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Some criteria of cyclically pure injective modules

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Abstract

The structure of cyclically pure injective modules over a commutative ring R is investigated and several characterizations for them are presented. In particular, we prove that a module D is cyclically pure injective if and only if D is isomorphic to a direct summand of a module of the form $\operatorname{Hom}_R(L, E)$ where L is the direct sum of a family of finitely presented cyclic modules and E is an injective module. Also, we prove that over a quasi-complete Noetherian ring $(R, \mathfrak{m})$ an R-module D is cyclically pure injective if and only if there is a family $\{C_\lambda\}_{\lambda \in \Lambda}$ of cocyclic modules such that D is isomorphic to a direct summand of $\prod_{\lambda \in \Lambda} C_\lambda$. Finally, we show that over a complete local ring every finitely generated module which has small cofinite irreducibles is cyclically pure injective.

Keywords: Cocyclic modules; Cyclic exact sequences; Cyclically pure injective modules; Quasi-complete rings; Small cofinite irreducibles

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1. Introduction

Throughout this paper, let R denote a commutative ring with identity. All modules are assumed to be left unitary. The notion of pure injective modules has a substantial role in commutative algebra and model theory. Even in model theory [8], the notion of pure injective modules is much more useful than that of injective modules. Also, there are some nice applications of this notion in theory of flat covers (see e.g. [11]).

There are several generalizations of the notion of pure injective modules. One of these generalizations is the notion of cyclically pure injective modules which has attracted more attention in recent years. Following his investigations on "direct summand conjecture," M. Hochster [6] studied the structure of Noetherian rings that are pure in every module in which it is cyclically pure. He showed that a Noetherian ring R is pure in every module in which it is cyclically pure if and only if R has small cofinite irreducibles. Using the notion of cyclically pure injective modules, L. Melkersson [7] provided some characterizations for a finitely generated module M over a Noetherian local ring which is pure in every cyclically pure extension of M. In this paper, our aim is to present some criterions of cyclically pure injective modules, through a systematic investigation of their structure.

There are several variants of the notion of purity (see e.g. [10]). More generally, let $\mathcal{S}$ be a class of R-modules. An exact sequence $0 \to A \to B \to C \to 0$ is $\mathcal{S}$-pure if for all $M \in \mathcal{S}$ the induced homomorphism $\operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C)$ is surjective. An R-module D is said to be $\mathcal{S}$-pure injective if for any $\mathcal{S}$-pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism $\operatorname{Hom}_R(B, D) \to \operatorname{Hom}_R(A, D)$ is surjective. When $\mathcal{S}$ is the class of finitely presented R-modules, $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules are called pure exact sequences and pure injective modules, respectively. In this article, we consider the class $\mathcal{S}$ consisting of all R-modules M for which there are an integer $n \in \mathbb{N}$ and a cyclic submodule G of R^n such that M is isomorphic to R^n/G. In Section 2, a characterization of cyclically pure exact sequences is given. Among other things, this characterization implies that for the above class $\mathcal{S}$, $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules coincide with cyclically pure exact sequences and cyclically pure injective modules, respectively. Also, several elementary results will be presented in this section, to ease reading the remainder of the paper.

In Section 3, we present two characterizations of cyclically pure injective modules. The first one, in particular, asserts that an R-module D is cyclically pure injective if and only if D has no proper essential cyclically pure extension. Also, it is proved that an R-module D is cyclically pure injective if and only if D is isomorphic to a direct summand of a module of the form $\operatorname{Hom}_R(L, E)$ where E is an injective R-module and L is the direct sum of a family of finitely presented cyclic modules.

In Section 4, we show that every R-module possesses a unique, up to isomorphism, cyclically pure injective envelope.

In Section 5, we investigate the question when cocyclic modules are cyclically pure injective. As a result, we present our last characterization of pure injective modules. Namely, we prove that over a quasi-complete Noetherian local ring $(R, \mathfrak{m})$ an R-module D is cyclically pure injective if and only if there is a family $\{C_\lambda\}_{\lambda \in \Lambda}$ of cocyclic modules such that D is isomorphic to a direct summand of $\prod_{\lambda \in \Lambda} C_\lambda$. Also, we prove that over a local Noetherian ring $(R, \mathfrak{m})$ every finitely generated R-module M that has small cofinite irreducibles is pure in every cyclically pure extension of M. As a result, we deduce that over a complete local ring every finitely generated module which has small cofinite irreducibles is cyclically pure injective.

2. Cyclically pure extensions of modules

Let $\mathcal{S}$ denote the class of all R-modules M such that there are an integer $n \in \mathbb{N}$ and a cyclic submodule G of R^n such that M is isomorphic to R^n/G. In the sequel, we show that cyclically pure exact sequences and cyclically pure injective modules are coincide with $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules, respectively.

Definition 2.1. (i) Recall that an exact sequence $0 \to A \to B \to C \to 0$ is said to be *cyclically pure* if the natural map $R/\mathfrak{a} \otimes_R A \to R/\mathfrak{a} \otimes_R B$ is injective for all finitely generated ideals $\mathfrak{a}$ of R. Also, an R-monomorphism $f : A \to B$ is said to be *cyclically pure*, if the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{\text{nat}} B/f(A) \to 0$ is cyclically pure. Moreover, a submodule A of an R-module B is called *cyclically pure submodule* if the inclusion map $A \hookrightarrow B$ is cyclically pure.

(ii) An R-module D is called *cyclically pure injective* if for any cyclically pure exact sequence $0 \to A \to B \to C \to 0$, the induced sequence

$$0 \to \operatorname{Hom}_R(C, D) \to \operatorname{Hom}_R(B, D) \to \operatorname{Hom}_R(A, D) \to 0$$

is exact.

