On a Recursive Approximation of Singularly Perturbed Parabolic Equations

J. Struckmeier and A. Unterreiter

FB Mathematik, Universität Kaiserslautern, Kaiserslautern 67653, Germany
E-mail: struckm@mathematik.uni-kl.de, unterreiter@mathematik.uni-kl.de

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The asymptotic analysis of IBVPs for the singularly perturbed parabolic PDE $\partial_t u + \partial_x u = \varepsilon \partial_{xx} u$ in the limit $\varepsilon \to 0$ motivates investigations of certain recursively defined approximative series ("ping-pong expansions"). The recursion formulae rely on operators assigning to a boundary condition at the left or the right boundary a solution of the parabolic PDE. Sufficient conditions for uniform convergence of ping-pong expansions are derived and a detailed analysis for the model problem $\partial_t u + \partial_x u = \varepsilon \partial_{xx} u$ is given. © 2000 Academic Press

1. INTRODUCTION

The recursive approximation derived in this paper arises from investigations on a singularly perturbed two-phase Stefan problem: If one of the two phases is characterized by slow diffusion, then a boundary layer at the phase change will yield a modified Stefan condition for the unperturbed one-phase problem.

Using matched asymptotic expansions a zeroth order correction term has been derived in [2, 7]. This correction term is sufficiently accurate as long as the moving interface stays away from a fixed boundary. If the moving interface approaches this fixed boundary, the whole problem will become—due to interacting layers—quite complicated. Moreover, the derivation of higher order correction terms for that singularly perturbed Stefan problem can seemingly not be performed in a straightforward manner by standard matching techniques [7]. To have a close insight to the singularly perturbed phase, it turned out to be necessary to develop a seemingly new (compare [1, 3, 6]) asymptotic analysis for the model
problem

\[ \partial_t u_x + \partial_x u_x = \varepsilon \partial_{xx} u_x, \quad u_x(0, x) = 1 - x, \quad u_x(t, 0) = 1, \quad u_x(t, 1) = 0, \quad (1.1) \]

with \( \varepsilon \ll 1 \).

We shall discuss the behavior of \( u_x \) as \( \varepsilon \to 0 \) in some detail right now (see Fig. 1). The following notations will be used.

- The time variable \( t \) ranges in the interval \((0, T] \), where \( T > 0 \) is fixed.
- \( C_b([0, T]) \) is the space of all bounded, continuous, real-valued functions whose domain is \([0, T]. We equip \( C_b([0, T]) \) with the canonical norm

\[ \|\alpha\|_T := \sup\{|\alpha(t)|; t \in ]0, T[\}, \quad \alpha \in C_b([0, T]). \]

- The spatial variable \( x \) ranges in \( \omega = ]0, 1[ \).
- \( C^2(\omega) \) is the space of all functions \( u : \omega \to \mathbb{R} \) having uniformly continuous derivatives up to order 2.
- \( \omega_T := ]0, T[ \times \omega. \)
- \( C_b(\omega_T) \) is defined in analogy to \( C_b([0, T]) \). We set

\[ \|u\|_\infty := \sup\{|u(t, x)|; (t, x) \in \omega_T\}, \quad u \in C_b(\omega_T). \]

\( C^2(\omega_T) \) is the space of all functions \( u : \omega_T \to \mathbb{R} \) having continuous partial derivatives up to order 2.

FIG. 1. Numerical solution \( u_x(t, x) \) for \( \varepsilon = 0.01 \) at different times \( t = 0.1, 0.5, 1.0, \) and 2.0 (from left to right).
We begin the asymptotic analysis of (1.1) by looking for an outer expansion of $u_\varepsilon$. Speaking phenomenologically for small values of $\varepsilon$ the boundary value at $x = 1$ is “far away” from the behavior of $u_\varepsilon$ on each sub-interval $]0, \theta[\cup ]\theta, 1[$, $0 < \theta < 1$, of $\omega$. Taking this idea literally, one may think of shifting the boundary condition from $x = 1$ to $x \to \infty$—and thus ignore it. Hence, we are led to consider the half-space problem

$$
\partial_t v_\varepsilon = -\partial_x v_\varepsilon + \varepsilon \partial_{xx} v_\varepsilon, \quad v_\varepsilon(0, x) = 1 - x, \quad v_\varepsilon(t, 0) = 1, \quad (1.2)
$$

with $(t, x) \in ]0, T[ \times \mathbb{R}^+$ and $v_\varepsilon(x \to \infty) = 0$. It may be reasonable to think that $v_\varepsilon$ will be a good approximation of $u_\varepsilon$, at least for $(t, x) \in ]0, T[ \times ]0, \theta[$. Equation (1.2) is explicitly solvable by means of the Sinus–Fourier transformation (see Section 3, compare [4]), which yields

$$
v_\varepsilon(t, x) = 1 - \frac{x - t}{2} \text{erfc}\left(\frac{t - x}{2\sqrt{\varepsilon t}}\right) - \frac{x + t}{2} e^{x/\varepsilon} \text{erfc}\left(\frac{t + x}{2\sqrt{\varepsilon t}}\right).
$$

We set $I_\varepsilon := v_\varepsilon|_{x=0}$ and for reasons that will become clear later on ($I_\varepsilon$ satisfies “by chance” the correct boundary conditions at $x = 0$), we set $I_\varepsilon^{[0]} := 0$. According to Fig. 2, $I_\varepsilon$ is an excellent approximation of $u_\varepsilon$ for small values of $\varepsilon$, naturally away from $x = 1$.

What should be done with the boundary condition at $x = 1$?

The canonical way to proceed would be to re-scale the equations close to $x = 1$ and to apply matching procedures afterwards. However, this standard procedure leads us astray from finding a new asymptotic analysis which allows for the derivation of higher correction terms for the singularly perturbed Stefan problem mentioned at the beginning.

![FIG. 2. Approximation $I_\varepsilon(t, x)$ for $\varepsilon = 0.01$ at different times $t = 0.1, 0.5, 1.0,$ and $2.0$ (from left to right).]
Let us check the difference $s_e^{-1} := u_e - I_e$ for small values of $\varepsilon$, indicated in Fig. 3.

Naturally, $s_e^{-1}$ has a boundary layer at $x = 1$. The equation satisfied by $s_e^{-1}$ is

$$\begin{cases} \partial_t s_e^{-1} = -\partial_x s_e^{-1} + \varepsilon \partial_{xx} s_e^{-1}, & s_e^{-1}(0, x) = 0, \\ s_e^{-1}(t, 0) = 0, & s_e^{-1}(t, 1) = -I_e(t, 1 -). \end{cases} \tag{1.3}$$

We deduce from Fig. 3 a very important property of (1.3): away from $x = 1$ the operator $A_e[v] := -v' + \varepsilon v''$ has the tendency to make $|s_e^{-1}|$ rather small. From this point of view it is not important to prescribe the boundary value $0$ at $x = 0$; i.e., we may replace (1.3) by the half-space problem (see [9])

$$\partial_t W_e = -\partial_x W_e + \varepsilon \partial_{xx} W_e, \quad W_e(0, x) = 0, W_e(t, 1) = -I_e(t, 1 -), \tag{1.4}$$

with $(t, x) \in [0, T[ \times ] - \infty, 1[$. Then (see Section 3)

$$W_e(t, x) = -\frac{1 - x}{\sqrt{4\pi\varepsilon}} \int_0^t s^{-3/2} \exp\left(-\frac{(1 - x + s)^2}{4\varepsilon s}\right) I_e(t - s, 1 -) \, ds. \tag{1.5}$$

Since $W_e$ solves a “right” boundary value problem, it is appropriate to introduce the notation

$$r_e^{[0]} := W_e|_{x=0}.$$
We put $u^{[0]}_e := I_e + r^{[0]}_e (= I_e + t^{[0]}_e + r^{[0]}_e)$ and observe that the function $\Sigma^{[0]}_e := u^{[0]}_e$ satisfies the correct equation, the correct initial condition, the correct boundary condition at $x = 1$, and $\Sigma^{[0]}_e(., 0 + ) = 1 + r^{[0]}_e(., 0 + )$, i.e. since $|r^{[0]}_e(., 0 + )|$ is rather small (see Fig. 4) “almost” the boundary condition at $x = 0$.

