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J. Differential Equations 244 (2008) 630–648

Journal of Differential Equations

www.elsevier.com/locate/jde

Studying discrete dynamical systems through differential equations

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Received 5 February 2007; revised 8 October 2007

Abstract

In this paper we consider dynamical systems generated by a diffeomorphism F defined on $\mathcal{U}$ an open subset of $\mathbb{R}^n$, and give conditions over F which imply that their dynamics can be understood by studying the flow of an associated differential equation, $\dot{x} = X(x)$, also defined on $\mathcal{U}$. In particular the case where F has $n-1$ functionally independent first integrals is considered. In this case X is constructed by imposing that it shares with F the same set of first integrals and that the functional equation $\mu(F(x)) = \det(DF(x))\mu(x)$, $x \in \mathcal{U}$, has some non-zero solution, μ. Several examples for $n = 2, 3$ are presented, most of them coming from several well-known difference equations.

MSC: 37C05; 37C27; 37E10; 39A20

Keywords: Conjugation of flows; Lie symmetries; Integrable vector fields; Integrable mappings; Difference equations

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.10.013

1. Introduction

In [17], Zeeman considered the two-dimensional map

$$F(x,y)=\left(y,\frac{a+y}{x}\right)\quad\text{with } a>0,$$

which is a diffeomorphism of $\mathcal{U}=\{(x,y)\in\mathbb{R}^2\colon\ x>0,\ y>0\}$. The study of this map is motivated from the study of the well-known Lyness difference equation $x_{k+2}=(a+x_{k+1})/x_k$. The map F has a unique fixed point in $\mathcal{U}$, and it has a first integral V, whose level curves are topological circles surrounding the unique critical point. By using classical tools of algebraic geometry, he proved that the action of F on each topological circle is conjugated to a rotation of the circle, see [17] and also the paper of Bastien and Rogalski [1]. Zeeman conjectured that the rotation number associated to each of these rotations, thought as a function of the energy levels of V is a monotonic function. In the end of this paper Zeeman says: "The conjecture was initially formulated in an attempt to give a dynamical systems proof of the result about rotations, but it was later bypassed by finding the geometric proof using the cubic nature of V. It might be possible to prove the conjecture by finding a Hamiltonian system of which F was the time-1 map; this would have the advantage of placing F in a larger context." Later, in [3], Beukers and Cushman proved the conjecture about monotonicity introducing the following integrable system:

$$\begin{cases}\dot{x}=-(x+1)\left(y-\dfrac{x+a}{y}\right),\\[2mm] \dot{y}=(y+1)\left(x-\dfrac{y+a}{x}\right).\end{cases}\tag{1}$$

Each energy level $\{V=h\}$ diffeomorphic to $\mathbb{S}^1$ (we write for shortness, $\{V=h\}\cong\mathbb{S}^1$) is a periodic orbit of the above system with period $T(h)$. They show that on each energy level $\{V=h\}\cong\mathbb{S}^1$, there exists a number $\tau(h)$, such that F is the map given by the solution of system (1) which passes through the point at this time $\tau(h)$. They identify the rotation number in each of these level sets as $\rho(h)=\tau(h)/T(h)$ and in order to prove the monotonicity of $\rho(h)$ they use the Abelian integrals–Picard–Fuchs type equation technique.

Motivated by the approach of [3], we follow the advice of Zeeman by placing F in a larger context. Before stating our main results we introduce some notation and definitions.

Along this paper $\mathcal{U}\subset\mathbb{R}^n$ will be an open connected set, F will be a diffeomorphism, defined from $\mathcal{U}$ onto $\mathcal{U}$, and $\dot{x}=X(x)$ will be a $\mathcal{C}^1$-differential equation also defined on $\mathcal{U}$. As usual, we denote by $\varphi(t,p)$ its solution satisfying that $\varphi(0,p)=p$, defined on its maximal interval $I_p\subset\mathbb{R}$. Given $p\in\mathcal{U}$ we write $\gamma_p:=\{\varphi(t,p)\in\mathbb{R}^n,\ t\in I_p\}$ and $\mathcal{O}_p:=\{F^k(p),\ k\in\mathbb{Z}\}$.

Given a diffeomorphism $F:\mathcal{U}\to\mathcal{U}$, we will say that:

- The map F satisfies *condition* X if there exists a $\mathcal{C}^1$-vector field X, also defined in $\mathcal{U}$, such that for any $p\in\mathcal{U}$,

$$X\big(F(p)\big)=\big(DF(p)\big)X(p).$$

  Such a vector field is called a *Lie symmetry* of the map F, see [10].