In the sequel, we use the abbreviation CP for the term “cyclically pure.”

Proposition 2.2. *Suppose* $0 \to A \hookrightarrow B \xrightarrow{\varphi} C \to 0$ *is an exact sequence of R-modules and R-homomorphisms. The following are equivalent*:

- (i) *For any $M \in \mathcal{S}$, the induced homomorphism* $\operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C)$ *is surjective.*
- (ii) *If the linear equation $\sum_{i=1}^n r_i x_i = a$, $a \in A$, $r_1, r_2, \ldots, r_n \in R$, is solvable in B, then it is also solvable in A.*
- (iii) $\mathfrak{a}B \cap A = \mathfrak{a}A$ *for any finitely generated ideal $\mathfrak{a}$ of R.*
- (iii′) $\mathfrak{a}B \cap A = \mathfrak{a}A$ *for any ideal $\mathfrak{a}$ of R.*
- (iv) *The exact sequence* $0 \to A \hookrightarrow B \xrightarrow{\varphi} C \to 0$ *is cyclically pure.*
- (iv′) *The natural map $R/\mathfrak{a} \otimes_R A \to R/\mathfrak{a} \otimes_R B$ is injective for all ideals $\mathfrak{a}$ of R.*

Proof. (ii) $\Rightarrow$ (i) Let $M = R^n/G$ where $G = R(r_1, r_2, \ldots, r_n)$ is a cyclic submodule of R^n and take an element f in $\operatorname{Hom}_R(M, C)$. For each $1 \leqslant i \leqslant n$, set $m_i = e_i + G$ where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of R^n. There is $y_i \in B$ such that $\varphi(y_i) = f(m_i)$ for all $1 \leqslant i \leqslant n$. One can see easily that $a := \sum_{i=1}^n r_i y_i \in A$. Hence, by the assumption there are $z_1, z_2, \ldots, z_n \in A$ such that $\sum_{i=1}^n r_i z_i = a$. We define $g : M \to B$, by $g(\sum_{i=1}^n s_i m_i) = \sum_{i=1}^n s_i(y_i - z_i)$ for all $s_1, s_2, \ldots, s_n \in R$. Suppose that $\sum_{i=1}^n s_i m_i = 0$

for some $s_1, s_2, \dots, s_n \in R$. Then $(s_1, s_2, \dots, s_n) \in G$. Hence there is $b \in R$ such that $(s_1, s_2, \dots, s_n) = b(r_1, r_2, \dots, r_n)$, and so

$$\sum_{i=1}^{n} s_i(y_i - z_i) = b\left(\sum_{i=1}^{n} r_i y_i - \sum_{i=1}^{n} r_i z_i\right) = 0.$$

Therefore g is well-defined. It is easy to see that $\varphi g = f$.

(i) $\Rightarrow$ (ii) Suppose that $n \in \mathbb{N}$, $r_1, r_2, \dots, r_n \in R$ and $y_1, y_2, \dots, y_n \in B$ are such that $a := \sum_{i=1}^{n} r_i y_i$ is an element of A. Set $M = R^n / R(r_1, r_2, \dots, r_n)$ and $m_i = e_i + R(r_1, r_2, \dots, r_n)$ for all $1 \leqslant i \leqslant n$. Define $f : M \to C$, by $f(\sum_{i=1}^{n} s_i m_i) = \sum_{i=1}^{n} s_i \varphi(y_i)$. It is a routine check that f is a well-defined R-homomorphism. By the assumption, there exists an R-homomorphism $g : M \to B$ such that $\varphi g = f$. We have $\varphi(y_i) = f(m_i) = \varphi(g(m_i))$, and so $y_i - g(m_i) \in A$ for all $1 \leqslant i \leqslant n$. Now, we have

$$\begin{aligned}\sum_{i=1}^{n} r_i\big(y_i - g(m_i)\big) &= \sum_{i=1}^{n} r_i y_i - \sum_{i=1}^{n} r_i g(m_i) = a - g\left(\sum_{i=1}^{n} r_i m_i\right) \\ &= a - g\big((r_1, r_2, \dots, r_n) + (r_1, r_2, \dots, r_n)R\big) \\ &= a.\end{aligned}$$

Next, the equivalences (ii) $\Rightarrow$ (iii′), (iii′) $\Rightarrow$ (iv′), and (iv′) $\Rightarrow$ (iv) are clear. Also, the implications (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are obvious, and so the proof is complete. □

Let $\{E_i\}_{i \in I}$ be a class of R-modules. It is known that $\prod_{i \in I} E_i$ is an injective R-module if and only if E_i is injective for all $i \in I$. By using the standard argument of this classical result, we can deduce the following analogue conclusion for CP-injective modules.

Lemma 2.3. *Let $\{D_i\}_{i \in I}$ be a class of R-modules. Then $\prod_{i \in I} D_i$ is a CP-injective R-module if and only if D_i is CP-injective for all $i \in I$.*

Lemma 2.4. *Let $\mathfrak{b}$ be an ideal of R. Then any CP-injective $R/\mathfrak{b}$-module is also CP-injective as an R-module.*

Proof. Let D be a CP-injective $R/\mathfrak{b}$-module. Assume that M and N are two R-modules. Let $\psi : N \to M$ be a CP-homomorphism and let $f \in \operatorname{Hom}_R(N, D)$. Since $\mathfrak{b}D = 0$, the map f induces the $R/\mathfrak{b}$-homomorphism $f^* : N/\mathfrak{b}N \to D$, defined by $f^*(x + \mathfrak{b}N) = f(x)$ for all $x + \mathfrak{b}N \in N/\mathfrak{b}N$. From our assumption on ψ, we deduce that the induced $R/\mathfrak{b}$-homomorphism $\psi^* : N/\mathfrak{b}N \to M/\mathfrak{b}M$ is cyclically pure. Thus there is an $R/\mathfrak{b}$-homomorphism $h : M/\mathfrak{b}M \to D$ with $h\psi^* = f^*$. Let $g = h\pi$ where π is the natural epimorphism $M \to M/\mathfrak{b}M$. Then $g\psi = f$. □

