N-Extremal Matrices of Measures for an Indeterminate Matrix Moment Problem

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In this paper we study the N-extremal matrices of measures associated to a completely indeterminate matrix moment problem, i.e., those matrices of measure \( W \), solutions of a completely indeterminate matrix moment problem for which the linear space of matrix polynomials is dense in the corresponding \( L^2(W) \).

1. INTRODUCTION

Let \( \mathcal{M}^* \) denote the set of positive Borel measures on the real line having moments of every order and infinite support. With \( \mu \in \mathcal{M}^* \) we can associate the sequence \( (p_n) \) of orthonormal polynomials. A measure \( \mu \in \mathcal{M}^* \) is determinate if no other measure has the same moments as those of \( \mu \), otherwise \( \mu \) is indeterminate. Let \( V_\mu \) denote the set of positive measures on the real line having the same sequence of moments as \( \mu \); a well-known theorem by M. Riesz [R] establishes that for \( v \in V_\mu \), the linear space of polynomials is dense in \( L^2(v) \) if and only if for some (and then for any) \( z \in \mathbb{C} \setminus \mathbb{R} \), \( \int_{\mathbb{R}} \frac{dz}{v(z)} \) is an extreme point (in the sense of convexity) of the convex set \( \{ \mu \cdot \frac{dz}{v(z)} : \rho \in V_\mu \} \). These measures are called N-extremal measures.

Associated to an indeterminate moment problem there is a one-dimensional set of N-extremal measures having a number of interesting properties:

1. An N-extremal measure is discrete with mass in countably many points, which are the zeros of a certain entire function of minimal exponential type (cf. [A, Theorem 2.4.3]).

2. For every real number \( t \) there is one and only one N-extremal measure \( \mu_t \) having a mass point at \( t \) (cf. [A, Theorem 3.4.1]).

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(3) If $\mu$ is the $N$-extremal measure having a mass point at $t$ then the measure $\mu - \delta_t$ is determinate (it follows from [A, Theorem 3.4]).

(4) The $N$-extremal measure $\mu$ having a mass point at $t$ reaches the maximum mass which can be concentrated in $t$ for any solution of the indeterminate moment problem, i.e.,

$$\mu_1(\{t\}) = \sup \{ \nu(\{t\}) : \nu \in V_\mu \}.$$ 

Moreover, this maximum is uniquely attained by $\mu$, and $\mu_1(\{t\}) = 1/\sum_{n=0}^{\infty} |p_n(t)|^2$ (cf. [A, Theorem 3.4.1]).

The aim of this paper is to study the $N$-extremal matrices of measures associated to a completely indeterminate matrix moment problem of size $N \times N$. We will show that these $N$-extremal matrices of measures also satisfy a number of good properties although the matrix structure creates important divergences; for instance, for every real number $t$, and for any natural number $m$, $0 \leq m < N$, there are infinitely many $N$-extremal measures having a mass point at $t$ of rank $m$, but only one having a mass point at $t$ of rank $N$. To present the results in full, we need some definitions and previous results.

Given $\mu = (\mu_{i,j})_{1 \leq i,j \leq N}$ a positive definite matrix of measures (for any Borel set $A$ the numerical matrix $\mu(A)$ is positive semidefinite) with finite matrix moments $S_k = \int_A t^k d\mu(t)$ of any order $k \geq 0$, we also denote by $V_\mu$ the set of positive definite matrices of measures having the same matrix moments as those of $\mu$.

By $(P_n)_{n=0}^{\infty}$ we denote the sequence of orthonormal matrix polynomials with respect to $\mu$, $P_n$ of degree $n$, and with non-singular leading coefficients (this sequence of orthonormal matrix polynomials is uniquely determined up to multiplication to the left by unitary matrices).

These polynomials $(P_n)_{n=0}^{\infty}$ satisfy a three-term recurrence relation of the form

$$tP_n(t) = A_{n+1} P_{n+1}(t) + B_n P_n(t) + A_n^* P_{n-1}(t), \quad n \geq 0, \quad (1.1)$$

($A_n$ and $B_n$ being $N \times N$ matrices such that $\det(A_n) \neq 0$ and $B_n^* = B_n$, with initial condition $P_{-1}(t) = \theta$. (Here and in the rest of this paper, we write $\theta$ for the null matrix, the dimension of which can be determined from the context. For instance, here $\theta$ is the $N \times N$ null matrix.) It is well-known that this recurrence relation is equivalent to the orthogonality with respect to a positive definite matrix of measures: This is the matrix version of Favard's Theorem (see [AN] or [DL1]).
We denote by \( Q_n(t) \) the corresponding sequence of polynomials of the second kind,

\[
Q_n(t) = \int_\mathbb{R} \frac{P_n(t) - P_n(x)}{t-x} \, dq(x), \quad n \geq 0,
\]

which also satisfy the recurrence relation (1.1), with initial conditions \( Q_0(t) = 0 \) and \( Q_1(t) = A_1^{-1} \).

The determinacy or indeterminacy of the matrix moment problem is related to the deficiency indices \( \delta^+ \) and \( \delta^- \) of the operator \( J \) defined by the infinite \( N \)-Jacobi matrix

\[
J = \begin{pmatrix}
B_0 & A_1 \\
A_1^* & B_1 & A_2 \\
& A_2^* & B_2 & A_3 \\
&&&& \ddots & \ddots & \ddots
\end{pmatrix}
\]

on the space \( \ell^2 \), where \( A_n \) and \( B_n \) are the coefficients which appear in the three-term recurrence relation (1.1).

The deficiency indices of a matrix of measures are by definition the deficiency indices of the operator defined on the space \( \ell^2 \) by its associated \( N \)-Jacobi matrix. In \([L2, \text{Theorem 3.1}]\) (see also \([B, \text{Theorem 2.6}]\)) it is proved that the rank of the limit matrix \( R(\lambda) \) defined by

\[
R(\lambda) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} P_k(\lambda) P_k(\lambda)^* \right)^{-1}
\]

is constant in every half-plane \( \text{Im} \lambda > 0 \) and \( \text{Im} \lambda < 0 \), and it coincides with the deficiency indices of \( J \). As a consequence of this, these deficiency indices do not depend on the sequence of orthonormal polynomials we take.

Thus the deficiency indices can be any natural number from 0 to \( N \), both being equal to 0 in the determinate case and both being equal to \( N \) in the so-called completely indeterminate case.

In this paper we assume that the matrix moment problem is completely indeterminate, that is, the matrix moment problem has the highest possible degree of indetermination, or equivalently, the deficiency indices of the operator defined by \( J \) on \( \ell^2 \) are both equal to \( N \) (it is enough to assume that one of these deficiency indices is equal to \( N \); see \([L2, \text{Theorem 3.2}]\)).

In this case, the two series

\[
\sum_{k=0}^{\infty} Q_k^*(\lambda) P_k(\eta) \quad \text{and} \quad \sum_{k=0}^{\infty} P_k^*(\lambda) P_k(\eta) \quad (1.2)
\]
converge uniformly in the variables $\lambda$ and $\eta$ on every bounded set of the complex plane.

