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On the Enumeration of Finite Maximal Connected Topologies*

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A connected topology $\mathcal{T}$ is said to be maximal connected if $\mathcal{U}$ strictly finer than $\mathcal{T}$ implies that $\mathcal{U}$ is disconnected. In this paper, it is shown that the number of homeomorphism classes of maximal connected topologies defined on a set with n points is equal to twice the number of n point trees minus the number of n point trees possessing a symmetry line. An enumeration of a class of topologies, called critical connected topologies, which includes the maximal connected spaces is then carried out with the help of Pólya's theorem. Another result is that a chain of connected n point T_0 topologies, linearly ordered by strict fineness, can contain a maximum of $\frac{1}{2}(n^2 - 3n + 4)$ topologies, and, moreover, this number is the best possible upper bound for the length of such a chain.

1. Introduction and Problem Definition

A topology $\mathcal{T}$ on a set N is said to be *maximal connected* iff it is connected, and if each space on the set N that is strictly finer than $\mathcal{T}$ is disconnected. Maximal connected topologies were introduced by J. P. Thomas in [1], where he raised the question of enumerating the maximal connected spaces definable on a finite set. In the present article, this and related questions are answered and some of the results of [1] are obtained (for finite spaces) in a slightly different way by considering the enumeration problem for a larger class of spaces.

It is shown in [1] that maximal connected spaces are necessarily T_0. In this article, all spaces under consideration are assumed to be finite and T_0. The notation $|A|$ will be used to specify the cardinality of the set A. If $\mathcal{T}$ and $\mathcal{U}$ are topologies, then the relation $\mathcal{T} \geqslant \mathcal{U}$ will express the fact that $\mathcal{T}$ is finer than $\mathcal{U}$, and $\mathcal{T} > \mathcal{U}$ will mean that $\mathcal{T}$ is strictly finer than $\mathcal{U}$. All n point topologies will be assumed to have been defined on the set N of cardinality n. If $\mathcal{T}$ is a topology on N and $\alpha \in N$, then $\alpha^*(\mathcal{T})$, or less

* The results of this paper are presented, with compliments, to Kalpana and Alpana Das.

explicitly α^*, when there is no risk of confusion, will denote the minimal open set of $\mathscr{T}$ that contains α, i.e., $\alpha^*(\mathscr{T}) = \bigcap \{O \mid \alpha \in O \in \mathscr{T}\}$. It is a consequence of the T_0 property that there exists a non-empty subset of N, henceforth denoted by $F(\mathscr{T})$, such that $\alpha^* = \alpha$ iff $\alpha \in F(\mathscr{T})$.

A n point topology $\mathscr{T}$ partitions N uniquely into pairwise disjoint subsets $F_1(\mathscr{T}), F_2(\mathscr{T}),..., F_m(\mathscr{T})$, $m \leqslant n$, where $F_1(\mathscr{T}) = F(\mathscr{T})$ and the $F_i(\mathscr{T})$ for $i > 1$ are determined recursively by setting $F_i(\mathscr{T}) = F(\mathscr{T}_i)$, $\mathscr{T}_i$ being the topology induced by $\mathscr{T}$ on the set $N - F_1(\mathscr{T}) \cup \cdots \cup F_{i-1}(\mathscr{T})$, and terminating the process at that value m of i for which $N - F_1(\mathscr{T}) \cup \cdots \cup F_m(\mathscr{T}) = \varnothing$. $\mathscr{T}$ is then called a *m level* topology and the ordered sequence of integers $|F_1(\mathscr{T})|$,..., $|F_m(\mathscr{T})|$ is called the *level distribution* of $\mathscr{T}$. Conversely, corresponding to every ordered partition N_1 ,..., N_m of N, there exists a m level space $\mathscr{T}$ on N such that $N_i = F_i(\mathscr{T})$, $i \leqslant m$. One example of such a topology is described most simply by specifying that $\alpha^* = \alpha$ if $\alpha \in N_1$ and $\alpha^* = N_1 \cup \cdots \cup N_{i-1} + \alpha$ if $\alpha \in N_i$, $1 < i \leqslant m$. If $\mathscr{T}$ is a topology and $\alpha \in F_i(\mathscr{T})$ then α is a point of *level i* in $\mathscr{T}$. A space with only level 1 points is clearly discrete. A topology $\mathscr{T}$ is *level connected* provided it is connected and $\alpha^* = \alpha + \bigcup\{\beta^* \mid \beta \in \alpha^* \cap F_{i-1}(\mathscr{T})\}$ for every point α of level $i > 1$. The example that was described just above is an instance of a level connected space. There exist connected spaces that are not level connected. For example, the connected space $\{\varnothing, \{a\}, \{b\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}\}$ is not level connected since $d^* \neq d \cup c^*$. A level connected topology $\mathscr{T}$ is defined to be *critical connected* iff there does not exist a level connected space $\mathscr{U}$ such that the level distributions of $\mathscr{T}$ and $\mathscr{U}$ are identical and $\mathscr{U} > \mathscr{T}$. Using a slightly different language, the critical connected spaces with level distribution n_1 ,..., n_m are the maximal elements of the collection of all level connected topologies with level distribution n_1 ,..., n_m , under the ordering relation of strict fineness. The main aim of this paper is to enumerate the homeomorphism classes of critical connected topologies. This enumeration is carried out with the help of certain results in the theory of finite trees.

An n point, m colored, $1 \leqslant m \leqslant n$, *descriptor graph* $\mathscr{G}$ is a graph on the point set N, where each of the points has assigned to it one of the m given *colors* 1, 2,..., m. α^+ will denote the color of the point α in the graph $\mathscr{G}$. The line set of $\mathscr{G}$ must satisfy the following requirements:

(1) Each point α with $\alpha^+ > 1$ is adjacent to at least one point β with $\beta^+ = \alpha^+ - 1$.

(2) Any two adjacent points have different colors.

(3) If $\alpha^+ > \beta^+ + 1$ and there exists a path of lines

$$(\alpha, \gamma_1)(\gamma_1, \gamma_2) \cdots (\gamma_i, \beta),\ i \geqslant 1,$$

between α and β where $\alpha^+ > \gamma_1{}^+ > \gamma_2{}^+ > \cdots > \gamma_i{}^+ > \beta^+$, then there does *not* exist a line (α, β) connecting α and β.