Theorem 2.5. *Let M be an R-module. Then there are a CP-injective R-module D and a CP-homomorphism $\varphi : M \to D$.*

Proof. Let R^* denote the set of all finitely generated ideals of R. Set $D = \prod_{\mathfrak{a} \in R^*} E_{R/\mathfrak{a}}(M/\mathfrak{a}M)$ where $E_{R/\mathfrak{a}}(M/\mathfrak{a}M)$ denotes the injective envelope of the $R/\mathfrak{a}$-module $M/\mathfrak{a}M$. Define $\varphi : M \to D$, by $\varphi(x) = (x + \mathfrak{a}M)_{\mathfrak{a} \in R^*}$ for all $x \in M$. It follows, by Lemmas 2.3 and 2.4 that D is CP-injective. Clearly, φ is injective. Next, we prove that φ is cyclically pure. To this end, let $\mathfrak{b}$ be an arbitrary finitely generated ideal of R and let y be an arbitrary element of $\mathfrak{b}D \cap \varphi(M)$. Then $y = \varphi(x)$ for some $x \in M$. Since $\varphi(x) \in \mathfrak{b}D$, it turns out that $x + \mathfrak{b}M \in \mathfrak{b}E_{R/\mathfrak{b}}(M/\mathfrak{b}M) = 0$. Thus $x \in \mathfrak{b}M$, and so $y \in \mathfrak{b}\varphi(M)$, as required. □

Corollary 2.6. *Let M be an R-module. There is an extension D of M such that D is CP-injective and it contains M as a CP-submodule.*

Proof. Let M be an R-module. By Theorem 2.5, there are a CP-injective R-module D' and a CP-homomorphism $\varphi : M \to D'$. Using [9, Proposition 1.1], it turns out that there are an extension D of M and an isomorphism $\psi : D \to D'$ which is such that $\psi(x) = \varphi(x)$ for all $x \in M$. It is easy to see that the inclusion map $M \hookrightarrow D$ is cyclically pure. □

Remark 2.7. (i) The analogue of some of our results for RD-purity were proved by R.B. Warfield (see e.g. [5, Chapter XIII.1]).

(ii) One can adapt the method of Warfield's proof of existence of RD-injective envelopes for proving the existence of CP-injective envelopes. We present a different proof for existence of CP-injective envelopes in Section 4.

3. Two characterizations

In this section, we present two characterizations of CP-injective modules. First, we bring a definition.

Definition 3.1. Let M be an R-module and N a CP-submodule of M. Then M is called *essential CP-extension* of N, if there is not any nonzero submodule K of M such that $K \cap N = 0$ and $(K + N)/K$ is a CP-submodule of M/K.

For a submodule N of an R-module M, it is known that M is an essential extension of N if and only if for any R-module L an R-homomorphism $\varphi : M \to L$ is injective whenever $\varphi|_N$ is injective. Similarly, for essential CP-extensions we have the following characterization.

Lemma 3.2. *Let N be a CP-submodule of an R-module M. Then M is essential CP-extension of N if and only if for any homomorphism $\varphi : M \to L$ such that $\varphi|_N$ is a CP-homomorphism, it follows that φ is injective.*

Proof. Suppose that M is an essential CP-extension of N. Let $\varphi : M \to L$ be a homomorphism such that $\varphi|_N$ is a CP-homomorphism. Let $K = \ker \varphi$ and let $\varphi^* : M/K \to L$ denote the natural monomorphism which induced by φ. Also, let $\rho : (K + N)/K \to N$ denote the natural isomorphism. Note that, because $\varphi|_N$ is injective, it turns out that $K \cap N = 0$. From the commutative diagram

$$\begin{array}{ccc} (K+N)/K & \hookrightarrow & M/K \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \varphi^*} \\ N & \xrightarrow{\ \varphi|_N\ } & L \end{array}$$

we deduce that $(K+N)/K$ is a CP-submodule of M/K. Therefore, it follows that $K = 0$. The proof of the converse is easy and we leave it to the reader. □

Lemma 3.3. *Let N be a CP-submodule of an R-module M. Then, there exists a submodule K of M such that*

(i) $K \cap N = 0$,
(ii) $(K+N)/K$ *is a CP-submodule of* M/K, *and*

that K is maximal with respect to inclusion among all submodules of M which satisfy the conditions (i) *and* (ii). *In particular, M/K is an essential CP-extension of $(K+N)/K$.*

Proof. Let Σ denote the class of all submodules of M which satisfy the conditions (i) and (ii). Then Σ is not empty, because $0 \in \Sigma$. Let Ω be a totally ordered subclass of Σ. Set $K = \bigcup_{K_\alpha \in \Omega} K_\alpha$. We show that K satisfies the conditions (i) and (ii). Clearly, $K \cap N = 0$. In view of Proposition 2.2, it is enough to show that $(K + \mathfrak{a}M) \cap (K + N) \subseteq (\mathfrak{a}N + K)$ for any ideal $\mathfrak{a}$ of R. But, it is a routine check, because by Proposition 2.2 $(K_\alpha + \mathfrak{a}M) \cap (K_\alpha + N) = \mathfrak{a}N + K_\alpha$ for any ideal $\mathfrak{a}$ of R and all $K_\alpha \in \Omega$. Thus the conclusion follows by Zorn's lemma. Now, we prove the last assertion. Assume there is a submodule L/K of M/K such that $L/K \cap ((N+K)/K) = 0$ and that $(L+N)/L$ is a CP-submodule of M/L. Then

$$L \cap N \subseteq L \cap (N + K) = K,$$

and so $L \cap N \subseteq K \cap N = 0$. Thus $L \in \Sigma$ and so $L = K$, by the assumption on K. Therefore, M/K is an essential CP-extension of $(K+N)/K$, as required. □

Now, we present our first characterization of CP-injective modules.