Using (1.2), we associate to a completely indeterminate matrix moment problem the following four entire matrix functions $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$:

$$A(\lambda) = \sum_{k=0}^{\infty} Q_k^*(0) Q_k(\lambda), \quad B(\lambda) = -I + \lambda \sum_{k=0}^{\infty} Q_k^*(0) P_k(\lambda),$$

$$C(\lambda) = I + \lambda \sum_{k=0}^{\infty} P_k^*(0) Q_k(\lambda), \quad D(\lambda) = \lambda \sum_{k=0}^{\infty} P_k^*(0) P_k(\lambda).$$

These four entire matrix functions are very useful for giving the Nevanlinna parametrization of the set $V_\mu$ of solutions to the matrix moment problem. This was done in [L2], providing a homeomorphism between the set $V_\mu$ of analytic matrix functions $V(\lambda)$ in the upper half plane $\mathbb{H}$ such that $V(\lambda)^* V(\lambda) \leq I$ and the set of positive definite matrices of measures $\nu$ solutions to the matrix moment problem. This homeomorphism is given by

$$\int_{\mathbb{R}} \frac{d\nu(t)}{t - \lambda} = -\{ C^*(\lambda)[I + V(\lambda)] - iA^*(\lambda)[I - V(\lambda)] \}$$

$$\times \{ D^*(\lambda)[I + V(\lambda)] - iB^*(\lambda)[I - V(\lambda)] \}^{-1}. \quad (1.3)$$

Indeed, it is not the set of measures which is parametrized, but the set of its Stieltjes transforms, which is just as good since the Stieltjes transform is invertible.

Riesz's theorem for an indeterminate matrix moment problem has been proved recently by the second author in [L1]. As in the scalar case, the linear set of matrix polynomials is dense in $L^2(\mu)$ if and only if $\int_{\mathbb{R}} \frac{d\mu(t)}{t - \lambda}$ is an extreme point (in the sense of convexity) of the convex set of matrices $\{ \int_{\mathbb{R}} \frac{d\mu(t)}{t - \lambda} : \rho \in V_\mu \}$. These matrices of measures $\mu$ for which the linear set of matrix polynomials is dense in $L^2(\mu)$ are again called $N$-extremal matrices of measures.

The $N$-extremal matrices of measures are obtained from the Nevanlinna parametrization when $V(\lambda)$ is taken to be a constant unitary matrix $U$; that is, the Stieltjes transform of an $N$-extremal matrix of measures is given by

$$\int_{\mathbb{R}} \frac{d\nu(t)}{t - \lambda} = -\{ C^*(\lambda)[I + U] - iA^*(\lambda)[I - U] \}$$

$$\times \{ D^*(\lambda)[I + U] - iB^*(\lambda)[I - U] \}^{-1}, \quad (1.4)$$

for a certain unitary matrix $U$.

We are now ready to give the main results of this paper.
As in the scalar case and as a consequence of the representation theorem, every \(N\)-extremal matrix of measures \(\mu\) associated to a completely indeterminate matrix moment problem is a discrete matrix of measures supported on the zeros of the analytic matrix function \(D^*(\lambda)[I + U] - iB^*(\lambda)[I - U]\). It is proved in [L2] that all of these zeros are real.

Let \(t_0\) be a real number. We will prove that the matrix \(D^*(t_0) + iB^*(t_0)\) is non-singular and that the matrix

\[
U_\mu = -(D^*(t_0) + iB^*(t_0))^{-1} (D^*(t_0) - iB^*(t_0))
\]  
\[\text{(1.5)}\]

is unitary. For any unitary matrix \(U\) we write

\[
A_U = \{u \in \mathbb{C}^N : Uu^* = U_\mu u^*\}.
\]

We have the following theorem which characterizes the \(N\)-extremal matrices of measures having a mass point at \(t_0\):

**Theorem 1.** The Nevanlinna parametrization \((1.4)\) establishes a bijective mapping between the sets

\[
\{ U : U \text{ is an unitary matrix, } \dim (A_U) = m \}
\]

and

\[
\{ v : v \text{ is an } N \text{-extremal matrix of measures with rank } (\nu(\{t_0\})) = m \}.
\]

Moreover:

1. If \(v\) is \(N\)-extremal, the matrix \(\nu(\{t_0\})\) is the inverse of the positive definite matrix \(\sum_{k=0}^\infty P_k^*(t_0) P_k(t_0)\) on the range of \(\nu(\{t_0\})\), that is, if \(u, v \in \text{Ker}^+(\nu(\{t_0\}))\) then

\[
\begin{align*}
\nu(\{t_0\}) \left( \sum_{k=0}^\infty P_k^*(t_0) P_k(t_0) \right) v^* \\
= u \left( \sum_{k=0}^\infty P_k^*(t_0) P_k(t_0) \right) (\nu(\{t_0\})) v^* = uv^*.
\end{align*}
\]

\[\text{(1.6)}\]

1. If \(v\) is \(N\)-extremal, the matrix \(\nu(\{t_0\})\) attains, on the range of \(\nu(\{t_0\})\), the maximum mass which can be concentrated at \(t_0\) for any solution of the indeterminate matrix moment problem; that is, if \(u\) belongs to the range of \(\nu(\{t_0\})\) and \(\mu\) has the same matrix moments as those of \(v\) then:

\[
u(\{t_0\}) u^* \geq u \mu(\{t_0\}) a^*.
\]
As a consequence, there is only one $N$-extremal matrix of measures with a non-singular mass at the point $t_0$: the $N$-extremal matrix of measures associated to the unitary matrix $U_{t_0}$ defined by (1.5). In this case the mass at $t_0$ is $(\sum_{k=0}^{\infty} P_k(t_0) P_k(t_0)^{-1})$.

We stress the important differences between our Theorem 1 and Property 2 of $N$-extremal measures pointed out earlier. Property 3 also has a more complicated interpretation in the matrix case which depends on the rank of the mass that the $N$-extremal solution supports on $t$:

**Theorem 2.** If $\nu$ is an $N$-extremal matrix of measures then the deficiency index of the matrix of measures $\nu - \nu(t)$ is less than or equal to $N - \text{rank}(\nu(t))$.

Finally, Property 4 (for the scalar case) has an analogue in the matrix case:

**Corollary 3.** The $N$-extremal matrix of measures associated to the unitary matrix $U_{t_0}$ is the only solution of the indeterminate matrix moment problem having maximum mass at the point $t_0$.

The paper is completed with some examples illustrating the relation (nonexistence of relation) between $N$-extremal matrices of measures and $N$-extremal measures: we show an $N$-extremal matrix of measures having measures on its diagonal which are not $N$-extremal.

