JOURNAL OF ALGEBRA 79, 169–205 (1982)

# Equivalence between Colocalization and Localization in Abelian Categories with Applications to the Theory of Modules

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*Communicated by P. M. Cohn*

Received June 13, 1978

## INTRODUCTION

In [34] Tachikawa and the author studied colocalization and localization for torsion theories in abelian categories and relations between them. In this paper we consider somewhat concrete cases and prove equivalances between colocalization and localization in abelian categories which generalize the Gabriel–Popescu theorem and results of Fuller [7] and Kato [12].

Let $R$ and $S$ be rings with identities and Mod-$R$ and Mod-$S$ the categories of unitary right $R$-, $S$-modules, respectively. Recently the Morita theorem has been generalized by many authors to equivalences between full subcategories of Mod-$R$ and Mod-$S$.

The first result which we would like to mention is that of Fuller. Let $\mathscr{C}$ be a complete additive subcategory of Mod-$R$, i.e., $\mathscr{C}$ is a full subcategory of Mod-$R$ closed under submodules, factor modules, arbitrary direct sums and isomorphic images. Then he proved that $\mathscr{C}$ and Mod-$R$ are category equivalent if and only if there exists a bimodule ${}_SU_R$ such that $U_R$ is a quasi-progenerator (i.e., a finitely generated quasi-projective self-generator), $S \cong \mathrm{End}(U_R)$ canonically and $\mathrm{Gen}(U_R) = \mathscr{C}$, where $\mathrm{Gen}(U_R)$ means the full subcategory of Mod-$R$ consisting of all homomorphic images of direct sums of copies of $U_R$.

Before describing other results it should be noted that in Fuller's result $U_R$ may have a zero trace ideal. In fact if $R$ is a primitive ring, we can take as $U_R$ a finite direct sum of simple modules which are not isomorphic to right ideals of $R$, and then $U_R$ is a quasi-progenerator with a zero trace ideal.

The second result is that of Azumaya [2]. Let ${}_RP$ be a projective module with $S = \mathrm{End}({}_RP)$. Let $J$ be the trace ideal of $P_S$ and $\mathscr{T}_J = \{N \in \text{Mod-}S \mid NJ = N\}$. Then he proved that $\mathrm{Hom}_S(P, -)$ and $- \otimes_R P$ induce an equivalence $\mathscr{T}_J \sim \mathrm{Im}\, \mathrm{Hom}_S(P, -)$.



0021-8693/82/110169-37$02.00/0

For convenience let $P_R$ be a projective right $R$-module with the trace ideal $I$. Then $I$ defines a torsion theory $(\mathscr{T}_I, \mathscr{F}_I)$ in Mod-$R$, where $\mathscr{T}_I = \{M \in \text{Mod-}R \mid MI = M\}$ and $\mathscr{F}_I = \{M \in \text{Mod-}R \mid MI = 0\}$. $M_R$ is said to be $I$-projective if $\text{Hom}_R(M, -)$ is exact on all short exact sequences $0 \to X'_R \to X_R \to X''_R \to 0$ with $X' \in \mathscr{F}_I$. A homomorphism $f\colon C(M) \to M$ is called an $I$-colocalization if $\text{Ker}\, f$, $\text{Cok}\, f \in \mathscr{F}_I$, $C(M) \in \mathscr{T}_I$ and $C(M)$ is $I$-projective. Then McMaster [17] proved that the canonical homomorphism $\phi_M$: $\text{Hom}_R(P, M) \otimes_S P \to M$ is the $I$-colocalization for all $M \in \text{Mod-}R$, where $S = \text{End}(P_R)$.

The third result is that of Kato. He has generalized both results of Azumaya and McMaster as follows: Let $\langle {}_SU_R, {}_RV_S \rangle$ be a Morita context with the trace ideals $I \subset R$ and $J \subset S$. For convenience we assume $UI = U$ (so does $JU = U$). By a manner similar to the above we can define $I$-projective modules and $I$-colocalizations for right $R$-modules. On the other hand, since $J$ is an idempotent ideal, it defines a hereditary torsion theory $(\mathscr{F}_J, \mathscr{D}_J)$ in Mod-$S$, where $\mathscr{F}_J = \{N \in \text{Mod-}S \mid NJ = 0\}$ and $\mathscr{D}_J = \{N \in \text{Mod-}S \mid \text{Ann}_N(J) = 0\}$. Then $J$-injective modules and $J$-localizations are defined dually. Let $\mathscr{C}_I = \{M \in \text{Mod-}R \mid M \in \mathscr{T}_I$ and $M_R$ is $I$-projective$\}$ and $\mathscr{L}_J = \{N \in \text{Mod-}S \mid N \in \mathscr{D}_J$ and $N_S$ is $J$-injective$\}$. Then he proved that if $U \in \mathscr{C}_I$, $\text{Hom}_R(U, -)$ and $- \otimes_S U$ induce an equivalence $\mathscr{C}_I \sim \mathscr{L}_J$. Moreover in this case the canonical homomorphisms $\phi_M$: $\text{Hom}_R(U, M) \otimes_S U \to M$ and $\psi_N\colon N \to \text{Hom}_R(U, N \otimes_S U)$ are the $I$-colocalization and the $J$-localization for all $M \in \text{Mod-}R$ and $N \in \text{Mod-}S$, respectively.

There is an important difference between the results of Fuller and Kato: In Kato's result the condition $UI = U$ is essential. But it does not hold generally in Fuller's result as was pointed before. So Tachikawa suggested that we obtain a theorem which contains both results of Fuller and Kato as special cases.

Before stating our results some further comments are necessary. Let $\mathscr{A}$ be an abelian category and $(\mathscr{T}, \mathscr{F})$ a torsion theory in $\mathscr{A}$ in the sense of Dickson [5]. An object $A \in \mathscr{A}$ is said to be divisible with respect to $(\mathscr{T}, \mathscr{F})$ if $\mathscr{A}(-, A)$ is exact on all short exact sequences $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathscr{T}$. A morphism $f\colon A \to B$ is said to be a localization of $A$ if $\text{Ker}\, f$, $\text{Cok}\, f \in \mathscr{T}$, $B \in \mathscr{F}$ and $B$ is divisible. $(\mathscr{T}, \mathscr{F})$ is called hereditary if $\mathscr{T}$ is closed under subobjects and is called strongly hereditary if every object of $\mathscr{A}$ has its localization. The dual of these definitions are obtained as translations of the same ones with respect to the torsion theory $(\mathscr{F}, \mathscr{T})$ in the dual category $\mathscr{A}^*$. In [34] we have proved that there is a 1–1 correspondence between strongly hereditary torsion theories and reflective subcategories of $\mathscr{A}$ such that reflectors preserve kernels. Such a reflective subcategory is called Giraud (and its dual is called Co-Giraud).

Let $U \in \text{Mod-}R$ and let $I$ be the trace ideal of $U_R$. For $M \in \text{Mod-}R$ let us set $t(M) = \sum \{\text{Im}\, f \mid f \in \text{Hom}_R(U, M)\}$. Then $t$ is an idempotent preradical.

It is known that $U_R$ is trace accessible (i.e., $UI = U$) if and only if $t$ preserves epimorphisms. If $t$ is epi-preserving, the corresponding torsion theory is $(\mathscr{T}_I, \mathscr{F}_I)$, which is cohereditary. Conversely every cohereditary torsion theory in a category of modules is obtained by a trace accessible modules by the same way of the above. Examples of trace accessible modules are (1) generators, (2) projective modules, (3) locally projective modules [35], (4) the module in the result of Kato.

Dually let $V \in$ Mod-$R$ and for any $M \in$ Mod-$R$ put $r(M) = \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(M, V)\}$. Then $r$ is a radical. It is well known that $r$ is left exact if and only if $V_R$ cogenerates its injective envelope. The name QF-3′ modules has come from this property [3, 4]. But in this paper we call $V_R$ QF-3 provided that $r$ is left exact. The name of QF-3′ modules will be left for another type of modules (Section 6). So trace accessible modules and QF-3 modules are relatively dual notions. On the other hand, recently Wakamatsu has made a categorical characterization for QF-3 modules as follows: $V_R$ is QF-3 if and only if for a monomorphism $f: M'_R \to M_R$, $\operatorname{Hom}(f, V) = 0$ implies $\operatorname{Hom}_R(M', V) = 0$. It should be noted that the notion of QF-3 modules generalizes both cogenerators and injective modules. So in any category an object which possesses the property described above we call QF-3 and its dual CQF-3.

Now let $\mathscr{B}$ be a complete additive subcategory of Mod-$R$ and $U \in \mathscr{B}$ a CQF-3 object with $S = \operatorname{End}(U_R)$ and $(\mathscr{T}, \mathscr{F})$ a torsion theory in $\mathscr{B}$ generated by $U$. Then from the result of Section 2 we can prove the following theorem.

THEOREM. *The following statements are equivalent.*

(i) *$U$ is codivisible with respect to $(\mathscr{T}, \mathscr{F})$.*

(ii) *The canonical homomorphism $\phi_B: \operatorname{Hom}_R(U, B) \otimes_S U \to B$ is the colocalization with respect to $(\mathscr{T}, \mathscr{F})$ for all $B \in \mathscr{B}$.*

(iii) *${}_SU$ is weakly flat (in the sense of Wakamatsu) and the canonical homomorphism $\psi_N: N \to \operatorname{Hom}_R(U, N \otimes_S U)$ is the localization for all $N \in$ Mod-$S$ with respect to the Gabriel topology $\mathbf{T} = \{J_S \subset S \mid JU = U\}$, where ${}_SU$ is weakly flat if for a monomorphism $f: {}_SX' \to {}_SX$, $f \otimes N = 0$ implies $X' \otimes_S U = 0$.*

If the above statements hold, $\operatorname{Hom}_R(U, -)$ and $- \otimes_S U$ induce an equivalence $\mathscr{C} \sim \mathscr{L}$, where $\mathscr{C}$ is the Co-Giraud subcategory of $\mathscr{B}$ associated with $(\mathscr{T}, \mathscr{F})$ and $\mathscr{L}$ is the Giraud subcategory of Mod-$S$ associated with $\mathbf{T}$.

In the above theorem if $\mathscr{C} = \mathscr{B}$ and $\mathscr{L} =$ Mod-$S$, we get the result of Fuller, and if $\mathscr{B} =$ Mod-$R$, we get that of Kato in the case of the derived context $\langle {}_SU_R, {}_R\operatorname{Hom}_R(U, R) \rangle$.

Fuller's result seems to be concerned with equivalences between a cocomplete abelian category and a category of modules. So in Section 2 we consider torsion theories in a cocomplete abelian category and obtain generalizations of Fuller's result and the Gabriel–Popescu theorem [26]. As a result we can prove that a cocomplete abelian category with a small generator is a Grothendieck category.

A weakly flat object plays an important role. So in Section 1 we consider the localization with respect to the hereditary torsion theory in Mod-$R$ which is determined by a weakly flat $R$-object ${}_RU$ in a cocomplete additive category $\mathscr{A}$. It will be shown that in complete additive subcategories of module categories, every CQF-3 object is weakly flat over its endomorphism ring.

Again let $\mathscr{A}$ be an abelian category and $\mathscr{B}$ a full subcategory of $\mathscr{A}$. Then $\mathscr{B}$ is said to be an exact subcategory if it is abelian and the inclusion functor is exact. In this paper we call $\mathscr{B}$ a strongly exact subcategory if it is closed under subobjects, quotient objects and finite coproducts. As an example of strongly exact subcategories, we have the following:

EXAMPLE (Robert [27, Proposition 1]). Let $U \in \mathscr{A}$ and put $\mathscr{P}(U) = \{A \in \mathscr{A} \mid U \text{ is } A\text{-projective}\}$. Then $\mathscr{P}(U)$ is a strongly exact subcategory.

In Section 3 we consider the colocalization in strongly exact subcategories of cocomplete abelian categories. In this section we establish the lattice isomorphism between the lattice of torsion subobjects of a CQF-3 codivisible object and the lattice of right ideals of its endomorphism ring such that the object is faithful to factor modules of the ring factored by those right ideals.

Section 4 is devoted to obtain the dual of results in Section 3. Let $W_R$ be a cogenerator of Mod-$R$. Then we can prove that every finitely $W_R$-cogenerated module is $W$-reflexive, in particular, if $W_R$ finitely cogenerates $R_R$, $W_R$ is balanced. As one more example, let $V_R$ be a quasi-injective module and $\mathscr{I}(V_R) = \{M \in \text{Mod-}R \mid V_R \text{ is } M\text{-injective}\}$. Then we can construct the hereditary torsion theory in $\mathscr{I}(V_R)$. It can be shown that for $X \in \mathscr{I}(V_R)$, if $X/r(X)$ is finitely $V$-cogenerated, the canonical homomorphism $\eta_X: X \to \text{Hom}_S(\text{Hom}_R(X, V), V)$ gives the localization, where $S = \text{End}(V_R)$ and $r$ is the torsion radical in $\mathscr{I}(V_R)$ associated with the torsion theory cogenerated by $V$.

The above example induces a duality between the category of $V_R$-copresented and finitely $V_R$-cogenerated module and the category of finitely generated and ${}_SV$-cogenerated left $S$-modules (cf. Lambek and Rattray [14]). In Section 5 we give category equivalence and duality as applications of Section 3 and 4.