Theorem 3.4. *Let D be an R-module. Then the following are equivalent*:

(i) *D is CP-injective.*
(ii) *For any CP-homomorphism $f : A \to B$, every homomorphism from A to D can be extended to a homomorphism from B to D.*
(iii) *Every CP-exact sequence* $0 \to D \to M \to N \to 0$ *splits.*
(iv) *D is a direct summand of every R-module L which is such that D is a CP-submodule of L.*
(v) *D has no proper essential CP-extension.*

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (iv) are clear.

(iv) $\Rightarrow$ (v) Let M be a essential CP-extension of D. Then, there is a submodule L of M such that $M = L + D$ and $L \cap D = 0$. Since M is essential CP-extension of D and $(L + D)/L = M/L$, we deduce that $L = 0$, and so $M = D$.

(v) $\Rightarrow$ (vi) Suppose L is a CP-extension of D. It may be assumed that L is a proper CP-extension of D. By Lemma 3.3, there is a submodule K of L such that L/K is an essential CP-extension of $(D + K)/K$ and that $D \cap K = 0$. But D has no proper essential CP-extension, so $D + K = L$, from which it follows that $L = D \oplus K$.

(iv) $\Rightarrow$ (i) By Corollary 2.6, there exists a CP-injective extension L of D. Therefore, D is a direct summand of L, and so it is CP-injective, by Lemma 2.3. □

Let D be an R-module. In [3, Corollary 2.12], we proved that D is pure injective if and only if D is isomorphic to a direct summand of a module of the form $\operatorname{Hom}_R(L, E)$ where E is an injective R-module and L is the direct sum of a family of finitely generated modules. Next, we will establish a similar characterization for CP-injective modules. First, we need to the following lemma.

Lemma 3.5. *Let $\mathfrak{a}$ be an ideal of R. Then an $R/\mathfrak{a}$-module D is injective as an $R/\mathfrak{a}$-module if and only if there is an injective R-module E such that D is equal to $0 :_E \mathfrak{a}$.*

Proof. The "if" part is known and it is easy to check. For the converse, one only need to note that for an injective $R/\mathfrak{a}$-module D, we have

$$D = E_{R/\mathfrak{a}}(D) = 0 :_{E_R(D)} \mathfrak{a}.$$

Here $E_R(D)$ (respectively $E_{R/\mathfrak{a}}(D)$) denotes the injective envelope of D as an R-module (respectively $R/\mathfrak{a}$-module). □

Theorem 3.6. *Suppose D is an R-module. Then the following are equivalent*:

(i) *D is CP-injective.*
(ii) *There is a family $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ of finitely generated ideals of R such that D is isomorphic to a direct summand of an R-module of the form $\prod_{\lambda \in \Lambda} E_\lambda$ where E_λ is an injective $R/\mathfrak{a}_\lambda$-module for all $\lambda \in \Lambda$.*
(iii) *D is isomorphic to a direct summand of a module of the form* $\operatorname{Hom}_R(L, E)$ *where E is an injective module and L is the direct sum of a family of finitely presented cyclic modules.*

Proof. In view of the proof of Theorem 2.5, the equivalence (i) $\Leftrightarrow$ (ii) follows by Lemmas 2.3 and 2.4.

(ii) $\Rightarrow$ (iii) Suppose that there is a family $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ of finitely generated ideals of R such that D is isomorphic to a direct summand of an R-module of the form $\prod_{\lambda \in \Lambda} E_\lambda$ where E_λ is an injective $R/\mathfrak{a}_\lambda$-module for all $\lambda \in \Lambda$. By Lemma 3.5, for each $\lambda \in \Lambda$, there is an injective R-module D_λ such that $E_\lambda = 0 :_{D_\lambda} \mathfrak{a}_\lambda$. Hence $\prod_{\lambda \in \Lambda} E_\lambda = \prod_{\lambda \in \Lambda} \operatorname{Hom}_R(R/\mathfrak{a}_\lambda, D_\lambda)$. Now, let $L = \bigoplus_{\lambda \in \Lambda} R/\mathfrak{a}_\lambda$ and $E = \prod_{\lambda \in \Lambda} D_\lambda$. Thus E is an injective R-module and

$\prod_{\lambda\in\Lambda} E_\lambda$ is a direct summand of the R-module $\mathrm{Hom}_R(L, E)$. Therefore, D is isomorphic to a direct summand of the R-module $\mathrm{Hom}_R(L, E)$.

(iii) $\Rightarrow$ (ii) It is clear, by Lemma 3.5. □

4. CP-injective envelops

In this section, we show that every R-module possesses a unique, up to isomorphism, CP-injective envelope.

Definition 4.1. (i) Let N be an R-module. A CP-essential extension M of N is said to be maximal if there is no proper extension of M which is a CP-essential extension of N.

(ii) Let M be a CP-submodule of a CP-injective R-module D. We say that D is a *minimal CP-injective extension* of M, if there is not any proper CP-injective submodule of D containing M.