During the 1940’s a few soviet authors obtained some results on the matrix moment problem from an operator theory approach. Especially important is the paper [K] where M. G. Krein develops a representation theory for hermitian operators with deficiency index $(m, m)$, with special emphasis on the entire operators. As a consequence of his results Krein stated (without proof) a weaker version of Corollary 3 (see p. 132 of [K]): for every real number $t$ there is only one solution of the indeterminate matrix moment problem supporting maximal mass at $t$ (see also [Z, p. 42]). As far as we know our Theorems 1 and 2 have not appeared in any other Krein papers on the matrix moment problem.

2. ENTIRE MATRIX FUNCTIONS

We review here some results about entire matrix functions which we will use in this paper. A $N \times N$ matrix function $F(\lambda) = (F_{i,j}(\lambda))_{1 \leq i, j \leq N}$ in $\mathbb{C}$ with values in the set $M_{N \times N} (\mathbb{C})$ of the $N \times N$ complex matrices is said to be entire if every entry $F_{i,j}(\lambda)$ is an entire function.
A point \( \lambda_0 \) in \( \mathbb{C} \) is a zero of \( F(\lambda) \) if it is a zero of \( \det F(\lambda) \), and the multiplicity of \( \lambda_0 \) as a zero of \( F(\lambda) \) is the multiplicity of \( \lambda_0 \) as a zero of \( \det F(\lambda) \).

An important notion we will need is that of Jordan chains. We list now the basic facts on Jordan chains which we will use later. They can be found in [GLR, Sect. 1.4, 1.6] for matrix polynomials; for entire matrix functions the definitions and results work in exactly the same way.

A sequence of vectors \( v_0, v_1, \ldots, v_k \) is called a (left) Jordan chain of length \( k + 1 \) of the entire matrix function \( F(\lambda) \) corresponding to \( \lambda_0 \) if

\[
\sum_{i=0}^{k} \frac{1}{i!} v_{i-1} F^{(i)}(\lambda_0) = \theta, \quad 1 = 0, 1, \ldots, k.
\]

If \( \lambda_0 \) is a zero of \( F(\lambda) \) of multiplicity \( m \), there exists a set of Jordan chains,

\[
\{ v_{1,0}, \ldots, v_{1, \mu_1-1} \}, \ldots, \{ v_{r,0}, \ldots, v_{r, \mu_r-1} \}
\]

such that

1. \( v_{1,0}, \ldots, v_{r,0} \) are linearly independent.
2. \( \mu_1 + \cdots + \mu_r = m. \)
3. \( r \) is the dimension of \( \ker F(\lambda_0) \) (see [GLR, Sect. 1.6]).

Such a set of Jordan chains is called a canonical set of Jordan chains.

For a given matrix \( A \), we denote by \( \text{Adj}(A) \) the classical adjoint, i.e., the matrix uniquely defined by the property

\[
A \text{Adj}(A) = \text{Adj}(A) A = \det(A) I.
\]

We finally include the following lemma which can be proved as Lemma 2.2 of [D3]:

**Lemma 2.1.** Let \( F(\lambda) \) be a \( N \times N \) entire matrix function and let \( \lambda_0 \) be a zero of \( F(\lambda) \) of multiplicity \( p \), i.e., a zero of multiplicity \( p \) of the scalar polynomial \( \det F(\lambda) \). We put

\[
L(\lambda_0, F) = \{ v \in \mathbb{C}^N : v F(\lambda_0) = 0 \}, \quad R(\lambda_0, F) = \{ v \in \mathbb{C}^N : F(\lambda_0) v^{*} = 0 \}.
\]

If \( \dim(L(\lambda_0, F)) = \dim(R(\lambda_0, F)) = p \), then \( (\text{Adj}(F(\lambda_0)))^{(p)}(\lambda_0) = \theta, \) for \( l = 0, \ldots, p - 2 \), and \( (\text{Adj}(F(\lambda_0)))^{(p-1)}(\lambda_0) \neq \theta \). Moreover, \( \text{rank}(\text{Adj}(F(\lambda_0)))^{(p-1)}(\lambda_0) = p \) and

\[
(\text{Adj}(F(\lambda_0)))^{(p-1)}(\lambda_0)
\]

defines a linear mapping from \( \mathbb{C}^N \) onto \( L(\lambda_0, F) \) which is an isomorphism from \( R(\lambda_0, F) \) into \( L(\lambda_0, F) \). Furthermore,

\[
F(\lambda_0)(\text{Adj}(F(\lambda_0)))^{(m-1)}(\lambda_0) = (\text{Adj}(F(\lambda_0)))^{(m-1)}(\lambda_0) F(\lambda_0) = \theta, \quad (2.1)
\]
\[ F'(\lambda_0)(\text{Adj } F(\lambda))^{(m-1)}(\lambda_0) + F'(\lambda_0)(\text{Adj } F(\lambda))^{(m)}(\lambda_0) = (\det F(\lambda))^{(m)}(\lambda_0) I. \]  

(2.2)

3. ENTIRE MATRIX FUNCTIONS ASSOCIATED TO A MATRIX MOMENT PROBLEM

In what follows, if \( P(\lambda) \) is a matrix polynomial we denote by \( P^*(\lambda) \) the polynomial obtained from \( P(\lambda) \) by replacing each of its matrix coefficients by its hermitian conjugate, so that \( P(\lambda)^* = P^*(\lambda) \). If \( F(\lambda) \) is a holomorphic function on a domain \( \Omega \) we denote by \( F^*(\lambda) \) the matrix function obtained from \( F(\lambda) \) by replacing each of the matrix coefficients in its power series expansion at 0 by its hermitian conjugate, and similarly we have \( F(\lambda)^* = F^*(\lambda) \).

In this section we study some properties of the entire matrix functions associated to a completely indeterminate matrix moment problem which we need to prove the main results on \( N \)-extremal matrices of measures.

The entire matrix functions defined by (1.3) satisfy some useful algebraic identities,

\[
\begin{align*}
A(\lambda) D^*(\lambda) &- B(\lambda) C^*(\lambda) = I, \quad \text{for } \lambda \in \mathbb{C}, \\
C(\lambda) D^*(\lambda) &- D(\lambda) C^*(\lambda), \quad \text{for } \lambda \in \mathbb{C}, \\
A(\lambda) B^*(\lambda) &- B(\lambda) A^*(\lambda), \quad \text{for } \lambda \in \mathbb{C}, \\
D^*(\lambda) B(\lambda) &- B^*(\lambda) D(\lambda), \quad \text{for } \lambda \in \mathbb{C},
\end{align*}
\]

(3.1) – (3.4)

and

\[
D^*(\lambda) B'(\lambda) - B^*(\lambda) D'(\lambda) = \sum_{k=0}^{\infty} P_k^*(\lambda) P_k(\lambda), \quad \text{for } \lambda \in \mathbb{C}. \tag{3.5}
\]

Formulae (3.1)-(3.3) can be found in [L2, (2.17-2.19)] and (3.4) can be proved similarly; (3.5) follows straightforwardly from (2.4) and (2.14) of [L2].