What is the reason for the excellent approximation properties of $u^{[0]}_e$?

Due to the maximum principle the difference $s^{[0]}_e := u_e - u^{[0]}_e = u_e - \Sigma^{[0]}_e$ satisfies

$$
\|u_e - u^{[0]}_e\|_\infty \leq \|r^{[0]}_e(., 0 + )\|_T.
$$

From (1.5) we have

$$
r^{[0]}_e(t, 0 + ) = \frac{1}{\sqrt{4\pi\varepsilon}} \int_0^t s^{-3/2} \exp \left( -\frac{(1 + s)^2}{4\varepsilon s} \right) I_e(t - s, 1) \, ds,
$$

$$
t \in [0, T[,
$$

where we note that the term

$$
I_e(\sigma, 1 - ) = 1 - \frac{1 - \sigma}{2} \text{erfc} \left( \frac{\sigma - 1}{2\sqrt{\varepsilon}\sigma} \right) - \frac{1 + \sigma}{2} e^{1/\varepsilon} \text{erfc} \left( \frac{\sigma + 1}{2\sqrt{\varepsilon}\sigma} \right)
$$

is uniformly bounded; i.e., there is $K \in [0, \infty[$ such that $|I_e(\sigma, 1 - )| \leq K$ for all $\varepsilon \in \mathbb{R}^+$, $\sigma \in [0, T[$. Hence we obtain for all $\varepsilon \in \mathbb{R}^+$ the estimate

$$
\|u_e - u^{[0]}_e\|_\infty \leq \|r^{[0]}_e(., 0 + )\|_T \leq 2 K \text{erfc} \left( \frac{1}{2\sqrt{\varepsilon}} \right);
$$

![FIG. 4. Function $W(t, x)$ for $\varepsilon = 0.01$ at time $t = 2.0$.](image-url)
thus
\[ \|u_\varepsilon - u^{[0]}_\varepsilon\|_\infty = O\left( \text{erfc}\left( \frac{1}{2\sqrt{\varepsilon}} \right) \right) = O\left( \sqrt{\varepsilon} \exp\left( -\frac{1}{4\varepsilon} \right) \right), \quad \text{as } \varepsilon \to 0. \]

(1.6)

Now let us try to derive higher correction terms: the two functions \(u_\varepsilon\) and \(u^{[0]}_\varepsilon\) differ from each other according to the boundary condition at \(x = 0\), so one may think of subtracting from \(u^{[0]}_\varepsilon\) a function satisfying (1.1) with vanishing initial values and with boundary condition \(u^{[0]}_\varepsilon(\cdot,0 + ) - 1 = r^{[0]}_\varepsilon(\cdot,0 + )\) at \(x = 0\). Our experiences with half-space solutions are so far excellent and it seems appropriate to consider the restriction to \(\omega_T\) of a half-space solution as a correction for \(u^{[0]}_\varepsilon\), i.e., we shall solve the half-space problem
\[
\partial_t U_\varepsilon = -\partial_x U_\varepsilon + \varepsilon \partial_{xx} U_\varepsilon, \quad U_\varepsilon(0,x) = 0, U_\varepsilon(t,0) = -u^{[0]}_\varepsilon(t,0 +),
\]

(1.7)

with \((t,x) \in [0,T] \times [0,\infty[\). We obtain (see Section 3)
\[
U_\varepsilon(t,x) = -\frac{x}{\sqrt{4\varepsilon \pi}} \int_0^t s^{-3/2} \exp\left( -\frac{(x - s)^2}{4\varepsilon s} \right) u^{[0]}_\varepsilon(t-s,0 +) \, ds. \quad (1.8)
\]

Now it is nearby to set
\[
u^{[1/2]}_\varepsilon := f^{[1]}_\varepsilon := U_\varepsilon|_{\omega_T}.
\]

But \(\Sigma^{[1/2]}_\varepsilon := u^{[0]}_\varepsilon + f^{[1/2]}_\varepsilon\) is not necessarily a better approximation than \(u^{[0]}_\varepsilon\). To verify this let us consider the IBVP satisfied by \(\Sigma^{[1/2]}_\varepsilon\),
\[
\begin{align*}
\partial_t \Sigma^{[1/2]}_\varepsilon &= -\partial_x \Sigma^{[1/2]}_\varepsilon + \varepsilon \partial_{xx} \Sigma^{[1/2]}_\varepsilon, \\
\Sigma^{[1/2]}_\varepsilon(t,0) &= 1,
\end{align*}
\]

so the difference \(s^{[1/2]}_\varepsilon := u_\varepsilon - \Sigma^{[1/2]}_\varepsilon\) satisfies
\[
\begin{align*}
\partial_t s^{[1/2]}_\varepsilon &= -\partial_x s^{[1/2]}_\varepsilon + \varepsilon \partial_{xx} s^{[1/2]}_\varepsilon, \\
s^{[1/2]}_\varepsilon(t,0) &= 0,
\end{align*}
\]

From the maximum principle we obtain
\[
\|u_\varepsilon - (u^{[0]}_\varepsilon + f^{[1/2]}_\varepsilon)\|_\infty \leq \|f^{[1]}(\cdot,1 -)\|_{\mathcal{F}}.
\]

1 In principle one may wish to fulfill the boundary condition at \(x = 1\) as well, but then one would run into the same troubles as for \(u_\varepsilon\); no “easy-to-handle” representation of this function would be available.
Since
\[
I^{[1]}_\varepsilon(t, 1 -) = -\frac{1}{\sqrt{4\varepsilon\pi}} \int_0^t s^{3/4} \exp\left(-\frac{(1-s)^2}{4\varepsilon s}\right) r^{[0]}_\varepsilon(t-s, 0 +) \, ds,
\]
t \in [0, T[,
we obtain the estimate \(\|I^{[1]}_\varepsilon(. , 1 -)\|_T = O(\|r^{[0]}_\varepsilon(. , 0 +)\|_T) = O(\text{erfc}(1/2\sqrt{\varepsilon}))\) as \(\varepsilon \to 0\), and therefore \(\|u_\varepsilon - (u^{[0]}_\varepsilon + u^{[1/2]}_\varepsilon)\|_\infty = O(\text{erfc}(1/2\sqrt{\varepsilon}))\) as \(\varepsilon \to 0\); i.e., it may happen that \(\|u_\varepsilon - (u^{[0]}_\varepsilon + u^{[1/2]}_\varepsilon)\|_\infty\) is of the same order of magnitude as \(\|u_\varepsilon - u^{[0]}_\varepsilon\|_\infty\).

This intermediate result is not entirely surprising: the norm \(\|u_\varepsilon - u^{[0]}_\varepsilon\|_\infty\) is determined by the difference of \(u_\varepsilon\) and \(u^{[0]}_\varepsilon\) at the left boundary. This order of magnitude is also the order of magnitude of \(I^{[1]}_\varepsilon\) at the right boundary. (As \(\varepsilon \to 0\) the PDE \(\partial_t u_\varepsilon = -\partial_x u_\varepsilon + \varepsilon \partial_{xx} u_\varepsilon\) becomes more transport equation "transporting" left boundary values into \(\omega\).) Hence the difference between \(u_\varepsilon\) and \(u^{[1/2]}_\varepsilon = u^{[0]}_\varepsilon + I^{[1]}_\varepsilon\) is at least of the order of magnitude of \(I^{[1]}_\varepsilon(. , 1 -)\), i.e., the order of magnitude of \(u_\varepsilon - u^{[0]}_\varepsilon\).

These considerations show that there is no chance to obtain higher order corrections just by adding half-space terms, which compensate the wrong boundary condition at \(x = 0\). One has to take into account corrections at the right boundary as well.