Lemma 4.2. *Let N be an R-module and M a CP-essential extension of N. There exists a maximal CP-essential extension C of N containing M.*

Proof. Suppose the contrary is true. By induction on ordinal numbers, we show that for any ordinal β, there is a CP-essential extension C_β of N containing M. Let β be an ordinal and assume that C_α is defined for all $\alpha < \beta$. Assume β is a predecessor $\beta - 1$. Since $C_{\beta-1}$ is not a maximal CP-essential extension of N, there is a proper extension C_β of $C_{\beta-1}$ such that C_β is a CP-essential extension of N. If β is a limit ordinal, then it is routine check that $C_\beta := \bigcup_{\alpha<\beta} C_\alpha$ is a CP-essential extension of N. By Corollary 2.6, there is an extension D of N such that D is CP-injective and it contains N as a CP-submodule. Let β be an ordinal with $|\beta| > |D|$. Then, by Lemma 3.2 the inclusion map $N \hookrightarrow D$ can be extended to a monomorphism $\psi : C_\beta \to D$. Hence $|\beta| \leqslant |C_\beta| \leqslant |D|$, which is a contradiction. □

Lemma 4.3. *Let M and M' be two R-modules and let $f : M \to M'$ be an isomorphism. Let N be a submodule M and $N' = f(N)$.*

- (i) *N is a CP-submodule of M if and only N' is a CP-submodule of M'.*
- (ii) *M is a CP-essential extension of N if and only M' is a CP-essential extension of N'.*
- (iii) *M is a maximal CP-essential extension of N if and only M' is a maximal CP-essential extension of N'.*

Proof. (i) is clear.

(ii) Assume that M is an CP-essential extension of N. By (i), N' is a CP-submodule of M'. Let K' be a submodule of M' such that $K' \cap N' = 0$ and that $(K' + N')/K'$ is a CP-submodule of M'/K'. Let $K = f^{-1}(K')$. Then $K \cap N = 0$, because f is monic. On the other hand, if $f^* : M/K \to M'/K'$ denotes the natural isomorphism induced by f, then (i) yields that $(K + N)/K$ is a CP-submodule of M/K. Note that $f^*((K + N)/K) = (K' + N')/K'$. Thus $K = 0$, and so $K' = 0$. Hence M' is a CP-essential extension of N'.

The converse follows by the symmetry. Note that $f^{-1}: M \to M'$ is an isomorphism with $f^{-1}(N') = N$.

(iii) By the symmetry, it is enough to show the "only if" part. Suppose that M is a maximal CP-essential extension of N. By (ii), M' is a CP-essential extension of N'. Let L' be an extension of M' such that it is a CP-essential extension of N'. By [9, Proposition 1.1], there are an extension L of M and an isomorphism $g: L \to L'$ such that the following diagram commutes:

$$\begin{array}{ccccc} N & \hookrightarrow & M & \hookrightarrow & L \\ \downarrow{\scriptstyle f|_N} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N' & \hookrightarrow & M' & \hookrightarrow & L' \end{array}$$

It follows by (ii), that L is a CP-essential extension of N. Hence $L = M$, by the maximality assumption on M. Therefore $L' = M'$, as required. □

Corollary 4.4. *Let M be a CP-injective R-module and N a CP-submodule of M. There is a submodule D of M which is maximal CP-essential extension of N.*

Proof. By Lemma 4.2, there exists a maximal CP-essential extension L of N. In view of Lemma 3.2, there is a monomorphism $\psi: L \to M$ such that $\psi|_N$ is equal to the inclusion map $N \hookrightarrow M$. Let $D = \psi(L)$. Since $\psi: L \to D$ is an isomorphism, it follows by Lemma 4.3(iii), that D is also a maximal CP-essential extension of N. □

Proposition 4.5. *Suppose that M is an R-module and that D is a maximal CP-essential extension of M. Then D is a CP-injective R-module.*

Proof. In view of Theorem 3.4, it is enough to show that D is a direct summand of every R-module which contains D as a CP-submodule. Let D be a CP-submodule of an R-module L. By Lemma 3.3, there exists a submodule K of L such that $K \cap M = 0$ and that L/K is a CP-essential extension of $(K + M)/K$. We show that L is the direct sum of K and D. First, we show that $K \cap D = 0$. Let $K_1 = K \cap D$. Then $K_1 \cap M = 0$ and since the natural embedding $M \cong (K + M)/K$ in L/K can be factored throw the natural embedding $M \cong (K_1 + M)/K_1$ in D/K_1, it follows that $(K_1 + M)/K_1$ is a CP-submodule of D/K_1. Thus $K_1 = 0$, as required. Now, let $f: D \to (K + D)/K$ denote that the natural isomorphism. Then $f(M) = (K + M)/K$. Thus, it follows by Lemma 4.3(iii) that $(K + D)/K$ is a maximal CP-essential extension of $(K + M)/K$. But L/K is a CP-essential extension of $(K + M)/K$ and $(K + D)/K \subseteq L/K$. Thus $L = K + D$. □

Now, we are ready to prove the main result of this section.

Theorem 4.6. *Let D be an R-module and M a submodule of D. The following are equivalent*:

(i) *D is a maximal CP-essential extension of M.*

(ii) *D is a CP-essential extension of M which is CP-injective.*
(iii) *D is a minimal CP-injective extension of M.*

Proof. (i) $\Rightarrow$ (ii) is clear by Proposition 4.5.

(ii) $\Rightarrow$ (iii) Suppose D' is a submodule of D containing M such that D' is CP-injective. By Corollary 4.4, there exists a submodule D'' of D' which is a maximal CP-essential extension of M. Since D is a CP-essential extension of M, it turns out that $D'' = D$. Hence $D' = D$.

