the values of $m''_n$. We will prove it in a little more general fashion:

**Proposition 5.2.** For any $n \geq 4$ and for any $(\alpha_2, \ldots, \alpha_n) \in F_2^{n-1}$, we have

$$S'(0, \alpha_2, \ldots, \alpha_n) = S'(\alpha_2, \ldots, \alpha_n) + S'(\alpha_3, \ldots, \alpha_n) + 1. \quad (8)$$

$$S'(1, \alpha_2, \ldots, \alpha_n) = S'(\alpha_2, \ldots, \alpha_n) - S'(\alpha_3, \ldots, \alpha_n) - 1. \quad (9)$$

In particular the relation $m''_n = m''_{n-1} + m''_{n-2} + 1$ holds for any $n \geq 4$.

**Proof.** This is a direct consequence of the decomposition (2).

**Remark 5.1.** By employing another decomposition (1), we have the following equalities:

$$S'(\alpha_1, \ldots, \alpha_{n-1}, 0) = S'(\alpha_1, \ldots, \alpha_{n-1}) + S'(\alpha_1, \ldots, \alpha_{n-2}) + 1, \quad (10)$$

$$S'(\alpha_1, \ldots, \alpha_{n-1}, 1) = S'(\alpha_1, \ldots, \alpha_{n-1}) - S'(\alpha_1, \ldots, \alpha_{n-2}) - 1. \quad (11)$$

Table 5.1 illustrates the values of $m''_n$ for small values of $n$. We also tabulate some other values which will be used later:

It follows from this table and Proposition 5.2 that

$$m''_n = S'(0^n - 1^2) = S'(0^n - 1^3).$$

**Proposition 5.3.** For any $n \geq 3$, we have

$$\min_{\alpha \in F_2^n} \chi_{A'_n}(\alpha) = m''_n.$$

**Proof.** We prove this by induction on $n$. For $n \leq 4$, the assertion follows from Tables 3.2 and 5.1, so we assume that $n \geq 5$ and that it holds for any integer smaller than $n$. When $\alpha \in F_2$ begins with $0$, it follows from (8) that

$$S'(\alpha) = S'(0, \alpha_2, \ldots, \alpha_n)$$

$$= S'(\alpha_2, \ldots, \alpha_n) + S'(\alpha_3, \ldots, \alpha_n) + 1.$$
equal to \(-S'(\alpha_4, \ldots, \alpha_n) - 2\) by (9). The maximum of \(S'(\alpha_4, \ldots, \alpha_n)\) being equal to \(b'_{n-3}\) by Proposition 4.1, we see that \(S'(\alpha) \geq -b'_{n-3} - 2\). Therefore the proof will be completed if we can show that the equality

\[ m'_n = -b'_{n-3} - 2 \]  

holds for any \(n \geq 5\). To show (12), notice that by Proposition 2.1 the sequence \([-b'_{n} - 2\] satisfies the recurrence relation

\[ -b'_{n} - 2 = (-b'_{n-1} - 2) + (-b'_{n-2} - 2) + 1, \]

which is the same as the recurrence relation satisfied by the sequence \(m'_n\) in Proposition 5.2. Furthermore from the tables we know that \(m'_n = -b'_n - 2 = -4\) and \(m'_n = -b'_n - 2 = -6\). Hence the equality (12) holds for any \(n \geq 5\). This finishes the proof of Proposition 5.3. \(\square\)

The second largest eigenvalue of \(A'_n\) is determined as follows:

**Proposition 5.4.** For any \(n \geq 3\), the character sum \(S'(010^{n-2})\) gives rise to the second largest eigenvalue of \(A'_n\).

**Proof.** We prove this by induction on \(n\). For \(n \leq 4\), the assertion follows from the table, so we assume that \(n \geq 5\) and that it holds for any integer smaller that \(n\). Let \(\alpha \in F_2^n\) be an arbitrary element. If \(\alpha = (0^n)\), then it gives us the largest eigenvalue, as is shown in the proof of Proposition 4.1. So we assume that \(\alpha \neq (0^n)\) and \(\alpha \neq (010^{n-2})\), and show that \(S'(\alpha) \leq S'(010^{n-2})\).