## 1. Weakly Flat Objects and Localization

Throughout this section $\mathscr{A}$ denotes a cocomplete additive category, i.e., a preadditive category with coproducts and cokernels. First of all it is convenient for us to review the existence of a left adjoint of a Hom-functor (see [19, p. 143, Theorem 3.1] for detail). Let $C \in \mathscr{A}$. Then $C$ is called an $R$-object if there is a ring homomorphism $R \to \operatorname{End}_{\mathscr{A}}(C)$. Then for any $A \in \mathscr{A}$, $\mathscr{A}(C, A)$ can be considered a right $R$-module. We can construct the left adjoint of the functor $\mathscr{A}(C, -)\colon \mathscr{A} \to$ Mod-$R$ as follows. Let $M \in$ Mod-$R$ and let $R^{(K)} \xrightarrow{\lambda} R^{(M)} \xrightarrow{\mu} M \to 0$ be an exact sequence, where $\mu$ is defined by $\mu((r_m)) = \sum r_m m$, $K = \operatorname{Ker} \mu$ and $\lambda$ is defined by $\lambda((r_k)) = \sum r_k k$. Then $\lambda$ induces a morphism $\hat{\lambda}\colon C^{(K)} \to C^{(M)}$. So $M \otimes_R C$ is defined by $\operatorname{Cok} \hat{\lambda}$. The isomorphism $\eta\colon \operatorname{Hom}_R(M, \mathscr{A}(C, A)) \cong \mathscr{A}(M \otimes_R C, A)$ is given as follows: Let $\varphi\colon M_R \to \mathscr{A}(C, A)_R$ be a homomorphism. For $m \in M$, let $C_m$ be the $m$th $C$ in $C^{(M)}$. Then $\varphi(m)\colon C_m \to A$ induces $\bar{\eta}(\varphi)\colon C^{(M)} \to A$. It is checked that $\bar{\eta}(\varphi)\hat{\lambda} = 0$. Hence $\bar{\eta}(\varphi)$ induces $\eta(\varphi)\colon M \otimes_R C \to A$. $\eta$ is an isomorphism and is natural in $M$ and $A$. This shows that $- \otimes_R C$ is a left adjoint of $\mathscr{A}(C, -)$. Let $\phi\colon \mathscr{A}(C, -) \otimes_R C \to 1_{\mathscr{A}}$ and $\psi\colon 1_{\text{Mod-}R} \to \mathscr{A}(C, - \otimes_R C)$ be natural transformations induced by $\eta$. They are called the right and left adjunctions, respectively. Suppose $\mathscr{A}$ has images. Then it is easily shown that $\operatorname{Im} \phi_A = \bigcup \{\operatorname{Im} f \mid f \in \mathscr{A}(C, A)\}$. Let $I$ be a right ideal of $R$. Put $IC = \operatorname{Im}(I \otimes_R C \to R \otimes_R C \cong C)$. Then it is also easy to see that $IC = \bigcup \{\operatorname{Im} x \mid x \in I\}$.

Now let us fix an $R$-object $U \in \mathscr{A}$ and adjunctions $\phi\colon \mathscr{A}(U, -) \otimes_R U \to 1_{\mathscr{A}}$, $\psi\colon 1_{\text{Mod-}R} \to \mathscr{A}(U, - \otimes_R U)$. For any $M \in$ Mod-$R$, put $r(M) = \operatorname{Ker} \psi_M$. Then $r$ is a radical.

*Proof.* First note that $\psi_M \otimes U\colon M \otimes_R U \to \mathscr{A}(U, M \otimes_R U) \otimes_R U$ is monomorphism. Since $r(M) \to M \to \mathscr{A}(U, M \otimes_R U)$ is a zero sequence, $(r(M) \otimes_R U \to M \otimes_R U) = 0$. Thus by the uniqueness of cokernels, we get an isomorphism $M \otimes_R U \cong M/r(M) \otimes_R U$ canonically. Now it follows $r(M/r(M)) = 0$ from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & r(M) & \longrightarrow & M & \longrightarrow & M/r(M) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathscr{A}(U, M \otimes_R U) & \cong & \mathscr{A}(U, M/r(M) \otimes_R U) & & \end{array}$$

Definition 1.1 (Wakamatsu). ${}_R U$ is weakly flat if for a monomorphism $f\colon X'_R \to X_R$, $f \otimes U = 0$ implies $X' \otimes_R U = 0$.

Lemma 1.1. *The following conditions are equivalent.*

(1) ${}_RU$ *is weakly flat.*

(2) $r$ *is a left exact radical.*

*Proof.* (1) ⇒ (2): Let $X \in$ Mod-$R$. Then $(r(X) \otimes_R U \to X \otimes_R U) = 0$ by the above. Hence $r(X) \otimes_R U = 0$ by the assumption. Thus $r$ is idempotent. It should be noted that $X \otimes_R U = 0$ if and only if $r(X) = X$. Hence if $r(X) = X$ and $X' \subset X$ $(X' \otimes_R U \to X \otimes_R U) = 0$ implies $r(X') = X'$. Therefore $r$ is left exact.

(2) ⇒ (1): Let $f: X'_R \to X_R$ be a monomorphism such that $f \otimes U = 0$. By the assumption, $\bar{f}: X'/r(X') \to X/r(X)$ is a monomorphism. Thus it follows $X' = r(X')$ from the commutative diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X'/r(X') & \longrightarrow & X/r(X) \\ & & \downarrow & & \downarrow \\ & & \mathscr{A}(U, X'/r(X') \otimes_R U) & \xrightarrow{0} & \mathscr{A}(U, X/r(X) \otimes_R U). \end{array}$$

Put $\mathscr{T} = \{M \in \text{Mod-}R \mid r(M) = M\}$ and $\mathscr{F} = \{M \in \text{Mod-}R \mid r(M) = 0\}$.

DEFINITION 1.2. ${}_RU$ is $\mathscr{T}$-flat if for a monomorphism $f$ in Mod-$R$ such that Cok $f \in \mathscr{T}$, $f \otimes U$ is an *isomorphism.*

For the rest of this section we assume that ${}_RU$ is weakly flat. In this case the Gabriel topology associated with $(\mathscr{T}, \mathscr{F})$ is $\mathbf{T} = \{I_R \subset R \mid IU = U\}$.

LEMMA 1.2. *For any* $A \in \mathscr{A}$, $\mathscr{A}(U, A)_R \in \mathscr{F}$.

*Proof.* Let $X \in \mathscr{T}$. Then $\text{Hom}_R(X, \mathscr{A}(U, A)) \cong \mathscr{A}(X \otimes_R U, A) = 0$.

LEMMA 1.3. *If* ${}_RU$ *is* $\mathscr{T}$*-flat,* $\mathscr{A}(U, A)_R$ *is divisible for all* $A \in \mathscr{A}$.

*Proof.* Let $0 \to X' \to X \to X'' \to 0$ be exact in Mod-$R$ with $X'' \in \mathscr{T}$. Then by the assumption, $X' \otimes_R U \cong X \otimes_R U$. Hence by the adjoint relation, $\text{Hom}_R(X, \mathscr{A}(U, A)) \cong \text{Hom}_R(X', \mathscr{A}(U, A))$ for all $A \in \mathscr{A}$.

PROPOSITION 1.4. *Suppose* ${}_RU$ *is weakly flat and* $\mathscr{T}$*-flat. For any* $M \in$ Mod-$R$, *put* $L(M) = \{f \in \mathscr{A}(U, M \otimes_R U) \mid (\text{Im}\, \psi_M : f) \in \mathbf{T}\}$. *Then* $\theta_M$: $M \to L(M)$ *is the localization, where* $\theta_M$ *is the homomorphism induced by* $\psi_M$.

*Proof.* This follows clearly from Lemmas 1.2 and 1.3.

COROLLARY 1.5. *Suppose* ${}_RU$ *is weakly flat and* $\mathscr{T}$*-flat. If* Cok $\psi_M \otimes_R U = 0$, *then* $\psi_M$ *is the localization* (*with respect to* $\mathbf{T}$).

Let ${}_SV_R$ be a bimodule. Then the left adjoint of the covariant functor $\mathrm{Hom}_{(S\text{-Mod})^{\mathrm{op}}}(V, -)$: $(S\text{-Mod})^{\mathrm{op}} \to \text{Mod-}R$ is $\mathrm{Hom}_R(-, V)$: $\text{Mod-}R \to (S\text{-Mod})^{\mathrm{op}}$. Hence we get:

EXAMPLE 1.1. Let ${}_SV_R$ be a bimodule such that $V_R$ is QF-3 and divisible with respect to the hereditary torsion theory cogenerated by $V_R$. Let $\mathbf{T}$ be the Gabriel topology corresponding to the torsion theory. For each $M \in \text{Mod-}R$, put $L(M) = \{f \in \mathrm{Hom}_S(\mathrm{Hom}_R(M, V), V) \mid (\mathrm{Im}\, \psi_M : f) \in \mathbf{T}\}$. Then $\theta_M$: $M \to L(M)$ is the localization, where $\psi_M$: $M \to \mathrm{Hom}_S(\mathrm{Hom}_R(M, V), V)$ is the canonical homomorphism and $\theta_M$ is canonically induced from $\psi_M$.

EXAMPLE 1.2 (Wakamatsu). Let ${}_RU_S$ be a bimodule and let $r$ be the radical in Mod-$R$ being similar to the one defined at the beginning of this section. Then the following conditions are equivalent.

(1) ${}_RU$ is weakly flat.

(2) $r$ is a left exact radical.

(3) For any cogenerator $W_S$ of Mod-$S$, $V_R = \mathrm{Hom}_S(U, W)$ is a QF-3 module.

*Proof.* (1) $\Leftrightarrow$ (2): Already proved.

(2) $\Leftrightarrow$ (3): Let $W_S$ be a cogenerator and put $V = \mathrm{Hom}_S(U, W)$. For all $M \in \text{Mod-}R$, let us set $r'(M) = \bigcap \{\mathrm{Ker}\, f \mid f \in \mathrm{Hom}_R(M, V)\}$. Then it is enough to show that $r = r'$. Since $W_S$ is a cogenerator, there is a monomorphism $\mathrm{Hom}_S(U, M \otimes_R U) \subsetneq \prod V_R$. Thus $r'(M) \subset r(M)$. Suppose $r(M) \not\subset r'(M)$. Then there exists $x \in r(M)$ such that there exists $f \in \mathrm{Hom}_R(M, V)$ with $f(x) \neq 0$. We have an isomorphism $\eta$: $\mathrm{Hom}_R(M, V) \cong \mathrm{Hom}_S(M \otimes_R U, W)$ defined by $\eta(f)(x \otimes u) = (f(x))u$. On the other hand $x \otimes u = 0$ for all $u \in U$ in $M \otimes_R U$. Therefore $(f(x))u = 0$ for all $u \in U$, i.e., $f(x) = 0$. This is a contradiction, proving $r = r'$.

## 2. EQUIVALENCE BETWEEN COLOCALIZATION AND LOCALIZATION

LEMMA 2.1. *Let $\mathscr{A}$ be a cocomplete abelian category with exact direct limits and $(\mathscr{T}, \mathscr{F})$ a cohereditary torsion theory in $\mathscr{A}$ with the idempotent radical $t$. Then the following conditions are equivalent.*

(1) *$\mathscr{F}$ is closed under coproducts.*

(2) *For any $A \in \mathscr{F}$ and a direct union $A = \bigcup A_\alpha$, we have $A = \bigcup t(A_\alpha)$.*

(3) *For any $A \in \mathscr{A}$ and a direct union $A = \bigcup A_\alpha$, we have $t(A) = \bigcup t(A_\alpha)$.*

*Proof.* (1) ⇒ (2): Let $A = \bigcup A_\alpha$ be the same as the condition of (2). Then we have a short exact sequence

$$0 \to \bigcup t(A_\alpha) \to \bigcup A_\alpha \to \varinjlim A/t(A_\alpha) \to 0.$$

Since $\mathscr{F}$ is closed under coproducts and quotient objects, $\varinjlim A/t(A_\alpha) \in \mathscr{F}$. On the other hand $A \in \mathscr{T}$. Hence $\varinjlim A/t(A_\alpha) = 0$. Therefore $A = \bigcup t(A_\alpha)$.

(2) ⇒ (3): Let $A = \bigcup A_\alpha$ be the same as the condition of (3). Then by AB-5 property, $t(A) = \bigcup (A_\alpha \cap t(A))$. By the assumption, $t(A) = \bigcup t(A_\alpha \cap t(A))$. On the other hand, $t(A_\alpha \cap t(A)) \subset t(A_\alpha)$. Hence $t(A) = \bigcup t(A_\alpha)$.

(3) ⇒ (1): Let $X_i \in \mathscr{F}$ $(i \in I)$. Let **J** be the family of finite subsets of $I$. We must show that $\oplus X_i \in \mathscr{F}$. By the assumption,

$$t\left(\oplus X_i\right) = \bigcup_{J \in \mathbf{J}} t\left(\bigoplus_{j \in J} X_j\right).$$

On the other hand, $t(\oplus_{j \in J} X_j) = 0$ since $\mathscr{F}$ is always closed under finite coproducts. This proves that $\oplus X_i \in \mathscr{F}$.

Now for the rest of this section, unless otherwise specified, $\mathscr{A}$ means a cocomplete abelian category. We fix $U \in \mathscr{A}$ with $R = \mathrm{End}_{\mathscr{A}}(U)$ and the adjunctions $\phi\colon \mathscr{A}(U,-) \otimes_R U \to 1_{\mathscr{A}}$, $\psi\colon 1_{\text{Mod-}R} \to \mathscr{A}(U, - \otimes_R U)$. For any $A \in \mathscr{A}$, let $t(A) = \mathrm{Im}\, \phi_A$. Then since $\mathrm{Im}\, \phi_A = \bigcup \{\mathrm{Im}\, f \mid f \in \mathscr{A}(U, A)\}$, $t$ is an idempotent preradical in $\mathscr{A}$.

DEFINITION 2.1. $U$ is CQF-3 if for an epimorphism $f\colon A \to A''$, $\mathscr{A}(U, f) = 0$ implies $\mathscr{A}(U, A'') = 0$.

LEMMA 2.2. *The following conditions are equivalent.*

(1) *$U$ is* CQF-3.

(2) *$t$ preserves epimorphisms.*

*Proof.* (1) ⇒ (2): Let $A \in \mathscr{A}$, Then $A \to A/t(A) \to 0$ induces $(\mathscr{A}(U, A) \to \mathscr{A}(U, A/t(A))) = 0$. Hence by the assumption, $\mathscr{A}(U, A/t(A)) = 0$. This implies $t(A/t(A)) = 0$, i.e., $t$ is a radical. Let $(\mathscr{T}, \mathscr{F})$ be the torsion theory in $\mathscr{A}$ associated with $t$. Then to show that $t$ is epi-preserving, it is sufficient to show that $(\mathscr{T}, \mathscr{F})$ is cohereditary by [34, Proposition 2.1]. Let $A \in \mathscr{F}$ and $A'$ a subobject of $A$. Then since $\mathscr{A}(U, A) = 0$, the canonical morphism $(\mathscr{A}(U, A) \to \mathscr{A}(U, A/A')) = 0$. Hence by the assumption, $\mathscr{A}(U, A/A') = 0$. This proves that $A/A' \in \mathscr{F}$.