- The map F satisfies *condition* μ if there exists a smooth map $\mu:\mathcal{U}\to\mathbb{R}$ such that for any $p\in\mathcal{U}$,

$$\mu\big(F(p)\big)=\det\big(DF(p)\big)\mu(p).$$

Notice that condition X says that the differential equation $\dot{x}=X(x)$ is invariant by the change of variables $u=F(x)$. On the other hand, condition μ implies several properties for the dynamical system generated by F. Before listing some of them we recall a well-known fact: any continuous positive function $\nu:\mathcal{W}\to\mathbb{R}^+$ induces an absolute continuous measure m_ν over the open set $\mathcal{W}\subset\mathcal{U}$ defined as $m_\nu(\mathcal{B})=\int_{\mathcal{B}}\nu(p)\,dp$, where $\mathcal{B}\subset\mathcal{W}$ is any Lebesgue measurable set. Recall also that it is said that m_ν is *an invariant measure* for the dynamical system generated by F if for all measurable sets $\mathcal{B}$, $m_\nu(F^{-1}(\mathcal{B}))=m_\nu(\mathcal{B})$. These are the properties:

1. The set $\mathcal{M}^0:=\{p\mid\mu(p)=0\}$ is invariant by F.
2. If $\det(DF(p))>0$ for all p, the sets $\mathcal{M}^+:=\{p\mid\mu(p)>0\}$ and $\mathcal{M}^-:=\{p\mid\mu(p)<0\}$ are also invariant by F and $\nu(p):=\pm\frac{1}{\mu(p)}$ induces, on each of them, an invariant absolutely continuous measure, m_ν, for the dynamical system generated by F.
3. If $\det(DF(p))<0$ for all p, the sets $\mathcal{M}^+$ and $\mathcal{M}^-$ are invariant by F^2 and $\nu(p)$ also induces on them an invariant absolutely continuous measure, m_ν, for the dynamical system generated by F^2.

Property 1 is straightforward and the last two properties are easy consequences of the Change of Variables Theorem. Notice also that some of the above sets $\mathcal{M}^0$, $\mathcal{M}^+$ or $\mathcal{M}^-$ can be empty. Moreover, if we know the existence of an absolutely continuous invariant measure for F, its density gives us a way of checking that F satisfies condition μ.

Observe also that if instead of condition μ, there exists a function μ such that F satisfies the following condition

$$\mu\big(F(p)\big)=-\det\big(DF(p)\big)\mu(p),$$

then the map F^2 satisfies condition μ, see also Lemma 13.

Finally recall that a first integral of the dynamical system generated by a map $F:\mathcal{U}\to\mathcal{U}$ is a non-constant $\mathbb{R}$-valued function V which is constant on the orbits of F. That is, $V(p)=V(F(p))$ for all $p\in\mathcal{U}$. A set $V_1,V_2,\dots,V_\ell$ of first integrals of F is said to be *functionally independent* if the rank of the matrix $(DV)(p)$ is ℓ for almost all $p\in\mathcal{U}$, where $V=(V_1,V_2,\dots,V_\ell)$. Notice that for any $\alpha\in\mathbb{R}^\ell$ the set $\{p\mid V(p)=\alpha\}$ is invariant by F.

Clearly, the maximum number ℓ of functionally independent first integrals for F is n. When $\ell=n$ it can be proved that in most cases there exists some $k\in\mathbb{N}$ such that $F^k=\mathrm{Id}$, see [5]. The case $\ell=n-1$ is sometimes called in the literature *the integrable case*, and will be specially studied in this paper. The values $\alpha\in\mathbb{R}^\ell$ such that there are points in $\{p\mid V(p)=\alpha\}$ where the matrix $(DV)(p)$ has rank smaller than ℓ are called *critical values*. The subset $\mathcal{C}$ of $\mathcal{U}$ formed by the points belonging to level sets given by critical values will be called *critical set*. It is clear that $\mathcal{C}$ is invariant by F and that

$$\mathcal{C}=\bigcup_{\{\alpha\ \text{critical}\}}\big\{p\;\big|\;V(p)=\alpha\big\}.$$

Our first result is

Theorem 1. *Assume that $F:\mathcal{U}\to\mathcal{U}$ is a diffeomorphism satisfying condition X and that an orbit γ_p, solution of the differential equation $\dot{x}=X(x)$, is invariant by F. Then*:

(a) *If γ_p reduces to a singular point of the differential equation then p is a fixed point of F.*
(b) *If γ_p is a periodic orbit of the differential equation then F restricted to γ_p is conjugated to a rotation of the circle. Moreover its rotation number is τ/T, where T is the period of γ_p and τ is defined by the equation $\varphi(\tau,p)=F(p)$.*
(c) *If γ_p is diffeomorphic to the real line then F restricted to γ_p is conjugated to a translation of the line.*

Notice that in the above result, when p is a critical point of X it is not necessarily a fixed point of F. For instance $X(x)\equiv 0$ satisfies condition X and gives no information about F.