We know from the Introduction that every \( N \)-extremal matrix of measures \( \nu \) has associated to it a unique unitary matrix \( U \) such that its Stieltjes transform is given by

\[
\begin{align*}
\int_{\mathbb{R}} \frac{d\nu(t)}{t - \lambda} &\sim \{ C^*(\lambda)[I + U] \\
&\quad - iA^*(\lambda)[I - U] \} \{ D^*(\lambda)[I + U] - iB^*(\lambda)[I - U] \}^{-1}.
\end{align*}
\]
We call
\[ T_0, U(\lambda) = C(\lambda)[I + U] - iA^*(\lambda)[I - U], \tag{3.6} \]
\[ T_1, U(\lambda) = D^*(\lambda)[I + U] - iB^*(\lambda)[I - U], \tag{3.7} \]
so that
\[ \int \frac{dv(t)}{t - \lambda} = -T_0, U(\lambda) T_1, U(\lambda)^{-1}. \]

These two entire matrix functions are the key to studying the N-extremal matrices of measures, and as one can easily see the zeros of the entire matrix function \( T_1, U(\lambda) \) are going to play an important role: their properties are included in the following lemma.

**Lemma 3.1.**
1. The entire matrix function \( T_1, U(\lambda) \) has only real zeros.
2. If \( t_0 \) is a zero of \( T_1, U(\lambda) \) and \( u \in \mathbb{C}^N \) is a left eigenvector associated to \( 0 (uT_1, U(t_0) = \theta) \), then
\[ T_1, U(t_0)(-B(t_0) + iD(t_0)) u^* = \theta \tag{3.8} \]
and
\[ T_1, U(t_0)(-B(t_0) + iD(t_0)) u^* = 2 \left( \sum_{k=0}^{\infty} P_k(t_0) P_k(t_0) \right) u^*. \tag{3.9} \]
3. The real number \( t_0 \) is a zero of \( T_1, U(\lambda) \) of multiplicity \( m \) if and only if \( \text{rank } (T_1, U(t_0)) = N - m \) and in this case \( m \leq N \).

**Proof.**
1. It is contained in the proof of Theorem 1 of [L2].
2. From \( uT_1, U(t_0) = \theta \), a direct calculation gives
\[ U(-B(t_0) + iD(t_0)) u^* = -(B(t_0) + iD(t_0)) u^*. \tag{3.10} \]
We then have from (3.7) that
\[
\begin{align*}
T_1, U(t_0)(-B(t_0) + iD(t_0)) u^* &= (D^*(t_0)(I + U) - iB^*(t_0)(I - U))(-B(t_0) + iD(t_0)) u^* \\
&= (D^*(t_0) - iB^*(t_0))(-B(t_0) + iD(t_0)) u^* \\
&= (D^*(t_0) + iB^*(t_0)) U(-B(t_0) + iD(t_0)) u^*;
\end{align*}
\]
Using (3.10) this expression is equal to

\[(D^*(t_0) - iB^*(t_0))(-B(t_0) + iD(t_0)) u^*\]
\[= -(D^*(t_0) + iB^*(t_0))(B(t_0) + iD(t_0)) u^*,\]

which reduces to \(\theta\) with the help of (3.4).

For the second assertion, using now (3.5), a similar calculation gives

\[T_{1, U}(t_0)(-B(t_0) + iD(t_0)) u^* = 2(-D^*(t_0) B(t_0) + B^*(t_0) D(t_0)) u^*\]
\[= 2\left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) u^*.\]

(3) To prove this we first prove that the numbers \(\mu_i\) in any canonical set of Jordan chains for \(T_{1, U}\) associated to \(t_0\) are all equal to 1 (see Section 2 for the notion of a canonical set of Jordan chains). Suppose on the contrary that this is not true. Then there exist two vectors \(v_0\) and \(v_1\), \(v_0 \neq \theta\), such that

1. \(v_0 T_{1, U}(t_0) = \theta\),
2. \(v_1 T_{1, U}(t_0) + v_0 T_{1, U}(t_0) = \theta\).

Multiplying (2) to the right by \((-B(t_0) + iD(t_0)) v_0^*\) we get that

\[v_1 T_{1, U}(t_0)(-B(t_0) + iD(t_0)) v_0^* + v_0 T_{1, U}(t_0)(-B(t_0) + iD(t_0)) v_0^* = 0.\]

Since \(v_0 T_{1, U}(t_0) = \theta\), (3.8) and (3.9) hold and so

\[0 = v_1 T_{1, U}(t_0)(-B(t_0) + iD(t_0)) v_0^* + v_0 T_{1, U}(t_0)(-B(t_0) + iD(t_0)) v_0^* = 2v_0 \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) v_0^* > 0.\]

Hence, the numbers \(\mu_i\) in any canonical set of Jordan chains for \(T_{1, U}\) associated to \(t_0\) must all be equal to 1.

Since \(\mu_1 + \cdots + \mu_r = m\) and \(\mu_i = 1\), \(1 \leq i \leq r\), we deduce that \(r = m\), that is, the multiplicity \(m\) is the dimension of \(\text{Ker}(T_{1, U}(t_0))\), hence \(m \leq N\) and \(\text{rank}(T_{1, U}(t_0)) = N - m\). The converse is immediate.

We include here two more formulas which we will use later. By using (3.1) it is straightforward to see that for any unitary matrix \(U\) one has

\[T_{1, U}^*(x) T_{1, U}(x) - T_{1, U}^*(x) T_{1, U}(x) T_{1, U}(x) = \theta, \quad \text{for } x \in \mathbb{R}, \quad (3.11)\]
Also, for any unitary matrix \( U \), using (3.1), (3.2), and (3.3), it is straightforward to see that
\[
\begin{align*}
(-B(\lambda) + iD(\lambda)) T_{0, \nu}(\lambda) + (A(\lambda) - iC(\lambda)) T_{1, \nu}(\lambda) &= 2I, \quad \text{for } \lambda \in \mathbb{C},
\end{align*}
\]
(3.12)

4. N-EXTREMAL MATRICES OF MEASURES.

We are now ready to prove Theorems 1 and 2 and Corollary 3 from the Introduction.

First of all, we observe that for every real number \( t_0 \) the matrix \((D^*(t_0) + iB^*(t_0))\) is invertible.

Suppose on the contrary that there exists a non-zero vector \( v \) such that
\[
v(D^*(t_0) + iB^*(t_0)) v^* = \mathbf{0}.
\]
By using (3.4) we get
\[
0 = v(D^*(t_0) + iB^*(t_0))(D(t_0) - iB(t_0)) v^*
= v(D^*(t_0) D(t_0) + B^*(t_0) B(t_0)) v^*.
\]
This gives \( vD^*(t_0) = vB^*(t_0) = 0 \), which together with (3.1) gives a contradiction.

By using (3.4) it is straightforward to prove that the matrix \( U_{t_0} \) defined by (1.5) is unitary.