(2) ⇒ (1): Let $f\colon A \to A''$ be an epimorphism such that $\mathscr{A}(U, f) = 0$.

By the assumption, $t(f): t(A) \to t(A'')$ is an epimorphism. Therefore clearly $t(A'') = 0$, i.e., $\mathscr{A}(U, A'') = 0$.

LEMMA 2.3. *Suppose $\mathscr{A}$ has exact direct limits and $U$ is CQF-3. Let $(\mathscr{T}, \mathscr{F})$ be the cohereditary torsion theory associated with $t$. Suppose $\mathscr{F}$ is closed coproducts. Then ${}_R U$ is weakly flat.*

*Proof.* Let $f: X'_R \to X_R$ be a monomorphism. We can construct the commutative diagram

$$\begin{array}{ccccccc}
R^{(K')} & \xrightarrow{\lambda'} & R^{(X')} & \xrightarrow{\mu'} & X' & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f} & & \\
R^{(K)} & \xrightarrow{\lambda} & R^{(X)} & \xrightarrow{\mu} & X & \longrightarrow & 0.
\end{array}$$

The above diagram induces a commutative diagram

$$\begin{array}{ccccccc}
U^{(K')} & \xrightarrow{\hat{\lambda}'} & U^{(X')} & \xrightarrow{\hat{\mu}'} & X' \otimes_R U & \longrightarrow & 0 \\
\downarrow{\scriptstyle \hat{\psi}} & & \downarrow{\scriptstyle \hat{\varphi}} & & \downarrow{\scriptstyle f \otimes U} & & \\
U^{(K)} & \xrightarrow{\hat{\lambda}} & U^{(X)} & \xrightarrow{\hat{\mu}} & X \otimes_R U & \longrightarrow & 0.
\end{array}$$

Put $A = \operatorname{Ker} \hat{\mu}\hat{\varphi}$ and $B = \operatorname{Im} \hat{\lambda}$. Let

$$\begin{array}{ccc}
P & \longrightarrow & t(A) \\
\downarrow & & \downarrow \\
U^{(K)} & \longrightarrow & B
\end{array}$$

be the pull back diagram. Then we further get the commutative diagram

$$\begin{array}{ccccc}
t(P) & \xrightarrow{\tau} & t(A) & \longrightarrow & U^{(X')} \\
\downarrow{\scriptstyle \kappa} & & \downarrow & & \downarrow \\
U^{(K)} & \longrightarrow & B & \longrightarrow & U^{(X)}.
\end{array}$$

Let **J** and **E** be the families of finite subsets of $K$ and $X'$ respectively. Then since $\mathscr{A}$ has exact direct limits,

$$\bigcup_{J \in \mathbf{J}} \kappa^{-1}(U^J) = t(P) \qquad \text{and} \qquad \bigcup_{E \in \mathbf{E}} \tau^{-1}(t(A) \cap U^E) = t(P).$$

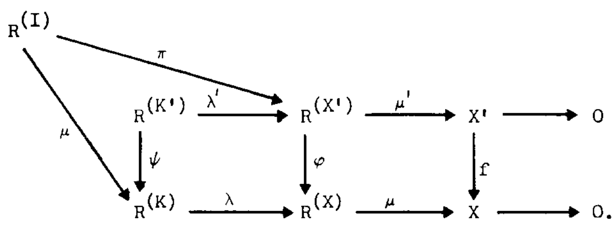
Now by Lemma 2.1,

$$\begin{aligned} t(P) &= \bigcup_{J \in \mathbf{J}} \kappa^{-1}(U^J) \\ &= \bigcup_{J \in \mathbf{J}} t\left(\bigcup_{E \in \mathbf{E}} (\kappa^{-1}(U^J) \cap \tau^{-1}(t(A) \cap U^E))\right) \\ &= \bigcup_{J \in \mathbf{J}} \left(\bigcup_{E \in \mathbf{E}} t(\kappa^{-1}(U^J) \cap \tau^{-1}(t(A) \cap U^E))\right) \\ &= \bigcup_{(J,E) \in \mathbf{J} \times \mathbf{E}} t(\kappa^{-1}(U^J) \cap \tau^{-1}(t(A) \cap U^E)). \end{aligned}$$

For each $(J, E)$, there is an epimorphism $\oplus U \to t(\kappa^{-1}(U^J) \cap \tau^{-1}(t(A) \cap U^E))$. So let $\eta\colon U^{(I)} \to t(P)$ be the epimorphism induced by those morphisms. Then $\eta$ induces a commutative diagram

$$\begin{array}{ccc} U^{(I)} & \xrightarrow{\hat{\pi}} & U^{(X')} \\ \downarrow{\scriptstyle \hat{\mu}} & & \downarrow{\scriptstyle \hat{\varphi}} \\ U^{(K)} & \xrightarrow{\hat{\lambda}} & U^{(X)}. \end{array}$$

By the construction of $\eta$, compositions $(U \to U^{(I)} \to U^{(X')})$ and $(U \to U^{(I)} \to U^{(K)})$ factor through finite subcoproducts of $U^{(X')}$ and $U^{(K)}$ respectively. Hence we get a commutative diagram




Since $f$ is a monomorphism, $\mu'\pi = 0$. Therefore $\hat{\mu}'\hat{\pi} = 0$. This proves that $t(A) \subset \operatorname{Ker} \hat{\mu}'$. Thus $\hat{\mu}'$ induces an epimorphism $A/t(A) \to \operatorname{Ker} f \otimes U$. On the other hand, $\mathscr{F}$ is closed under quotient objects. Hence $\operatorname{Ker} f \otimes U \in \mathscr{F}$. Now suppose $f \otimes U = 0$. Then $\operatorname{Ker} f \otimes U = X' \otimes_R U \in \mathscr{F}$. Hence $\mathscr{A}(U, X' \otimes_R U) = 0$. This implies that $X' \otimes_R U = 0$. Therefore ${}_RU$ is weakly flat.

Following Lambek and Rattray [14], we call $U$ weakly small if every morphism $U \to \oplus U$ factors through a finite subcoproduct of $\oplus U$.

LEMMA 2.4. *Suppose $U$ is CQF-3 and weakly small. Then ${}_RU$ is weakly flat.*

*Proof.* The proof is easier than that of the preceding lemma.

If $t$ is an idempotent radical, we call the associated torsion theory generated by $U$.

THEOREM 2.5. *Suppose $\mathscr{A}$ has exact direct limits and $U$ is CQF-3. Let $(\mathscr{T}, \mathscr{F})$ be the torsion theory generated by $U$ and $(\mathscr{T}', \mathscr{F}')$ the hereditary torsion theory in* Mod-$R$ *determined by the weakly flatness of ${}_RU$. Moreover suppose $\mathscr{F}$ is closed under coproducts. Then the following statements are equivalent.*

(i) *$U$ is codivisible with respect to $(\mathscr{T}, \mathscr{F})$.*

(ii) *$\phi_A : \mathscr{A}(U, A) \otimes_R U \to A$ is the colocalization of $A$ with respect to $(\mathscr{T}, \mathscr{F})$ for all $A \in \mathscr{A}$.*

(iii) *$\psi_M : M \to \mathscr{A}(U, M \otimes_R U)$ is the localization of $M$ with respect to $(\mathscr{T}', \mathscr{F}')$ for all $M \in$ Mod-$R$.*

If the above statements hold, $\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C} \sim \mathscr{L}$, where $\mathscr{C}$ is the Co-Giraud subcategory of $\mathscr{A}$ associated with $(\mathscr{T}, \mathscr{F})$ and $\mathscr{L}$ is the Giraud subcategory of Mod-$R$ associated with $(\mathscr{T}', \mathscr{F}')$.

This time we prove the following theorem. The proof of Theorem 2.5 will be obtained by some minor modifications (using the method of Lemma 2.3) of the next proof I.

THEOREM 2.6. *Suppose $U$ is CQF-3 and weakly small. Let $(\mathscr{T}, \mathscr{F})$ be the torsion theory in $\mathscr{A}$ generated by $U$ and $(\mathscr{T}', \mathscr{F}')$ the hereditary torsion theory in* Mod-$R$ *determined by ${}_RU$.*

I. *The following statements are equivalent.*

(i) *$U$ is codivisible with respect to $(\mathscr{T}, \mathscr{F})$.*

(ii) *$\phi_A : \mathscr{A}(U, A) \otimes_R U \to A$ is the colocalization of $A$ with respect to $(\mathscr{T}, \mathscr{F})$ for all $A \in \mathscr{A}$.*

(iii) *$\psi_M : M \to \mathscr{A}(U, M \otimes_R U)$ is the localization of $M$ with respect to $(\mathscr{T}', \mathscr{F}')$ for all $M \in$ Mod-$R$.*

II. *Suppose the above statements hold. Let $\mathscr{C}$ be the Co-Giraud subcategory of $\mathscr{A}$ associated with $(\mathscr{T}, \mathscr{F})$ and $\mathscr{L}$ the Giraud subcategory of* Mod-$R$ *associated with $(\mathscr{T}', \mathscr{F}')$. Then*

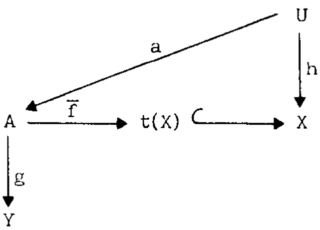
(iv) *$\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C} \sim \mathscr{L}$.*

(v) *$\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C} \sim$* Mod-$R$ *if and only if every epimorphism of the type $\oplus U \to U$ splits.*

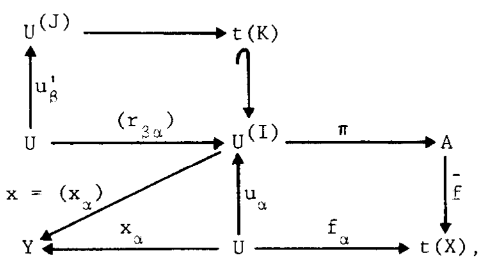
*Proof.* I. (i) $\Rightarrow$ (ii): First we show that $(\mathscr{T}, \mathscr{F})$ is strongly cohereditary. Let $X \in \mathscr{A}$ and let $f = (f_\alpha): U^{(I)} \to t(X)$ be an epimorphism. Then $f$ induces a short exact sequence

$$0 \longrightarrow K/t(K) \longrightarrow U^{(I)}/t(K) \xrightarrow{\bar{f}} t(X) \longrightarrow 0,$$

where $K = \operatorname{Ker} f$. Put $A = U^{(I)}/t(K)$. Then $(A \xrightarrow{\bar{f}} t(X) \subsetneq X)$ is the colocalization of $X$. Hence $(\mathscr{T}, \mathscr{F})$ is strongly cohereditary. For $Y \in \mathscr{A}$, we define the homomorphism $\mathscr{E}: \mathscr{A}(A, Y) \to \operatorname{Hom}_R(\mathscr{A}(U, X), \mathscr{A}(U, Y))$. Let $g$: $A \to Y$ and $h: U \to X$ be morphisms. Then $\operatorname{Im} h \subset t(X)$. Hence there exists a unique morphism $a: U \to A$ such that the diagram



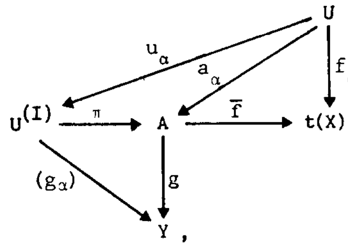

is commutative. So we define $\mathscr{E}$ via $(\mathscr{E}(g))h = ga$. It is easy to check that $\mathscr{E}$ is a group homomorphism. Next we define the homomorphism $\theta$: $\operatorname{Hom}_R(\mathscr{A}(U, X), \mathscr{A}(U, Y)) \to \mathscr{A}(A, Y)$. Let $\varphi: \mathscr{A}(U, X)_R \to \mathscr{A}(U, Y)_R$ be a homomorphism such that $\varphi(jf_\alpha) = x_\alpha$, where $j$ is the inclusion $t(X) \subsetneq X$. First note that there is an epimorphism $U^{(J)} \to t(K)$. Since $U$ is weakly small we may put $(r_{\beta\alpha}) = (U \xrightarrow{u'_\beta} U^{(J)} \to t(K) \subsetneq U^{(I)})$ ($r_{\beta\alpha} \in R$ and for each $\beta \in J$, $r_{\beta\alpha} = 0$ for all but a finite number of $\alpha \in I$), where $u'_\beta$'s are the injections. Then we have a commutative diagram




where $\pi$ is the canonical epimorphism and $u_\alpha$'s are the injections. We show that $x$ factors through $\pi$. To see this it is enough to show that $\operatorname{Ker} x \supset t(K)$. Since $\operatorname{Ker} f \supset t(K)$, $\sum f_\alpha r_{\beta\alpha} = 0$ for each $\beta \in J$. Hence

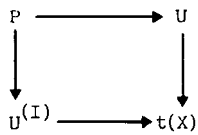
$$\varphi\left(\sum f_\alpha r_{\beta\alpha}\right) = \sum x_\alpha r_{\beta\alpha} = 0$$

for all $\beta \in J$. Thus by the property of coproducts, $(U^{(J)} \to t(K) \to U^{(I)} \xrightarrow{x} Y) = 0$. On the other hand $U^{(J)} \to t(K)$ is an epimorphism. Hence $\operatorname{Ker} x \supset t(K)$. So we define $\theta$ via $\theta(\varphi) = \bar{x}$, where $\bar{x}$ is defined by $x = \bar{x}\pi$. Then it is also easy to see that $\theta$ is a group homomorphism. Next we show $\theta\mathscr{E} = 1$. Let $g: A \to Y$ be given. Put $(g_\alpha) = g\pi$. Then we get the commutative diagram

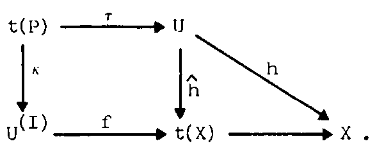



where we put $a_\alpha = \pi u_\alpha$. Hence $ga_\alpha = g_\alpha$, i.e., $(\mathscr{E}(g))f_\alpha = g_\alpha$. Thus by the definition of $\theta$, $\theta\mathscr{E}(g) = g$. Therefore $\theta\mathscr{E} = 1$.