For short, in the next three results, when on a regular solution γ_p of $\dot{x}=X(x)$, the map F restricted to it satisfies one of the above last two possibilities, namely (b) or (c), we will say that the dynamics of F restricted to γ_p is *translation-like*.

As a consequence of Theorem 1 we prove

Theorem 2. *Let $F:\mathcal{U}\to\mathcal{U}$ be a diffeomorphism having $n-1$ functionally independent first integrals $V_1,\dots,V_{n-1}$ and satisfying condition μ. Let X_μ be the vector field defined as*

$$X_\mu(x)=\mu(x)\left(-\frac{\partial V_1(x)}{\partial x_2},\frac{\partial V_1(x)}{\partial x_1}\right)\quad \textit{if } n=2,\quad \textit{and} \tag{2}$$

$$X_\mu(x)=\mu(x)\big(\nabla V_1(x)\times\nabla V_2(x)\times\cdots\times\nabla V_{n-1}(x)\big)\quad \textit{if } n>2, \tag{3}$$

where $\times$ means the cross product in $\mathbb{R}^n$. Then for each regular orbit γ_p of $\dot{x}=X_\mu(x)$, such that the number of connected components of $\{x\mid V(x)=V(p)\}$ is $M<\infty$ there is a natural number m, $1\leqslant m\leqslant M$, such that γ_p is invariant by F^m, and the dynamics of F^m restricted to γ_p is translation-like.

Notice that the hypothesis that γ_p is a regular orbit implies that $p\notin\mathcal{M}^0\cup\mathcal{C}$. Observe also that the above result forces that the periodic points of F are either contained in the set of critical points of the associated vector field X or in the periodic orbits of X (which of course have to be contained in the level sets of V diffeomorphic to $\mathbb{S}^1$). Moreover they correspond to the periodic orbits where the rotation number of F is rational. In consequence these type of periodic points never appear isolated.

We also point out that Example 4 of Section 4.1 shows that the bijectivity of F cannot be removed from the hypotheses of the above results.

The above result can be applied to the following family of difference equations

$$x_{k+n}=\frac{R(x_{k+1},x_{k+2},\dots,x_{k+n-1})}{x_k}, \tag{4}$$

where $R:\mathcal{U}\to\mathbb{R}^+$, $\mathcal{U}=\{y\in\mathbb{R}^n\mid y_i>0,\ i=1,2,\dots,n\}$ and $n\geqslant 2$, under the hypothesis that they have $n-1$ functionally independent rational invariants. Several examples for $n=3$ of this situation are presented in [11,14]. We prove

Corollary 3. *Consider $\mathcal{U} = \{x \in \mathbb{R}^n \mid x_i > 0,\ i = 1, 2, \dots, n\}$. Let $F : \mathcal{U} \to \mathcal{U}$ be the map*

$$F(x_1, x_2, \dots, x_n) = \left(x_2, x_3, \dots, x_n, \frac{R(x_2, x_3, \dots, x_n)}{x_1}\right),$$

associated to the difference equation (4) *and assume that it has $n-1$ functionally independent rational first integrals in $\mathcal{U}$. Let γ_p be a regular solution of the differential equation given in* (2) *or* (3)*, according whether $n = 2$ or $n > 2$, with $\mu(x) = x_1 x_2 \dots x_n$. Then there exists $m \in \mathbb{N}$, which is at most the number of connected components of $\{x \mid V(x) = V(p)\}$, such that the dynamics of F^m restricted to γ_p is translation-like. Moreover, when n is odd m has to be even.*

In particular, notice that the above corollary says that when n is odd, all the periodic points of F of odd period have to be contained in the critical set $\mathcal{C}$, which is an algebraic set of measure zero.