**Proof of Theorem 1**
Suppose \( U \) is a unitary matrix such that \( \dim(A_{\nu}) = m \). From (1.5) and (3.7) an easy computation shows that
\[
A_{\nu} = \{ u \in \mathbb{C}^N : T_{1, \nu}(t_0) u^* = \mathbf{0} \}, \quad (4.1)
\]
where \( T_{1, \nu} \) is defined by (3.7). Consequently, the hypothesis means that \( \text{rank}(T_{1, \nu}(t_0)) = N - m \). Lemma 3.1(3) now gives that \( t_0 \) is a zero of \( T_{1, \nu}(\lambda) \) of multiplicity \( m \). The Nevanlinna parametrization (1.4) and Eqs. (3.6), (3.7) give that
\[
\int_{\mathbb{R}} \frac{dr_{\nu}(t)}{t - \lambda} = -T_{0, \nu}(\lambda) T_{1, \nu}(\lambda)^{-1}. \quad (4.2)
\]
Following the proof of [D3, Theorem 3.1], it is straightforward to see that 
\(-T_{0, \nu}(\lambda) T_{1, \nu}(\lambda)^{-1}\) admits a simple fraction decomposition whose residue \( r_{\nu}(\{t_0\}) \) at \( t_0 \) is given by
\[
T_{0, \nu}(t_0) \text{Adj}(T_{1, \nu}(t_0))^{(m-1)}(t_0) \bigg/ \det(T_{1, \nu}(t_0))^{m_1}(t_0). \quad (4.3)
\]
Lemma 3.1(3) and Lemma 2.1 give that \( \text{rank} (\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) = m. \) We then deduce that the rank of the matrix \( v_{1,0}(\{t_0\}) \) is at least \( m. \) To see that this rank is exactly \( m, \) it is enough to prove that if \( (\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) u^* = v^* \neq \emptyset, \) then \( v_{1,0}(\{t_0\}) u^* \neq \emptyset. \) Since \( T_{1, 1}(t_0)(\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) u^* = \emptyset \) (Lemma 2.1), by using (3.12) we get

\[
(-B(t_0) + iD(t_0)) T_{1, 1}(t_0) v^* = 2v^*.
\]

Since \( (\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) u^* = v^* \) we have that

\[
(-B(t_0) + iD(t_0)) v_{1,0}(\{t_0\}) u^* = \frac{2}{\det(\text{Adj } T_{1, 1}(\lambda))^{(m)}(t_0)} v^* \neq \emptyset,
\]

from which we obtain \( v_{1,0}(\{t_0\}) u^* \neq \emptyset. \)

So we have proved that if \( U \) is unitary and \( \text{dim}(A_{U}) = m \) then \( \text{rank}(v_{1,0}(\{t_0\})) = m. \)

Conversely, suppose now that \( U \) is a unitary matrix such that the mass \( v_{1,0}(\{t_0\}) \) of the corresponding \( N \)-extremal matrix of measures has rank \( m. \) From (4.2) \( T_{1, 1}(\lambda) \) has a zero at \( t_0. \) Write \( r \) for its multiplicity. Lemma 3.1(3) gives that rank \( (T_{1, 1}(t_0)) = N-r \) and so from (4.1) we have \( \text{dim}(A_{U}) = r. \) We have just proved that if \( \text{dim}(A_{U}) = r \) then rank \( (v_{1,0}(\{t_0\})) = r; \) since rank \( (v_{1,0}(\{t_0\})) = m, \) we deduce that \( r = m. \)

We now prove Part 1 of Theorem 1. Since \( v_{1,0}(\{t_0\}) \) is positive semidefinite, it is enough to prove (1.6) for the basis \( \{u_1, ..., u_m\} \) of the range of \( v_{1,0}(\{t_0\}) \) formed by an orthonormal system of eigenvectors of \( v_{1,0}(\{t_0\}) \) associated to the positive eigenvalues \( \lambda_i, i = 1, ..., m, \) of \( v_{1,0}(\{t_0\}). \) We then have to prove that

\[
u_i(v_{1,0}(\{t_0\})) \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) u_j^* = \delta_{i,j}, \quad \text{for} \quad 1 \leq i, j \leq m. \quad (4.4)
\]

To do that we need the following lemma:

**Lemma 4.1.** If \( u \) is an eigenvector of \( v_{1,0}(\{t_0\}) \) associated to the eigenvalue \( \lambda > 0, \) then \( u T_{1, 1}(t_0) = \emptyset \) and

\[2(\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) u^* = \alpha(\text{det } T_{1, 1}(\lambda))^{(m)}(t_0)(-B(t_0) + iD(t_0)) u^* \quad (4.5)\]

**Proof.** Multiplying (4.3) to the right by \( u^* \) and using that \( v_{1,0}(\{t_0\}) u^* = \alpha u \) we obtain

\[\alpha(\text{det } T_{1, 1}(\lambda))^{(m)}(t_0) u^* = T_{0, 0}(t_0)(\text{Adj } T_{1, 1}(\lambda))^{(m-1)}(t_0) u^*. \quad (4.6)\]
From (3.11) we have
\[(T_{\phi}^* v(t_0) - T_{\phi}^* v(t_0)) T_{\phi} v(t_0) (\text{Adj } T_{\phi} v(\lambda))^{(m-1)}(\lambda_0) u^* = \theta. \tag{4.7}\]

Using (4.6) and (2.2) in Lemma 2.1 this expression reduces to
\[\alpha(\det T_{\phi} v(\lambda))^{(m)}(t_0) T_{\phi}^* v(t_0) u^* = \theta,\]
and since \(\alpha(\det T_{\phi} v(\lambda))^{(m)}(t_0) \neq 0\) we get \(T_{\phi}^* v(t_0) u^* = \theta,\) or equivalently, \(u T_{\phi} v(t_0) = \theta.\)

Now (4.5) follows using (4.6), (2.1) in Lemma 2.1, and (3.12) successively:
\[\alpha(\det T_{\phi} v(\lambda))^{(m)}(t_0) (-B(t_0) + iD(t_0)) u^* = \frac{1}{(2\text{Adj } T_{\phi} v(\lambda))^{(m-1)}(t_0)} u^*.\]

We now return to the proof of (4.4). Since \(u_i v(t_0) = \lambda_i u_i,\) for \(1 \leq i \leq m,\)
with \(\lambda_i > 0,\) Lemma 4.1 gives \(u_i T_{\phi} v(t_0) = \theta,\) and we can then apply (3.9) to get
\[u_i (\lambda_i u_i) \left( \sum_{k=0}^{\infty} P_k^{(m)}( t_0 ) \right) u^* = \frac{1}{2} \lambda_i u_i T_{\phi} v(t_0) (-B(t_0) + iD(t_0)) u^*.\]