When \(\alpha \in F_2^n\) begins with 0, it follows from (8) and (11) that

\[ S'(\alpha) = S'(\alpha_2, \ldots, \alpha_n) + S'(\alpha_3, \ldots, \alpha_n) + 1, \]

\[ S'(010^{n-2}) = S'(010^{n-3}) + S'(010^{n-4}) + 1. \]

By the induction hypothesis and since \((\alpha_3, \ldots, \alpha_n) \neq (0^{n-2})\) by our assumption, we have

\[ S'(\alpha_2, \ldots, \alpha_n) \leq S'(010^{n-3}), \]

\[ S'(\alpha_3, \ldots, \alpha_n) \leq S'(010^{n-4}). \]

Hence the assertion follows from (13) and (14). When \(\alpha \in F_2^n\) begins with 1, it follows from (9) that

\[ S'(\alpha) = S'(\alpha_2, \ldots, \alpha_n) - S'(\alpha_3, \ldots, \alpha_n) - 1. \]

If \((\alpha_3, \ldots, \alpha_n) = (0, \ldots, 0)\), then \(\alpha = 10^{n-1}\) or \(110^{n-2}\). Since both of the sequences \(S'(10^{n-1})\) and \(S'(1010^{n-2})\) obey the same recurrence by Remark, the table shows that \(S'(10^{n-1}) < S'(010^{n-2})\) holds for any \(n \geq 3\). Furthermore if \(\alpha = 110^{n-2}\), then \(S'(\alpha)\) gives rise to the least eigenvalue by Proposition 5.3. Therefore we may assume that \((\alpha_3, \ldots, \alpha_n) \neq (0, \ldots, 0)\). Then it follows from the induction hypothesis and Proposition 5.3 that

\[ S'(\alpha_2, \ldots, \alpha_n) \leq S'(010^{n-3}) \]

\[ S'(\alpha_3, \ldots, \alpha_n) \geq S'(110^{n-4}). \]

These two inequalities imply by (15) that

\[ S'(\alpha) \leq S'(010^{n-3}) - S'(110^{n-4}) - 1. \]

Let \(p_n = S'(010^{n-2})\) and \(q_n = S'(110^{n-2})\). Then by (16) our assertion will be a consequence of the following claim:

\[ p_{n-1} - q_{n-2} < p_n \]

for any \(n \geq 5\). \(\square\)

We prove this by induction on \(n\). Since we see the validity for \(n \leq 6\) by the table, we assume that \(n \geq 7\), and that it holds for smaller values of \(n\). By (10), we know that

\[ p_{n-1} - q_{n-2} = (p_{n-2} - q_{n-3}) + (p_{n-3} - q_{n-4}) \]

holds for any \(n \geq 7\). By the induction hypothesis we have

\[ p_{n-2} - q_{n-3} < p_{n-1}, \]

\[ p_{n-3} - q_{n-4} < p_{n-2}, \]

hence the inequality (18) implies that

\[ p_{n-1} - q_{n-2} < p_{n-1} - p_{n-2} = p_n - 1 < p_n. \]

Thus we see that claim (17) holds true. This finishes the proof of Proposition 5.4. \(\square\)

Next we consider min \(\text{Spec} \ A_n\). Let \(m_n = S(0^{n-3}1^3)\). The following table shows the values for small \(n\):

We will show that this character sum accomplishes the minimum of \(\text{Spec} \ A_n\). In order to prove the minimality, we need a recurrence relation among the values of \(m_n\):

**Proposition 5.5.** For any \(n \geq 5\), we have \(m_n = m_{n-1} + m_{n-2} + 1\).
**Proof.** Let a map $jux_{10:01} : B_{n-4} \to B_n$ be defined by the rule

$$jux_{10:01}(a_1, \ldots, a_{n-4}) = (1, 0, a_1, \ldots, a_{n-4}, 0, 1).$$

Then we have $B'_n(0) = B_n(0) \cup jux_{10:01}(B'_{n-4}(0) \cup \{0\})$, and it follows that

$$S'(0^{n-3}1^3) = S(0^{n-3}1^3) - (S'(0^{n-5}1) + 1),$$

hence we have

$$m_n = S(0^{n-3}1^3) = S'(0^{n-3}1^3) + S'(0^{n-5}1) + 1.$$  \hfill (19)

By (8) the sequence $\{m'_n\} = \{S'(0^{n-3}1^3)\}$ satisfies the recurrence relation $m'_n = m'_{n-1} + m'_{n-2} + 1$. One can check also that if we put $m''_n = S'(0^{n-1}1)$ then the sequence $\{m''_n\}$ obeys the same recurrence $m''_n = m''_{n-1} + m''_{n-2} + 1$ by (8). Therefore it follows from (19) that the sequence $\{m_n\}$ must satisfy the same recurrence relation. This finishes the proof of Proposition 5.5. \hfill \Box