Theorem 2 can also be applied to the area preserving maps F, giving

Corollary 4. *Let $F : \mathcal{U} \to \mathcal{U}$, $U \subset \mathbb{R}^n$ be an area preserving map, i.e.* $\det(DF(x)) \equiv 1$*, and assume that it has $n-1$ functionally independent rational first integrals, $V_1, V_2, \dots, V_{n-1}$, in $\mathcal{U}$. Take any smooth function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ and let γ_p be a regular solution of the differential equation given in* (2) *or* (3)*, according whether $n = 2$ or $n > 2$, with $\mu(x) = \Phi(V_1(x), V_2(x), \dots, V_{n-1}(x))$. Then there exists $m \in \mathbb{N}$, which is at most the number of connected components of $\{x \mid V(x) = V(p)\}$, such that the dynamical system generated by F^m restricted to γ_p is translation-like.*

From the proof of the above results we will see that for the maps F satisfying the corresponding hypotheses and when the orbit γ_p associated to X is diffeomorphic to a circle then the rotation number of F^m, for some $m \in \mathbb{N}$, can be studied analytically through X, see the proof of Theorem 1, or [3,6]. Even more, these results open the possibility for the numerical explorations of the rotation numbers by only using the standard numerical methods of integration of ordinary differential equations, see again [6].

This paper is organized as follows: Section 2 is devoted to give some preliminary results. Section 3 includes the proofs of all the results stated in this introduction. Section 4 deals with some concrete applications of our results, mainly to difference equations. In particular, Section 4.1 deals with the case $n = 2$ and includes a revised version of the study of the planar Lyness map, the study of the so-called Gumovski–Mira-type maps and two more examples. Section 4.2 includes two three-dimensional maps: the Todd map, studied much extensively in [6], and an example extracted from [11] and [14]. Finally Section 5 contains some figures.

2. Preliminary results

Next results give some interpretations of the fact that F satisfies condition X.

Lemma 5. (i) *Let $F(x, y)$ be a diffeomorphism from U to U and assume that it satisfies condition X, i.e. $X(F(p)) = (DF(p))X(p)$ for all $p \in U$. Then $I_p = I_{F(p)} = I$ and for all $t \in I$, $F(\varphi(t, p)) = \varphi(t, F(p))$.*

(ii) *Conversely, if for each $p \in \mathcal{U}$ it holds that $F(\varphi(t, p)) = \varphi(t, F(p))$ for $|t|$ small then $X(F(p)) = (DF(p))X(p)$.*

Proof. (i) Consider the change of variables $u = F(x)$. Since F satisfies condition X, the system $\dot{x} = X(x)$ becomes

$$\dot{u} = \big(DF(x)\big)\dot{x} = \big(DF(x)\big)X(x) = X\big(F(x)\big) = X(u).$$

Since a change of coordinates gives us a conjugation of the corresponding flows, we get $F(\varphi(t,p)) = \varphi(t, F(p))$ whenever the equality has sense, i.e., for all $t \in I_p \cap I_{F(p)}$. But since for all $t \in I_p$, $F(\varphi(t,p))$ is well defined and it is equal to $\varphi(t, F(p))$ we get that $I_p \subset I_{F(p)}$. On the other hand since F is a homeomorphism we get that calling $q = F(p)$, $(DF(F^{-1}(q)))(DF^{-1}(q)) = \mathrm{Id}$ and hence $X(F^{-1}(q)) = (DF^{-1}(q))X(q)$, that is, F^{-1} also satisfies condition X. So, $I_{F(p)} \subset I_p$ and the equality holds.

(ii) Taking derivatives with respect to t in $F(\varphi(t,p)) = \varphi(t, F(p))$ and substituting at $t = 0$ we get the desired result. □

From Lemma 5 we see that if F is a diffeomorphism satisfying condition X then F maps orbits of X into orbits of X. Next proposition proves that in the case that an orbit γ of X is invariant by F, then the action of F on γ can be thought as the flow of X at a certain fixed time that only depends on γ.

Proposition 6. *Assume that γ_p is invariant by F and let $\tau(p)$ be defined by $\varphi(\tau(p), p) = F(p)$. Then*

$$\varphi\big(\tau(p), q\big) = F(q) \quad \textit{for all } q \in \gamma_p,$$

if and only if

$$X\big(F(q)\big) = (DF)_q\, X(q) \quad \textit{for all } q \in \gamma_p.$$

Proof. In the following we denote $\tau = \tau(p)$. Let $q \in \gamma_p$, that is, $q = \varphi(t,p)$ for some $t \in I_p$. Then if $\varphi(\tau, q) = F(q)$ for all $q \in \gamma_p$ we have that

$$\varphi\big(\tau, \varphi(t,p)\big) = F\big(\varphi(t,p)\big) \quad \text{for all } t \in I_p.$$

Taking derivatives with respect to t we get

$$X\big(\varphi(t+\tau, p)\big) = \big(DF\big(\varphi(t,p)\big)\big)\, X\big(\varphi(t,p)\big) \quad \text{for all } t \in I_p,$$

which implies that $X(F(q)) = (DF(q))X(q)$ for all $q \in \gamma_p$.