A point β is said to be *accessible* to the point α if $\beta^+ < \alpha^+$ and either α and β are adjacent, or there exists a path $(\alpha, \gamma_1)(\gamma_1, \gamma_2) \cdots (\gamma_i, \beta)$, $i \geqslant 1$, with $\alpha^+ > \gamma_1{}^+ > \cdots > \gamma_i{}^+ > \beta^+$. Define, for each $\alpha \in N$, $\alpha^*(\mathscr{G}) = \alpha + \{\beta \mid \beta$ accessible to $\alpha\}$. It is easy to verify that the set $\{\alpha^*(\mathscr{G}) \mid \alpha \in N\}$ satisfies the requirements for the collection of minimal open neighborhoods of the points of a T_0 topology on N. The topology *described* by the descriptor graph $\mathscr{G}$, is denoted $T(\mathscr{G})$, and is defined to be that T_0 topology $\mathscr{T}$ on N for which $\alpha^*(\mathscr{T}) = \alpha^*(\mathscr{G})$ for each $\alpha \in N$. Clearly, the color of any point in $\mathscr{G}$ is identical to its level in $T(\mathscr{G})$. Hence an m colored descriptor graph describes an m-level topology. It should be observed that $\alpha^* = \alpha + \cup\{\beta^* \mid \beta^+ < \alpha^+$ and β is adjacent to $\alpha\}$.

2. Basic Propositions

Lemma 1. *Let $\mathscr{T}$ be a topology on N. Then there exists one, and only one, n-point descriptor graph which describes $\mathscr{T}$.*

Comment. This uniquely defined graph is called the descriptor graph of $\mathscr{T}$ and is denoted by $G(\mathscr{T})$.

Proof. Assume that $\mathscr{T}$ is an m-level topology. The existence of the graph $G(\mathscr{T})$ will be demonstrated by actual construction according to the following "program":

START 1. Assign the color i to each point of $F_i(\mathscr{T})$, for all $i \leqslant m$.

2. Set the value of a variable p equal to 1.

3. If $p = m$, then go to 10; otherwise go to 4.

4. Increase p by 1.

5. Draw all possible lines (α, β) such that $\alpha \in F_p(\mathscr{T})$, $\beta \in F_{p-1}(\mathscr{T})$ and $\beta \in \alpha^*$.

6. Set the value of a variable k equal to $p - 2$.

7. If $k = 0$, then go to 3; otherwise go to 8.

8. Draw all possible lines (α, β) such that $\alpha \in F_p(\mathscr{T})$, $\beta \in F_k(\mathscr{T})$, $\beta \in \alpha^*$ but β is not accessible to α in the graph as it now stands prior to the execution of this instruction.

9. Reduce k by 1 and go to 7.

END 10. Construction of $G(\mathscr{T})$ is complete.

It is a straightforward matter to prove that the graph $G(\mathscr{T})$ constructed above is in fact a descriptor graph. Moreover $G(\mathscr{T})$, by condition 3 in the definition of descriptor graphs, has the minimum number of adjacencies needed to characterize $\mathscr{T}$. That $G(\mathscr{T})$ is the only possible descriptor graph which describes $\mathscr{T}$ can also be easily demonstrated.

The results in the next lemma, presented without proof, will be useful in the subsequent discussions:

LEMMA 2. (a) *A topology $\mathscr{T}$ is homeomorphic to a topology $\mathscr{U}$ iff there exists an isomorphism between the descriptor graphs $G(\mathscr{T})$ and $G(\mathscr{U})$ which preserves adjacencies as well as colors.*

(b) *If $\mathscr{T}$ and $\mathscr{U}$ are topologies, then $\mathscr{T} \geqslant \mathscr{U}$ iff $\alpha^*(\mathscr{T}) \subseteq \alpha^*(\mathscr{U})$ for all $\alpha \in N$, and $\mathscr{T} > \mathscr{U}$ iff in addition to the previous condition, $\gamma^*(\mathscr{T}) \subset \gamma^*(\mathscr{U})$ for at least one $\gamma \in N$.*

(c) *If $\mathscr{G}$ and $\mathscr{H}$ are descriptor graphs such that every point has the same color in both graphs, and if the line set of $\mathscr{G}$ is a proper subset of the line set of $\mathscr{H}$, then $T(\mathscr{G}) > T(\mathscr{H})$.*

(d) *A descriptor graph describes a level connected space iff the graph is connected and $| \alpha^+ - \beta^+ | = 1$ for any pair of adjacent points α, β.*

(e) *If $\mathscr{T}$ and $\mathscr{U}$ are level connected spaces with identical level distributions and if $\mathscr{T} > \mathscr{U}$, then the line set of $G(\mathscr{T})$ is a proper subset of the line set of $G(\mathscr{U})$.*

(f) *A topology $\mathscr{T}$ is connected iff the descriptor graph $G(\mathscr{T})$ is connected.*

LEMMA 3. *If $\mathscr{T}$ is a connected topology such that there does not exist any connected topology $\mathscr{U} > \mathscr{T}$, where $\mathscr{U}$ and $\mathscr{T}$ have identical level distributions, then $G(\mathscr{T})$ is a tree. The statement remains valid if the term "connected" is replaced by level "connected."*

Proof. Since $\mathscr{T}$ is connected, the level of $\mathscr{T}$ is at least 2. If $\mathscr{G} = G(\mathscr{T})$ is not a tree, then there exists an α, with $\alpha^+ > 1$, and a cycle $(\alpha, \beta_1) \cdots (\beta_k, \alpha)$ such that $\beta_1{}^+, \beta_k{}^+ < \alpha^+$. Let $\mathscr{H}$ be the graph in which each point retains the color it had in $\mathscr{G}$, and suppose that the line set of $\mathscr{H}$ is obtained from $\mathscr{G}$ by deleting the line (α, β_k) if $\beta_1{}^+ = \alpha^+ - 1$. Otherwise the line set of $\mathscr{H}$ is obtained by removing the line (α, β_1). Since $\mathscr{G}$ is a descriptor graph, therefore, in either case, α is adjacent in $\mathscr{H}$ to at least one γ such that $\gamma^+ = \alpha^+ - 1$. Clearly, $\mathscr{H}$ is also a descriptor graph. As each point has the same color in both graphs, therefore the level distributions of $\mathscr{T}$ and $\mathscr{U} = \mathscr{T}(\mathscr{H})$ are identical. Since $\mathscr{H}$ has been obtained from $\mathscr{G}$ by removing a single line from a cycle, $\mathscr{H}$ is connected. Lemma 2c and f now imply that $\mathscr{U}$ is connected and $\mathscr{U} > \mathscr{T}$. This contradicts the hypothesis of the lemma, and so $G(\mathscr{T})$ is a tree. The second part of the proposition is established by observing that, as a consequence of Lemma 2d, the constructed space $\mathscr{U}$ is level connected if $\mathscr{T}$ is level connected.