(iii) $\Rightarrow$ (i) By Corollary 4.4, there is a submodule D' of D such that D' is a maximal CP-essential extension of M. Now, the module D' is CP-injective, by Proposition 4.5, thus $D' = D$, by the minimality assumption on D. $\square$

Corollary 4.7. *Let M be an R-module. Then there exists an R-module D satisfying the equivalent conditions* (i), (ii) *and* (iii) *in Theorem* 4.6. *Moreover, if D_1, D_2 are both minimal CP-injective extensions of M and $i : M \to D_1$ and $j : M \to D_2$ denote the related inclusion maps, then there is an isomorphism $\theta : D_1 \to D_2$ such that $\theta i = j$.*

Proof. The existence of a such R-module D follows by Lemma 4.2. Now, assume that the R-modules D_1 and D_2 are minimal CP-injective extensions of M. In view of Lemma 3.2, there exists a monomorphism $\theta : D_1 \to D_2$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xhookrightarrow{\;i\;} & D_1 \\ \big\downarrow{\scriptstyle \mathrm{id}_M} & & \big\downarrow{\scriptstyle \theta} \\ M & \xhookrightarrow{\;j\;} & D_2 \end{array}$$

The module $\theta(D_1)$ is a CP-injective submodule of D_2 that contains M. Hence $\theta(D_1) = D_2$, by the minimality assumption on D_1. This completes the proof. $\square$

Definition 4.8. Let M be an R-module and D an extension of M. If D satisfies one of the equivalent conditions of Theorem 4.6, then D is said to be the *CP-injective envelope* of M.

Let χ denote a class of R-modules. We recall the notion of χ-envelope from [11].

Definition 4.9. Let M be an R-module. An R-module $D \in \chi$ is called χ-envelope of M if there is a homomorphism $\varphi : M \to D$ such that

(i) for any homomorphism $\varphi' : M \to D'$ with $D' \in \chi$, there is a homomorphism $f : D \to D'$ such with $\varphi' = f\varphi$, and
(ii) if a homomorphism $f : D \to D$ is such that $\varphi = f\varphi$, then f must be an automorphism.

By [11, Proposition 1.2.1], the χ-envelope of an R-module is unique up to isomorphism. Now, we present our last result.

Theorem 4.10. *Let χ be the class of all CP-injective R-modules and M an R-module. Let D be a CP-injective envelope of M. Then D is isomorphic to the χ-envelope of M.*

Proof. Let $\varphi : M \to D$ denote the inclusion homomorphism. Let $D' \in \chi$ and $\psi : M \to D'$ be a homomorphism. By Theorem 3.4, there exists a homomorphism $f : D \to D'$ such that $\psi = f\varphi$.

Now, suppose a homomorphism $f : D \to D$ is such that $\varphi = f\varphi$. By Lemma 3.2, the map f is injective. By Lemma 4.3(iii), $f(D)$ is also a maximal CP-essential extension of M. Hence $f(D)$ is CP-injective, by Proposition 4.5. On the other hand, we have $M \subseteq f(D) \subseteq D$. Therefore, by using Theorem 4.6, we deduce that $f(D) = D$, and so f is an automorphism, as required. □

5. Cocyclic modules

In [3], we showed that over a Noetherian ring R an R-module D is pure injective if and only if D is isomorphic to a direct summand of the direct product of a family of Artinian modules. In this section, we intent to provide an analogue characterization for CP-injective modules, by using cocyclic modules instead of Artinian modules. It is known that any Artinian module is pure injective, but it is not the case that every cocyclic module is CP-injective (see Example 5.6). Thus, it is interesting to know when a cocyclic modules is CP-injective. First, we recall some definitions.

Definition 5.1. (i) (See [4, p. 4].) An R-module M is called *cocyclic* if M is isomorphic to a submodule of the injective envelope of a simple module.

(ii) (See [1].) An R-module M is called *subdirectly irreducible* if for any family $\{M_i\}_{i\in I}$ of R-modules and any monomorphism $f : M \to \prod_{i\in I} M_i$, there exists $i \in I$ such that the map $\pi_i f : M \to M_i$ is injective, where $\pi_i : \prod_{i\in I} M_i \to M_i$ denotes the ith projection map.

In the following result, we collect some other conditions that are equivalent to the definition of a cocyclic module.

Proposition 5.2. *Let M be a nonzero R-module. Then the following are equivalent*:

(i) *M is cocyclic.*
(ii) *$E_R(M) = E_R(S)$ where S is a simple module.*
(iii) *The socle of M is simple and M is an essential extension of its socle.*
(iv) *The intersection of all nonzero submodules of M is nonzero.*
(v) *There exists an element $c \in M$ such that for every R-module N and every R-homomorphism $f : M \to N$, it follows that f is injective if and only if $c \notin \ker f$.*
(vi) *The intersection of all nonzero submodules of M is a simple submodule of M.*
(vii) *M is subdirectly irreducible.*

Proof. The equivalence (i) ⇔ (ii) follows by [9, Proposition 2.28]. Also, the equivalences (iv) ⇔ (v) and (iv) ⇔ (vi) are clear.

(i) $\Rightarrow$ (iii) Suppose the simple R-module S is such that M is isomorphic to a submodule of $E_R(S)$. Then M possesses a simple submodule S' such that every nonzero submodule of M contains S'. Hence, the socle of M is simple. On the other hand, M is essential extension of its socle, by [9, Proposition 3.17].

Next, the equivalence (iii) $\Leftrightarrow$ (vi) and the implication (iii) $\Rightarrow$ (ii) both are deduced, by [9, Proposition 3.17].