Keeping the idea of considering half-space solutions we define \(r^{[1]}_\varepsilon\) to be the (restriction to \(\omega_T\) of the) solution of a half-space problem as in (1.4) but with prescribed boundary condition \(-I^{[1]}(., 1 -)\) at \(x = 1\). Introducing \(u^{[1]}_\varepsilon := I^{[1]}_\varepsilon + r^{[1]}_\varepsilon\), the difference \(s^{[1]}_\varepsilon := u_\varepsilon - (u^{[0]}_\varepsilon + u^{[1]}_\varepsilon)\) will satisfy
\[
\begin{align*}
\partial_t s^{[1]}_\varepsilon &= -\partial_x s^{[1]}_\varepsilon + \varepsilon \partial_{xx} s^{[1]}_\varepsilon, \\
s^{[1]}_\varepsilon(0, x) &= 0, \\
s^{[1]}_\varepsilon(t, 0 +) &= r^{[1]}_\varepsilon(t, 0 +), \\
s^{[1]}_\varepsilon(t, 1) &= 0,
\end{align*}
\]
so \(\|u_\varepsilon - (u^{[0]}_\varepsilon + u^{[1]}_\varepsilon)\|_\infty = O(\|r^{[1]}_\varepsilon(. , 0 +)\|_T)\) as \(\varepsilon \to 0\) by the maximum principle. On the other hand it follows as before that
\[
\|r^{[1]}_\varepsilon(. , 0 +)\|_T \leq 2\|I^{[1]}_\varepsilon(. , 1 -)\|_T \text{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right),
\]
and therefore \(\|u_\varepsilon - (u^{[0]}_\varepsilon + u^{[1]}_\varepsilon)\|_\infty = O(\text{erfc}(1/2\sqrt{\varepsilon})^2)\) as \(\varepsilon \to 0\); i.e., \(u^{[1]}_\varepsilon\) is a higher order correction term.

Let us repeat the single steps for the construction of \(u^{[1]}_\varepsilon\) out of \(r^{[0]}_\varepsilon\):

(1) Set \(\tilde{I}^{[1]}_\varepsilon := \Gamma^{[1]}_\varepsilon[r^{[0]}_\varepsilon\] where for each sufficiently smooth function \(v : \omega_T \to \mathbb{R}\) the image \(\Gamma^{[1]}_\varepsilon[v]\) of the operator \(\Gamma^{[1]}_\varepsilon\) is the (restriction to \(\omega_T\) of the) solution of the half-space problem (1.7) with prescribed boundary condition \(-v(. , 0 +)\) at \(x = 0\).
(2) Set \( r_0^{[1]} := \Gamma_0'[l_0^{[1]}] \), where for each sufficiently smooth function \( w : \omega_\tau \rightarrow \mathbb{R} \) the image \( \Gamma_0'[w] \) of the operator \( \Gamma_0' \) is the (restriction to \( \omega_\tau \) of the) solution of the half-space problem \((1.4)\) with prescribed boundary condition \(-w(.,1-)\) at \( x = 1 \).

(3) Set \( u_0^{[1]} := l_0^{[1]} + r_0^{[1]} \).

It is a straightforward to deduce from \((1)-(3)\) a recursive definition of \( l_0^{[k]}, r_0^{[k]}, \) and \( u_0^{[k]} \) with corresponding error estimates

\[
\|u_\varepsilon - u_\varepsilon^{[k]}\|_\infty = O \left( \left( \text{erfc} \left( \frac{1}{2\sqrt{\varepsilon}} \right) \right)^{k+1} \right), \quad k \in \mathbb{N}_0;
\]

see also Section 3.

The construction of the sequence \((u_\varepsilon^{[k]})_{k \in \mathbb{N}_0}\) heavily relies on the “one-sided” operators \( \Gamma_\varepsilon', \Gamma_\varepsilon'' \) which assign to a given function a solution of \((1.1)\) subject to boundary conditions at the left or at the right, respectively. Furthermore, a close screening (the details can be found in Section 3) exhibits the distinctive importance of the norm \( \Theta[\varepsilon] \) of the composed operator \( \Gamma_\varepsilon' \circ \Gamma_\varepsilon'' \). Whenever \( \Theta[\varepsilon] < 1 \)—which is the case for all sufficiently small values of \( \varepsilon \)—then the sequence \((u_\varepsilon^{[k]})_{k \in \mathbb{N}_0}\) converges to \( u_\varepsilon \) as \( \varepsilon \to 0 \). Since \( \lim_{\varepsilon \to 0} \Theta[\varepsilon] = 0 \) we have convergence of \( u_\varepsilon^{[0]} \) to \( u_\varepsilon \) as \( \varepsilon \to 0 \); i.e., \( u_\varepsilon^{[0]} \) is an approximation of \( u_\varepsilon \) for small values of \( \varepsilon \), and—as a “side-product”—the functions \( u_\varepsilon^{[1]}, u_\varepsilon^{[2]}, \ldots \) are higher correction terms.

Hence, we have developed a seemingly new asymptotic analysis of \((1.1)\) by employing solution operators for half-line problems.

The paper is organized as follows. In Section 2 the preliminary investigations are re-visited from a theoretical point of view. A general class of parabolic initial-boundary value problems is considered. It is assumed that “one-sided” operators \( \Gamma_\varepsilon', \Gamma_\varepsilon'' \) as stated above exist. “Ping-pong expansions” are defined and convergence results are established. In Section 3 the theory of Section 2 is applied to \((1.1)\) and rigorous convergence results are obtained. Several conclusions are drawn in Section 4. The proofs of the results of Section 2 are rather straight-forward and can be found in [8].

### 2. PING-PONG EXPANSIONS AND THEIR CONVERGENCE

In this section we analyse the preliminary investigations of the previous section from an abstract point of view. A close screening of the argumentation exhibits the fact that rather few properties of \((1.1)\) are relevant for the excellent convergence properties of the proposed asymptotic expansion. These properties (assumptions (A.1)–(A.6) and (B1)–(B3) of Theorem 2 below) can be expected to hold for a broad class of parabolic IBVPs (initial-boundary value problems); see [5].
In order to give the analysis an appropriate framework—that is, we shall try not to be led astray by peculiarities of (1.1)—we shall impose as few as possible assumptions on the class of IBVPs considered here.

We consider IBVPs of the form

$$\begin{cases}
\partial_t u_\varepsilon - A_\varepsilon[u_\varepsilon] = f_\varepsilon, \\
u_\varepsilon(0, x) = u_\varepsilon'(x), \quad u_\varepsilon(t, 0) = \alpha_\varepsilon(t), \quad u_\varepsilon(t, 1) = \beta_\varepsilon(t),
\end{cases}$$

(2.1)

where $\varepsilon > 0$ is a “small” real parameter. The $\varepsilon$-dependent\(^2\) operator $A_\varepsilon$ acts on $C^2(\omega_T)$ via

$$A_\varepsilon[v] = a_\varepsilon v + b_\varepsilon \partial_x v + c_\varepsilon \partial_{xx} v,$$

where

(A.1) $a_\varepsilon, b_\varepsilon, c_\varepsilon \in C_b(\omega_T)$.

We assume

(A.2) $f_\varepsilon \in C_b(\omega_T)$.

(A.3) $\alpha_\varepsilon, \beta_\varepsilon \in C_b([0, T])$.

(A.4) $u_\varepsilon' \in C^2(\omega_T)$.

We shall make use of the concept of “$C_2$-solutions” of (2.1).

**DEFINITION 1.** $u_\varepsilon$ is a $C_2$-solution of (2.1) iff

1. $u_\varepsilon \in C_2 := \{v \in C^2(\omega_T) \cap C_b(\omega_T) : v(t, 0 +), v(t, 1 -) \text{ exist for all } t \in \[0, T]\}$, and $v(\cdot, 0 +), v(\cdot, 1 -)$ belong to $C_b([0, T])$,
2. for all $(t_0, x_0) \in \omega_T$, $P_t[u_\varepsilon](t_0, x_0) = f_\varepsilon(t_0, x_0)$,
3. $u_\varepsilon(\cdot, 0 +) = \alpha_\varepsilon$ and $u_\varepsilon(\cdot, 1 -) = \beta_\varepsilon$,
4. $\lim_{t \to 0} \int_\omega |u_\varepsilon(t, x) - u_\varepsilon'(x)| \, dx = 0$.