LEMMA 4. *A topology $\mathscr{T}$ is critical connected iff $G(\mathscr{T})$ is a tree in which $| \alpha^+ - \beta^+ | = 1$ for any pair of adjacent points α, β.*

Proof. The "only if" part is a restatement of Lemma 3. To prove sufficiency, assume that the level connected space $\mathscr{U}$ has the same level distribution as $\mathscr{T}$ and $\mathscr{U} > \mathscr{T}$. By Lemma 2d, $\mathscr{T}$ is level connected. Therefore a consequence of Lemma 2e is that the line set of $G(\mathscr{U})$ can be obtained by deleting some lines from the tree $G(\mathscr{T})$. Therefore $G(\mathscr{U})$ is disconnected and so, by Lemma 2f, $\mathscr{U}$ is also disconnected. Hence $\mathscr{T}$ is critical connected.

Comment. It should be observed, however, that a tree can be colored such that, for any pair of adjacent points α, β, $| \alpha^+ - \beta^+ | = 1$ and still fail to be a descriptor graph.

LEMMA 5. *If $\mathscr{T}$ is a maximal connected topology, then the level of $\mathscr{T}$ is equal to* 2.

Proof. Let m, $m > 2$, be the level of $\mathscr{T}$ and suppose that α is a point of level m in $\mathscr{T}$. Then two situations are possible:

(A) There exists only one $\beta \in F_{m-1}(\mathscr{T})$ such that $\beta \in \alpha^*$. In this event, let γ be any point of $F_{m-2}(\mathscr{T})$ such that $\gamma \in \beta^*$,

(B) $| \alpha^* \cap F_{m-1}(\mathscr{T})| \geqslant 2$. In this situation, let β, γ be arbitrary points of $F_{m-1}(\mathscr{T})$ and $F_{m-2}(\mathscr{T})$, respectively, such that $\beta \in \alpha^*$ and $\gamma \in \beta^*$.

In either case, because $(\alpha, \beta)(\beta, \gamma)$ is a path in $G(\mathscr{T})$ between α and γ and $\alpha^+ > \beta^+ > \gamma^+$, therefore α and γ are not adjacent. Let $\mathscr{U}$ be the topology whose descriptor graph is obtained from $G(\mathscr{T})$ by deleting the line (α, β), then adding the line (α, γ) and finally, in case (A) assigning the color $m - 1$ to α, and in case (B) allowing α to retain its original color m. Then $\mathscr{U}$ is connected and $\mathscr{U} > \mathscr{T}$, so that $\mathscr{T}$ cannot be maximal connected. Since a level 1 topology is always discrete, the result follows.

Comment. The arguments used in the proofs of Lemmas 3 and 5 provide a constructive technique for finding a maximal connected topology finer than any given connected topology.

THEOREM 1. *A topology is maximal connected iff its descriptor graph is a bicolored tree.*

Proof. First, it will be shown that, if $\mathcal{T}$ is a connected topology of level 2, and $\mathcal{U}$ is a connected space strictly finer than $\mathcal{T}$, then the level distributions of $\mathcal{T}$ and $\mathcal{U}$ are the same. Clearly $F_1(\mathcal{T}) \subseteq F_1(\mathcal{U})$. Suppose $\alpha \in F_1(\mathcal{U})$ but $\alpha \notin F_1(\mathcal{T})$. Since $\mathcal{U}$ is connected, there exists a $\beta \in N - F_1(\mathcal{U})$ such that $\alpha \in \beta^*(\mathcal{U})$. As $N - F_1(\mathcal{U}) \subseteq N - F_1(\mathcal{T}) = F_2(\mathcal{T})$, therefore both α, $\beta \in F_2(\mathcal{T})$ so that $\alpha \notin \beta^*(\mathcal{T})$. Therefore $\beta^*(\mathcal{U}) \not\subseteq \beta^*(\mathcal{T})$, which is impossible since $\mathcal{U}$ is finer than $\mathcal{T}$. Hence $F_1(\mathcal{T}) = F_1(\mathcal{U})$. Therefore $F_2(\mathcal{U}) \subseteq F_2(\mathcal{T})$. Now suppose that $F_2(\mathcal{U}) \subset F_2(\mathcal{T})$, i.e., the level of $\mathcal{U}$ is greater than 2. Then there exists a β such that $\beta \in F_2(\mathcal{T})$ and $\beta \in N - F_1(\mathcal{U}) \cup F_2(\mathcal{U})$. By the second condition, there exists an $\alpha \in F_2(\mathcal{U})$ such that $\alpha \in \beta^*(\mathcal{U})$. Clearly, $\alpha \in F_2(\mathcal{T})$ so that $\alpha \notin \beta^*(\mathcal{T})$ and so $\beta^*(\mathcal{U}) \not\subseteq \beta^*(\mathcal{T})$, which again contradicts the fact that $\mathcal{U} > \mathcal{T}$. Therefore the level distributions of $\mathcal{T}$ and $\mathcal{U}$ are identical. Since all connected level 2 spaces are level connected, this result and Lemma 5 imply that a topology is maximal connected iff it is a critical connected, level 2 space. Theorem 1 now follows as a direct consequence of Lemma 4.

Comment. Theorem 1 is a completely equivalent restatement, using a different jargon, of Theorem (5) in [1] specialized to the case of finite topologies.

THEOREM 2. *The number of homeomorphism classes of maximal connected n-point topologies is equal to twice the number of n-point trees minus the number of n-point trees having a symmetry line.*

Proof. Every uncolored tree without a symmetry line can be bicolored to produce precisely two non-isomorphic descriptor graphs. For the case of a tree possessing a single central point, one descriptor graph has its central point colored 1, and the other has its center colored 2. For the case of a bicentral tree possessing two non-similar points α and β as the central points, one graph has $\alpha^+ = 1$, $\beta^+ = 2$ and the other graph has $\alpha^+ = 2$ and $\beta^+ = 1$. Since adjacent points cannot have the same color, and only the two colors 1 and 2 are to be used, it is clear that the colors of the centers uniquely determine the colors of the remaining points. In the case of a bicentral tree possessing two similar central points α and β, the colorings with $\alpha^+ = 1$, $\beta^+ = 2$ and $\alpha^+ = 2$, $\beta^+ = 1$ produce two isomorphic descriptor graphs. Since each descriptor graph describes a topology, the proposition now follows as a consequence of Theorem 1.

In [2], tree diagrams for trees on up to 10 points are provided. Using these diagrams, the number of homeomorphism classes of maximal connected topologies on 2 to 10 points can be computed, using Theorem 2, to be 1, 2, 3, 6, 10, 22, 42, 94, and 203, respectively.