In order to show the minimality of $m_n$, we need some lemmas:

**Lemma 1.** For any $n \geq 6$, we have

$$S(1, 1, 0, \alpha_4, \ldots, \alpha_n) = S(\alpha_5, \ldots, \alpha_n) - S'(0, \alpha_4, \ldots, \alpha_{n-1}) - 1.$$

**Proof.** Let us use three maps $jux_{00} : B_{n-2} \to B_n$, $jux_{010} : B_{n-3} \to B_n$, $jux_{100} : B_{n-3} \to B_n$, each of which is defined by

$$jux_{00}(a_1, \ldots, a_{n-2}) = (0, 0, a_1, \ldots, a_{n-2})$$

$$jux_{010}(a_1, \ldots, a_{n-3}) = (0, 1, 0, a_1, \ldots, a_{n-3})$$

$$jux_{100}(a_1, \ldots, a_{n-3}) = (1, 0, a_1, \ldots, a_{n-3}, 0).$$

One can see that they provide us with a decomposition

$$B_n(0) = jux_{00}(B'_{n-2}(0)) \cup jux_{010}(B'_{n-3}(0) \cup \{0\}) \cup jux_{100}(B'_{n-3}(0) \cup \{0\}).$$

It follows from this and formula (8) that

$$S(1, 1, 0, \alpha_4, \ldots, \alpha_n) = S'(0, \alpha_4, \ldots, \alpha_n) - (S'(\alpha_4, \ldots, \alpha_n) + 1) - (S'(0, \alpha_4, \ldots, \alpha_{n-1}) + 1)$$

$$= S'(\alpha_5, \ldots, \alpha_n) - S'(0, \alpha_4, \ldots, \alpha_{n-1}) - 1.$$

This concludes the proof of Lemma 1. \hfill \Box

**Lemma 2.** For any $n \geq 4$, we have

$$S(0, 0, a_3, \ldots, a_n) = S(0, a_3, \ldots, a_n) + S(a_3, \ldots, a_n) + 1,$$

$$S(1, 1, a_3, \ldots, a_n) = S(1, a_3, \ldots, a_n) - S(a_3, \ldots, a_n) - 1.$$

**Proof.** As is noted in the proof of Proposition 2.1, we have

$$B_n(0) = jux_{00}(B'_{n-1}(0)) \cup jux_{100}(B'_{n-3} \cup \{0\}).$$

Therefore for any $n \geq 7$, we have

$$S(a_1, \ldots, a_n) = S'(a_2, \ldots, a_n) + (-1)^{a_1}(S'(a_3, \ldots, a_{n-1}) + 1)$$

$$S(a_2, \ldots, a_n) = S'(a_3, \ldots, a_n) + (-1)^{a_2}(S'(a_4, \ldots, a_{n-1}) + 1)$$

$$S(a_3, \ldots, a_n) = S'(a_4, \ldots, a_n) + (-1)^{a_3}(S'(a_5, \ldots, a_{n-1}) + 1).$$  \hfill (20)

(21)

(22)

By using (21) and (22), we have

$$S(a_2, \ldots, a_n) + (-1)^{a_2}(S(a_3, \ldots, a_n) + 1) = (S'(a_3, \ldots, a_n) + (-1)^{a_2}(S'(a_4, \ldots, a_{n-1}) + 1))$$

$$+ (-1)^{a_2}(S'(a_4, \ldots, a_n) + (-1)^{a_3}(S'(a_5, \ldots, a_{n-1}) + 1)) + (-1)^{a_2}$$

$$= (S'(a_3, \ldots, a_n) + (-1)^{a_2}(S'(a_4, \ldots, a_n) + 1))$$

$$+ (-1)^{a_2}(S'(a_4, \ldots, a_{n-1}) + (-1)^{a_3}(S'(a_5, \ldots, a_{n-1}) + 1)) + (-1)^{a_2}$$

$$= S'(a_2, \ldots, a_n) + (-1)^{a_2}(S'(a_3, \ldots, a_{n-1}) + 1).$$

Comparing the rightmost side with (20), we see that if $a_1 = a_2$, then

$$S(a_1, \ldots, a_n) = S(a_2, \ldots, a_n) + (-1)^{a_2}(S(a_3, \ldots, a_n) + 1).$$