In order to keep things simple we assume

(A.5) There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, the IBVP (2.1) has exactly one $C_2$-solution.

We shall focus our attention on the distinguished recursively defined series

$$\left( \sum_{k=0}^\nu u^{(k)}_\varepsilon \right)_{\nu \in \mathbb{N}_0}$$

\(^2\) The subsequent investigations do not require explicit properties of the dependence on $\varepsilon$. Hence, in order to keep the investigations as general as possible, the functional dependence on $\varepsilon$ remains unspecified.
to approximate \(u_\varepsilon\). The recursions rely on linear operators \(\Gamma_{\varepsilon}^l, \Gamma_{\varepsilon}^r : \mathbf{C}_2 \to \mathbf{C}_2\), and sequences
\[
(u_\varepsilon^{[0]}, u_\varepsilon^{[1]}, u_\varepsilon^{[2]}, \ldots) = (I_{\varepsilon}, t_\varepsilon^{[0]} + r_\varepsilon^{[0]}, t_\varepsilon^{[1]} + r_\varepsilon^{[1]}, t_\varepsilon^{[2]} + r_\varepsilon^{[2]}, \ldots)
\]
such that

(E1) \(I_{\varepsilon} \in \mathbf{C}_2\) satisfies \(P_{\varepsilon}[I_{\varepsilon}] = f_{\varepsilon}\), \(\lim_{t \to 0} \int_{\omega} |I_{\varepsilon}(t,x) - u_{\varepsilon}^{l}(x)| \, dx = 0\),

(E2) \(t_\varepsilon^{[0]}, r_\varepsilon^{[0]} \in \mathbf{C}_2\) satisfy
\[
P_{\varepsilon}[t_\varepsilon^{[0]}] = 0, \quad \lim_{t \to 0} \int_{\omega} |t_\varepsilon^{[0]}(t,x)| \, dx = 0,
\]
\[
l_\varepsilon^{[0]}(t,0+) = \alpha_{\varepsilon}(t) - I_{\varepsilon}(t,0+)
\]
\[
P_{\varepsilon}[r_\varepsilon^{[0]}] = 0, \quad \lim_{t \to 0} \int_{\omega} |r_\varepsilon^{[0]}(t,x)| \, dx = 0,
\]
\[
r_\varepsilon^{[0]}(t,1-) = \beta_{\varepsilon}(t) - I_{\varepsilon}(t,1-) - l_\varepsilon^{[0]}(t,1-),
\]
and for all \(k \in \mathbb{N}_0\),

(E3) \(l_{\varepsilon}^{[k]} = \Gamma_{\varepsilon}^l[r_{\varepsilon}^{[k-1]}]\) satisfies
\[
P_{\varepsilon}[l_{\varepsilon}^{[k]}] = 0, \quad \lim_{t \to 0} \int_{\omega} |l_{\varepsilon}^{[k]}(t,x)| \, dx = 0,
\]
\[
l_{\varepsilon}^{[k]}(t,0+) = -r_{\varepsilon}^{[k-1]}(t,0+)
\]

(E4) \(r_{\varepsilon}^{[k]} = \Gamma_{\varepsilon}^r[l_{\varepsilon}^{[k]}]\) satisfies
\[
P_{\varepsilon}[r_{\varepsilon}^{[k]}] = 0, \quad \int_{\omega} |r_{\varepsilon}^{[k]}(t,x)| \, dx = 0,
\]
\[
r_{\varepsilon}^{[k]}(t,1-) = -l_{\varepsilon}^{[k]}(t,1-),
\]

(E5) \(\|u_{\varepsilon}^{[k-1]}\|_x \leq g_{k-1}(\varepsilon)\) and \(\|u_{\varepsilon} - (u_{\varepsilon}^{[0]} + u_{\varepsilon}^{[1]} + \ldots + u_{\varepsilon}^{[k-1]})\|_x \leq g_k(\varepsilon)\),

where \((g_k)_{k \in \mathbb{N}_0}\) is a sequence of order functions \(g_k : \mathbb{R}^+ \to \mathbb{R}^+\) with
\[
\lim_{\varepsilon \to 0} g_{k+1}(\varepsilon) = 0, \quad g_{k+1} = o(g_k)\text{ as } \varepsilon \to 0, k \in \mathbb{N}_0.
\]

**Remark 1.** (a) According to (E1) the function \(I_{\varepsilon}\) satisfies \(P[I_{\varepsilon}] = f_{\varepsilon}\) and fulfills the initial conditions. It is not assumed that \(I_{\varepsilon}\) satisfies the boundary conditions.

(b) Due to (E1), (E2) the function \(u_{\varepsilon}^{[0]} = I_{\varepsilon} + t_{\varepsilon}^{[0]} + r_{\varepsilon}^{[0]}\) is a \(C_2\)-solution of
\[
\begin{cases}
P[u_{\varepsilon}^{[0]}] = f_{\varepsilon}, & u_{\varepsilon}^{[0]}(0,x) = u_{\varepsilon}^{l}(x), \\
u_{\varepsilon}^{[0]}(t,0) = \alpha_{\varepsilon}(t) + r_{\varepsilon}^{[0]}(t,0+) , & u_{\varepsilon}^{[0]}(t,1) = \beta_{\varepsilon}(t),
\end{cases}
\]
i.e., $u^{[0]}_x$ satisfies the parabolic PDE (2.1), the correct initial condition, and the correct boundary condition at $x = 1$. By (E5) we have

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u^{[0]}_x\|_\infty \leq \lim_{\varepsilon \to 0} g_1(\varepsilon) = 0.$$ 

Hence $u^{[0]}_x$ is for “small” values of $\varepsilon$ an approximation for $u_\varepsilon$. Furthermore, since

$$\|r^{[0]}_x\|_T = \|u_\varepsilon(\cdot , 0 +) - u^{[0]}_x(\cdot , 0 +)\|_T \leq \|u_\varepsilon - u^{[0]}_x\|_\infty = O(g_1(\varepsilon))$$

as $\varepsilon \to 0$,

we obtain $\lim_{\varepsilon \to 0} \|r^{[0]}_x\|_T = 0$, so $u^{[0]}_x$ satisfies approximately the boundary condition at $x = 0$.

(c) The recursion formulae for $l^{[k]}_x, r^{[k]}_x, k \in \mathbb{N}$ imply

$$l^{[k]}_x = \left(1^{[k]}_x \ast (\Gamma^{[k]}_x \circ \Gamma^{[k]}_x)^{k-1}\right)\left[r^{[0]}_x\right], \quad r^{[k]}_x = (\Gamma^{[k]}_x \circ \Gamma^{[k]}_x)^k\left[r^{[0]}_x\right],$$

(2.2)

which shows the distinctive importance of $r^{[0]}_x$.

(d) Putting for $k \in \mathbb{N}_0$

$$\Sigma^{[k]}_x := \sum_{\xi=0}^k r^{[\xi]}_x = r^{[0]}_x + \sum_{\xi=1}^k (\Gamma^{[\xi]}_x \circ \Gamma^{[\xi]}_x)^\xi \left[r^{[0]}_x\right],$$

(2.3)

we see as in (b) that $\Sigma^{[k]}_x$ is a $C^*_x$-solution of

$$P\left[\Sigma^{[k]}_x\right] = f_\varepsilon, \quad \Sigma^{[k]}_x(0, x) = u^{[0]}_x(x), \quad \Sigma^{[k]}_x(t, 0) = \alpha_\varepsilon(t) + r^{[k]}_x(t, 0 +), \quad \Sigma^{[k]}_x(t, 1) = \beta_\varepsilon(t),$$

(2.4)

with $\|r^{[k]}_x(\cdot , 0 +)\|_T = O(g_{k+1})$ as $\varepsilon \to 0$. Hence we deduce from (E5) and the properties assumed for the sequence $(g_k)_{k \in \mathbb{N}_0}$ that the functions

$$u^{[0]}_x = \Sigma^{[0]}_x, \Sigma^{[1]}_x, \Sigma^{[2]}_x, \Sigma^{[3]}_x, \ldots, \Sigma^{[k]}_x, \ldots$$

are approximations of zeroth, first, second, third, . . . , $k$th, . . . order for $u_\varepsilon$.