The problem of enumerating critical connected topologies, not necessarily of level 2, will now be examined. The approach to this problem will be very similar to the one used in [2] for counting uncolored trees.

The word *descriptor tree* will from now on mean a descriptor graph which is a tree, and which is so colored that $|\alpha^+ - \beta^+| = 1$ for any pair of adjacent points α and β. As in [2], a *rooted tree* is a tree with one distinguished point, called the *root*. The *distance* between two points in a tree is the number of lines in the path joining them. The *diameter* of a tree is the greatest distance between any two points. The *root-diameter* of a rooted tree is the greatest distance between the root and all other points. The cycle index, $Z(S_n)$ of the symmetric group of degree n is:

$$Z(S_n) = \frac{1}{n!} \sum_{(j)} \frac{n!}{1^{j_1} j_1! \cdots k^{j_k} j_k! \cdots n^{j_n} j_n!} f_1^{j_1} \cdots f_k^{j_k} \cdots f_n^{j_n},$$

where the sum is taken over all partitions $(j) = j_1, j_2, ..., j_k, ..., j_n$ such that $1 j_1 + 2 j_2 + \cdots + k j_k + \cdots + n j_n = n$. If $g(x_1, ..., x_j, ...)$ is a power series in the symbols $x_1, ..., x_j, ...$, then $Z(S_n, g)$ denotes the function obtained from $Z(S_n)$ by replacing each indeterminate f_k by $g(x_1{}^k, ..., x_j{}^k, ...)$. Further, $Z(S_\infty, g) = 1 + \sum_{n=1}^{\infty} Z(S_n, g)$. The enumeration theorem of Pólya, as stated in [2], will be used.

Two new terms—*complete rooted tree* and *incomplete rooted tree*—will now be defined by providing recursive prescriptions for constructing the class of these objects. A rooted tree with root color c and root diameter 0 consists of a single point, the root, colored c. The tree is complete if $c = 1$. If $c > 1$, then the tree is incomplete. There do not exist complete rooted trees with root color higher than 1 and root diamter 0. Neither do there exist incomplete rooted trees with root color 1 and root diameter 0. A rooted tree with root color c and root diameter $d \geqslant 1$ is constructed by connecting a point, the root, of color c to the roots of an arbitrary finite collection T of complete and incomplete trees such that every tree of T has root diameter $\leqslant d - 1$ and T contains at least one tree with root diameter $d - 1$. The constructed tree is incomplete if $c > 1$ and all the trees of T have root color $c + 1$. It is complete if either $c = 1$ and all the trees of T have root color 2, or $c > 1$ and T satisfies the following three conditions:

(#1) At least one tree of T is a complete tree with a root colored $c - 1$,

(#2) All the trees of T with root color $c - 1$ are complete,

(#3) Each tree in T has either $c - 1$ or $c + 1$ as the color of its root.

If complete rooted trees with root color $c - 1$ and root diameter $\leqslant d-1$ do not exist, then complete rooted trees with root color c and root diameter d also do not exist.

Let $r_{n_1 \cdots n_i \cdots}(c, d, +)$ and $r_{n_1 \cdots n_i \cdots}(c, d, -)$ be, respectively, the number of complete and incomplete rooted trees with root color c, root diameter $\leqslant d$, and n_i points colored i. The counting series for these objects are $R(c, d, \pm) = \sum r_{n_1 \cdots n_i \cdots}(c, d, \pm)\, x_1^{n_1} \cdots x_i^{n_i} \cdots$, where the summations are over all possible finite non-negative integer sequences $n_1, \ldots, n_i, \ldots$. Let $Q(c, d) = R(c, d, +) + R(c, d, -) =$ the counting series for the total number of rooted complete and incomplete trees with root color c and root diameter $\leqslant d$. For $d \geqslant 1$, the counting series for complete and incomplete trees with root color c and root diameter $= d$ are $R^*(c, d, \pm) = R(c, d, \pm) - R(c, d-1, \pm)$ and $Q^*(c, d) = Q(c, d) - Q(c, d-1)$. Obviously, $R^*(c, 0, \pm) = R(c, 0, \pm)$ and $Q^*(c, 0) = Q(c, 0)$.

Using Pólya's theorem and the definition of a complete rooted tree, it is evident that:

$$R(1, 0, +) = x_1 .$$

$$R(c, 0, +) = 0 \text{ for } c > 1.$$

$$R(1, d, +) = x_1[Z(S_\infty, Q(2, d-1))] \text{ for } d \geqslant 1.$$

$$R(c, d, +) = x_c \left[\sum_{m=1}^{\infty} Z(S_m, R(c-1, d-1, +))\right] \times [Z(S_\infty, Q(c+1, d-1))]$$

$$\text{for } c > 1 \text{ and } d \geqslant 1.$$

From the definition of an incomplete rooted tree, it follows that:

$$R(1, 0, -) = 0.$$

$$R(c, 0, -) = x_c \text{ for } c > 1.$$

$$R(1, d, -) = 0 \text{ for } d \geqslant 1.$$

$$R(c, d, -) = x_c[Z(S_\infty, Q(c+1, d-1))] \text{ for } c > 1 \text{ and } d \geqslant 1.$$

A *branch* $\langle \alpha, \beta \rangle$ of a tree, determined by a point α and a line (α, β), is defined to be that subtree which contains α and all points reachable by paths from α whose first line is (α, β). Now let $\mathscr{T}$ be a critical connected topology. Suppose that, for the moment, the tree $G(\mathscr{T})$ is rooted at some arbitrary point α. It is a straightforward matter to prove that:

(1) If the color of α is 1, then the branch $\langle \alpha, \beta \rangle$ is a complete rooted tree for every β adjacent to α.

(2) If $\alpha^+ > 1$, β is adjacent to α and

(a) $\beta^+ = \alpha^+ + 1$, then the branch $\langle\alpha, \beta\rangle$ is an incomplete rooted tree with root color α^+,

(b) $\beta^+ = \alpha^+ - 1$, then the branch $\langle\alpha, \beta\rangle$ is a complete rooted tree with root color α^+.