(vi) $\Rightarrow$ (vii) Consider the family $\{M_i\}_{i\in I}$ of R-modules and a monomorphism $f : M \to \prod_{i\in I} M_i$. For each $i \in I$, let $\pi_i : \prod_{i\in I} M_i \to M_i$ denote the ith projection map. Assume that the simple R-module S is equal to the intersection of all nonzero submodules of M and let x be a nonzero element of S. Since $f(x) \neq 0$, it follows that there is $i \in I$ such that $(\pi_i f)(x) \neq 0$. This implies that $\ker(\pi_i f) = 0$, because otherwise $S \subseteq \ker(\pi_i f)$, which is a contradiction.

Finally, we prove that (vii) implies (iv). Let $\{N_\lambda\}_{\lambda\in\Lambda}$ denote the set of all nonzero submodules of M and let $f : M \to \prod_{\lambda\in\Lambda} M/N_\lambda$ denote the natural homomorphism defined by $x \mapsto (x + N_\lambda)_{\lambda\in\Lambda}$. Denote $\bigcap_{\lambda\in\Lambda} N_\lambda$ by S. If $S = 0$, then f is injective, and so there is $\lambda \in \Lambda$ such that $\pi_\lambda f : M \to M/N_\lambda$ is injective. This implies that $N_\lambda = 0$, which is a contradiction. □

Proposition 5.3. *Let M be an R-module and let $\{N_i\}_{i\in I}$ denote the set of all submodules N of M, such that M/N is cocyclic. Then the natural map $\psi : M \to \prod_{i\in I} M/N_i$ is cyclically pure. In particular, if M is CP-injective then M is isomorphic to a direct summand of $\prod_{i\in I} M/N_i$.*

Proof. Let $L = \prod_{i\in I} M/N_i$ and for each $i \in I$ let $\pi_i : L \to M/N_i$ denote the ith natural projection map. Define $\psi : M \to L$ by $x \mapsto (\pi_i(x))_i$. We show that ψ is a CP-homomorphism. To this end, let $\mathfrak{a}$ be an ideal of R and consider the following commutative diagram in which all maps are natural ones.

$$\begin{array}{ccc} M \otimes_R R/\mathfrak{a} & \xrightarrow{\psi\otimes \mathrm{id}_{R/\mathfrak{a}}} & L \otimes_R R/\mathfrak{a} \\ \downarrow \cong & & \downarrow \\ M/\mathfrak{a}M & \xrightarrow{\theta} & \prod_{i\in I} M/(\mathfrak{a}M + N_i) \end{array}$$

It suffices to show that the bottom map is injective. Let $\alpha = x + \mathfrak{a}M$ be a nonzero element of $M/\mathfrak{a}M$. Using Zorn's lemma, we deduce that there is a submodule N of M such that $\mathfrak{a}M \subseteq N$ and $x \notin N$, but x belongs to any submodule of M which strictly contains N. Now, by Proposition 5.2, it turns out that M/N is cocyclic. So, there is $j \in I$ such that $N = N_j$. Since $\pi_j(x) \neq 0$, it follows that θ is monomorphism, as required. □

Definition 5.4. (See e.g. [7].) A Noetherian local ring $(R, \mathfrak{m})$ is called *quasi-complete* if for any decreasing sequence $\{\mathfrak{a}_i\}_{i\in I}$ of ideals of R and any $n \geqslant 0$, there exists $i \in I$ such that $\mathfrak{a}_i \subseteq (\bigcap_{i\in I} \mathfrak{a}_i) + \mathfrak{m}^n$.

Now, we are ready to present our last characterization of CP-injective modules.

Theorem 5.5. *Let $(R, \mathfrak{m})$ be a quasi-complete local ring. An R-module D is CP-injective if and only if D is isomorphic to a direct summand of the direct product of a family of cocyclic modules.*

Proof. Let $E = E_R(R/\mathfrak{m})$. By [7, Remark 3.2], every cocyclic R-module has the form $0:_E \mathfrak{a}$ for some ideal $\mathfrak{a}$ of R. Thus, by Lemma 2.4 every cocyclic R-module is CP-injective. Now, the conclusion follows by Lemma 2.3 and Proposition 5.3. □

Example 5.6. By [6, Theorem 6], a Prüfer domain R is locally almost maximal if and only if every cocyclic R-module is pure injective. On the other hand, by [2, Example 2.4] there exists a valuation domain R such that R is not almost maximal. Hence cocyclic modules are not CP-injective in general, and so the converse of Proposition 5.3 is not true.

In [6], M. Hochster investigated the structure of Noetherian rings R with the property that R is pure in each CP-extension of R. Let $(R, \mathfrak{m})$ be a Noetherian local ring, he defined a finitely generated R-module M to have *small cofinite irreducibles* if for every $n \in \mathbb{N}$ there is an irreducible submodule Q of M such that $Q \subseteq \mathfrak{m}^n M$ and M/Q is Artinian. He showed that a Noetherian ring R is pure in each CP-extension of R if and only if $R_\mathfrak{m}$ has small cofinite irreducibles for all maximal ideals $\mathfrak{m}$ of R. In this section, we will prove that over a local Noetherian ring R every finitely generated R-module M that has small cofinite irreducibles is pure in every CP-extension of M. As a result, we deduce that over a complete local ring every finitely generated module which has small cofinite irreducibles is CP-injective.

Lemma 5.7. *Let R be a Noetherian ring and D a finitely generated cocyclic R-module. Then D is CP-injective.*

Proof. There is a maximal ideal $\mathfrak{m}$ of R such that D is isomorphic to a submodule of $E := E_R(R/\mathfrak{m})$. Then it is easy to see that the natural map $D \to D_\mathfrak{m}$ is an isomorphism. Also, one can check easily that, if D is CP-injective as an $R_\mathfrak{m}$-module then it is also CP-injective as an R-module. So, we may and do assume that R is local with the maximal ideal $\mathfrak{m}$.