(c) Trivial choices for $l^{[0]}_x, r^{[0]}_x$, and $\Gamma^{[0]}_x$, $\Gamma^{[k]}_x$ would be $l^{[0]}_x = u^{[0]}_x$, $l^{[0]}_x = r^{[0]}_x = 0$, and $\Gamma^{[k]}_x = \Gamma^{[k]}_x = 0$, the zero-operator. Hence operators $\Gamma^{[k]}_x, \Gamma^{[k]}_x$ as proposed in (E3), (E4) are always available. (This choice is however not of interest here. We are not concerned with an existence and uniqueness theory for parabolic PDEs.)

The topic of discussion is to approximate “complicated” solutions $u_\varepsilon$ by more tractable functions.

(f) According to (2.2)–(2.4) the boundary conditions at $x = 0$ and $x = 1$ enter the recursion formula in different ways. Indeed a much more important role is played by $r^{[0]}_x$ and the boundary condition at $x = 1$, which is satisfied by each $\Sigma^{[k]}_x$, than by $l^{[0]}_x$ and the boundary condition at $x = 0$,
which will usually not be satisfied by any $\Sigma^{[k]}_n$. It is however not difficult to interchange the roles of the right and the left boundary.

(g) Loosely speaking the construction of the sequence $(u_n^{[0]}, u_n^{[1]}, u_n^{[2]}, \ldots)$ has something in common with “ping-pong”: consider for $k \in \mathbb{N}_0$ the function $u_n^{[k]}$. Then one constructs $f_n^{[k+1]}$ by solving a “left-hand boundary value problem,” whose purpose is the elimination of $u_n^{[k]}(., 0 +)$ at the left boundary (“ping”). After this intermediate step one finds $u_n^{[k+1]}$ by doing the very same thing, but now with $f_n^{[k+1]}$ at the right boundary: one solves a “right-hand boundary value problem,” whose purpose is the elimination of $f_n^{[k+1]}(., 1 -)$ at the right boundary (“pong”). Therefore, one is motivated to call the approximation defined by (E1)–(E5) a “ping-pong expansion.”

Due to (E3), (E4) the operators $\Gamma^l, \Gamma^r$ assign to a function $u \in C_2$, a solution $w_\varepsilon$ of the homogeneous PDE $P_\varepsilon[w_\varepsilon] = 0$ with vanishing initial data and with boundary conditions given by $u(., 0 +), u(., 1 -)$, respectively. Hence $\Gamma^l, \Gamma^r$ do not depend on the entire function $u$ but only on its trace $u(., 0 +), u(., 1 -)$ at the left (right) boundary, respectively.

We assume henceforth

(A.6) There are linear operators $L_\varepsilon : C_B(0, T) \to C_2$ and $R_\varepsilon : C_B(0, T) \to C_2$ such that

(A.6.a) $L_\varepsilon, R_\varepsilon$ are $\|\cdot\|_T - \|\cdot\|_\infty$-bounded, i.e.,

\[
\|L_\varepsilon[a]\|_\infty, \|a\|_T \leq 1 < \infty,
\]

\[
\|R_\varepsilon[a]\|_\infty, \|a\|_T \leq 1 < \infty.
\]

(A.6.b) For all $a \in C_B(0, T)$,

\[
P_\varepsilon[L_\varepsilon[a]] = P_\varepsilon[R_\varepsilon[a]] = 0.
\]

(A.6.c) For all $a \in C_B(0, T)$,

\[
L_\varepsilon[a](\cdot, 0 +) = a, \quad R_\varepsilon[a](\cdot, 1 -) = a.
\]

(A.6.d) For all $a \in C_B(0, T)$,

\[
\lim_{t \to 0} \int_\omega |L_\varepsilon[a](t, x)| \, dx = \lim_{t \to 0} \int_\omega |R_\varepsilon[a](t, x)| \, dx = 0.
\]

We set for $v \in C_2$

\[
\Gamma^l[v] := L_\varepsilon[-v(\cdot, 0 +)], \quad \Gamma^r[v] := R_\varepsilon[-v(\cdot, 1 -)].
\]

(2.5)

It is obvious that the operators $\Gamma^l : C_2 \to C_2$, $\Gamma^r : C_2 \to C_2$ of (2.5) are linear and $\|\cdot\|_\infty - \|\cdot\|_\infty$-bounded, i.e.,

\[
\|\Gamma^l[v]\|_\infty, \|v\|_\infty \leq 1 \leq \|L_\varepsilon\|_{T, \infty},
\]

\[
\|\Gamma^r[v]\|_\infty, \|v\|_\infty \leq 1 \leq \|R_\varepsilon\|_{T, \infty}.
\]
Now it is straightforward to prove

**Proposition 1.** Assume (A.1)-(A.6) and (E1). Let \( \Gamma_x, \Gamma'_x \) be as in (2.5), let

\[
l_x^{[0]} := L_x \left[ \alpha_x - L_x(\cdot, 0 +) \right], \quad r_x^{[0]} := R_x \left[ \beta_x - L_x(\cdot, 1) - l_x^{[0]}(\cdot, 1 -) \right],
\]

(2.6)

and for \( k \in \mathbb{N} \) let \( l_x^{[k]} := \Gamma_x^{[k]}(r_x^{[k-1]}), r_x^{[k]} := \Gamma'_x^{[k]}(l_x^{[k]}) \).

Then (E2)-(E4) hold.

Now we are in the position to formulate the main theoretical result (whose proof is essentially an application of properties of the geometric series [8]):

**Theorem 2.** Assume (A.1)-(A.6) and (E1). Let \( \Gamma_x, \Gamma'_x \) as in (2.5) and for \( k \in \mathbb{N}_0 \) let \( l_x^{[k]}, r_x^{[k]} \) be as in Proposition 1. Assume furthermore

\[
(1 + \| \Gamma_x^{[k]} \|_{\infty}) (\| R_x \|_{T, \infty} \| \beta_x \|_{T} + \| L_x \|_{T, \infty} + \| L_x \|_{T, \infty} \| \alpha_x - L_x(\cdot, 0 +) \|_{T}) \leq K.
\]

(B1) There is \( K \in ]0, \infty[ \) such that for all \( \varepsilon \in ]0, \varepsilon_0[ \),

\[
\left\| \left[ \Gamma_x^{[k]} \Gamma'_x \right]^{[k]} \|_{\infty, \infty} \leq \left[ \Theta(\varepsilon) \right]^{k}.
\]

(B2) There is for each \( \varepsilon \in ]0, \varepsilon_0[, \) a number \( \Theta(\varepsilon) \in ]0, 1[ \) such that

\[
\lim_{\varepsilon \to 0} \Theta(\varepsilon) = 0, \quad \text{and for all } k \in \mathbb{N}
\]

(B3) There is \( K_1 \in [0, \infty[ \) such that for all \( \varepsilon \in ]0, \varepsilon_0[, \) and for all \( a \in C_\beta(J), \) if \( v_x = v_x[a] \) is the unique \( C_\gamma \)-solution of

\[
P_x[v_x] = 0, \quad v_x(0, x) = 0, v_x(t, 0) = a, v_x(t, 1) = 0,
\]

then

\[
\int_0^t |v_x(\tau, z)| \, d\tau \, dz \leq K_1 \| a \|_{\infty}.
\]

Under these assumptions we have for all \( k \in \mathbb{N} \)

\[
\left\| u_x^{[k]} \right\|_{\infty} \leq K \left[ \Theta(\varepsilon) \right]^{k}, \quad \left\| u_x - \sum_{i=0}^{k-1} u_x^{[i]} \right\|_{\infty} \leq \frac{K}{1 - \Theta(\varepsilon)} \left[ \Theta(\varepsilon) \right]^{k}, \quad (2.7)
\]

i.e., (E5) is satisfied with

\[
g_0(\varepsilon) := \max \{ 1, \| u_x^{[0]} \|_{\infty} \}, \quad g_k(\varepsilon) := \frac{K}{1 - \Theta(\varepsilon)} \left[ \Theta(\varepsilon) \right]^{k}, \quad k \in \mathbb{N}.
\]

Some remarks will clarify the theorem above:

**Remark 2.** (1) In Theorem 2 no strong maximum principle is assumed. This is in accordance with [3] whose asymptotic analysis is settled only on a
weak maximum principle. In Theorem 2 assumptions (A.1)–(A.6) allow for a replacement of the maximum principle by the weaker assumption (B3).