Using (4.5), Eq. (4.8) becomes
\[\frac{\lambda_i}{\lambda_j} \left( \frac{1}{\det T_{\phi} v(\lambda))^{(m)}(t_0)} u_i T_{\phi} v(t_0) (\text{Adj } T_{\phi} v(\lambda))^{(m-1)}(t_0) u_j^*, \right.\]
and again using that \(u_i T_{\phi} v(t_0) = \theta\) this expression is equal to
\[\frac{\lambda_i}{\lambda_j} \left( \frac{1}{\det T_{\phi} v(\lambda))^{(m)}(t_0)} u_i \times \{ T_{\phi} v(t_0) (\text{Adj } T_{\phi} v(\lambda))^{(m-1)}(t_0) \right.\]
\[\left. + T_{\phi} v(t_0) (\text{Adj } T_{\phi} v(\lambda))^{(m)}(t_0) \} u_j^*, \right.\]
which by virtue of (2.2) in Lemma 2.1 reduces to
\[
\frac{\partial}{\partial t} \mathbf{u}_t \mathbf{u}_t^* = \delta_{i,j}, \quad \text{for} \quad 1 \leq i, j \leq m,
\]
which finishes the proof of formula (4.3).

Part 2 of Theorem 1 follows by observing that for any matrix of measures \( \mu \in V \), we have
\[
\sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) = \int_{\mathbb{R} \setminus \{t_0\}} \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t) \right) d\mu(t) \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t) \right)^* \\
= \int_{\mathbb{R} \setminus \{t_0\}} \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t) \right) d\mu(t) \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t) \right)^* \\
+ \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) \mu(\{t_0\}) \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right)^*,
\]
which gives
\[
\left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) \geq \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) \mu(\{t_0\}) \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right)^*,
\]
and hence
\[
\left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right) \left( \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right)^{-1} - \mu(\{t_0\}) \right) \\
\times \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right)^{-1} \geq \mu(\{t_0\}). \tag{4.9}
\]

From (1.6) it is clear that for any \( u \) in the range of \( \mathcal{v}(\{t_0\}) \) we have
\[
\mathcal{v}(\{t_0\}) u^* = u \left( \sum_{k=0}^{\infty} P_k^*(t_0) P_k(t_0) \right)^{-1} u^*.
\]

Consequently, if \( \mu \) is any matrix of measures in \( V \) and \( u \) is any vector in the range of \( \mathcal{v}(\{t_0\}) \), by using (4.9) we get
\[
\mathcal{v}(\{t_0\}) u^* \geq \mu(\{t_0\}) u^*.
\]
In the proof of Theorem 2, we can assume, without loss of generality, that \( t = 0 \).

To prove Theorem 2, we need to introduce the relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation (for more details about this relationship see [D1, D2, DV]).

Indeed, we associate to each matrix weight \( W \) an inner product \( B_W \) in the space of scalar polynomials \( P \) defined by

\[
B_W(p, q) = \sum_{m, m' = 0}^{N-1} R_{N, m}(p) R_{N, m'}(q) \, dt_{m, m'}, \tag{4.10}
\]

where the operators \( R_{N, m} : P \to P, 0 \leq m \leq N - 1 \), are defined by

\[
R_{N, m}(p) = \sum_{n} \frac{p^{(nN + m)}(0)}{(nN + m)!} t^n
\]

(i.e., for every \( m \), the operator \( R_{N, m} \) takes from the scalar polynomial \( p \) just those powers \( t^k \) for which \( k \equiv m (\text{mod } N) \) and then changes \( t^{nN + m} \) to \( t^n \)). This inner product \( B_W \) has an associated sequence of orthonormal polynomials \( (p_n) \). This sequence \( (p_n) \) satisfies a \((2N + 1)\)-term recurrence formula of the form

\[
t^N p_n(t) = c_{n, 0} p_n(t) + \sum_{k=1}^{N} [c_{n, k} p_{n-k}(t) + c_{n+k, n} p_{n+k}(t)], \tag{4.11}
\]

where \( p_k = 0 \) for \( k < 0 \), \( c_{n, 0} \) is a real sequence, and \( c_{n, k} \) are complex sequences with \( c_{n, N} \neq 0 \). Moreover, it is proved in [D2] that each sequence of polynomials satisfying this kind of \((2N + 1)\)-term recurrence relation is orthonormal with respect to an inner product defined by a \( N \times N \) positive definite matrix of measures in the way given by (4.10).

There is also a relationship between the scalar recurrence coefficients in (4.11) and the matrix recurrence coefficients in (1.1). Indeed, if we define the \( N \times N \) hermitian matrices \( B_n \), \( n \geq 0 \), as

\[
B_{n, i, l} = \begin{cases} c_{n, N + l, |i - l|} & \text{if } i \geq l, \\ c_{n, N + l, |i - l|} & \text{if } i \leq l. \\
\end{cases}
\]

and the \( N \times N \) lower triangular matrices \( A_n \), \( n \geq 1 \), as

\[
A_{n, i, l} = \begin{cases} 0 & \text{if } i < l, \\ c_{n, N + l, N + l - i} & \text{if } i \geq l, \\
\end{cases}
\]
then the matrix polynomials \((P_n)\) defined by
\[
 tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0,
\]
where \(P_{-1} = \theta\) and \(P_0 = (dW(t))^{-1/2}\), are orthonormal with respect to the matrix weight \(W\) (see [DV]). In particular, this implies that each positive definite matrix of measures \(W\) has associated to it an \(N\)-Jacobi matrix which is a \((2N+1)\)-banded infinite Hermitian matrix.

We are now ready to establish the two lemmas we need to prove Theorem 2.

**Lemma 4.2.** Let us consider the matrix of measures \(v = v(\{0\})\delta_0\), where \(v\) is an \(N\)-extremal matrix of measures with \(v(\{0\})\) a matrix of rank \(m\). Then, there exist integers \(0 \leq i_1 < \cdots < i_m \leq N-1\) such that for any initial conditions \(p_k, k = 0, \ldots, N-1, p_k\) a scalar polynomial of degree \(k\), the sequence of scalar polynomials \((p_n)\), defined by the \((2N+1)\)-term recurrence formula (4.10) associated to \(v = v(\{0\})\delta_0\), satisfies that \((p_n(0))_n \in \ell^2, j = 1, \ldots, m\).

**Lemma 4.3.** Let us consider a matrix of measures \(W\) with equal deficiency indices, say \(l, 0 \leq l \leq N\), and having 0 as a point of regular type for the operator defined by its \(N\)-Jacobi matrix \(J\) (see [AG, Vol. 2, p. 91] for the definition of point of regular type). Then, there exist integers \(0 \leq i_1 < \cdots < i_m \leq N-1\) and initial conditions \(p_k, k = 0, \ldots, N-1, p_k\) a scalar polynomial of degree \(k\), such that the sequence of scalar polynomials \((p_n)\), defined by the \((2N+1)\)-term recurrence formula (4.10) associated to \(W\), satisfies that \((p_n(0))_n \in \ell^2, j = 1, \ldots, l\).