Next let $\varphi: \mathscr{A}(U, X)_R \to \mathscr{A}(U, Y)_R$ be a homomorphism and $\varphi(jf_\alpha) = x_\alpha$. Let $h: U \to X$ be a morphism. Then $h$ induces $\hat{h}: U \to t(X)$. Let




be the pull back diagram. Then it induces a commutative diagram

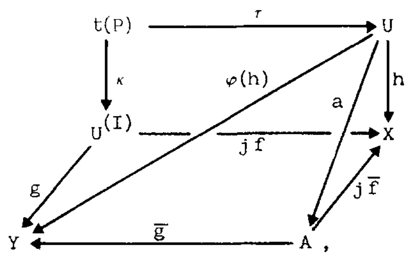



Note that $\tau$ is an epimorphism by the property of pull backs. Since $t(P) \in \mathscr{T}$, there is an epimorphism $\chi: U^{(L)} \to t(P)$. Put $r_\gamma = \tau\chi u''_\gamma$, where $u''_\gamma: U \to U^{(L)}$ is the injection for all $\gamma \in L$. Since $U$ is weakly small we may put $\kappa\chi u''_\gamma = (r_{\gamma\alpha})$ ($r_{\gamma\alpha} = 0$ for almost all $\alpha \in I$ for each $\gamma \in L$). Then

$$\begin{aligned} \varphi(h)\,\tau\chi u''_\gamma &= \varphi(h)\,r_\gamma \\ &= \varphi(hr_\gamma) \\ &= \varphi(h\tau\chi u''_\gamma) \\ &= \varphi(jf\kappa\chi u''_\gamma) \end{aligned}$$

$$= \varphi\left(\sum f_\alpha r_{\gamma\alpha}\right)$$
$$= \sum x_\alpha r_{\gamma\alpha}$$
$$= g\kappa\chi u''_\gamma,$$

where $g = (x_\alpha)\colon U^{(I)} \to Y$. Since $\gamma$ runs through all indices, $\varphi(h)\tau = g\kappa$. Now we get the diagram




where $a\colon U \to A$ is defined via $j\bar{f}a = h$ by the codivisibility of $U$. To see that $\mathscr{E}\theta = 1$, it is sufficient to show $\bar{g}a = \varphi(h)$ by the definition of $\mathscr{E}$.

$$j\bar{f}a\tau = h\tau$$
$$= jf\kappa$$
$$= j\bar{f}\pi\kappa.$$

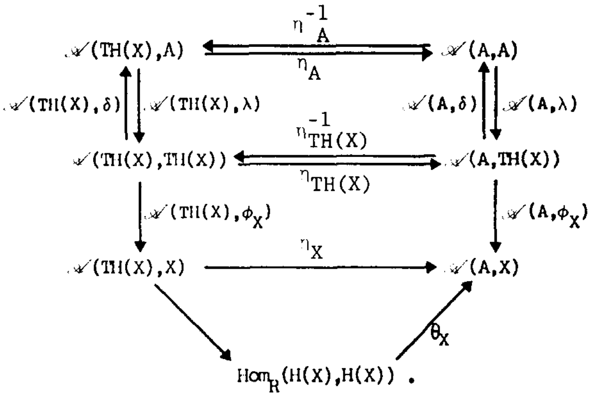
Since $j$ is the inclusion, $\bar{f}a\tau = \bar{f}\pi\kappa$. Hence $\bar{f}(a\tau - \pi\kappa) = 0$. Thus $(a\tau - \pi\kappa)$: $t(P) \to A$ factors through $t(P) \to K/t(K)$. Hence $a\tau = \pi\kappa$. Then

$$\bar{g}a\tau = \bar{g}\pi\kappa$$
$$= g\kappa$$
$$= \varphi(h)\tau.$$

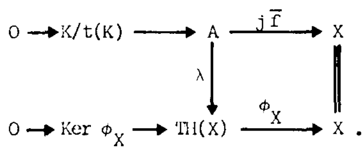
Since $\tau$ is an epimorphism, $\varphi(h) = \bar{g}a$. Therefore $\mathscr{E}\theta = 1$. The naturality of $\mathscr{E}$ on $Y$ is clear by the definition of itself. Now there exists a natural isomorphism

$$\eta_Y\colon \mathscr{A}(TH(X), Y) \cong \mathrm{Hom}_R(H(X), H(Y)) \cong \mathscr{A}(A, Y),$$

where we have put $H = \mathscr{A}(U, -)$ and $T = - \otimes_R U$ for convenience. Put $\lambda = \eta_{TH(X)}(1_{TH(X)})$ and $\delta = \eta_A^{-1}(1_A)$. Then we get the commutative diagram



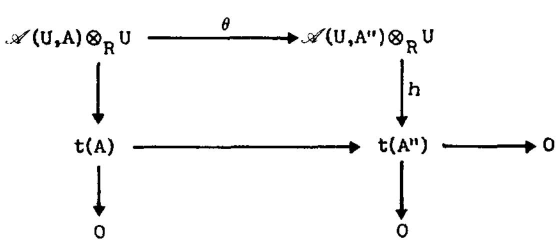

Then $\lambda$ is an isomorphism and we get the commutative diagram




Therefore $\phi_X$ is the colocalization.

(ii) ⇒ (iii): It is clear that $H$ and $T$ induce an equivalence Im $T \sim$ Im $H$. Hence for any $M \in$ Mod-$R$, Cok $\psi_M \otimes_R U = 0$. So it is sufficient to show that $_RU$ is $\mathcal{T}'$-flat by Corollary 1.5. Let $0 \to X' \to X \to X'' \to 0$ be exact in Mod-$R$ with $X'' \in \mathcal{T}'$. Then since $X \otimes_R U$ is codivisible, the exact sequence $0 \to K \to X' \otimes_R U \to X \otimes_R U \to 0$ splits, because $K \in \mathcal{F}$ by [34, Lemma 3.2]. Hence $K = 0$.

(iii) ⇒ (i): Let $g: A \to A''$ be an epimorphism in $\mathcal{A}$. Then we have a commutative diagram

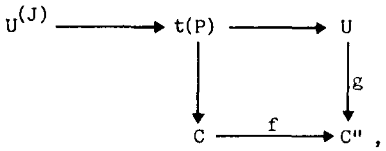



with exact rows and columns, where $\theta = \mathcal{A}(U, g) \otimes U$. Since $h$ is a minimal epimorphism by [34, Lemma 2.2], $\theta$ is an epimorphism. Let $0 \to A' \to A \to$

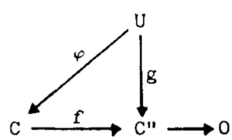
$A'' \to 0$ be exact in $\mathscr{A}$ with $A' \in \mathscr{F}$. Then we have a short exact sequence $0 \to \mathscr{A}(U, A) \to \mathscr{A}(U, A'') \to N \to 0$ with $N \otimes_R U = 0$ by the above. Then it splits since $\mathscr{A}(U, A)_R$ is divisible with respect to $(\mathscr{T}', \mathscr{F}')$. Thus $N = 0$. This proves that $U$ is codivisible.

II. (iv) is obvious.

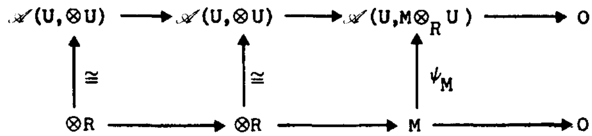
(v) Suppose every epimorphism of the type $\oplus U \to U$ splits. We show that $U$ is projective in $\mathscr{C}$. Let $f$: $C \to C''$ be an epimorphism in $\mathscr{C}$. Then $f$ is an epimorphism in $\mathscr{A}$, too. Let $g$: $U \to C''$ be any morphism. Then we get the commutative diagram in $\mathscr{A}$




which is induced from the pull back diagram. By the assumption there is a morphism $U \to U^{(J)}$ such that $(U \to U^{(J)} \to t(P) \to U) = 1_U$. Hence by putting $\varphi = (U \to U^{(J)} \to t(P) \to C)$, the diagram




is commutative. Thus $U$ is projective in $\mathscr{C}$. Let $M \in$ Mod-$R$ and let $\oplus R \to \oplus R \to M \to 0$ be exact. Then since $U$ is weakly small and projective in $\mathscr{C}$, we have the commutative diagram




with exact rows. Hence $\psi_M$ is an isomorphism. Therefore $\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C} \sim$ Mod-$R$. The converse is obvious.
This completes the proof.

COROLLARY 2.7. *A cocomplete abelian category with a weakly small generator is a Grothendieck category.*

*Remark.* If $U$ is a small generator in $\mathscr{A}$, then with respect to the corresponding hereditary torsion theory in Mod-$R$, every direct sum of torsion free divisible modules is again divisible.

COROLLARY 2.8 (Gabriel and Popescu [26]). *Let $\mathscr{A}$ be a Grothendieck category with a generator $U$. Let $R = \mathrm{End}_{\mathscr{A}}(U)$. Then the following assertions hold.*

(1) *The functor $\mathscr{A}(U, -): \mathscr{A} \to$ Mod-$R$ is full and faithful.*

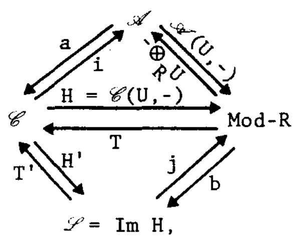
(2) *${}_RU$ is flat (i.e., $- \otimes_R U$ is an exact functor).*

*Proof.* This is a special case of Theorem 2.5.

If we assume the Gabriel–Popescu theorem, we can give an easier proof of Theorem 2.5 without assuming Lemma 2.3. We give this proof.

*Proof of Theorem* 2.5. Assume Corollary 2.8.

(i) $\Rightarrow$ (ii), (iii): Suppose $U$ is codivisible. Then $(\mathscr{T}, \mathscr{F})$ is strongly cohereditary by the same reason in Theorem 2.6. Let $\mathscr{C}$ be the Co-Giraud subcategory of $\mathscr{A}$ associated with $(\mathscr{T}, \mathscr{F})$, $i: \mathscr{C} \to \mathscr{A}$ the inclusion functor and $a: \mathscr{A} \to \mathscr{C}$ the coreflector. Then $\mathscr{C}$ is a cocomplete abelian category with a generator $U$. We show that $\mathscr{C}$ has exact direct limits. Since $i$ is right exact and $a$ is exact, direct limits are right exact in $\mathscr{C}$. Hence it is sufficient to show that they preserve monomorphisms. Let $0 \to X'_\alpha \to X_\alpha$ be exact in $\mathscr{C}$ and let $0 \to K_\alpha \to i(X'_\alpha) \to i(X_\alpha)$ be exact in $\mathscr{A}$. Then $K_\alpha \in \mathscr{F}$. By the assumption, $\mathscr{F}$ is closed under quotient objects and coproducts. Hence $\varinjlim K_\alpha \in \mathscr{F}$. Thus we have a monomorphism $a(\varinjlim i(X'_\alpha)) \to a(\varinjlim i(X_\alpha))$. On the other hand, $a(\varinjlim i(-)) = \varinjlim(-)$. Therefore $\mathscr{C}$ has exact direct limits. Since $M \otimes_R U \in \mathscr{C}$ for all $M \in$ Mod-$R$, we get the commutative diagram




where $T(M) = M \otimes_R U$ for $M \in$ Mod-$R$, $T'$ is the inverse equivalence of $H'$, $b$ is the kernel preserving reflector and $j$ is the inclusion functor. In particular $\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C} = \mathrm{Im}\, T \sim \mathscr{S} = \mathrm{Im}\, H$. Hence $\mathrm{Ker}\, \phi_A \in \mathscr{F}$ for all $A \in \mathscr{A}$. This proves (ii).

To prove (iii) we first show that ${}_RU$ is weakly flat. Let $f: X'_R \to X_R$ be a monomorphism such that $f \otimes U = 0$. Then since $jb = \mathscr{A}(U, - \otimes_R U)$ is left exact, $(\mathscr{A}(U, X' \otimes_R U) \to \mathscr{A}(U, X \otimes_R U)) = 0$ implies $\mathscr{A}(U, X' \otimes_R U) = 0$.

Hence $X' \otimes_R U = 0$. Therefore ${}_RU$ is weakly flat. Then the other part of (iii) is similar to that of Theorem 2.6.

(ii) $\Rightarrow$ (i): Obvious.

(iii) $\Rightarrow$ (i): Same as that of Theorem 2.6.

Now we are in a position to prove the theorem in the Introduction. To prove it it is enough to show the following easy lemma.

LEMMA 2.9. *Let $\mathscr{B}$ be a complete additive subcategory of* Mod-$R$ *and $(\mathscr{T}, \mathscr{F})$ a torsion theory in $\mathscr{B}$. Then $\mathscr{F}$ is closed under direct sums.*

*Proof.* Let $T \in \mathscr{T}$ and $F_i \in \mathscr{F}$. Then

$$\mathrm{Hom}_R\left(T, \bigoplus F_i\right) \subsetneq \mathrm{Hom}_R\left(T, \prod F_i\right) \cong \mathrm{Hom}_R(T, F_i) = 0.$$

Therefore $\mathrm{Hom}_R(T, \oplus F_i) = 0$. This implies that $\oplus F_i \in \mathscr{F}$.