Let $\hat{R}$ denote the completion of R with respect to $\mathfrak{m}$-adic topology. Each element of $E_R(R/\mathfrak{m})$ is annihilated by some power of $\mathfrak{m}$. Hence $E_R(R/\mathfrak{m})$ has a natural structure as an $\hat{R}$-module. Note that, if we regard this $\hat{R}$-module as an R-module by means of the natural ring homomorphism $R \to \hat{R}$, then we recover the original R-module structure on $E_R(R/\mathfrak{m})$. Note also that a subset of $E_R(R/\mathfrak{m})$ is an R-submodule if and only if it is an $\hat{R}$-submodule. Set $\mathfrak{a} := \operatorname{Ann}_R D$ and $E := E_R(R/\mathfrak{m})$. Since, D is finitely generated, it turns out that $\operatorname{Ann}_{\hat{R}} D = \mathfrak{a}\hat{R}$. Therefore, by [9, Corollary, p. 154] we have

$$D = (0:_E \operatorname{Ann}_{\hat{R}} D) = (0:_E \mathfrak{a}).$$

Thus the claim follows, by Lemma 2.4. □

Theorem 5.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. If M has small cofinite irreducibles, then M is pure in every CP-extension of M.*

Proof. Let $\{N_i\}_{i\in I}$ denote the set of all submodules N of M, such that M/N is cocyclic. Let L and $\psi : M \to L$ be as in the proof of Proposition 5.3. Let C be a CP-extension of M and let $i : M \hookrightarrow C$ denote the inclusion map. From Lemma 5.7, it follows that L is CP-injective, and so there is a homomorphism $f : C \to L$ such that $fi = \psi$. Therefore, to prove M is pure in C, it suffices to show that ψ is pure. So, we are going to show that for any finitely generated R-module N, the induced map

$$\psi \otimes \mathrm{id}_N : M \otimes_R N \to L \otimes_R N$$

is injective. Assume there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n N = 0$. Then, since M has small cofinite irreducibles, there is an irreducible submodule Q_0 of M such that $Q_0 \subseteq \mathfrak{m}^n M$ and M/Q_0 is Artinian. Then there is $j \in I$ such that $Q_0 = N_j$. For each $i \in I$, let $\pi_i : M \to M/N_i$ denote the natural epimorphism. Now, because the modules $M \otimes_R N$ and $M/N_j \otimes_R N$ are naturally isomorphic, it turns out that $\pi_j \otimes \mathrm{id}_N$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\psi\otimes \mathrm{id}_N} & L \otimes_R N \\ \Big\downarrow{\scriptstyle \mathrm{id}_{M\otimes_R N}} & & \Big\downarrow{\scriptstyle \cong} \\ M \otimes_R N & \xrightarrow{\prod(\pi_i\otimes \mathrm{id}_N)} & \prod_{i\in I}(M/N_i \otimes_R N). \end{array}$$

Hence $\psi \otimes \mathrm{id}_N$ is injective.

Next, assume that N is an arbitrary finitely generated R-module. Suppose that $\ker(\psi \otimes \mathrm{id}_N)$ contains a nonzero element x. Set $K = M \otimes_R N$. Since $\bigcap_{i\in\mathbb{N}} \mathfrak{m}^i K = 0$, it follows that there is $n \in \mathbb{N}$ such that $x \notin \mathfrak{m}^n K$. Set $\bar{N} = N/\mathfrak{m}^n N$ and let $\pi : N \to \bar{N}$ denote the natural epimorphism. Because the modules $K/\mathfrak{m}^n K$ and $M \otimes_R \bar{N}$ are naturally isomorphic, it turns out that the element $(\mathrm{id}_M \otimes \pi)(x)$ of the module $M \otimes_R \bar{N}$ is nonzero. From the commutative diagram

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\mathrm{id}_M\otimes\pi} & M \otimes_R \bar{N} \\ \Big\downarrow{\scriptstyle \psi\otimes\mathrm{id}_N} & & \Big\downarrow{\scriptstyle \psi\otimes\mathrm{id}_{\bar{N}}} \\ L \otimes_R N & \xrightarrow{\mathrm{id}_L\otimes\pi} & L \otimes_R \bar{N} \end{array}$$

we deduce that $\psi \otimes \mathrm{id}_{\bar{N}}$ is not injective, which is a contradiction in view of the first paragraph of the proof. □

Corollary 5.9. *Let $(R,\mathfrak{m})$ be a Noetherian complete local ring and M a finitely generated R-module. If M has small cofinite irreducibles, then M is CP-injective.*

Proof. By Theorem 5.8, M is pure in every CP-extension of M. Thus by [7, Theorem 3.3], M is CP-injective. □

Remark 5.10. (i) Let R be a field. Clearly, every monomorphism is split and so it is pure. We show that $M = R \oplus R$ does not have small cofinite irreducibles. Suppose the contrary is true. Then there is an irreducible submodule Q of M such that $Q \subseteq 0M = 0$. That is the zero submodule of M is irreducible. Therefore we achieved at a contradiction. This shows that the converse of Theorem 5.8 and Corollary 5.9 do not hold. Thus one may consider these results as generalizations of [6] and [7, Corollary 3.4], respectively.

(ii) It might be interesting to know when the converse of the last part of Proposition 5.3 holds. Clearly, this is the case when every cocyclic R-module is CP-injective. By Theorem 3.4 and Proposition 5.2, it is easy to see that if every cocyclic R-module is CP-injective, then the only CP-submodules of a cocyclic R-module are the trivial ones. Is the converse true?

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