Conversely, it is also true that every complete or incomplete rooted tree with root diameter $\geqslant 1$, is isomorphic to a branch at the root of some descriptor tree rooted at a suitable point. It is therefore evident that the collection of bicentral descriptor trees possessing the diameter $2\nu + 1$ and central points colored c and $c + 1$ is identical to the collection of trees that can be constructed by connecting the root of a complete rooted tree with root color c and root diameter ν to the root of another tree, which may be either complete or incomplete, with root color $c + 1$ and root diameter ν. It is also evident that the collection of descriptor trees possessing the diameter 2ν and a single central point colored c is precisely the set of trees that can be constructed by connecting a point of color c to the roots of an arbitrary finite collection T of complete and incomplete trees, with T satisfying the requirements #1 to #3 as well as the two conditions: (a) each tree of T has root diameter $\leqslant \nu - 1$ and (b) T contains at least two trees of root diameter $\nu - 1$. Let $V_{2\nu+1}(c, c + 1)$ be the counting series involving the symbols $x_1 ,..., x_i ,...$ such that the coefficient of $x_1^{n_1} \cdots x_i^{n_i} \cdots$ is equal to the number of bicentral descriptor trees possessing the diameter $2\nu + 1$, n_i points colored i, and center points colored c and $c + 1$. Also, let $V_{2\nu}(c)$ be a similar counting series for descriptor trees with diameter 2ν and a single central point colored c.

From the considerations of the last paragraph, it follows that:

$$V_{2\nu+1}(c, c + 1) = R^*(c, \nu, +)\, Q^*(c + 1, \nu), \; \nu \geqslant 0.$$

$$V_0(1) = x_1 .$$

$$V_0(c) = 0 \text{ if } c > 1.$$

$$V_2(1) = x_1x_2^2 + x_1x_2^3 + \cdots + x_1x_2^n + \cdots .$$

$$V_2(2) = x_2[(x_1^2 + x_1^3 + \cdots)(1 + x_3 + x_3^2 + \cdots) + x_1(x_3 + x_3^2 + \cdots)].$$

$$V_2(c) = 0 \text{ if } c > 2.$$

$$V_{2\nu}(1) = x_1 \Big[\sum_{m=2}^{\infty} Z(S_m , Q^*(2, \nu - 1))\Big] \times [Z(S_\infty , Q(2, \nu - 2))], \; \nu \geqslant 2.$$

$$V_{2\nu}(c) = x_c[L_{2\nu}(c)\, M_{2\nu}(c) + L'_{2\nu}(c)\, M'_{2\nu}(c) + L''_{2\nu}(c)\, M''_{2\nu}(c)], \; c > 1 \text{ and}$$
$$\nu \geqslant 2, \text{ where:}$$

$$L_{2\nu}(c) = \left[\sum_{m=2}^{\infty} Z(S_m, R^*(c-1, \nu-1, +))\right] \times [Z(S_\infty, R(c-1, \nu-2, +))],$$

$$L'_{2\nu}(c) = R^*(c-1, \nu-1, +) \times [Z(S_\infty, R(c-1, \nu-2, +))],$$

$$L''_{2\nu}(c) = \sum_{m=1}^{\infty} Z(S_m, R(c-1, \nu-2, +)),$$

$$M_{2\nu}(c) = Z(S_\infty, Q(c+1, \nu-1)),$$

$$M'_{2\nu}(c) = \left[\sum_{m=1}^{\infty} Z(S_m, Q^*(c+1, \nu-1))\right] \times [Z(S_\infty, Q(c+1, \nu-2))].$$

$$M''_{2\nu}(c) = \left[\sum_{m=2}^{\infty} Z(S_m, Q^*(c+1, \nu-1))\right] \times [Z(S_\infty, Q(c+1, \nu-2))].$$

As a consequence of Lemma 2a, the counting series for critical connected topologies is therefore

$$\sum_{\nu=0}^{\infty} \sum_{c=1}^{\infty} V_{2\nu}(c) + V_{2\nu+1}(c, c+1).$$

The coefficient of $x_1^{n_1} \cdots x_i^{n_i} \cdots$ in this series is the number of homeomorphism classes of critical connected topologies with level distribution $n_1, \ldots, n_i, \ldots$. This completes the formal solution to the problem enumerating critical connected spaces. The counting series for the various types of descriptor trees on up to 7 points appear at the end of this article. The counting series, up to the first few terms, in x and y which displays the number of homeomorphism classes of n-point critical connected spaces with level m as the coefficient of $x^m y^n$ is found to be:

$$\begin{array}{rl}
x^2y^2 + 2x^2y^3 + 3x^2y^4 + 6x^2y^5 + 10x^2y^6 + 22x^2y^7 + \cdots & \text{(maximal} \\
+\ x^3y^3 + 3x^3y^4 + 9x^3y^5 + 22x^3y^6 + 64x^3y^7 + \cdots & \text{connected} \\
+\ x^4y^4 + 4x^4y^5 + 14x^4y^6 + 44x^4y^7 + \cdots & \text{spaces)} \\
+\ x^5y^5 + 5x^5y^6 + 20x^5y^7 + \cdots & \\
+\ x^6y^6 + 6x^6y^7 + \cdots & \\
+\ x^7y^7 + \cdots & \\
\hline
\text{so that } y^2 + 3y^3 + 7y^4 + 20y^5 + 52y^6 + 157y^7 + \cdots &
\end{array}$$

is the counting series, up to the first seven terms, which displays the total number of homeomorphism classes of n-point critical connected spaces as the coefficient of y^n.

Having enumerated the maximal connected topologies, it is natural to try to count those spaces for which the connectedness requirement has been slightly relaxed. For example, define the *connectivity* of a topology to be 0 iff it is maximal connected. Then recursively define the connectivity of a connected space $\mathscr{T}$ to be k iff any connected topology strictly finer than $\mathscr{T}$ has connectivity at most $k-1$. One can now enquire about the number of n-point topologies with a specified connectivity. The substitution of connectedness in place of level-connectedness in the definition of a critical connected space also provides a class of enumeration problems. A discussion on the enumeration of finite topologies with connectivity 1, together with certain related topics, will appear elsewhere. The rest of this paper is devoted to setting up a simple procedure (Lemma 7) which will allow the determination of the connectivity of a given topology.