COROLLARY 2.10. *A CQF-3 object of a complete additive subcategory of* Mod-$R$ *is weakly flat over its endomorphism ring.*

COROLLARY 2.11. *Let the situation be the same as either Theorem* 2.5 *or* 2.6. *Suppose $U$ is codivisible. Then*

(1) *$U$ is projective in $\mathscr{A}$ if and only if the inclusion functor $j: \mathscr{L} \to$* Mod-$R$ *is exact, and*

(2) *${}_RU$ is flat if and only if the inclusion functor $i: \mathscr{C} \to \mathscr{A}$ is exact.*

Let $\mathrm{Gen}(U)$ be the full subcategory of $\mathscr{A}$ consisting of all $U$-generated objects and $\overline{\mathrm{Gen}}(U)$ the full subcategory of $\mathscr{A}$ consisting of all subobjects of objects of $\mathrm{Gen}(U)$. An object $A \in \mathscr{A}$ is said to be $U$-presented if there is an exact sequence $\oplus U \to \oplus U \to A \to 0$. Now let $U_R$ be a $\Sigma$-quasi-projective module. Then since $\mathscr{P}(U_R)$ contains arbitrary direct sums of copies of $U$, $\mathscr{P}(U_R) \supset \mathrm{Gen}(U_R)$. Thus by the theorem of the Introduction, we get:

EXAMPLE 2.1. Let $U_R$ be a $\Sigma$-quasi-projective module with $S = \mathrm{End}(U_R)$. Let $\mathscr{C}$ be the full subcategory of Mod-$R$ consisting of all $U_R$-presented modules. Then $\mathscr{C}$ is a Grothendieck category and ${}_SU$ is weakly flat.

## 3. CQF-3 OBJECTS AND COLOCALIZATION

Throughout this section $\mathscr{A}$ denotes a cocomplete abelian category and $\mathscr{B}$ a strongly exact subcategory of $\mathscr{A}$. We fix $U \in \mathscr{B}$ with $R = \mathrm{End}_{\mathscr{A}}(U)$ and the adjunctions $\phi: \mathscr{A}(U,-)\otimes_R U \to 1_{\mathscr{A}}$ and $\psi: 1_{\text{Mod-}R} \to \mathscr{A}(U, -\otimes_R U)$. For $A \in \mathscr{B}$, put $t(A) = \operatorname{Im} \phi_A$. Then since $\mathscr{B}$ is closed under subobjects, $t: \mathscr{B} \to \mathscr{B}$ is an idempotent preradical.

We define CQF-3 objects in $\mathscr{B}$ as in Section 2.

LEMMA 3.1. *The following conditions are equivalent.*

(1) *U is a CQF-3 object.*

(2) *t preserves epimorphisms.*

*Proof.* The proof is completely similar to that of Lemma 2.2.

THEOREM 3.2. *Suppose U is CQF-3 and codivisible with respect to the cohereditary torsion theory in $\mathscr{B}$ generated by U. Then for $X \in \mathscr{B}$, if $t(X)$ is finitely U-generated, $\phi_X: \mathscr{A}(U,X)\otimes_R U \to X$ is the colocalization.*

*Proof.* In the proof of (i) $\Rightarrow$ (ii) of Theorem 2.6, that $U$ is weakly small is necessary only if the set $I$ is infinite. Hence if $t(X)$ is finitely $U$-generated, we can take $I$ as finite. Therefore $\phi_X$ is the colocalization.

DEFINITION 3.1. Let $M \in$ Mod-$R$. Then $U$ is said to be $M$-faithful if $\psi_M$ is a monomorphism, and is said to be completely faithful if $U$ is $M$-faithful for all $M \in$ Mod-$R$.

LEMMA 3.3. *The following statements hold.*

(1) *If U is M-faithful, U is M'-faithful for all submodules M' of M.*

(2) *If U is $M_i$-faithful, then U is $\Pi M_i$-faithful.*

*Proof.* (1) is clear.

(2) Let $\pi_i: \Pi M_i \to M_i$ be projections. Consider the commutative diagram

$$\begin{array}{ccc}
\Pi M_i & \xrightarrow{\ \psi_{\Pi M_i}\ } & \mathscr{A}(U, \Pi M_i \otimes_R U) \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \mathscr{A}(U, \pi_i \otimes U)} \\
M_i & \xrightarrow{\ \psi_{M_i}\ } & \mathscr{A}(U, M_i \otimes_R U).
\end{array}$$

Let $(m_i) \in \operatorname{Ker} \psi_{\Pi M_i}$. Then $\pi_i(m_i) = m_i \in \operatorname{Ker} \psi_{M_i}$. Hence $m_i = 0$. Therefore $\psi_{\Pi M_i}$ is a monomorphism.

Now put $\mathbf{L}(U) = \{X \subset U \mid t(X) = X\}$ and $\mathbf{L}(R) = \{I_R \subset R \mid U$ is $R/I$-faithful$\}$. In this case we have identified $X \subset U$ as a class of all subobjects of $U$ equivalent to $X$. Then note that $\mathbf{L}(U)$ is a set.

LEMMA 3.4. *$\mathbf{L}(U)$ and $\mathbf{L}(R)$ are complete lattices by the usual order relations.*

*Proof.* Let $\{X_i\} \subset \mathbf{L}(U)$. Then it is clear that $\sum X_i \in \mathbf{L}(U)$. Hence $\mathbf{L}(U)$ is a complete lattice. Next let $\{I_j\} \subset \mathbf{L}(R)$. Then by the preceding lemma, $\bigcap I_j \in \mathbf{L}(R)$. Thus $\mathbf{L}(R)$ is also a complete lattice.

PROPOSITION 3.5. *Let the situation be the same as Theorem 3.2. Then $\mathbf{L}(U)$ and $\mathbf{L}(R)$ are lattice isomorphic by the assignment $F$: $\mathbf{L}(U) \to \mathbf{L}(R)$ via $F(X) = \mathscr{A}(U, X) = \{r \in R \mid \operatorname{Im} r \subset X\}$ and by the inverse assignment $G$: $\mathbf{L}(R) \to \mathbf{L}(U)$ via $G(I) = IU$.*

*Proof.* Let $X \in \mathbf{L}(U)$. Then we have the exact sequence $0 \to \mathscr{A}(U, X) \to R \to \mathscr{A}(U, U/X)$. Put $Y = \operatorname{Im}(R \to \mathscr{A}(U, U/X))$. Then since $U/X$ is codivisible, $U$ is $Y$-faithful by Lemma 3.3. This implies $\mathscr{A}(U, X) = F(X) \in \mathbf{L}(R)$. Next consider the commutative diagram

$$\begin{array}{ccc} \mathscr{A}(U,X)\otimes_R U & \longrightarrow & R\otimes_R U \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_U} \\ X & \longrightarrow & U. \end{array}$$

Then $\mathscr{A}(U, X)U = F(X)U = GF(X) = \operatorname{Im} \phi_X = X$ since $\operatorname{Im} \phi_X = t(X) = X$.

Conversely let $I \in \mathbf{L}(R)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \downarrow & & \\ & & \mathscr{A}(U, I\otimes_R U) & \longrightarrow & \mathscr{A}(U, R\otimes_R U) & \longrightarrow & \mathscr{A}(U, R/I\otimes_R U) & & \\ & & \downarrow & & \cong\downarrow & & & & \\ 0 & \longrightarrow & \mathscr{A}(U, IU) & \longrightarrow & \mathscr{A}(U,U) = R. & & & & \end{array}$$

Then by the definition of $IU$, $0 \to \mathscr{A}(U, IU) \to R \to \mathscr{A}(U, R/I \otimes_R U)$ is exact. Hence by five lemmas, $(I \to^{\phi_I} \mathscr{A}(U, I \otimes_R U) \to \mathscr{A}(U, IU)$ is an isomorphism. On the other hand, $(R \cong \mathscr{A}(U, R \otimes_R U) \cong \mathscr{A}(U, U)) = 1_R$. Hence $I = \mathscr{A}(U, IU) = FG(I)$.

LEMMA 3.6. *Suppose $U$ is projective in $\mathscr{B}$. Then $\mathbf{L}(R)$ contains all finitely generated right ideals.*

*Proof.* Let $I$ be a finitely generated right ideal of $R$. Then $R/I$ is fiitely presented. Hence by five lemma, $\psi_{R/I}$ is an isomorphism. In particular $U$ is $R/I$-faithful.

COROLLARY 3.7 (cf. Harada [9, Proposition 2.7]). *Suppose $U$ is projective in $\mathscr{B}$. Then* $\mathbf{L}(U)$ *is noetherian if and only if $R$ is right noetherian.*

The proof of next corollary is essentially due to [9, Proposition 2.3].

COROLLARY 3.8 (Harada). *Suppose $U$ is projective in $\mathscr{B}$. Then if* $\mathbf{L}(U)$ *is artinian, $R$ is semiprimary.*

*Proof.* Since $\mathbf{L}(U)$ is artinian and contains all principal right ideals, $R$ is left perfect by [1, Theorem 24.8]. Hence it is sufficient to show that Rad $R$ is nilpotent. By the assumption, there exists a positive integer $n$ such that $(\text{Rad } R)^n U = (\text{Rad } R)^{2n} U$. Put $I = (\text{Rad } R)^n$. Suppose $I \neq 0$. Then there is a minimal subobject $(xR)U$ with respect to the condition such that $x \in I$ and $xI \neq 0$. Then $xI \neq 0$ implies $xI^2 \neq 0$ (because $xI^2U = xIU \neq 0$). Thus there also exists $y \in I$ such that $xyI \neq 0$. Since $0 \neq (xyR)U \subset (xR)U$, we get $(xR)U = (xyR)U$ by the minimality of $(xR)U$. This implies $xR = xyR$ by Proposition 3.5. Therefore $x = xyr$ for some $r \in R$. Since $1 - yr$ by Proposition 3.5. Therefore $x = xyr$ for some $r \in R$. Since $1 - yr$ is invertible, $x = 0$. This is a contradiction. Therefore Rad $R$ is nilpotent.

LEMMA 3.9. *Let $U_R$ be a quasi-projective module with $S = \text{End}(U_R)$. Suppose $U$ is a CQF-3 object of some complete additive subcategory of* Mod-*$R$. Then $_SU$ is completely faithful if and only if $U_R$ is finitely generated.*

*Proof.* If $U_R$ is finitely generated, $U_R$ is $\sum$-quasi-projective. Hence by Theorem 2.6, II(2), $_SU$ is completely faithful.

Conversely suppose $_SU$ is completely faithful. Let $U_R = \sum X_\alpha$ be a direct union. Then $S = \bigvee \text{Hom}_R(U, X_\alpha)$ in $\mathbf{L}(S)$. Since $_SU$ is completely faithful, $\mathbf{L}(S)$ coincides with the lattice of all right ideals. Hence $S = \sum \text{Hom}_R(U, X_\alpha)$. Therefore there exists $X_\alpha$ such that $S = \text{Hom}_R(U, X_\alpha)$. Thus $U = X_\alpha$. This proves that $U_R$ is finitely generated.

COROLLARY 3.10. *Let $U_R$ be a quasi-projective artinian module with $S = \text{End}(U_R)$. Let $1_S = e_1 + \cdots + e_n$ be a decomposition of primitive idempotents and $U = U_1 + \cdots + U_n$ the corresponding decomposition. Then*:

(1) *If each $U_i$ is CQF-3 in some complete additive subcategory of* Mod-*$R$, $U_R$ is finitely generated.*

(2) *In addition to* (1), *if $U_R$ is trace accessible, $U_R$ is a finite direct sum of cyclic projective modules.*

*Proof.* Let $N = \text{Rad } S$. To prove (1), we may assume that $U_R$ is indecomposable. Let $t$ be the idempotent radical in $\overline{\mathrm{Gen}}(U_R)$ associated with the torsion theory generated by $U$ in $\overline{\mathrm{Gen}}(U_R)$. Let $U_R = \sum X_\alpha$ be a direct union. Then $U = \sum t(X_\alpha)$. Hence $S = \bigvee \mathrm{Hom}_R(U, t(X_\alpha))$. Since $N$ is nilpotent, $N \in \mathbf{L}(S)$. $N_S$ is a unique maximal submodule of $S_S$. Hence there exists $X_\alpha$ such that $S = \mathrm{Hom}_R(U, t(X_\alpha))$. Thus $U = t(X_\alpha) \subset X$. Therefore $U_R$ is finitely generated. This proves (1).

Suppose $U_R$ is trace accessible. Let $J$ be the trace ideal in $S$ of the derived context $\langle {}_SU_{R,R}\,\mathrm{Hom}_R(U, R)_S\rangle$. Then since ${}_SU$ is completely faithful, $J = S$. Thus by the dual basis lemma, $U_R$ is finitely generated projective. It only remains to show that each $U_i$ is cyclic. Let $U_i = x_1R + \cdots + x_kR$. Then $U_i = x_1I + \cdots + x_kI$, where $I$ is the trace ideal of $U_{i_R}$. Then there exists $x_j$ such that $\mathrm{Hom}_R(U_i, x_jI) = \mathrm{End}(U_{i_R})$. Hence $U_i = x_jR$. This completes the proof.

PROPOSITION 3.11. *Let $\mathscr{B}$ be a strongly exact subcategory of* Mod-$R$ *and $U \in \mathscr{B}$ a CQF-3 object with $S = \mathrm{End}(U_R)$. Let $(\mathscr{T}, \mathscr{F})$ be the torsion theory generated by U in $\mathscr{B}$ with the idempotent radical t. Then the following conditions are equivalent.*

(1) *$(\mathscr{T}, \mathscr{F})$ is hereditary.*

(2) *$U_R$ generates each of its submodules.*

(3) *${}_SU$ is flat and the canonical homomorphism $\phi_B\colon \mathrm{Hom}_R(U, B) \otimes_S U \to B$ is an isomorphism for all $B \in \mathscr{B}$.*

*Proof.* (1) ⇒ (2): Obvious.

(2) ⇒ (3): It is a general result that $\mathscr{T}$ is closed under group extensions in $\mathscr{B}$. Hence it is an easy consequence that $\mathrm{Gen}(U_R) = \overline{\mathrm{Gen}}(U_R)$. Thus $\mathrm{Gen}(U_R)$ is a Grothendieck category. On the other hand, $\mathrm{Gen}(U_R)$ is an exact subcategory. Therefore (3) holds.

(3) ⇒ (1): For any $B \in \mathscr{B}$, we have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Hom}_R(U, t(B)) \otimes_S U & \longrightarrow & \mathrm{Hom}_R(U, B) \otimes_S U \\ & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & t(B) & \longrightarrow & B. \end{array}$$

Hence $t \cong \mathrm{Hom}_R(U, -) \otimes_S U$, which is left exact by the assumption.