(2) According to (2.7) we have \( \lim_{k \to \infty} \| u_e - \sum_{\nu=0}^k u_e^{(\nu)} \|_\infty = 0 \) for each \( \varepsilon \in ]0, \varepsilon_0[ \), i.e., the series \( u_e^{(0)} + u_e^{(1)} + \cdots + u_e^{(k)} \) converges uniformly to \( u_e \) as \( k \to \infty \). Actually, this improves (E5) which gives for fixed \( \varepsilon \in ]0, \varepsilon_0[ \) the estimate \( \lim_{k \to \infty} \| u_e - \sum_{\nu=0}^k u_e^{(\nu)} \|_\infty \leq \limsup_{k \to \infty} g_k(\varepsilon) \), with a right-hand side perhaps larger than 0.

(3) That the assumption “\( \lim_{\varepsilon \to 0} \Theta(\varepsilon) = 0 \)” is not essential to obtain the estimates (2.7).

THEOREM 3. Assume (A.1)–(A.6) and (E1). Let \( \Gamma^i_e, \Gamma^j_e \) be as in (2.5) and for \( k \in \mathbb{N}_0 \) let \( l^k_{[1]}, r^k_{[1]} \) be as in Proposition 1. Assume (B1) and (B3) of Theorem 2 and furthermore

\[(B^*) \quad \text{For each } \varepsilon \in ]0, \varepsilon_0[ \text{ there is a number } \Theta(\varepsilon) \in ]0, 1[ \text{ such that for all } k \in \mathbb{N} \]

\[ \left\| \left[ \Gamma^i_e \circ \Gamma^j_e \right]^k \right\|_{\infty, \infty} \leq \left[ \Theta(\varepsilon) \right]^k. \]

Then we have for all \( k \in \mathbb{N} \)

\[ \left\| u^{[k]}_e \right\|_\infty \leq K \left[ \Theta(\varepsilon) \right]^k, \quad \left\| u_e - \sum_{\nu=0}^{[k-1]} u^{(\nu)}_e \right\|_\infty \leq \frac{K}{1 - \Theta(\varepsilon)} \left[ \Theta(\varepsilon) \right]^k. \quad (2.8) \]

Under these assumptions of Theorem 3 there is uniform convergence of the series \( u^{[0]}_e + u^{[1]}_e + \cdots + u^{[k]}_e \) to \( u_e \) as long as the estimate of \((B^*)\) holds. Hence \( u^{[0]}_e + u^{[1]}_e + \cdots + u^{[k]}_e \) is for fixed \( \varepsilon \) an approximation for \( u_e \). Here, three aspects are of interest:

(a) If \( \Theta : ]0, \varepsilon_0[ \to ]0, 1[ \) is increasing, there there is due to (2.8) for each \( \eta \in ]0, \varepsilon_0[ \) a number \( k(\eta) \in \mathbb{N} \)—independent of \( \varepsilon \in ]0, \varepsilon_0[ \)—such that

\[ \forall \varepsilon \in ]0, \varepsilon_0[ \}, \quad \left\| u_e - \sum_{\nu=0}^{k(\eta)} u^{(\nu)}_e \right\|_\infty \leq \eta, \]

i.e., we can choose independently of \( \varepsilon \in ]0, \varepsilon_0[ \) a fixed “order” of the expansion to achieve a prescribed accuracy of the approximation.

(b) If \( \Theta \) is not increasing, then the norm \( \| u_e - \sum_{\nu=0}^k u^{(\nu)}_e \|_\infty \) will possibly grow for fixed \( k \). In this case an increasing number (as \( \varepsilon \to 0 \)) of terms in the expansion of \( u_e \) may be necessary to achieve a certain accuracy of the approximation.

(c) The additional assumption “\( \lim_{\varepsilon \to 0} \Theta(\varepsilon) = 0 \)” ensures that \( u^{[0]}_e \) is for all sufficiently small \( \varepsilon \) an acceptable approximation for \( u_e \).
3. AN APPLICATION OF PING-PONG EXPANSIONS

In this section we reconsider the singularly perturbed IBVPs of Section 1,

\[
\begin{align*}
P_\varepsilon[u_\varepsilon] := & \partial_t u_\varepsilon + \partial_x u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = 0, \\
u_\varepsilon(t, 0) = & \alpha_\varepsilon(t), \\
u_\varepsilon(t, 1) = & \beta_\varepsilon(t),
\end{align*}
\]

with \( \varepsilon \in [0, \varepsilon_0] \), where \( \varepsilon_0 = \varepsilon_* \) is uniquely determined via

\[
4 \operatorname{erfc}\left( \frac{1}{2 \sqrt{\varepsilon_*}} \right) = 1, \quad \varepsilon_* \in \mathbb{R}, \text{i.e., } \varepsilon_* = 0.3778422150 \ldots .
\]

We assume

\[(D.1) \quad \text{For each } \varepsilon \in [0, \varepsilon_*], \alpha_\varepsilon, \beta_\varepsilon \in C_0([0, T]) \text{ and } u'_I \in C^2(\overline{\omega}).\]

We also assume

\[(D.2) \quad \text{There is } K_4 \in [0, \infty[ \text{ such that for all } \varepsilon \in [0, \varepsilon_*]:
\]

\[
\|u'_I\|_{\omega}, \quad \left\| (u'_I)' \right\|_{\omega}, \quad \left\| (u'_I)'' \right\|_{\omega}, \quad \|\alpha_\varepsilon\|_T, \quad \|\beta_\varepsilon\|_T \leq K_4.
\]

The validity of (A.6) (existence and uniqueness of smooth solutions) and of (B3) of Theorem 2 (weak version of the maximum principle) follows from the standard theory of parabolic PDEs [5].

According to (D.2) there are constants \( K_2, K_3 \in [0, \infty[ \) and functions \( G_\varepsilon : [0, \infty[ \to \mathbb{R}, \varepsilon \in [0, \varepsilon_*[, \) such that

\[
\text{for all } \varepsilon \in [0, \varepsilon_*[, \quad G_\varepsilon \text{ is twice differentiable and } G_\varepsilon|_\omega = u'_I, \quad (3.2)
\]

and

\[
\forall \varepsilon \in [0, \varepsilon_*[, \quad \forall \gamma \in [0, \infty[, \quad \left| (G_\varepsilon' - \gamma G_\varepsilon')'(y) \right| \leq K_5(1 + y^{K_1}). \quad (3.3)
\]

Remark 3. Due to assumption (D.2) there are extensions \( G_\varepsilon^+ \) of \( u'_I \) to \( [0, \infty[ \) which together with their first and second derivatives bounded independently of \( \varepsilon \in [0, \varepsilon_*[ \). However, if one chooses polynomials \( p_i(x), \ x \in \omega, \) as initial conditions for \( u'_I \) these functions \( G_\varepsilon^+ \) will not agree with the canonical extensions \( p_i(z), \ z \in [0, \infty[, \) In order to avoid this inconvenience polynomial growth of \( G_\varepsilon^+ \) is allowed in (3.3).