In order to prove the converse observe that from Lemma 5 we have that the function which assigns $F(\varphi(t,p))$ at each $t \in I_p$, is the solution which takes the value $F(p)$ at $t = 0$, that is,

$$\varphi\big(t, F(p)\big) = F\big(\varphi(t,p)\big) \quad \text{for all } t \in I_p.$$

Hence, since $\varphi(\tau, p) = F(p)$, if $q = \varphi(t,p)$ we get that $\varphi(t, \varphi(\tau, p)) = F(q)$ or, equivalently $\varphi(\tau, q) = F(q)$ for all $q \in \gamma_p$, as we wanted to see. □

Remark 7. Notice that by using the above result and Lemma 5 we obtain that if $F(x, y)$ is a diffeomorphism from U to U satisfying condition X, and it holds that $F(p) \in \gamma_p$ then $I_p = I_{F(p)} = (-\infty, \infty)$.

It can occur that for some point $p \in U$, the corresponding orbit solution of $\dot{x} = X(x)$, γ_p is not invariant by F, but γ_p is contained in a set formed by finitely many orbits which is an invariant set by F. To analyze this situation we introduce the following definition: We say that a set of orbits of X, $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$, is a *minimal invariant set of* F if Γ is invariant by F but none of its proper subsets is invariant by F.

Proposition 8. *Assume F is a diffeomorphism in $\mathcal{U}$ and that it satisfies condition X. Let Γ be a set formed by k different orbits of X, which is a minimal invariant set of F. Then each $\gamma \in \Gamma$ is invariant by F^k.*

Proof. From Lemma 5 we know that for all orbits γ, $F(\gamma)$ is also an orbit of X. Fix $\gamma_1 \in \Gamma$. Observe that $F^i(\gamma_1) \neq F^j(\gamma_1)$ for all $1 \leqslant i < j \leqslant k$, because if for some i and j the equality was true, we would get that $\gamma_1 = F^{j-i}(\gamma_1)$ and this would produce and invariant set formed by $j - i$ orbits, in contradiction with the fact that Γ is minimal. This observation allows us to rename the orbits in Γ by setting $\gamma_{i+1} = F^i(\gamma)$, for $i = 1, 2, \dots, k-1$. Furthermore $F(\gamma_k) = \gamma_1$ (otherwise $F(\gamma_k) = \gamma_i$ with $i > 1$, but this would give again a contradiction). Clearly $F(\gamma_k) = \gamma_1$ is equivalent to $F^k(\gamma_1) = \gamma_1$, hence $F^k(\gamma_i) = F^k(F^{i-1}(\gamma_1)) = F^{i-1}(F^k(\gamma_1)) = F^{i-1}(\gamma_1) = \gamma_i$ for all $i = 1, 2, \dots, k$, as we wanted to prove. $\square$

As a corollary we get the following result:

Proposition 9. *Assume that F satisfies condition X and that p is a k-periodic point of F. Then if $X(p) \neq 0$ all the points in $\{\gamma_p, \gamma_{F(p)}, \dots, \gamma_{F^{k-1}(p)}\}$ are k-periodic points of F.*

Proof. Since $F^k(p) = p$, $\Gamma := \{\gamma_p, \gamma_{F(p)}, \dots, \gamma_{F^{k-1}(p)}\}$ is a minimal invariant set. From the above proposition and Proposition 6, all the points in Γ are k-periodic points. $\square$

Recall that the cross product of $n-1$ vectors $W_1, \dots, W_{n-1}$ in $\mathbb{R}^n$, $n \geqslant 3$, is defined as the unique vector $W = W_1 \times \cdots \times W_{n-1} \in \mathbb{R}^n$ such that for all $U \in \mathbb{R}^n$:

$$U \cdot W = \det \begin{pmatrix} U \\ W_1 \\ \vdots \\ W_{n-1} \end{pmatrix},$$

where the dot indicates the usual scalar product, see [16]. It is an easy exercise to check that the expression of W is given by

$$W = W_1 \times \cdots \times W_{n-1} = \sum_{i=1}^{n} (-1)^{1+i} \det\bigl(M_i(W_1, \dots, W_{n-1})\bigr) \frac{\partial}{\partial x_i}, \tag{5}$$

where $M_i(W_1, \dots, W_{n-1})$ is the $(n-1)\times(n-1)$ matrix obtained extracting the ith column of the matrix

$$\begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \end{pmatrix},$$

that is, setting $W_k = (W_{k,1}, W_{k,2}, \dots, W_{k,n})$, the matrix given by

$$M_i(W_1, \dots, W_{n-1}) = \begin{pmatrix} W_{1,1} & W_{1,2} & \dots & \widehat{W_{1,i}} & \dots & W_{1,n-1} \\ W_{2,1} & W_{2,2} & \dots & \widehat{W_{2,i}} & \dots & W_{2,n-1} \\ \vdots & & & \dots & & \vdots \\ W_{n-1,1} & W_{n-1,2} & \dots & \widehat{W_{n-1,i}} & \dots & W_{n-1,n-1} \end{pmatrix}.$$