COROLLARY 3.12 (Fuller [7]). *Let $U_R$ be a quasi-projective module and generate each of its submodules. Then* $\mathrm{Gen}(U_R) = \overline{\mathrm{Gen}}(U_R)$.

## 4. QF-3 Objects and Localization

Throughout this section unless otherwise specified, $\mathscr{A}$ denotes a complete abelian category and $\mathscr{B}$ denotes a strongly exact subcategory of $\mathscr{A}$. We fix $V \in \mathscr{B}$ and $R = \mathrm{End}_{\mathscr{A}}(V)$. The contravariant functor $\mathscr{A}(-, V)\colon \mathscr{A} \to R\text{-Mod}$ has a colimit reversing contravariant adjoint $T\colon R\text{-Mod} \to \mathscr{A}$. Let $\phi\colon 1_{\mathscr{A}} \to T(\mathscr{A}(-, V))$ and $\psi\colon 1_{R\text{-Mod}} \to \mathscr{A}(T(\ ), V)$ be the natural transformations associated with the adjoint relation. Then $\phi$ has following properties:

(1) $(\mathscr{A}(A, V) \to \mathscr{A}(\mathrm{Ker}\, \phi_A, V)) = 0$, and

(2) $\mathrm{Coim}\, \phi_A$ is cogenerated by $V$.

Hence by putting $r(B) = \mathrm{Coim}\, \phi_B$, $r$ is an idempotent coradical in $\mathscr{B}$ in the sense of [34].

Definition 4.1. $V$ is a QF-3 object in $\mathscr{B}$ if for a monomorphism $f$: $B' \to B$ in $\mathscr{B}$, $\mathscr{A}(f, V) = 0$ implies $\mathscr{A}(B', V) = 0$.

Lemma 4.1 (cf. [34]). *The following conditions are equivalent.*

(1) *$V$ is QF-3.*

(2) *$r$ preserves monomorphisms.*

Theorem 4.2. *Suppose $V$ is QF-3 and divisible with respect to the hereditary torsion theory in $\mathscr{B}$ associated with $r$. Then for $X \in \mathscr{B}$, if $r(X)$ is finitely $V$-cogenerated, $\phi_X\colon X \to T(\mathscr{A}(X, V))$ is the localization.*

Let ${}_SQ_R$ be a bimodule. Then the adjoint of $\mathrm{Hom}_R(-, Q)$ is $\mathrm{Hom}_S(-, Q)$. Hence we get:

Example 4.1. Let $W_R$ be a cogenerator with $S = \mathrm{End}(W_R)$. Then every finitely $W_R$-cogenerated module is ${}_SW_R$-reflexive, hence in particular if $W_R$ finitely cogenerates $R$, $W_R$ is balanced.

In Theorem 4.2, that $r(X)$ is finitely $V$-cogenerated cannot be replaced by infinitely $V$-cogenerated, for if $R$ is a Morita ring such that ${}_SW_R$ defines a Morita duality, then no infinite direct sum of $W$-reflexive right $R$-modules is reflexive (Camillo).

Lemma 4.3. *The following statements hold.*

(1) *If $\psi_M$ is a monomorphism, then $\psi_{M'}$ is a monomorphism for all submodules $M'$ of $M$.*

(2) *If $\psi_{M_i}$ is a monomorphism for each $M_i$, then $\psi_{\Pi M_i}$ is also a monomorphism.*

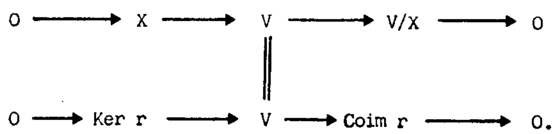
Now put $\mathbf{L}(V) = \{Y \mid Y$ is a quotient object of $V$ such that $r(Y) = Y\}$ and $\mathbf{L}(R) = \{{}_R I \subset R \mid \psi_{R/I}$ is a monomorphism$\}$.

LEMMA 4.4. *$\mathbf{L}(V)$ and $\mathbf{L}(R)$ are complete lattices.*

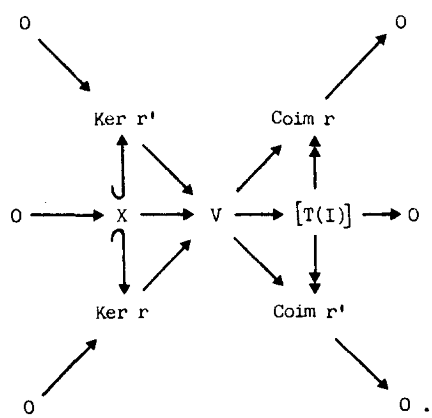
*Remark.* It should be noted that the order relation in $\mathbf{L}(V)$ is defined as follows: For $Y, Y' \in \mathbf{L}(V)$, $Y \leqslant Y'$ if and only if $Y \subset Y'$ in $\mathscr{A}^*$.

PROPOSITION 4.5. *Let the situation be the same as Theorem* 4.2. *Then* $\mathbf{L}(V)$ *and* $\mathbf{L}(R)$ *are lattice isomorphic by the assignment* $F: \mathbf{L}(V) \to \mathbf{L}(R)$ *via* $F(Y) = \{r \in R \mid \operatorname{Coim} r \leqslant Y\}$ *and by the inverse assignment* $G: \mathbf{L}(R) \to \mathbf{L}(V)$ *via* $G(I) = [T(I)]$, *where* $[T(I)] = \operatorname{Coim}(V = T(R) \to T(I))$.

Now we set $\hat{\mathbf{L}}(V) = \{X \subset V \mid r(V/X) = V/X\}$. Then by the usual order relation, $\hat{\mathbf{L}}(V)$ is a complete lattice. Moreover it is clear that $\mathbf{L}(V)$ and $\hat{\mathbf{L}}(V)$ are lattice anti-isomorphic by $X \leftrightarrow V/X$. Let $\hat{F}: \hat{\mathbf{L}}(V) \to \mathbf{L}(R)$ be the composition $\hat{\mathbf{L}}(V) \to \mathbf{L}(V) \to \mathbf{L}(R)$. Let $X \in \hat{\mathbf{L}}(V)$ and consider the diagram:




Hence $r \in \hat{F}(X)$ if and only if $X \subset \operatorname{Ker} r$. Thus $\hat{F}(X) = \{r \in R \mid \operatorname{Ker} r \supset X\} = \mathscr{A}(V/X, V)$. Furthermore the composition $\hat{G}: \mathbf{L}(R) \to \mathbf{L}(V) \to \hat{\mathbf{L}}(V)$ is a lattice anti-isomorphism. Let $I \in \mathbf{L}(R)$ and $r, r' \in I$, and let $X = \operatorname{Ker}(V \to T(I))$. Then it follows $\hat{G}(I) = \bigcap \{\operatorname{Ker} r \mid r \in I\}$ from the commutative diagram




Thus:

PROPOSITION 4.5′. *Let the situation be the sameq as Theorem* 4.2. *Then* $\hat{\mathbf{L}}(V)$ *and* $\mathbf{L}(R)$ *are lattice anti-isomorphic by the assignment* $F\colon \hat{\mathbf{L}}(V) \to \mathbf{L}(R)$ $(\hat{F}(X) = \mathscr{A}(V/X, V))$ *and by the inverse assignment* $\hat{G}\colon \mathbf{L}(R) \to \hat{\mathbf{L}}(V)$ $(\hat{G}(I) = \bigcap \{\operatorname{Ker} r \mid r \in I\})$.

COROLLARY 4.6 (Harada and Ishii [10]). *Suppose* $V$ *is injective in* $\mathscr{B}$. *Then* $\hat{\mathbf{L}}(V)$ *is artinian if and only if* $R$ *is left noetherian.*

COROLLARY 4.7 (Harada and Ishii [10]). *Suppose* $V$ *is injective in* $\mathscr{B}$. *Then if* $\hat{\mathbf{L}}(V)$ *is noetherian,* $R$ *is semiprimary.*

## 5. EQUIVALENCE AND DUALITY

Throughout this section let $\mathscr{A}$ be a cocomplete abelian category and $U$ a quasi-projective object with $R = \operatorname{End}_{\mathscr{A}}(U)$.

DEFINITION 5.1. An object $A \in \mathscr{A}$ is said to be $U$-presented if there exists an exact sequence $\oplus U \to \oplus U \to A \to 0$ (some authors call $A$ to be $U$-codominant dimension at least 2). In the above, if we can take $\oplus U$'s as finite coproducts, $A$ is said to be finitely $U$-presented.

We set $\mathscr{C}(U) = \{A \in \mathscr{A} \mid A$ is $U$-presented and finitely $U$-generated$\}$ and $\mathscr{D}(U) = \{M \in \text{Mod-}R \mid M_R$ is finitely generated and $U$ is $M$-faithful$\}$.

PROPOSITION 5.1. $\mathscr{A}(U, -)$ *and* $- \otimes_R U$ *induce an equivalence* $\mathscr{C}(U) \sim \mathscr{D}(U)$.

*Proof.* $U$ generates a cohereditary trosion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{P}(U)$. Let $\phi\colon \mathscr{A}(U, \cdot) \otimes_R U \to 1_{\mathscr{A}}$ and $\psi\colon 1_{\text{Mod-}R} \to \mathscr{A}(U, - \otimes_R U)$ be adjunctions. Let $A \in \mathscr{C}(U)$. Then $A \in \mathscr{P}(U)$ and $A$ is codivisible with respect to $(\mathscr{T}, \mathscr{F})$. Hence $\phi_A$ is an isomorphism. Thus $\psi_{\mathscr{A}(U,A)}$ is also an isomorphism. Therefore $\mathscr{A}(U, A) \in \mathscr{D}(U)$. Conversely let $M \in \mathscr{D}(U)$. Then it is easy to see that $\psi_M$ is an isomorphism. On the other hand, $\phi_{M \otimes_R U}$ is an isomorphism and $M \otimes_R U \in \mathscr{P}(U)$ imply that $M \otimes_R U \in \mathscr{C}(U)$. Therefore $\mathscr{A}(U, -)$ and $- \otimes_R U$ induce an equivalence $\mathscr{C}(U) \sim \mathscr{D}(U)$.

We give another proof of this proposition using the method originally introduced by Lambek and Rattray.

Let $A \in \mathscr{A}$. Consider classes $\mathscr{O}$ of subobjects of $A$ satisfying the following conditions:

(1) $0 \in \mathscr{O}$.

(2) If $X_1, X_2 \in \mathscr{O}$, then $X_1 \cup X_2 \in \mathscr{O}$.

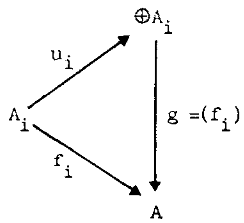
(3) If $X \in \mathscr{O}$ and $Y \subset X$, then $Y \in \mathscr{O}$.

We call such a class $\mathscr{C}$ a cofilter of $A$.

No we construct the new category $\mathscr{B}$ whose objects are the pairs $(A, \mathscr{C})$ and whose morphisms are the morphisms in $\mathscr{A}$ and map each member of the cofilter into another one.

LEMMA 5.2. *$\mathscr{B}$ is a cocomplete additive category.*

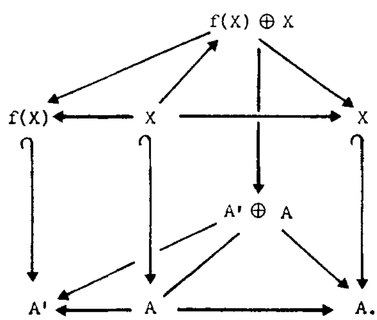
*Proof.* Let $f: (A, \mathscr{C}) \to (A', \mathscr{C}')$ be a morphism in $\mathscr{B}$. Let $\pi: A' \to \operatorname{Cok} f$ be a canonical morphism in $\mathscr{A}$. Then $\operatorname{Cok} f = (\operatorname{Cok} f, \mathscr{C}'')$, where $\mathscr{C}'' = \{\pi(X) \mid X \in \mathscr{C}'\}$. Next let $(A_i, \mathscr{C}_i) \in \mathscr{B}$. Put $\mathscr{C} = \{X \in \mathscr{A} \mid X \subset X_{i_1} \oplus \cdots \oplus X_{i_n}, X_{i_j} \in \mathscr{C}_{i_j}\}$. Then clearly $\mathscr{C}$ is a cofilter of $\oplus A_i$. We show that $\oplus (A_i, \mathscr{C}_i) = (\oplus A_i, \mathscr{C})$. Let $(A, \mathscr{C}') \in \mathscr{B}$ and $f_i: (A_i, \mathscr{C}_i) \to (A, \mathscr{C}')$ be morphisms in $\mathscr{B}$. Then there exists a unique morphism $g: \oplus A_i \to A$ such that the diagram



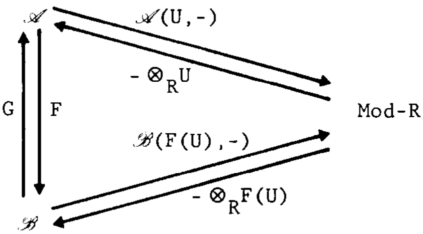

is commutative, where $u_i$'s are injections. Note that $u_i$'s are morphisms in $\mathscr{B}$. Let $X \in \mathscr{C}$. Then there exist $X_{i_j} \in \mathscr{C}_{i_j}$ $(j = 1,\dots, n)$ such that $X \subset X_{i_1} \oplus \cdots \oplus X_{i_n}$. Then $g(X) \subset f_{i_1}(X_{i_1}) \cup \cdots \cup f_{i_n}(X_{i_n}) \in \mathscr{C}'$ since $f_{i_j}$'s are morphisms in $\mathscr{B}$. Therefore $g$ is a morphism in $\mathscr{B}$. Finally we show that $\mathscr{B}$ is additive. Let $f, g: (A, \mathscr{C}) \to (A', \mathscr{C}')$ be morphisms in $\mathscr{B}$. Then $f + g$ can be expressed by the composition

$$A \xrightarrow{\binom{f}{1}} A' \oplus A \xrightarrow{(1,g)} A'.$$

Hence it is enough to show that $\binom{f}{1}$ is a morphism in $\mathscr{B}$. Let $X \in \mathscr{C}$. We show that $\binom{f}{1}(X) \subset f(X) \oplus X$. We get the commutative diagram



This implies that $\binom{f}{1}(X) \subset f(X) \oplus X$. Therefore $\binom{f}{1}$ is a morphism in $\mathscr{B}$.
We get the diagram of functors




where $F(A) = (A, \mathscr{O}_A)$, $\mathscr{O}_A$ is a class of all subobjects of $A$ and $G$ is a forgetful functor. Then $F$ is full and faithful and is a right adjoint of $G$. Note that $\mathscr{B}(F(U), -) \cdot F = \mathscr{A}(U, -)$ and $G(- \otimes_R F(U)) = (- \otimes_R U)$.