For any topology $\mathscr{T}$ on N, the quantity $\mathscr{T}^*$ is defined by the relation

$$\mathscr{T}^* = \sum_{\alpha} |\alpha^*(\mathscr{T})|, \qquad \alpha \in N.$$

THEOREM 3. *Let $\mathscr{T}$ and $\mathscr{V}$ be topologies on N and suppose that $\mathscr{T} \geqslant \mathscr{V}$. Then the situation $\mathscr{T} > \mathscr{U} > \mathscr{V}$, for some space $\mathscr{U}$ on N, occurs iff*

$$\mathscr{V}^* - \mathscr{T}^* \geqslant 2.$$

Proof. Suppose that $\mathscr{T} > \mathscr{U} > \mathscr{V}$ for some space $\mathscr{U}$. Then clearly, $\alpha^*(\mathscr{T}) \subseteq \alpha^*(\mathscr{U}) \subseteq \alpha^*(\mathscr{V})$ for all $\alpha \in N$, $\beta^*(\mathscr{T}) \subset \beta^*(\mathscr{U})$ for at least one $\beta \in N$, and $\gamma^*(\mathscr{U}) \subset \gamma^*(\mathscr{V})$ for at least one $\gamma \in N$. Therefore $\mathscr{U}^* - \mathscr{T}^* \geqslant 1$ and $\mathscr{V}^* - \mathscr{U}^* \geqslant 1$, so that $\mathscr{V}^* - \mathscr{T}^* \geqslant 2$. Now suppose that $\mathscr{T} \geqslant \mathscr{V}$ and $\mathscr{V}^* - \mathscr{T}^* \geqslant 2$. This ensures that $S = \{\alpha \mid \alpha^*(\mathscr{T}) \subset \alpha^*(\mathscr{V})\}$ is nonvoid. (A) Let β be the only point in S, in case S contains a single point and (B) in case S contains more than one point, let β be any point of S such that, if $\gamma \in S - \beta$, then $\beta \notin \gamma^*(\mathscr{T})$. Such a β will always exist because of the T_0 nature of the spaces under consideration. Then $R = \beta^*(\mathscr{V}) - \beta^*(\mathscr{T}) \neq \varnothing$. Therefore there exists a $\delta \in R$ such that $\delta^*(\mathscr{T}) \cap (R - \delta) = \varnothing$. Now define $\tilde{\alpha} = \alpha^*(\mathscr{T})$ if $\alpha \neq \beta$ and $\tilde{\beta} = \beta^*(\mathscr{T}) + \delta$. Then there exists a topology $\mathscr{U}$ defined by the requirement that $\alpha^*(\mathscr{U}) = \tilde{\alpha}$ for all $\alpha \in N$. To prove this, it is sufficient to show that, if $\mu \in \tilde{\lambda}$, then $\tilde{\mu} \subseteq \tilde{\lambda}$ for all $\mu, \lambda \in N$. First suppose that both μ and λ are different from β. Then

$$\mu \in \tilde{\lambda} \rightarrow \mu \in \lambda^*(\mathscr{T}) \rightarrow \mu^*(\mathscr{T}) \subseteq \lambda^*(\mathscr{T}) \rightarrow \tilde{\mu} \subseteq \tilde{\lambda}.$$

Similarly, if $\mu \neq \delta$, then $\mu \in \tilde{\beta}$ implies that $\tilde{\mu} \subseteq \tilde{\beta}$. As $\mathscr{T} \geqslant \mathscr{V}$, therefore $\delta^*(\mathscr{T}) \subseteq \delta^*(\mathscr{V})$ and so $\delta \in R \rightarrow \delta \in \beta^*(\mathscr{V}) \rightarrow \delta^*(\mathscr{V}) \subseteq \beta^*(\mathscr{V}) \rightarrow \delta^*(\mathscr{T}) \subseteq \beta^*(\mathscr{V})$. Also, it is clear that $(\delta^*(\mathscr{T}) - \delta) \cap (\beta^*(\mathscr{V}) - \beta^*(\mathscr{T})) = \varnothing$.

Therefore $\delta^*(\mathscr{T}) - \delta \subseteq \beta^*(\mathscr{T})$, which is equivalent to $\tilde{\delta} - \delta \subseteq \beta^*(\mathscr{T})$ and hence $\tilde{\delta} \subseteq \tilde{\beta}$. Finally, suppose that $\lambda \neq \beta$ and $\beta \in \tilde{\lambda}$. Then $\beta \in \lambda^*(\mathscr{T})$. Therefore, in either of the cases, A and B, $\lambda \notin S$. Hence $\lambda^*(\mathscr{V}) = \lambda^*(\mathscr{T}) = \tilde{\lambda}$. Therefore $\beta \in \lambda^*(\mathscr{T}) \rightarrow \beta \in \lambda^*(\mathscr{V}) \rightarrow \beta^*(\mathscr{V}) \subseteq \lambda^*(\mathscr{V}) \rightarrow \beta^*(\mathscr{T}) \cup R \subseteq \lambda^*(\mathscr{V}) \rightarrow \tilde{\beta} \subseteq \tilde{\lambda}$. The existence of $\mathscr{U}$ has thus been established. It will now be demonstrated that $\mathscr{U}$ is a T_0 topology. To prove this, it is sufficient to show that, if $\mu \neq \lambda$, then $\mu \in \tilde{\lambda}$ implies $\lambda \notin \tilde{\mu}$ for all $\mu, \lambda \in N$. First suppose that μ, λ, β are all different. Then $\mu \in \tilde{\lambda} \rightarrow \mu \in \lambda^*(\mathscr{T}) \rightarrow \lambda \notin \mu^*(\mathscr{T}) \rightarrow \lambda \notin \tilde{\mu}$. Similarly, if $\beta \neq \mu \neq \delta$, then $\mu \in \tilde{\beta}$ implies that $\beta \notin \tilde{\mu}$. Now $\delta \in R$, i.e., $\delta \in \beta^*(\mathscr{V})$, so that the T_0 property of $\mathscr{V}$ implies $\beta \notin \delta^*(\mathscr{V})$, and, since $\mathscr{T} \geqslant \mathscr{V}$, $\beta \notin \delta^*(\mathscr{T}) = \tilde{\delta}$. Finally suppose that $\lambda \neq \beta$ and $\beta \in \tilde{\lambda}$. Then $\beta \in \lambda^*(\mathscr{T})$ and so, as a consequence of the last result, $\beta \notin \delta^*(\mathscr{T})$, it follows that $\lambda \neq \delta$. Therefore $\beta \in \tilde{\lambda} \rightarrow \lambda \notin \beta^*(\mathscr{T}) \rightarrow \lambda \notin \tilde{\beta}$. Hence $\mathscr{U}$ is a T_0 space. From the definition of $\mathscr{U}$ it is clear that $\mathscr{T} \geqslant \mathscr{U} \geqslant \mathscr{V}$. Now $|\gamma^*(\mathscr{U})| = |\gamma^*(\mathscr{T})|$ for all $\gamma \neq \beta$ and $|\beta^*(\mathscr{U})| = |\beta^*(\mathscr{T})| + 1$, so that $\mathscr{U}^* = \mathscr{T}^* + 1$. Therefore $\mathscr{T} > \mathscr{U} > \mathscr{V}$. This completes the proof.

For the next two theorems assume, for the sake of concreteness, that the n-point set $N = \{1, 2, \ldots, n\}$.

THEOREM 4. *If $\mathscr{T}$ is a maximal connected space on a set of n points, then* $\mathscr{T}^* = 2n - 1$.