We will need the following lemmas:

Lemma 10. *Let A be an $n \times n$ matrix, and $W_1, \dots, W_{n-1}$ be vectors of $\mathbb{R}^n$. Then*

$$A\big(A^t W_1 \times \cdots \times A^t W_{n-1}\big) = \det(A)(W_1 \times \cdots \times W_{n-1}).$$

Proof. Set $\tilde{W} = \times_{i=1}^{n} A^t W_i$ and $W = \times_{i=1}^{n} W_i$, then, for each $U \in \mathbb{R}^n$, some simple computations show that

$$U \cdot (A\tilde{W}) = \big(A^t U\big) \cdot \tilde{W} = \det \begin{pmatrix} A^t U \\ A^t W_1 \\ \vdots \\ A^t W_{n-1} \end{pmatrix} = \det \left(\begin{pmatrix} U \\ W_1 \\ \vdots \\ W_{n-1} \end{pmatrix} \cdot A \right)$$

$$= \det(A) \det \begin{pmatrix} U \\ W_1 \\ \vdots \\ W_{n-1} \end{pmatrix} = \det(A)(U \cdot W). \qquad \square$$

Notice that in the next lemma, since the first integrals are functionally independent, they can be labeled so that condition (6) is satisfied in a neighborhood of most points of $\mathcal{U}$.

Lemma 11. *Let $F : U \to U$ be a diffeomorphism where U is an open connected set $U \subset \mathbb{R}^n$ and assume that F has $n-1$ functionally independent first integrals $V_1, \dots, V_{n-1}$. Assume that*

$$\det \begin{pmatrix} V_{1,1} & \dots & V_{1,n-1} \\ \vdots & & \vdots \\ V_{n-1,1} & \dots & V_{n-1,n-1} \end{pmatrix} (p) \neq 0 \tag{6}$$

for all $p \in \mathcal{W} \subset \mathcal{U}$ ($\mathcal{W}$ an open set), where $V_{i,j} = \frac{\partial V_i}{\partial x_j}$. Then any vector field in $\mathcal{W}$ sharing with F the same set of functionally independent first integrals writes as:

$$X_\mu(p) = \mu(p)(-V_{1,y}, V_{1,x}) \quad \textit{if } n = 2, \quad \textit{and}$$
$$X_\mu(p) = \mu(p)\big(\nabla V_1(p) \times \nabla V_2(p) \times \cdots \times \nabla V_{n-1}(p)\big) \quad \textit{if } n > 2,$$

where $\mu : \mathcal{W} \to \mathbb{R}$ is an arbitrary smooth function.

Proof. Set $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$. Recall that if X has V_k as a first integral then $X(V_k) = 0$. Thus X must satisfy the system of equations

$$(X_1 V_{k,1} + X_2 V_{k,2} + \cdots + X_n V_{k,n})(p) = 0, \quad \text{for each } k = 1, \dots, n-1.$$

Since condition (6) is satisfied in $\mathcal{W}$, solving this system of equations by Cramer's method we have

$$X_i(p) = \frac{X_n(p)}{\det(V(p))} \det \begin{pmatrix} V_{1,1} & \dots & \overbrace{-V_{1,n}}^{i\text{th column}} & \dots & V_{1,n-1} \\ \vdots & & \vdots & & \vdots \\ V_{n-1,1} & \dots & -V_{n-1,n} & \dots & V_{n-1,n-1} \end{pmatrix}(p) \quad \text{for } i = 1, \dots, n-1,$$

where

$$V(p) = \begin{pmatrix} V_{1,1} & \dots & V_{1,n-1} \\ \vdots & & \vdots \\ V_{n-1,1} & \dots & V_{n-1,n-1} \end{pmatrix}(p).$$