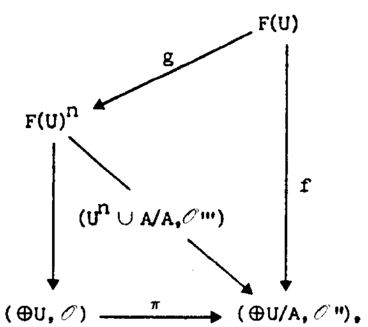
LEMMA 5.3. *$F(A)$ is weakly small in $\mathscr{B}$ for all $A \in \mathscr{A}$.*

*Proof.* Let $F(A) \to \oplus F(A)$ be a morphism in $\mathscr{B}$. Since $A \in \mathscr{O}_A$, $f(A)$ lies in the cofilter of $\oplus F(A)$. Thus $f$ factors through a finite subcoproduct of $\oplus F(A)$.

DEFINITION 5.2 (Lambek and Rattray [14]). $P \in \mathscr{B}$ is weakly projective if for every set $I$, every cokernel $f: P^{(I)} \to B$ and every morphism $g: P \to B$, there exists $h: P \to P^{(I)}$ such that $fh = g$.

LEMMA 5.4. *Suppose $U$ is quasi-projective in $\mathscr{A}$. Then $F(U)$ is weakly projective in $\mathscr{B}$.*

*Proof.* Let $\oplus F(U) = (\oplus U, \mathscr{O})$ and $A \subset \oplus U$. Let $f: F(U) \to (\oplus U, \mathscr{O})/(A, \mathscr{O}') = (\oplus U/A, \mathscr{O}'')$ be a morphism. Since $U$ lies in the cofilter of $F(U)$, $\operatorname{Im} f \subset X_1 \oplus \cdots \oplus X_n \cup A/A$ $(X_i \subset U)$. Hence there exists $g: F(U) \to F(U)^n$ such that the diagram




where $\pi$ is a canonical morphism. Therefore $F(U)$ is weakly projective.

Now let $U$ be quasi-projective in $\mathscr{A}$. Then by [14, Theorem 4], $\mathscr{B}(F(U), -)$ and $- \otimes_R F(U)$ induce an equivalence $\mathscr{B}' \sim \operatorname{Im} \mathscr{B}(F(U), -)$, where $\mathscr{B}'$ is a full subcategory consisting of all $F(U)$-presented objects. So in particular, we get Proposition 5.1.

The dual of the above is easy by using the notion of filters. We leave it to the reader for detail.

PROPOSITION 5.5. *Let the situation be the same as Proposition* 5.1. *Then the following statements are equivalent.*

(1) *$\mathscr{C}(U)$ is abelian.*

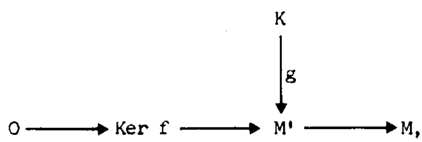
(2) *$\mathscr{C}(U)$ consists of all finitely $U$-presented objects.*

(3) *$\mathscr{D}(U)$ consists of all finitely presented right $R$-modules.*

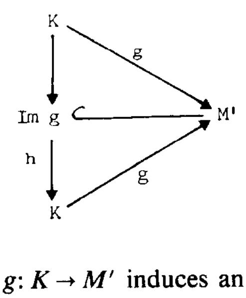
*Hence if $\mathscr{C}(U)$ is abelian, $R$ is right coherent.*

*Proof.* (2) $\Leftrightarrow$ (3) is easily verified. Before showing (1) $\Rightarrow$ (3), we show that $\mathscr{D}(U)$ is closed under cokernels. Let $f: M \to M''$ be a morphism in $\mathscr{D}(U)$ and $\operatorname{Cok} f$ the cokernel of $f$ in Mod-$R$. Then by five lemma, $\operatorname{Cok} f \in \mathscr{D}(U)$.

(1) $\Rightarrow$ (3): Let $f: M' \to M$ be a morphism in $\mathscr{D}(U)$ and $g: K \to M'$ the kernel of $f$ in $\mathscr{D}(U)$. Consider the diagram in Mod-$R$




where $\operatorname{Ker} f$ is the kernel of $f$ in Mod-$R$. It is obvious that $\operatorname{Im} g \subset \operatorname{Ker} f$ and $\operatorname{Im} g \in \mathscr{D}(U)$, where $\operatorname{Im} g$ is the image of $g$ in Mod-$R$. Hence there exists a unique homomorphism $h$: $\operatorname{Im} g \to K$ such that the diagram




is commutative. Therefore $g: K \to M'$ induces an isomorphism $K \cong \operatorname{Im} g$. On the other hand, every finitely generated submodule of $\operatorname{Ker} f$ is in $\mathscr{D}(U)$. This implies $\operatorname{Im} g = \operatorname{Ker} f$. Thus $\mathscr{D}(U)$ is closed under kernels. Let $N \in \mathscr{D}(U)$ and let $R^n \to N \to 0$ be exact. Then $\mathrm{Ker}(R^n \to N) \in \mathscr{D}(U)$. Therefore $N_R$ is finitely presented.

(3) $\Rightarrow$ (1): Suppose $\mathscr{D}(U)$ consists of all finitely presented right $R$-modules. Since every finitely generated right ideal of $R$ is finitely presented, $R$ is right coherent. Hence $\mathscr{D}(U)$ coincides with the category of coherent right $R$-modules, which is abelian.

In the preceding corollary, if $R$ is right noetherian, clearly (3) holds. But it is not a necessary condition in order to (3) hold.

EXAMPLE 5.1. Let $R$ be a semisimple ring and $V_R$ an infinitely generated cogenerator with $S = \mathrm{End}(V_R)$. Then $\mathscr{C}^*(V_R) = \{M \in \text{Mod-}R \mid M_R$ is $V$-copresented and finitely $V$-cogenerated$\}$ is abelian. But $S$ is not left noetherian since $\mathbf{L}(V_R)$ coincides with the lattice of all submodules of $V_R$ and $\mathbf{L}(V_R)$ is not artinian.

Also that $R$ is right coherent is not a sufficient condition in order to (2) hold, for let $R$ be right coherent but not right noetherian. Then clearly $\mathscr{C}(R) = \mathscr{D}(R) = \{M \in \text{Mod-}R \mid M_R$ is finitely generated$\}$. Hence

$$\mathscr{C}(R) \text{ is abelian} \Leftrightarrow \text{every finitely generated right } R\text{-module is finitely presented} \Leftrightarrow R \text{ is right noetherian.}$$

Hence $\mathscr{C}(R)$ is not abelian.

EXAMPLE 5.2. Let $R$ be any ring, $I$ an infinite set such that $\mathrm{card}\, I \geqslant \mathrm{card}\, R$. Let $P_R = R^{(I)}$. Then $\mathscr{C}(P_R)$ is abelian.

*Proof.* First we show that $\mathrm{card}\, P = \mathrm{card}\, I$. Put $F_n = \{(r_i) \in P \mid \mathrm{card}\{(r_i, i) \mid r_i \neq 0\} \leqslant n\}$. Then it is clear that $P = \bigcup F_n$. We define the map $\theta\colon R^n \times I^n \to F_n$ via $\theta((r_1,\dots, r_n), (i_1,\dots, i_n)) = \sum u_{i_j}(r_{i_j})$, where $u_{i_j}\colon R \to P$ are injections. Then clearly $\theta$ is surjective. Hence $\mathrm{card}\, F_n \leqslant \mathrm{card}\, R^n \times I^n \leqslant \mathrm{card}\, I^{2n} = \mathrm{card}\, I$. Thus $\mathrm{card}\, I \leqslant \mathrm{card}\, P \leqslant \mathrm{card}\, I \times \mathbb{N} = \mathrm{card}\, I$. Now let $X_R$ be finitely $P$-generated. Let $f\colon P^n \to X$ be an epimorphism. Then since $\mathrm{card}\, P^n = \mathrm{card}\, I$, $\mathrm{Ker}\, f$ can be generated by $I$ elements. So put $\mathrm{Ker}\, f = \sum_{i \in I} x_i R$. Then there is an epimorphism $\varphi\colon P \to \mathrm{Ker}\, f$ such that $\varphi(r_i) = \sum x_i r_i$. This implies that $X_R$ is finitely $P$-presented. Hence $\mathscr{C}(P_R)$ is abelian.

Finally we consider coherent objects. Again let $U \in \mathscr{A}$ be a quasi-projective object with $R = \mathrm{End}_{\mathscr{A}}(U)$. Let $\mathbf{U}$ be a category of all finitely $U$-generated projective objects in $\mathscr{P}(U)$. Let $X \in \mathscr{P}(U)$. Then $X$ is said to be $\mathbf{U}$-coherent if $X$ is finitely $U$-presented and every finitely $U$-generated subobject of $X$ is finitely $U$-presented. If every object of $\mathbf{U}$ is $\mathbf{U}$-coherent, $\mathbf{U}$ is said to be coherent. It is easily shown that $\mathbf{U}$ is coherent if and only if $U$ is $\mathbf{U}$-coherent.

PROPOSITION 5.6. *If $U$ is* **U**-*coherent, $R$ is right coherent.*

*Proof.* Let $f: R^n \to R$ be a homomorphism in Mod-$R$. Then since $U$ is **U**-coherent, we have the exact sequence $U^m \to U^n \xrightarrow{f \otimes U} U$. Thus we have an exact sequence $R^m \to R^n \xrightarrow{f} R$ since $U$ is projective in $\mathscr{P}(U)$. Therefore $R$ is right coherent.

Now let $R$ be any ring and **P** an additive category of all $\chi$-generated projective right $R$-modules. Let $P_R = R^{(I)}$ and $S = \operatorname{End}(P_R)$, where $I$ is a set such that $\operatorname{card} I = \chi$. $R$ is said to be right $\chi$-coherent if every $\chi$-generated right ideal of $R$ is $\chi$-related. It is known that $R$ is right $\chi$-coherent if and only if **P** is coherent (hence $P_R$ is **P**-coherent).

EXAMPLE 5.3. Under the same situation of the above, the following conditions are equivalent.

(1) $R$ is right $\chi$-coherent.

(2) $P_R$ is **P**-coherent.

(3) $S$ is right coherent.

In Example 5.2, $R$ is right card $I$-coherent.

## 6. SUPPLEMENTS

In this section we give a couple of applications. Let $R$ be a ring and $E_R$ the injective envelope of $R_R$. Then Sato called $R$ right QF-3 if every finitely generated submodule of $E_R$ is torsionless, and he showed that if $R$ is left and right noetherian, $R$ is left QF-3 if and only if $R$ is right QF-3. In this section we generalize his result. But we call $R$ right QF-3′ if $R$ is right QF-3 in the sense of Sato. First we introduce QF-3′ modules.

Let $V \in$ Mod-$R$ and $\overline{\operatorname{cog}}(V_R)$ the full subcategory of Mod-$R$ consisting of all submodules of homomorphic images of finite direct sums of copies of $V_R$. In other words $\overline{\operatorname{cog}}(V_R)$ is the smallest strongly exact subcategory containing $V_R$.

PROPOSITION 6.1. *Let $E_R$ be the injective envelope of $V_R$. Then the following conditions are equivalent.*

(1) *$V$ is QF-3 in $\overline{\operatorname{cog}}(V_R)$.*

(2) *Every finitely $V_R$-generated submodule of $E_R$ is $V_R$-torsionless.*

(3) *Every finitely $V_R$-generated torsion free module with respect to the torsion theory cogenerated by $E_R$ in* Mod-$R$ *is $V_R$-torsionless.*

*Proof.* (1) $\Rightarrow$ (3): Let $X_R$ be a submodule of $\prod E_R$ and $V_R^n \to X_R \to 0$ be exact. Let $r(X) = \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(X, V)\}$. We must show $r(X) = 0$. Suppose $r(X) \neq 0$. Then there exists $0 \neq f: X_R \to E_R$ such that $f(r(X)) \neq 0$. Then since $\operatorname{Im} f \hookrightarrow \prod V$, there exists $0 \neq g: X_R \to V_R$ such that $g(r(X)) \neq 0$. This is a contradiction.

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): Let $f: X'_R \to X_R$ be a monomorphism in $\overline{\operatorname{cog}}(V_R)$ such that $\operatorname{Hom}(f, V) = 0$. We may assume that $X_R$ is finitely $V$-generated. Suppose $\operatorname{Hom}_R(X', V) \neq 0$. Then there exists $0 \neq g: X'_R \to E_R$. Since $E_R$ is injective, $g$ can be extended to $h: X_R \to E_R$. Put $N = \operatorname{Im} h$. Then $N_R$ is finitely $V$-generated. Hence $N_R$ is $V$-torsionless by the assumption. Then there exist $x \in X'$ and $u: N_R \to V_R$ such that $ug(x) \neq 0$. But $ug$ is the image of $uh$ by $\operatorname{Hom}(f, V)$. This is a contradiction. Therefore $V$ is QF-3 in $\overline{\operatorname{cog}}(V_R)$.

DEFINITION 6.1. $V_R$ is a QF-3′ module if $V$ is QF-3 in $\overline{\operatorname{cog}}(V_R)$.

PROPOSITION 6.2. *Let $V_R$ be a QF-3′ module and $E_R$ the injective envelope of $V_R$. Then the following statements are equivalent.*

(1) *$V$ is divisible with respect to the torsion theory cogenerated by $V$ in $\overline{\operatorname{cog}}(V_R)$.*

(2) *For every finitely $V_R$-generated submodule $X$ of $E_R$ such that $X \supset V$, $X/V$ is $V_R$-torsionless.*

*Moreover if $R \in \overline{\operatorname{cog}}(V_R)$, these are equivalent to:*

(3) *$V_R$ is divisible with respect to the torsion theory cogenerated by $E_R$ in* Mod-$R$.