Proof. An inductive proof is presented. The two possible connected 2 point spaces are both homeomorphic to $\mathscr{V} = \{\varnothing, \{1\}, \{1, 2\}\}$. Clearly $\mathscr{V}^* = 3$ and so the result in question is true for $n = 2$. Now assume, as the hypothesis of induction, that the theorem holds for all n point maximal connected spaces. It is an obvious consequence of Theorem 1 that the descriptor tree of any $n + 1$ point maximal connected space $\mathscr{T}$ can be obtained from the descriptor tree of a suitable n point maximal connected space $\mathscr{U}$ by either (A) connecting a point α colored 1 to a suitable point β with color 2 in $G(\mathscr{U})$ or (B) connecting a point α colored 2 to a suitable point β with color 1 in $G(\mathscr{U})$. Remembering that in a bicolored descriptor tree a point μ is accessible to a point λ iff $\mu^+ = 1$, $\lambda^+ = 2$ and μ, λ are adjacent, it becomes evident that, in case (A), $|\alpha^*(\mathscr{T})| = 1$, $|\beta^*(\mathscr{T})| = |\beta^*(\mathscr{U})| + 1$ and, in case (B), $|\alpha^*(\mathscr{T})| = 2$, $|\beta^*(\mathscr{T})| = |\beta^*(\mathscr{U})|$. For any other point $\gamma \neq \alpha, \beta$, $|\gamma^*(\mathscr{T})| = |\gamma^*(\mathscr{U})|$ in both cases. Therefore, in any case, $\mathscr{T}^* = \mathscr{U}^* + 2$. Hence, by the induction hypothesis,

$$\mathscr{T}^* = 2(n + 1) - 1.$$

THEOREM 5. *A chain of connected n point T_0 topologies, linearly ordered* *by strict fineness, can contain a maximum of* $\frac{1}{2}(n^2 - 3n + 4)$ *topologies. Moreover, there exist chains of connected n point* T_0 *spaces with this length.*

Proof. It is a straightforward matter to show that, if $\mathscr{T}$ is a T_0 space on N, then $\mathscr{T}$ is finer than a space homeomorphic to $\mathscr{U} = \{\varnothing, \{1\}, \{1, 2\},...; \{1, 2,..., i\},..., N\}$. Thus $\mathscr{U}$, in a sense, represents the coarsest possible T_0 space on N. Further, a connected space which is not maximal connected is bounded above by some maximal connected space. Therefore a typical chain of connected T_0 spaces whose length cannot be increased further looks like:

some homeomorphic image of $\mathscr{U} = \mathscr{T}_1 < \mathscr{T}_2 < \cdots < \mathscr{T}_{m-1} < \mathscr{T}_m =$ some maximal connected space.

An implication of Theorem 3 is that $\mathscr{T}_i^* - \mathscr{T}_{i+1}^* = 1$ for $i = 1,..., m-1$. Since $\mathscr{U}^* = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, it follows from Theorem 4 that $m = \frac{1}{2}n(n+1) - (2n-1) + 1 = \frac{1}{2}(n^2 - 3n + 4)$. The existence of such chains with length $\frac{1}{2}(n^2 - 3n + 4)$ is now an obvious consequence of Theorem 4.

The next two results follow immediately from the discussions of the previous three Theorems.

LEMMA 6. *A necessary condition for an n point space* $\mathscr{T}$ *to be a connected* T_0 *space is that* $2n - 1 \leqslant \mathscr{T}^* \leqslant \frac{1}{2}n(n+1)$

LEMMA 7. *The connectivity of a given connected n point* T_0 *topology* $\mathscr{T}$ *is* $\mathscr{T}^* - 2n + 1$.

3. Counting Series

There now follow the counting series for the various kinds of descriptor trees on up to 7 points:

$$V_1(1, 2) = x_1x_2 .$$

$$V_1(c, c+1) = 0 \text{ if } c > 1.$$

$$\begin{aligned} V_3(1, 2) = {} & (x_1^2x_2^2 + x_1x_2^2x_3) + (x_1^2x_2^3 + x_1^3x_2^2 + x_1x_2^2x_3^2 + x_1x_2^3x_3 \\ & + x_1^2x_2^2x_3) + (x_1^2x_2^4 + x_1^3x_2^3 + x_1^4x_2^2 + x_1x_2^2x_3^3 \\ & + x_1x_2^3x_3^2 + x_1x_2^4x_3 + x_1^2x_2^2x_3^2 + x_1^2x_2^3x_3 + x_1^3x_2^2x_3) \\ & + (x_1^2x_2^5 + x_1^3x_2^4 + x_1^4x_2^3 + x_1^5x_2^2 + x_1x_2^2x_3^4 \\ & + x_1x_2^3x_3^3 + x_1x_2^4x_3^2 + x_1x_2^5x_3 + x_1^2x_2^2x_3^3 + x_1^2x_2^3x_3^2 \\ & + x_1^2x_2^4x_3 + x_1^3x_2^2x_3^2 + x_1^3x_2^3x_3 + x_1^4x_2^2x_3) + \cdots . \end{aligned}$$

$$V_3(2,3) = (x_1x_2x_3x_4) + (x_1x_2x_3x_4^2 + x_1x_2x_3^2x_4 + x_1^2x_2x_3x_4) + (x_1x_2x_3x_4^3 + x_1x_2x_3^2x_4^2 + x_1x_2x_3^3x_4 + x_1^2x_2x_3x_4^2 + x_1^2x_2x_3^2x_4 + x_1^3x_2x_3x_4) + (x_1x_2x_3x_4^4 + x_1x_2x_3^2x_4^3 + x_1x_2x_3^3x_4^2 + x_1x_2x_3^4x_4 + x_1^2x_2x_3x_4^3 + x_1^2x_2x_3^2x_4^2 + x_1^2x_2x_3^3x_4 + x_1^3x_2x_3x_4^2 + x_1^3x_2x_3^2x_4 + x_1^4x_2x_3x_4) + \cdots.$$

$$V_3(c, c+1) = 0 \text{ if } c > 2.$$

$$V_5(1,2) = (x_1^3x_2^3 + x_1^2x_2^3x_3 + x_1x_2^2x_3^2x_4 + x_1^2x_2^2x_3x_4) + (2x_1^3x_2^4 + 2x_1^4x_2^3 + 2x_1^2x_2^3x_3^2 + 2x_1^2x_2^4x_3 + 3x_1^3x_2^3x_3 + x_1x_2^2x_3^2x_4^2 + 2x_1x_2^2x_3^3x_4 + x_1x_2^3x_3^2x_4 + x_1^2x_2^2x_3x_4^2 + 3x_1^2x_2^2x_3^2x_4 + x_1^2x_2^3x_3x_4 + 2x_1^3x_2^2x_3x_4) + \cdots.$$