Our aim is to apply Theorem 2. Having checked already several of the assumptions it remains to provide \( I_\varepsilon, R_\varepsilon, L_\varepsilon \) satisfying (E1), (A.6) and (B1), (B2) of Theorem 2.
(a) The Initial Function $I$. We consider the half-space problem

$$\partial_t \Phi_e = -\partial_x \Phi_e + \varepsilon \partial_x \Phi_e, \quad \Phi_e(0, x) = G_e(x), \Phi_e(t, 0) = u'_e(0 +), \quad (t, x) \in [0, T] \times [0, \infty[. \quad (3.4)$$

Introducing

$$\phi_e(t, x) := \exp\left(-\frac{x}{2\varepsilon}\right)(\Phi_e(t, x) - G_e(x)), \quad (t, x) \in [0, T] \times [0, \infty[$$

one obtains the IBVP

$$\partial_t \phi_e = -\frac{1}{4\varepsilon} \phi_e + \varepsilon \partial_x \phi_e + p_e, \quad \phi_e(0, x) = 0, \phi_e(t, 0) = 0, \quad (t, x) \in [0, T] \times [0, \infty[. \quad (3.5)$$

where for $x \in [0, \infty[$,

$$p_e(x) = \exp\left(-\frac{x}{4\varepsilon}\right)H_e(x), \quad \text{with} \quad H_e(x) = \varepsilon G''_e(x) - G'_e(x).$$

The IBVP (3.5) is easily solvable by means of the Sine–Fourier transformation [10]. By setting

$$I_e(t, x) := \exp\left(\frac{x}{4\varepsilon}\right) \phi_e(t, x) + G_e(x), \quad (t, x) \in \omega_T,$$

one obtains [8]

$$I_e(t, x) := \varepsilon \left(\left(u'_L\right)'(x) - \left(u'_L\right)'(0)\right) + u'_L(0 +)
+ \int_0^\infty \exp\left(-\frac{y}{\varepsilon}\right)\left[H_e(x + y) - H_e(y)\right] dy
+ \frac{1}{2} \int_0^\infty H_e(x - y) \left[ \text{erfc}\left(\frac{t - y}{2\sqrt{\varepsilon t}}\right) + \exp\left(\frac{y}{2\varepsilon}\right) \text{erfc}\left(\frac{t + y}{2\sqrt{\varepsilon t}}\right) \right] dy
+ \frac{1}{2} \int_0^\infty H_e(x + y) \left[ \exp\left(-\frac{y}{\varepsilon}\right) \text{erfc}\left(\frac{t + y}{2\sqrt{\varepsilon t}}\right) \right] dy
- \frac{1}{2} \int_0^\infty H_e(y) \exp\left(-\frac{y}{\varepsilon}\right) \text{erfc}\left(\frac{t - x - y}{2\sqrt{\varepsilon t}}\right) dy
- \frac{1}{2} \int_0^\infty H_e(y) \exp\left(\frac{x}{\varepsilon}\right) \text{erfc}\left(\frac{t + x + y}{2\sqrt{\varepsilon t}}\right) dy. \quad (3.6)$$
Now it is straightforward to deduce [8].

**Proposition 4.** Assume (D.1) and (D.2). Let \( G_\varepsilon \) be as in (3.2), (3.3) and let \( I_\varepsilon \) be as in (3.6). Then \( I_\varepsilon \) is a \( C_2 \)-solution of

\[
\partial_t I_\varepsilon = -\partial_x I_\varepsilon + \varepsilon \partial_{xx} I_\varepsilon,
I_\varepsilon(0, x) = u_\varepsilon'(x), I_\varepsilon(t, 0) = u_\varepsilon'(0 + ), (t, x) \in \omega_T.
\]

Furthermore, there is a constant \( K_5 = K_1(K_2, K_3, K_4, \varepsilon, \epsilon) \in ]0, \infty[ \) such that for all \( \varepsilon \in ]0, \varepsilon_0[ \), \( \| I_\varepsilon \|_\infty \leq K_5 \).

**Remark 4.** The estimate on \( \| I_\varepsilon \|_\infty \) does not depend on \( T \).

(b) **The Operator \( L_\varepsilon \).** The purpose of \( L_\varepsilon \) is to map left-boundary data \( a \) into \( C_2 \) such that \( P_\varepsilon[L_\varepsilon[a]] = 0 \) and \( L_\varepsilon[a](0, \cdot) = 0 \). We take \( a \in C_0^\infty(]0, T[) \) and consider the half-space problem (1.7) with “\( a(t) \)” replacing \( -u_\varepsilon^{[0]}(t, 0 + ) \).” Proceeding in analogy to the construction of \( I_\varepsilon \) one solves this IBVP by means of the Sine–Fourier transformation [10]. Denoting the solution by “\( L_\varepsilon[a] \)” we have [8]

\[
L_\varepsilon[a](t, x) = \frac{x}{\sqrt{4\pi \varepsilon \pi}} \int_0^t s^{-3/2} \exp \left(-\frac{(x-s)^2}{4\varepsilon s}\right) a(t-s) \, ds, \quad (t, x) \in \omega_T. \tag{3.7}
\]

Via (3.7) a unique mapping \( L_\varepsilon : C_0^\infty(]0, T[) \to C_2 \), \( a \mapsto L_\varepsilon[a] \) is defined. It is easy to prove [8]

**Proposition 5.** The operator \( L_\varepsilon \) is for all \( \varepsilon \in ]0, \varepsilon_0[ \) linear and satisfies (A.6), in particular \( \| L_\varepsilon \|_{T, \infty} \leq 2 \).

(c) **The Operator \( R_\varepsilon \).** The purpose of \( R_\varepsilon \) is to map right-boundary data \( a \) onto \( C_2 \) such that \( P_\varepsilon[R_\varepsilon[a]] = 0 \) and \( R_\varepsilon[a](0, \cdot) = 0 \). We take \( a \in C_0^\infty(]0, T[) \) and consider the half-space problem (1.4) with “\( a(t) \)” replacing \( -L_\varepsilon(t, 1 - \cdot) \)” Proceeding in analogy to the construction of \( L_\varepsilon \) one solves this IBVP by means of the Sine–Fourier transformation [10]. Denoting the solution by “\( R_\varepsilon[a] \)” we have [8]

\[
R_\varepsilon[a](t, x) = \frac{1-x}{\sqrt{4\pi \varepsilon \pi}} \int_0^t s^{-3/2} \exp \left(-\frac{(1-x+s)^2}{4\varepsilon s}\right) b(t-s) \, ds, \quad (t, x) \in \omega_T. \tag{3.8}
\]

Via (3.8) a unique mapping \( R_\varepsilon : C_0^\infty(]0, T[) \to C_2 \), \( a \mapsto R_\varepsilon[a] \) is defined. It is easy to prove [7]
PROPOSITION 6. The operator $R_\varepsilon$ is for all $\varepsilon \in [0, \varepsilon_\star]$ linear and satisfies (A.6), in particular $\|R_\varepsilon\|_{L^\infty} \leq 1$. Furthermore, for all $\varepsilon \in [0, \varepsilon_\star]$, for all $a \in C_B(J)$, and for all $x \in \overline{\omega} = [0, 1]$,

$$\|R_\varepsilon (a)(\cdot, x)\|_2 \leq 2\|a\|_T \operatorname{erfc}\left(\sqrt{\frac{1-x}{4\varepsilon}}\right). \quad (3.9)$$

(d) Ping-Pong Asymptotics for (3.1). We introduce the operators $\Gamma^r_\varepsilon$, $\Gamma^l_\varepsilon : \mathbf{C}_2 \to \mathbf{C}_2$ as in (2.5); i.e., we set for $v \in \mathbf{C}_2$

$$\Gamma^l_\varepsilon[v] := L_\varepsilon[-v(\cdot, 0 + )], \quad \Gamma^r_\varepsilon[v] := R_\varepsilon[-v(\cdot, 1 - )]. \quad (3.10)$$

We readily deduce from (3.10), from Lemma 5, from Lemma 6, and from (3.9) the estimate

$$\forall \varepsilon \in ]0, \varepsilon_\star[, \quad \|\Gamma^l_\varepsilon \circ \Gamma^r_\varepsilon\|_{L^\infty} \leq 4 \operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right). \quad (3.11)$$

Estimate (3.11) is the last requirement to apply Theorem 2. We deduce

COROLLARY 1. Assume (D.1), (D.2). Let $\varepsilon \in ]0, \varepsilon_\star[$. Let $I_\varepsilon$ be as in (3.6), let $L_\varepsilon$ be as in (3.7), and let $R_\varepsilon$ be as in (3.8). For $k \in \mathbb{N}_0$ let $l^{(k)}_\varepsilon$ and $r^{(k)}_\varepsilon$ be as in Proposition 1. Let $K_4, K_5 \in ]0, \infty[$ be as in (D.2) and as in Lemma 4, respectively.