We omit the proof, because there is no new idea for the proof other than in categories of modules.

What we are most interested in is a QF-3′ ring. A famous example of QF-3′ rings is $\mathbb{Z}$, the ring of integers. Thus we get an example which states that in abelian categories "hereditary torsion theory" does not necessarily mean "strongly hereditary."

PROPOSITION 6.3. *Let $R$ be right QF-3′. Suppose $R$ is either left noetherian or right linearly compact with essential socle. Then $R$ is left QF-3′.*

*Proof.* (cf. Lemma 2.3). Let $f: {}_RM' \to {}_RM$ be a monomorphism such that

${}_RM$ is finitely generated and $\mathrm{Hom}(f, R) = 0$. Let ${}_RN$ be an arbitrary finitely generated submodule of $M'$ and $f': N \to M$ the restriction of $f$. Then we get the commutative diagram

$$\begin{array}{ccccccc}
R^{(I)} & \longrightarrow & R^n & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \\
R^{(J)} & \longrightarrow & R^m & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

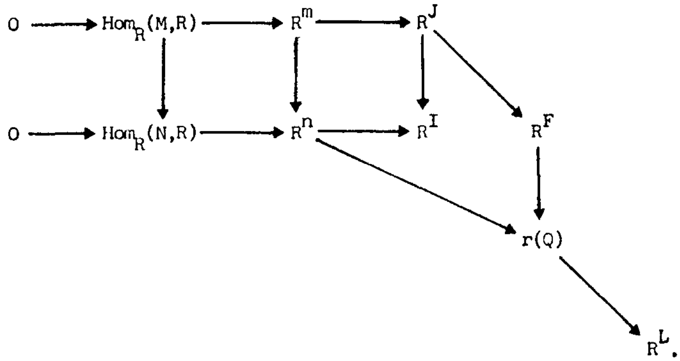
Thus we further get the commutative diagram of right $R$-modules

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M,R) & \xrightarrow{\pi'^*} & R^m & \xrightarrow{\varphi'^*} & R^J \\
& & \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h^*} \\
0 & \longrightarrow & \mathrm{Hom}_R(N,R) & \xrightarrow{\pi^*} & R^n & \xrightarrow{\varphi^*} & R^I.
\end{array}$$

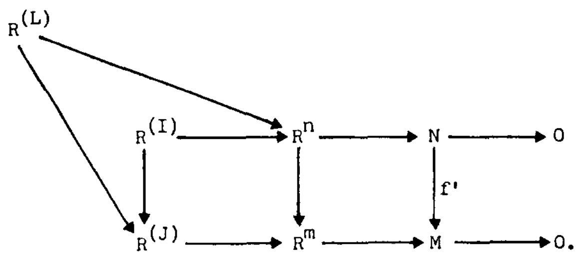
If $R$ is left noetherian, $J$ can be taken as a finite set. So suppose $R$ is right linearly compact with essential right socle. Put $K = \mathrm{Ker}\ g^*$. Then note that $\mathrm{Hom}_R(M, R) \subset K$. Let $\mathbf{F}$ be the family of all finite subsets of $J$. Let us denote $\hat{F}$ as the complement of $F \in \mathbf{F}$. Then since $\bigcap_{F \in \mathbf{F}} R^{\hat{F}} = 0$, $\mathrm{Hom}_R(M, R) = \bigcap_{F \in \mathbf{F}} \varphi'^{*-1}(R^{\hat{F}})$. By the assumption it is clear that $K_R$ is a finitely completely meet irreducible submodule of $R^m$. Hence by Müller [23], there exists $F \in \mathbf{F}$ such that $\varphi'^{*-1}(R^{\hat{F}}) \subset K$. Let $R^J \to R^F$ be the projection. Then $\varphi'^*$ induces a monomorphism $R^m/\varphi'^{*-1}(R^{\hat{F}}) \to R^F$. Now let $r$ be the idempotent coradical (in the sense of [34]) associated with the hereditary torsion theory cogenerated by $R_R$ in $\overline{\mathrm{cog}(R_R)}$. Then we know that $r$ preserves monomorphisms. Let

$$\begin{array}{ccc}
R^m/\varphi'^{*-1}(R^{\hat{F}}) & \longrightarrow & R^F \\
\downarrow & & \downarrow \\
R^n & \longrightarrow & Q
\end{array}$$

be the push out diagram. Then $Q \in$ mod-$R$, where mod-$R = \overline{\mathrm{cog}(R_R)}$. Hence $r(Q)$ is defined and $R^n \to r(Q)$ is a monomorphism. Let $r(Q) \to R^L$ be a monomorphism. Then we get the commutative diagram




Thus it induces the further one.



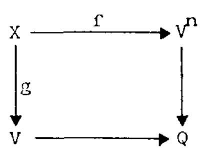

Thus we conclude $\mathrm{Hom}_R(N, R) = 0$. Since ${}_RN$ is an arbitrary finitely generated submodule of $M'$, $\mathrm{Hom}_R(M', R) = 0$. Therefore $R$ is left QF-3$'$.

LEMMA 6.4. *Let $\mathscr{B}$ be a strongly exact subcategory of* Mod-$R$ *and $V \in \mathscr{B}$ a QF-3 object with $S = \mathrm{End}(V_R)$. Let $f: X_R \to V_R^n$ be a monomorphism. Then* $\mathrm{Hom}_S(\mathrm{Cok}\, \mathrm{Hom}(f, V), V) = 0$.

*Proof.* Let $\pi_i: V^n \to V$ $(i = 1,..., n)$ be the projections and put $f_i = \pi_i f$. Then we have an exact sequence

$$S^n \xrightarrow{\mathscr{E}} \mathrm{Hom}_R(X, V) \longrightarrow N \longrightarrow 0,$$

where $(s_1,..., s_n)\mathscr{E} = s_1 f_1 + \cdots + s_n f_n$ for all $s_i \in S$. Let $\theta$: ${}_S\mathrm{Hom}_R(X, V) \to {}_SV$ be a homomorphism such that $\mathscr{E}\theta = 0$. Then we have to show $\theta = 0$. It is easy to see that $\mathscr{E}\theta = 0$ if and only if $f_1\theta = \cdots = f_n\theta = 0$. Suppose there exists $g \in \mathrm{Hom}_R(X, V)$ such that $g\theta \neq 0$. Let



be the push out diagram (in $\mathscr{B}$). Let $r$ be the idempotent coradical associated with the hereditary torsion theory cogenerated by $V$ in $\mathscr{B}$. Since $r$ preserves monomorphisms, $(V \to Q \to r(Q))$ is a monomorphism. Let $r(Q) \to \Pi V_\alpha$ be a monomorphism with $V_\alpha = V$. Let $j_i: V \to V^n$ $(i = 1,\dots, n)$ be the injections and $\pi_\alpha: \Pi V_\alpha \to V$ the projections. Put $k = (V \to r(Q) \to V)$, $s_\alpha = \pi_\alpha k$, $h_\alpha = (V^n \to^h r(Q) \to \Pi V_\alpha \to^{\pi_\alpha} V)$ and $s_{\alpha_i} = h_\alpha j_i$. Then for each $x \in X$,

$$\begin{aligned} h_\alpha f(x) &= h_\alpha(f_1 x,\dots, f_n x) \\ &= h_\alpha j_1(f_1 x) + \cdots + h_\alpha j_n(f_n x) \\ &= s_{\alpha_1}(f_1 x) + \cdots + s_{\alpha_n}(f_n x) \\ &= (s_{\alpha_1} f_1 + \cdots + s_{\alpha_n} f_n)x. \end{aligned}$$

Hence $h_\alpha f = s_{\alpha_1} f_1 + \cdots + s_{\alpha_n} f_n$. Then for all $\alpha$, since $\theta$ is $S$-linear,

$$\begin{aligned} k(g\theta) &= s_\alpha(g\theta) \\ &= (s_\alpha g)\theta \\ &= (h_\alpha f)\theta \\ &= (s_{\alpha_1} f_1 + \cdots + s_{\alpha_n} f_n)\theta \\ &= (s_{\alpha_1} f_1)\theta + \cdots + (s_{\alpha_n} f_n)\theta \\ &= s_{\alpha_1}(f_1\theta) + \cdots + s_{\alpha_n}(f_n\theta) \\ &= 0. \end{aligned}$$

Therefore $k(g\theta) = 0$. But $k$ is a monomorphism, hence $g\theta = 0$. This is a contradiction. Thus $\theta = 0$.

THEOREM 6.5. *Let $R$ be left noetherian, right QF-3′ and contain all simple right modules. Then $R$ is QF.*

*Proof.* The proof is immediate from the preceding lemmas.

Next corollary is well known (e.g. [11]).

COROLLARY 6.6. *Let $R$ be either left noetherian or right artinian, and a cogenerator in* Mod-$R$. *Then $R$ is QF.*

For the rest of this section our intention is set upon to characterize right hereditary rings. A module $M_R$ is said to be semi-injective if $\mathrm{Hom}_R(-, M)$ is exact on all short exact sequences $0 \to I \to R \to R/I \to 0$ with finitely generated right ideal I of $R$, and is said to be a self-cogenerator if $M$ is a cogenerator in $\overline{\mathrm{cog}}(M_R)$. Let $Q_R$ be an injective self-cogenerator with $S = \mathrm{End}(Q_R)$. Then Müller and Turnidge [14] proved that $S$ is semihereditary if and only if every finitely $Q$-cogenerated factor of $Q_R$ is injective. We generalize a little bit this result.

Let $\mathscr{A}$ be a cocomplete abelian category and $U \in \mathscr{A}$ a quasi-projective object with $R = \mathrm{End}_{\mathscr{A}}(U)$.

DEFINITION 6.2. ${}_RU$ is semiflat if the canonical morphism $0 \to I \otimes_R U \to R \otimes_R U$ is exact for all finitely generated right ideals $I$ of $R$.

The next lemma is essential for the later discussion.

LEMMA 6.7. *The following conditions are equivalent.*

(1) *$R$ is right semihereditary and ${}_RU$ is semiflat.*

(2) *Every finitely $U$-generated subobject of $U$ is progective in $\mathscr{P}(U)$.*

*Proof.* (1) $\Rightarrow$ (2): Let $X \subset U$ and $U^n \to X \to 0$ be exact. Then $U/X$ is codivisible with respect to the torsion theory generated by $U$ in $\mathscr{P}(U)$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A}(U,X)\otimes_R U & \longrightarrow & R\otimes_R U & \longrightarrow & \mathscr{A}(U,U/X)\otimes_R U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & X & \longrightarrow & U & \longrightarrow & U/X & \longrightarrow & 0 \end{array}$$

with exact rows because $\mathscr{A}(U, X)$ is a finitely generated right ideal of $R$ and ${}_RU$ is semiflat. Hence $\mathscr{A}(U, X) \otimes_R U \cong X$. On the other hand, by the assumption, $R^n \cong \mathscr{A}(U, X) \oplus Y$ for some $Y_R$. Thus $U^n \cong X \oplus (Y \otimes_R U)$. This proves that $X$ is projective in $\mathscr{P}(U)$.

(2) $\Rightarrow$ (1): Let $I_R$ be a finitely generated right ideal of $R$. Then note that $\mathscr{A}(U, I \otimes_R U) \cong I$. We have an exact sequence $0 \to N \to I \otimes_R U \to U$. Put $L = \mathrm{Im}(I \otimes_R U \to U)$. Then by the assumption, $L$ is projective in $\mathscr{P}(U)$. Thus $I \otimes_R U \cong L \oplus N$. On the other hand the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathscr{A}(U,N) & \longrightarrow & \mathscr{A}(U,I\otimes_R U) & \longrightarrow & \mathscr{A}(U,R\otimes_R U) \\ & & & & \uparrow & & \uparrow \\ & & 0 & \longrightarrow & I & \longrightarrow & R \end{array}$$

implies $\mathscr{A}(U, N) = 0$. Hence $N = 0$. This implies that ${}_RU$ is semiflat. Since $I \otimes_R U$ is finitely generated by $U$ and projective in $\mathscr{P}(U)$, we conclude that $I_R$ is projective. Therefore $R$ is right semihereditary.

COROLLARY 6.8. *Let $U_R$ be a projective module with $S = \mathrm{End}(U_R)$. Then the following conditions are equivalent.*

(1) *$S$ is right semihereditary and $_SU$ is flat.*

(2) *Every finitely $U$-generated submodule of $U_R$ is projective.*

COROLLARY 6.9. *Let $V_R$ be an injective module with $S = \mathrm{End}(V_R)$. Then the following conditions are equivalent.*

(1) *$S$ is left semihereditary and $_SV$ is semi-injective.*

(2) *Every finitely $V$-cogenerated factor of $V_R$ is injective.*

THEOREM 6.10. *The following statements are equivalent.*

(1) *$R$ is right hereditary.*

(2) *An endomorphism ring of any injective right $R$-module is left semihereditary.*

(3) *An endomorphism ring of any projective right $R$-module is right semihereditary.*

*Proof.* $(1) \Rightarrow (2)$: This has been proved in Corollary 6.9.

$(2) \Rightarrow (1)$: Let $V_R$ be an injective module and $X$ a submodule of $V_R$. Take an injective cogenerator $W_R$ such that $V \subsetneq W$ and $V/X \subsetneq W$. Since $V_R$ is isomorphic to a direct summand of $W_R$, $V/X$ is isomorphic to a factor module of $W_R$. By the assumption, $\mathrm{End}(W_R)$ is left semihereditary. Since $W_R$ is an injective cogenerator, $W$ is semi-injective over $\mathrm{End}(W_R)$ (by Lemma 6.4 or [28, Theorem 3.3]). Therefore by Corollary 6.9, $V/X_R$ is injective. Hence $R$ is right hereditary.

$(1) \Rightarrow (3) \Rightarrow (1)$: This is the dual of $(1) \Rightarrow (2) \Rightarrow (1)$.

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