Before proving these lemmas, we prove Theorem 2.

**Proof of Theorem 2.** Let \(J\) denote the \(N\)-Jacobi matrix associated to \(v = v(\{0\})\delta_0\). Since \(v\) is discrete we deduce that there exists \(\varepsilon > 0\) such that \((-\varepsilon, \varepsilon) \cap \text{supp}(v = v(\{0\})\delta_0) = \emptyset\). Hence, for any \(f \in L^2(v = v(\{0\})\delta_0)\) with \(tf \in L^2(v = v(\{0\})\delta_0)\) we have that
\[
\|tf\|_{L^2(v = v(\{0\})\delta_0)} \geq \varepsilon \|f\|_{L^2(v = v(\{0\})\delta_0)}.
\]
In particular, for any polynomial \(P\) we have
\[
\|tP(t)\|_{L^2(v = v(\{0\})\delta_0)} \geq \varepsilon \|P(t)\|_{L^2(v = v(\{0\})\delta_0)}.
\]
Taking into account that the operator associated to the \(N\)-Jacobi matrix \(J\) in \(\ell^2\) (with domain the linear space of finite sequences) represents in \(\ell^2\) the operator of multiplication by \(t\) in \(L^2(v = v(\{0\})\delta_0)\), we have that for \(x = (x_n)_n, x_n = 0\) for \(n\) large enough, then \(\|xJ\|_2 \geq \varepsilon \|x\|_2\). That is, 0 is a
point of regular type of the operator $J$, and thus the deficiency indices of $J$ are equal (see [AG, Vol. 2, p. 93]). Write $l$ for these deficiency indices.

If we now apply Lemmas 4.2 and 4.3 to $v - v(\{0\}) \delta_0$, we deduce that $l + m \leq N$ and Theorem 2 is proved. 

We prove Lemma 4.2.

Proof of Lemma 4.2 Given the initial conditions $p_k$, $k = 0, \ldots, N - 1$, $p_k$ a scalar polynomial of degree $k$, for the recurrence relation (4.10), we consider the $N \times N$ non-singular upper triangular matrices $A$ and $B$ defined by

$$A_{i,j} = R_{N,i}(p_j), \quad B_{i,j} = R_{N,i}(r_j), \quad i, j = 0, \ldots, N - 1,$$

where $(r_n)_n$ is the sequence of scalar polynomials orthonormal for the inner product $B_{n-n}(v(0))$. If we write $S = A^{-1}B$ and $W = S(v - v(\{0\}) \delta_0) S^*$, it is easy to check that the sequence $(p_n)_n$ is orthonormal with respect to the inner product $B_W$.

Since $m = \operatorname{rank}(v(\{0\}))$, we can then take $m$ linearly independent eigenvectors $u_1, \ldots, u_m \in \mathbb{C}^N$ associated to non-null eigenvalues of $v(\{0\})$, so that $u_i(v(\{0\})) u_i^* > 0$, $i = 1, \ldots, m$. Since the vectors $u_i$, $i = 1, \ldots, m$, are linearly independent, the vectors $v_i = u_i S^{-1}$, $i = 1, \ldots, m$, are linearly independent too; we can then find integers $0 \leq l_1, \ldots, l_m \leq N - 1$ such that

$$v_j I_{l_i, l_j} v_j^* > 0, \quad j = 1, \ldots, m,$$

(4.12)

where $I_{l_i, l_j}$ denotes the $N \times N$ matrix with zero entries except the $(k, k)$ equal to 1.

We first prove that if $W$ has the same matrix moments as those of $W$ then $v_i W(\{0\}) \delta_0 = \theta$, $i = 1, \ldots, m$. Indeed, this is equivalent to proving that if $W$ has the same matrix moments as those of $v - v(\{0\}) \delta_0$ and $u_i W(\{0\}) \delta_0 = \theta$, $i = 1, \ldots, m$. If not, there is $i$, $1 \leq i \leq m$, such that $u_i W(\{0\}) \neq \theta$, then

$$u_i W(\{0\}) u_i^* > 0.$$

(4.13)

Since $u_i \in \ker^+ (v(\{0\}))$ and $v$ is $N$-extremal, we deduce from Theorem 1(b) that if $\mu$ has the same moments as those of $v$ then $u_i W(\{0\}) u_i^* \geq u_i W(\{0\}) u_i^*$. But (4.13) gives that $u_i (W(\{0\}) + v(\{0\})) u_i^* > u_i W(\{0\}) u_i^*$, which is a contradiction because $W + v(\{0\}) \delta_0$ has the same moments as those of $v$.

We finally prove that $(B_m^{l,j}(0))_n \neq 0$, $j = 1, \ldots, m$, where the sequence of scalar polynomials $(p_n)_n$ is defined by the $(2N + 1)$-term recurrence formula (4.11) associated to $v - v(\{0\}) \delta_0$ with initial conditions $p_k$, $k = 0, \ldots, N - 1$.

We consider the set $T$ of real numbers $M$ for which the bilinear form $B_M$ defined by $B_M(p, q) = B_M(p, q) - M p^{(l,j)}(0) q^{(l,j)}(0)$ satisfies $B_M(p, p) \geq 0$
for any scalar polynomial \( p \). Taking into account that (see Lemma 5.1 of [D2])
\[
\left( \sum_{n} |p^{(n)}(0)|^2 \right)^{-1} = \sup T,
\]
we deduce that if \( (p^{(n)}(0))_n \in \ell^2 \) then there exists \( M > 0 \) such that \( B_M(p, q) = B_M(p, q) + M p^{(n)}(0) q^{(n)}(0) \). Since the operator of multiplication by \( t^N \) is symmetric for \( B_M \), i.e., \( B_M(t^N p, q) = B_M(p, t^N q) \), it follows from Theorem 3.1 of [D2] that there exists a positive definite matrix of measures \( \mu \) such that \( B_M = B_\mu \). That is, \( B_M(p, q) = B_\mu(p, q) + M p^{(n)}(0) q^{(n)}(0) \), or equivalently, the positive definite matrix of measures \( \mu + M(ij)^2 \delta_0 I_{\delta_i} \) has the same matrix moments as those of \( W \). But then, for \( f = 1, ..., m \), \( v_j(\mu + M(ij)^2 \delta_0 I_{\delta_i})(\{0\}) = 0 \). But this is a contradiction because from (4.12) we get that:
\[
v_j(\mu(\{0\}) + M(ij)^2 I_{\delta_i}) v_j^* \geq M(ij)^2 v_j I_{\delta_i} v_j^* > 0.
\]

We now prove Lemma 4.3.