$$V_5(2,3) = (x_1^2x_2^3x_3 + x_1x_2^2x_3x_4x_5) + (2x_1^2x_2^3x_3^2 + x_1^2x_2^4x_3 + 2x_1^3x_2^3x_3 + x_1^2x_2^2x_3^2x_4 + x_1^2x_2^3x_3x_4 + x_1x_2x_3^2x_4^2x_5 + x_1x_2^2x_3x_4x_5^2 + x_1x_2^2x_3x_4^2x_5 + x_1x_2^2x_3^2x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5) + \cdots.$$

$$V_5(3,4) = x_1x_2x_3x_4x_5x_6 + (x_1x_2x_3x_4x_5x_6^2 + x_1x_2x_3x_4x_5^2x_6 + x_1x_2x_3x_4^2x_5x_6 + x_1x_2x_3^2x_4x_5x_6 + x_1^2x_2x_3x_4x_5x_6) + \cdots.$$

$$V_5(c, c+1) = 0 \text{ if } c > 3.$$

Descriptor trees with $\leqslant 7$ points are not involved in any other non-zero $V_{2\nu+1}(c, c+1)$,

$$V_2(1) = x_1x_2^2 + x_1x_2^3 + x_1x_2^4 + x_1x_2^5 + x_1x_2^6 + \cdots.$$

$$V_2(2) = (x_1^2x_2 + x_1x_2x_3) + (x_1^3x_2 + x_1x_2x_3^2 + x_1^2x_2x_3) + (x_1^4x_2 + x_1x_2x_3^3 + x_1^2x_2x_3^2 + x_1^3x_2x_3) + (x_1^5x_2 + x_1x_2x_3^4 + x_1^2x_2x_3^3 + x_1^3x_2x_3^2 + x_1^4x_2x_3) + (x_1^6x_2 + x_1x_2x_3^5 + x_1^2x_2x_3^4 + x_1^3x_2x_3^3 + x_1^4x_2x_3^2 + x_1^5x_2x_3) + \cdots.$$

$$V_2(c) = 0 \text{ if } c > 2.$$

$$V_4(1) = (x_1^3x_2^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_3) + (x_1^3x_2^3 + x_1^4x_2^2 + x_1x_2^2x_3^3 + x_1x_2^3x_3^2 + 2x_1^2x_2^2x_3^2 + x_1^2x_2^3x_3 + 2x_1^3x_2^2x_3) + (x_1^3x_2^4 + 2x_1^4x_2^3 + 2x_1^5x_2^2 + 2x_1x_2^2x_3^4 + 2x_1x_2^3x_3^3 + x_1x_2^4x_3^2 + 3x_1^2x_2^2x_3^2 + 3x_1^2x_2^3x_3^2 + x_1^2x_2^4x_3 + 4x_1^3x_2^2x_3^2 + 3x_1^3x_2^3x_3 + 3x_1^4x_2^2x_3) + \cdots.$$

$$V_4(2) = (x_1^2x_2^3 + x_1x_2^2x_3x_4) + (x_1^2x_2^4 + x_1^3x_2^3 + x_1^2x_2^3x_3$$
$$+ x_1x_2x_3^2x_4^2 + x_1x_2^2x_3x_4^2 + x_1x_2^2x_3^2x_4 + x_1x_2^3x_3x_4$$
$$+ x_1^2x_2^2x_3x_4) + (2x_1^2x_2^5 + 2x_1^3x_2^4 + x_1^4x_2^3 + x_1^2x_2^3x_3^2$$
$$+ x_1^2x_2^4x_3 + x_1^3x_2^3x_3 + x_1x_2x_3^2x_4^3 + x_1x_2x_3^3x_4^2$$
$$+ x_1x_2^2x_3x_4^3 + 2x_1x_2^2x_3^2x_4^2 + x_1x_2^2x_3^3x_4 + x_1x_2^3x_3x_4^2$$
$$+ x_1x_2^3x_3^2x_4 + x_1x_2^4x_3x_4 + x_1^2x_2x_3^2x_4^2 + x_1^2x_2^2x_3x_4^2$$
$$+ x_1^2x_2^2x_3^2x_4 + 2x_1^2x_2^3x_3x_4 + x_1^3x_2^2x_3x_4) + \cdots .$$

$$V_4(3) = (x_1^2x_2^2x_3 + x_1x_2x_3x_4x_5) + (x_1^2x_2^2x_3^2 + x_1^3x_2^2x_3$$
$$+ x_1^2x_2^2x_3x_4 + x_1x_2x_3x_4x_5^2 + x_1x_2x_3x_4^2x_5 + x_1x_2x_3^2x_4x_5$$
$$+ x_1^2x_2x_3x_4x_5) + (2x_1^2x_2^2x_3^3 + 2x_1^3x_2^2x_3^2 + x_1^3x_2^3x_3$$
$$+ 2x_1^4x_2^2x_3 + x_1^2x_2^2x_3x_4^2 + x_1^2x_2^2x_3^2x_4 + x_1^3x_2^2x_3x_4$$
$$+ x_1x_2x_3x_4x_5^3 + 2x_1x_2x_3x_4^2x_5^2 + x_1x_2x_3x_4^3x_5$$
$$+ x_1x_2x_3^2x_4x_5^2 + x_1x_2x_3^2x_4^2x_5 + x_1x_2x_3^3x_4x_5$$
$$+ x_1^2x_2x_3x_4x_5^2 + x_1^2x_2x_3x_4^2x_5 + x_1^2x_2x_3^2x_4x_5$$
$$+ x_1^2x_2^2x_3x_4x_5 + x_1^3x_2x_3x_4x_5) + \cdots .$$

$$V_4(c) = 0 \text{ if } c > 3.$$

$$V_6(1) = x_1^3x_2^4 + x_1x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3x_4 + \cdots .$$

$$V_6(2) = x_1^4x_2^3 + 2x_1^2x_2^3x_3^2 + 2x_1^3x_2^3x_3 + x_1x_2^2x_3^2x_4x_5$$
$$+ x_1^2x_2^2x_3x_4x_5 + \cdots .$$

$$V_6(3) = x_1^2x_2^4x_3 + x_1x_2^2x_3x_4x_5x_6 + \cdots .$$

$$V_6(4) = x_1^2x_2^2x_3^2x_4 + x_1x_2x_3x_4x_5x_6x_7 + \cdots .$$

$$V_6(c) = 0 \text{ if } c > 4.$$

Descriptor trees with $\leqslant 7$ points are not involved in any other non-zero $V_{2\nu}(c)$.

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