Thus we obtain the equalities in Lemma 2. \hfill \Box
Table 5.2

<table>
<thead>
<tr>
<th>Character sum $m_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$m_n = S(0^{n-3}1^3)$</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
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<tr>
<td>5</td>
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<td>8</td>
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<td>9</td>
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<tr>
<td>10</td>
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</tbody>
</table>

Table 5.3

<table>
<thead>
<tr>
<th>Some character sums.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$m_n = S(0^{n-3}1^3)$</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>$S(0^{n-2}1^2)$</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>$m_n' = S'(0^{n-3}1^3)$</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>$S'(010^{n-2})$</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

Now we can prove the following:

**Theorem 5.1.** For any $n \geq 3$, we have $\min \text{Spec } A_n = m_n$.

**Proof.** We prove this by induction on $n$. For $n \leq 6$, the assertion follows from Table 5.2, so we assume that $n \geq 7$ and that it holds for any integer smaller than $n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_2^n$, and we divide our argument into four cases.

**Case 1:** When $\alpha_1 = \alpha_2 = 0$.

In this case, it follows from Lemma 2 and the induction hypothesis that

$$S(\alpha) = S(0, \alpha_3, \ldots, \alpha_n) + S(\alpha_3, \ldots, \alpha_n) + 1 \geq m_{n-1} + m_{n-2} + 1.$$  

The right hand side being equal to $m_n$ by Proposition 5.5, the assertion is proved.

**Case 2:** When $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0$.

Table 5.3 will help the reader to confirm our argument by numerical values. Looking at the values of $S(0^{n-3}1^3)$ and $S(0^{n-2}1^2)$ in this table, we see by Lemma 2 that $S(0^{n-3}1^3) \leq S(0^{n-2}1^2)$ for any $n \geq 5$. Therefore if $(\alpha_4, \ldots, \alpha_n) = (0, \ldots, 0)$, then

$$S(1, 1, 0, \alpha_4, \ldots, \alpha_n) = S(0^{n-2}1^2) \geq S(0^{n-3}1^3) = m_n,$$

and if $(\alpha_4, \ldots, \alpha_n) = (0, \ldots, 0, 1)$, then

$$S(1, 1, 0, \alpha_4, \ldots, \alpha_n) = S(0^{n-3}1^3) = m_n.$$  

Therefore we may assume that $(\alpha_4, \ldots, \alpha_{n-1}) \neq (0, \ldots, 0)$. Then it follows from Proposition 5.4 that

$$S'(0, \alpha_4, \ldots, \alpha_{n-1}) \leq S'(010^{n-5}).$$  

Hence Lemma 1 and the induction hypothesis imply that for any $n \geq 7$ we have

$$S(1, 1, 0, \alpha_4, \ldots, \alpha_n) - S'(0, \alpha_4, \ldots, \alpha_{n-1}) - 1 \geq S'(0^{n-7}1^3) - S'(010^{n-5}) - 1.$$  

Thus we are reduced to showing the following claim:

**Claim.** $S'(0^{n-7}1^3) - S'(010^{n-5}) - 1 \geq S(0^{n-3}1^3)$ holds for any $n \geq 7$.

Let us set $r_{n-4} = S'(0^{n-7}1^3), s_{n-3} = S'(010^{n-5})$, and $t_n = S(0^{n-3}1^3)$. Furthermore we put $u_n = r_{n-4} - s_{n-3} - t_n$, so that our task is to prove that $u_n$ is positive for any $n \geq 7$. We prove this by induction on $n$. The table above shows the validity of the claim for $n = 7, 8$, so we assume that $n \geq 9$ and that the assertion holds for smaller values of $n$. It follow from (8) and (10) that for any $n \geq 9$ we have

$$u_{n-1} + u_{n-2} - 1 = (r_{n-5} - s_{n-4} - t_{n-1}) + (r_{n-6} - s_{n-5} - t_{n-2}) - 1 = (r_{n-5} + r_{n-6} + 1) - (s_{n-4} + s_{n-5} + 1) - (t_{n-1} + t_{n-2} + 1) = r_{n-4} - s_{n-3} - t_n = u_n.$$  

Since the leftmost side is positive by the induction hypothesis, we see that the rightmost side $u_n$ is positive too. Thus the proof for Case 2 is finished.

Since $S(\alpha)$ is invariant under cyclic transformations of coordinates, we are reduced to considering the cases when $\alpha = (1^n), \alpha = ((01)^n)$. 

**Lemma 3.** We have

\[ v_{n+2} = v_{n+1} + v_n + 1, \]

for any \( n \geq 1 \).