Setting $X_n(p) = \mu(p)\det(V(p))$, we have that for each $i = 1, \dots, n-1$,

$$\begin{aligned} X_i(p) &= \mu(p) \det \begin{pmatrix} V_{1,1} & \dots & \overbrace{-V_{1,n}}^{i\text{th column}} & \dots & V_{1,n-1} \\ \vdots & & \vdots & & \vdots \\ V_{n-1,1} & \dots & -V_{n-1,n} & \dots & V_{n-1,n-1} \end{pmatrix}(p) \\ &= \mu(p)(-1)^{1+i} \det \begin{pmatrix} V_{1,1} & V_{1,2} & \dots & \widehat{V_{1,i}} & \dots & V_{1,n} \\ \vdots & & & \dots & & \vdots \\ V_{n-1,1} & V_{n-1,2} & \dots & \widehat{V_{n-1,i}} & \dots & V_{n-1,n} \end{pmatrix}(p) \\ &= (-1)^{1+i} \det\big(M_i(\nabla V_1, \dots, \nabla V_{n-1})\big)(p). \end{aligned}$$

Therefore from expression (5), we have that $X(p) = \mu(p)(\nabla V_1 \times \cdots \times \nabla V_{n-1})(p)$, and the statement is proved. □

By using Lemma 11, next result gives a large family of vector fields preserving the foliation induced by the first integrals of F. It also gives an easy way of checking whether some of these vector fields X are such that F satisfies condition X.

Theorem 12. *Let $F : U \to U$ be a diffeomorphism where U is an open connected set $U \subset \mathbb{R}^n$ and assume that F has $n-1$ functionally independent first integrals $V_1, \dots, V_{n-1}$. Then the following statements hold*:

(i) *For any smooth function $\mu:\mathcal{U}\to\mathbb{R}$, the family of vector fields*

$$X_\mu(p)=\mu(p)(-V_{1,y},V_{1,x})\quad \textit{if } n=2,\quad \textit{and}$$
$$X_\mu(p)=\mu(p)\big(\nabla V_1(p)\times\nabla V_2(p)\times\cdots\times\nabla V_{n-1}(p)\big)\quad \textit{if } n>2,$$

shares with F the same set of functionally independent first integrals.

(ii) *The map F satisfies condition X_μ if and only if the map F satisfies condition μ.*

Proof. (i) Obvious from the definition of the cross product of the vectors ∇V_i.

(ii) Set $Y(p)=(\nabla V_1\times\cdots\times\nabla V_{n-1})(p)$, so that $X(p)=\mu(p)Y(p)$. We have to prove that for all $p\in U$,

$$\text{(a)}\quad \mu\big(F(p)\big)=\det\big(\big(DF(p)\big)\big)\mu(p)\quad\Leftrightarrow\quad \text{(b)}\quad X\big(F(p)\big)=\big(DF(p)\big)X(p).$$

Prior to proving this, observe that $V_i(F(p))=V(p)$ for all $i=1,\dots,n-1$. Then we have that $\nabla V_i(F(p))^t(DF(p))=\nabla V_i(p)^t$ and so

$$\nabla V_i(p)=\big(\nabla V_i\big(F(p)\big)^t\big(DF(p)\big)\big)^t=\big(DF(p)\big)^t\nabla V\big(F(p)\big).$$

Hence, we can write $Y(p)=((DF(p))^t\nabla V_1(F(p))\times\cdots\times(DF)_p^t\nabla V_{n-1}(F(p)))$.

Using Lemma 10 we obtain

$$\begin{aligned}\big(DF(p)\big)Y(p)&=(DF)_p\big((DF)_p^t\nabla V_1\big(F(p)\big)\times\cdots\times\big(DF(p)\big)^t\nabla V_{n-1}\big(F(p)\big)\big)\\&=\det\big(\big(DF(p)\big)\big)\big(\nabla V_1\big(F(p)\big)\times\cdots\times\nabla V_{n-1}\big(F(p)\big)\big)\\&=\det\big(\big(DF(p)\big)\big)Y\big(F(p)\big).\end{aligned}$$

So we have proved the following identity

$$\big(DF(p)\big)Y(p)=\det\big(\big(DF(p)\big)\big)Y\big(F(p)\big).\tag{7}$$

Assume that condition (a) holds. Using Eq. (7) we have

$$\begin{aligned}X\big(F(p)\big)&=\mu\big(F(p)\big)Y\big(F(p)\big)=\det\big(\big(DF(p)\big)\big)\mu(p)Y\big(F(p)\big)\\&=\big(DF(p)\big)\mu(p)Y(p)=\big(DF(p)\big)X(p),\end{aligned}$$

hence condition (a) implies condition (b).