**Proof of Lemma 4.3** Let \( J \) be the operator defined by the \( N \)-Jacobi ((2N + 1)-banded infinite Hermitian) matrix associated to \( W \). The deficiency indices of \( J \) are equal to \( l, 0 \leq l \leq N \), and since 0 is a point of regular type for \( J \) we have from Theorem 3 of [AG, Vol. 2, p. 108] that 0 is an eigenvalue of a selfadjoint extension \( \hat{J} \) of \( J \) of multiplicity \( k \). It is not difficult to see that in this case
\[
\dim \{ x = (x_n)_n \in \ell^2 : xJ = 0 \} = l.
\]
Expanding the equation \( xJ = 0 \), we get
\[
0 = c_{n,0} x_n + \sum_{k=1}^{N} \left[ c_{n,k} x_{n-k} + c_{n+k,n} x_{n+k} \right], \quad n \geq 0, \quad (4.14)
\]
where \( x_k = 0 \), for \( k < 0 \). We remark that any solution of this difference equation is characterized by its initial conditions \( x_k, k = 0, ..., N-1 \). This means that
\[
\dim \{ (x_k)_{k=0, ..., N-1} : (x_n)_n \in \ell^2 \} = l,
\]
where \( (x_n)_n \) is the sequence defined by (4.14) with initial conditions \( (x_k)_{k=0, ..., N-1} \).

We can then take a basis \( u_0, u_1, ..., u_{N-1} \) of \( \mathbb{C}^N \) such that the matrix \( T \) whose rows are the vectors \( u_i \) is upper triangular and such that the vectors \( u_0, ..., u_i \) form a basis of \( \{ (x_k)_{k=0, ..., N-1} : (x_n)_n \in \ell^2 \} \). We now consider
polynomials \( p_k, k = 0, ..., N-1 \), with degree of \( p_k \) equal to \( k \) and such that 
\[ T_{i,j} = u_{i,j} = R_{N,i}(p_j) = R_{N,j}(p_i)(0), \quad i, j = 0, ..., N-1. \]

We prove that the sequence of scalar polynomials \( (p_n)_n \) defined by the 
\((2N+1)\)-term recurrence formula (4.11) associated to \( W \) with as its initial 
conditions these \( p_k, k = 0, ..., N-1 \), satisfies that \( (p_n^0(0))_n \in \ell^2, j = 1, ..., l \).

Indeed, from the recurrence formula (4.11) for \( (p_n)_n \) we deduce that for 
\( m = 0, ..., N-1 \), the sequence of polynomials \( (R_{N,m}(p_n))_n \) satisfies the 
recurrence formula:
\[
tr_{N,m}(p_n)(t) = c_{n,0} R_{N,m}(p_n)(t) 
+ \sum_{k=1}^{N} [\tilde{c}_{n,k} R_{N,m}(p_{n-k})(t) + c_{n+k,n} R_{N,m}(p_{n+k})(t)].
\]

For \( t = 0 \), this recurrence formula reduces to (4.14). Since \( T_{i,j} = u_{i,j} = \) \( R_{N,i}(p_j) = R_{N,j}(p_i)(0), \) \( i, j = 0, ..., N-1 \), and for \( i_1, ..., i_l \), the vectors \( u_{i_1}, ..., u_{i_l} \) 
are a basis for \( \{ (x_k)=a_1, ..., N-1; (x_n) \in \ell^2 \} \), we deduce that \( (R_{N,i}(p_n)(0))_n \in \ell^2, j = 1, ..., l \). But \( R_{N,i}(p_n)(0) = p_n^0(0). \]

We now prove Corollary 3.

**Proof of Corollary 3.** Theorem 1 gives that the mass point at \( t_0 \) of 
the \( N \)-extremal matrix of measures \( v_{U} \) associated to the unitary matrix 
\( U \) has rank equal to \( N \). Then, Theorem 2 gives that the matrix of 
measures \( v_{U} - v_{U} \{(t_0) \} \delta_{n} \) is determinate. Now, the proof follows 
straightforwardly.

To complete this paper we show an example of an \( N \)-extremal matrix of 
measures associated to a completely indeterminate matrix moment problem 
having measures on its diagonal which are not \( N \)-extremal.

Indeed, we take two \( N \)-extremal measures \( \mu, v \) with different but not 
disjoint supports, that is, there exist two real numbers \( t_0, t_1 \) for which 
\( \mu(\{t_0\}) > 0, \nu(\{t_0\}) > 0, \) and \( \mu(\{t_1\}) = 0, \nu(\{t_1\}) = 0. \) This implies that for any 
positive numbers \( a, b > 0 \) the measure \( a\mu + bv \) is not \( N \)-extremal: indeed, we 
can take the \( N \)-extremal measure \( \sigma \) having the same moments as those of 
\( v \) and mass point at \( t_1 \). Then the measure \( a\mu + b\sigma \) has the same moments 
as those of \( a\mu + bv \) but \( (a\mu + bv(\{t_1\}) < (a\mu + b\sigma)(\{t_1\}), \) hence \( a\mu + bv \) is 
not \( N \)-extremal. For any \( 2 \times 2 \) unitary matrix \( U \) for which \( U_{1,2} \neq 0 \), we take 
the positive definite matrix of measures defined by
\[
W_U = U \begin{pmatrix} \mu & 0 \\ 0 & v \end{pmatrix} \end{pmatrix} U^*.
\]

From what we have just proved the measures on the diagonal of \( W_U \) are 
not \( N \)-extremal. We now prove, however that \( W_U \) is an \( N \)-extremal matrix.
of measures corresponding to a completely indeterminate matrix moment problem.

Indeed, if we write

\[ P_n(t) = \begin{pmatrix} r_n(t) & 0 \\ 0 & s_n(t) \end{pmatrix}, \]

where \((r_n)_n\) and \((s_n)_n\) are the sequences of orthonormal polynomials for \(\mu\) and \(\nu\) respectively, we have straightforwardly that the matrix polynomials \(Q_n(t) = R_n(t) U^*\) are orthonormal with respect to \(W_U\). But

\[ \sum_n R_n^*(t) R_n(t) = U \left( \sum_n P_n^*(t) P_n(t) \right) U^*, \]

and since \(\mu\) and \(\nu\) are indeterminate measures we deduce that the rank of \(\sum_n R_n^*(t) R_n(t)\) is 2. That is, \(W_U\) is a solution of a completely indeterminate matrix moment problem. Now, the maximum mass which can be concentrated at \(t_0\) is given by

\[ \left( \sum_n R_n^*(t_0) R_n(t_0) \right)^{-1} = \left( U \left( \sum_n P_n^*(t_0) P_n(t_0) \right) U^* \right)^{-1}. \]

Since \(\mu\) and \(\nu\) are \(N\)-extremal and \(\mu(\{t_0\})\), \(\nu(\{t_0\}) > 0\), we have that

\[ \left( \sum_n P_n^*(t_0) P_n(t_0) \right)^{-1} = \begin{pmatrix} \mu(\{t_0\}) & 0 \\ 0 & \nu(\{t_0\}) \end{pmatrix}, \]

that is,

\[ \left( \sum_n R_n^*(t_0) R_n(t_0) \right)^{-1} = W_U(\{t_0\}). \]

and then Corollary 3 gives that \(W_U\) is \(N\)-extremal.

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