Then the ping-pong series $(u^{(0)}_\varepsilon, u^{(1)}_\varepsilon, u^{(2)}_\varepsilon, \ldots)$ with

$$u^{(0)}_\varepsilon = I_\varepsilon + l^{(0)}_\varepsilon + r^{(0)}_\varepsilon, \quad u^{(k)}_\varepsilon = l^{(k)}_\varepsilon + r^{(k)}_\varepsilon, \quad k \in \mathbb{N},$$

has the following properties:

(a) For all $k \in \mathbb{N}_0$, $\|u^{(k)}_\varepsilon\|_\infty \leq 6(K_4 + K_5)(4 \operatorname{erfc}(1/2\sqrt{\varepsilon}))^k$.

(b) For all $k \in \mathbb{N}$, $\|u_\varepsilon - \sum_{\ell=0}^{k-1} u^{(\ell)}_\varepsilon\|_\infty \leq (6(K_4 + K_5)/(1 - 4 \operatorname{erfc}(1/2\sqrt{\varepsilon})))^k/4 \operatorname{erfc}(1/2\sqrt{\varepsilon}))^k$; in particular, the ping-pong series $\left(\sum_{\ell=0}^{k-1} u^{(\ell)}_\varepsilon\right)_{k \in \mathbb{N}_0}$ converges uniformly to $u_\varepsilon$ as $k \to \infty$.

Remark 5. The ping-pong series $\left(\sum_{\ell=0}^{k-1} u^{(\ell)}_\varepsilon\right)_{k \in \mathbb{N}_0}$ of Corollary 1 converges for all $\varepsilon \in ]0, \varepsilon_\star[$ independently of the choices of $u^{(l)}_\varepsilon, \alpha_\varepsilon, \beta_\varepsilon$ as long as (D.1), (D.2) hold. The norms of these functions determine the rate of convergence of the ping-pong series, but not whether the ping-pong series converge at all.

(e) Discussion. In accordance with the discussion of the Introduction one might expect that $I_\varepsilon + l^{(0)}_\varepsilon$ is away from $x = 1$ for all sufficiently small $\varepsilon$ an excellent approximation for $u_\varepsilon$. This is indeed the case. We deduce from Lemma 6 for all $\theta \in ]0, 1[$,

$$\forall (t, x) \in ]0, T[ \times ]0, \theta[, \quad \left|u^{(0)}_\varepsilon\right|(t, x) \leq 6(K_4 + K_5) \operatorname{erfc}\left(\frac{\sqrt{1-\theta}}{2\sqrt{\varepsilon}}\right),$$
such that due to Theorem 1

$$V(t, x) \in ]0, T[ \times ]0, \theta],$$

$$\left| u_\varepsilon - (I_\varepsilon + l^{[0]}_\varepsilon)(t, x) \right| \leq 6(K_1 + K_3) \frac{2 - C}{1 - C} C, \quad C = \text{erfc}\left(\frac{\sqrt{1 - \theta}}{2\varepsilon}\right),$$

i.e., $I_\varepsilon + l^{[0]}_\varepsilon \to u_\varepsilon$ uniformly on $]0, T[ \times ]0, \theta]$ as $\varepsilon \to 0$.

Now let us discuss the behavior of $I_\varepsilon + l^{[0]}_\varepsilon$, as $\varepsilon \to 0$. Here the initial and boundary data will play a prominent role. In order to keep things simple let us assume that $u^I_\varepsilon = u^I$ and $\alpha_\varepsilon = \alpha \in C_p([0, T])$ are $\varepsilon$-independent. Then it is easy to deduce from Lemma 4 and from (3.7) with the aid of Lebesgue’s dominated convergence theorem

$$V(t, x) \in \omega_T, \quad \text{if } x - t \neq 0, \text{ then } \lim_{\varepsilon \to 0} \left(I_\varepsilon + l^{[0]}_\varepsilon\right)(t, x) = u_0(t, x),$$

where

$$u_0 : \omega_T \to \mathbb{R}, \quad u_0(t, x) = \begin{cases} u^I(x - t), & x - t > 0 \\ \frac{\alpha(0)}{2} + u^I(0), & x = t \\ \alpha(t - x) + u^I(0), & x - t < 0 \end{cases}$$

is a weak solution of the transport equation

$$\partial_t u_0 + \partial_x u_0 = 0, \quad u_0(0, x) = u^I(x), \quad u_0(t, 0) = \alpha(t).$$

A close screening of the estimates actually gives a more detailed result:

$I_\varepsilon + l^{[0]}_\varepsilon \to u_0$ uniformly on each compact $K \subset \subset \{(t, x) \in \omega_T : x - t \neq 0\}$.

Furthermore, since $I_\varepsilon + l^{[0]}_\varepsilon$ is uniformly (i.e., independent of $\varepsilon \in ]0, \varepsilon_a[$) bounded on $\omega_T$, we deduce from the point wise convergence almost everywhere

$$\forall p \in [1, \infty], \quad \lim_{\varepsilon \to 0} \int_{\omega_T} \left| u_0(t, x) - \left(I_\varepsilon + l^{[0]}_\varepsilon\right)(t, x) \right|^p dt dx = 0.$$

Uniform convergence on $\omega_T$ of $I_\varepsilon + l^{[0]}_\varepsilon$ to $u_0$ is usually not available because the limiting function $u_0$ is continuous iff the additional assumption $\alpha(0 + ) = 0$ holds.
4. CONCLUSIONS

In the previous sections we derived a recursive approximation for singularly perturbed parabolic equations of the form

\[ \partial_t u_\varepsilon = a_\varepsilon \partial_x u_\varepsilon + b_\varepsilon \partial_x u_\varepsilon + c_\varepsilon \partial_x \partial_x u_\varepsilon, \quad u_\varepsilon(0, x) = u_\varepsilon^0(x) \]  

(4.12)

for \( x \in [0, 1] \) with time-dependent boundary conditions at \( x = 0 \) and \( x = 1 \), respectively.

The peculiarity of the ping-pong expansions presented here is the employment of one-sided solution operators \( \Gamma_\varepsilon^l, \Gamma_\varepsilon^r \) which allow us to obtain solutions of (4.12) with vanishing initial data and prescribed boundary data at the left or at the right boundary, respectively.

The existence of such operators is not the crucial point for the definition of ping-pong extensions. Seemingly there are many operators with the required properties, compare Remark 1(e). The essential part is the fact that some of the operators \( \Gamma_\varepsilon^l, \Gamma_\varepsilon^r \) allow for a rather explicit representation. This is, e.g., the case for the model problem (3.1) where convolution-type representations for \( \Gamma_\varepsilon^l, \Gamma_\varepsilon^r \) are available by means of the Sine–Fourier transform.

It would go far beyond the scope of the present paper to determine criteria under which additional assumptions on the system’s parameters, in particular, on their dependence of \( \varepsilon \), a comparably explicit representation for \( \Gamma_\varepsilon^l, \Gamma_\varepsilon^r \) is available. Although a discussion of this question in broad generality is rather interesting it has to be postponed to future investigations.\(^3\)

The authors are indebted to one of the anonymous referees for pointing out the following fact stimulating further investigations of another kind. Seemingly the ping-pong expansion for (1.1) allows for an excellent approximation of the derivative of \( u_\varepsilon \), in particular at \( x = 1 \). The question arises whether there is also uniform convergence of the ping-pong expansion’s derivative. The discussion of this question seemingly either requires additional compatibility conditions at the boundary or has to be performed away from \( t = 0 \). In any case the calculations would become rather involved (even for the introductory example (1.1)) and are thus left for future work.

\(^3\) However, it is worth noting that preliminary investigations on the singularly perturbed Stefan problem mentioned in the Introduction are encouraging.
REFERENCES