**Proof.** Let \( v'_n = S'(n(0)^n) \) and \( x'_n = S'(1(0)^n) \). Furthermore we put \( V'_n = v'_n - v'_{n-1} - v'_{n-2} \) for \( n \geq 3 \), and show first by induction that \( V'_n = 1 \) for any \( n \geq 3 \). As is evident from Table 5.4, \( V'_n = 1 \) for \( n \leq 6 \). In order to prove \( V'_n = 1 \) for any \( n \geq 7 \), we use the double recurrence relations

\[
\begin{align*}
   v'_n &= x'_{n-1} + v'_{n-1} + 1, \\
   x'_n &= v'_n - x'_{n-1} - 1,
\end{align*}
\]

which are consequences of formulas (8) and (9). Combining these two, we obtain

\[ v'_n = 2v'_{n-1} - x'_{n-2} = 2v'_{n-1} - v'_{n-2} + x'_{n-3} + 1. \]

Repeating in this way, we have

\[ v'_{2m} = 2v'_{2m-1} - v'_{2m-2} + v'_{2m-3} - v'_{2m-4} + \cdots + v'_5 - v'_4 + x'_3 + 1. \]

Similarly we have

\[
\begin{align*}
   v'_{2m-1} &= 2v'_{2m-2} - v'_{2m-3} + v'_{2m-4} - v'_{2m-5} + \cdots + v'_4 - v'_3 + x'_2 + 1, \\
   v'_{2m-2} &= 2v'_{2m-3} - v'_{2m-4} + v'_{2m-5} - v'_{2m-6} + \cdots + v'_3 - v'_2 + x'_1 + 1.
\end{align*}
\]

Therefore we have

\[ V'_{2m} = 2V'_{2m-1} - V'_{2m-2} + V'_{2m-3} - V'_{2m-4} + \cdots + V'_5 - V'_4 + (x'_3 - x'_2 - x'_1) - 1. \]

Suppose that \( V'_k = 1 \) holds for any integer \( k < 2m \). Since Table 5.4 shows that \( x'_3 - x'_2 - x'_1 = 1 \), we see by the induction hypothesis and by the above formula that

\[ V'_{2m} = 2 - 1 + 1 - 1 + \cdots + 1 - 1 + 1 - 1 = 1. \]

Hence our assertion is shown to hold for any even integers. The case of odd integers are treated similarly. Hence we have \( V'_n = 1 \) for any \( n \geq 3 \). By using this result we can show that \( V'_n = 1 \) for any \( n \geq 3 \). By the decomposition

\[
B'_n(0) = B_n(0) \cup jux_{0101}(B'_{n-4}(0) \cup \{0\})
\]

established in the proof of Proposition 5.5, we have

\[ S'(0(0)^n) = S'(n(0)^n) - S'(0(0)^{n-2} + 1), \]

which implies that

\[ v_n = v'_n + v'_{n-2} + 1. \]

Therefore we have \( v_n = V'_n + V'_{n-2} - 1 \), and hence we see that \( v_n = 1 \) for any \( n \geq 3 \). This finishes the proof of Lemma 3. \( \square \)

Since we have \( S(01) = 0 \), \( S((01)^2) = 2 \), we see by Lemma 3 that \( S((01)^n) \geq 0 \) for any \( n \geq 1 \). This finishes the treatment of Case 4, and completes the proof of Theorem 5.1. \( \square \)
Remark 5.2. A part of the arguments used in this paper can be employed in a more general setting. For example, if we have a family \( \{C_n\} \) of subsets \( C_n \subset \{0, 1\}^n \), \( n \geq 2 \), which satisfies the equality
\[
C_n = jux_0(C_{n-1}) \sqcup jux_0(C_{n-2} \sqcup \{0\})
\]
similar to formula (1), then we are able to attach a family of matrices \( \{D_n\} \) to \( \{C_n\} \) and determine their spectra. Furthermore if we replace the embedding \( p' : B_n \to B_n^{n-1} \) by \( p'_3 : B_n \to B_n^{n-2} \) defined by \( p'_3(a) = ((a_1, a_2, a_3), (a_2, a_3, a_4), \ldots, (a_{n-2}, a_{n-1}, a_n)) \), and define \( p_3 : B_n \to B_n^3 \) similarly, then we come across another infinite family of graphs on \( \{0, 1\}^n \) with several amusing properties. We will investigate these matters elsewhere.

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References