Assume now that (b) holds. Then we have that $\mu(F(p))Y(F(p))=(DF(p))\mu(p)Y(p)$. From Eq. (7), and since F is a diffeomorphism, we have

$$Y(p)=\det\big(\big(DF(p)\big)\big)(DF)_p^{-1}Y\big(F(p)\big),$$

and so

$$\begin{aligned}\mu\big(F(p)\big)Y\big(F(p)\big)&=\big(DF(p)\big)\mu(p)\det\big(\big(DF(p)\big)\big)\big(DF(p)\big)^{-1}Y\big(F(p)\big)\\&=\mu(q)\det\big(\big(DF(p)\big)\big)Y\big(F(p)\big).\end{aligned}$$

Hence $\mu(F(p)) = \det((DF(p)))\mu(p)$, and thus condition (b) implies condition (a), and the proof of statement (ii) is completed. $\square$

In the previous theorem we have seen the role of condition μ for maps F having $n-1$ functionally independent first integrals. Next lemma gives some properties that can help to find solutions of the functional equations $\mu(F(p)) = \pm\det((DF(p)))\mu(p)$. Its proof is straightforward.

Lemma 13. *Fix a diffeomorphism F from $\mathcal{U}$ into itself, and consider the two sets of continuous functions $\Sigma_F^{\pm} := \{\mu : \mathcal{U} \to \mathbb{R} \mid \mu(F(x)) = \pm\det(DF(x))\mu(x)\}$. Then*:

1. *The spaces $\Sigma_F^{\pm}$ are vectorial spaces.*
2. *If $\mu \in \Sigma_F^{+}$ then $\mu \in \Sigma_{F^k}^{+}$ for any $k \in \mathbb{N}$.*
3. *If $\mu \in \Sigma_F^{-}$ then $\mu \in \Sigma_{F^{2k}}^{+}$ for any $k \in \mathbb{N}$.*
4. *If $\mu, \nu \in \Sigma_F^{+}$ then $\mu^{\ell}\nu^{1-\ell} \in \Sigma_F^{+}$ for any $\ell \in \mathbb{Z}$.*
5. *If $\mu \in \Sigma_F^{+}$ and $\nu \in \Sigma_F^{-}$ then $\mu^{\ell}\nu^{1-\ell} \in \Sigma_{F^{2k}}^{+}$ for any $k \in \mathbb{N}$ and any $\ell \in \mathbb{Z}$.*
6. *If $\mu \in \Sigma_F^{\pm}$ and V is a first integral of F then $\mu \cdot V \in \Sigma_F^{\pm}$.*
7. *If $\det(F(x)) \equiv 1$ then Σ_F^{+} is the set of first integrals of F plus the constant functions.*

3. Proof of the main results

Proof of Theorem 1. (a) It is trivial.

(b) If $\gamma_p \cong \mathbb{S}^1$, then γ_p is a periodic orbit of $\dot{x} = X(x)$. Let $T(p)$ be the minimal period of γ_p. By Proposition 6 we know that there exists $\tau(p)$ such that $\varphi(\tau(p), q) = F(q)$ for all $q \in \gamma_p$. We are going to prove that the restriction of F to γ_p is conjugated to a rotation of the circle with rotation number $\rho(p) = \tau(p)/T(p)$. Let $H : \mathbb{S}^1 \to \gamma_p$ and $R_\tau : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $H(e^{it}) = \varphi(\frac{T(p)}{2\pi}t, p)$ and $R_\tau(e^{it}) = e^{i(t + 2\pi\frac{\tau(p)}{T(p)})}$.

Then H is clearly exhaustive and if $H(e^{it}) = H(e^{is})$, then $p = \varphi(\frac{T(p)}{2\pi}(t-s), p)$ and hence, $\frac{T(p)}{2\pi}(t-s) = kT(p)$ for a certain $k \in \mathbb{Z}$. But then $t - s = 2k\pi$ and $e^{it} = e^{is}$ in $\mathbb{S}^1$. So, it is a diffeomorphism.

To see that H is the desired conjugation we have to show that

$$F \circ H = H \circ R_\tau.$$

As we can see, by using Lemma 5, the computation is straightforward. On the one hand,

$$(F \circ H)\big(e^{it}\big) = F\left(\varphi\left(\frac{T(p)}{2\pi}t, p\right)\right) = \varphi\left(\tau(p) + \frac{T(p)}{2\pi}t, p\right),$$

and on the other one

$$(H \circ R_\tau)\big(e^{it}\big) = H\left(e^{i(t+2\pi\frac{\tau(p)}{T(p)})}\right) = \varphi\left(\frac{T(p)}{2\pi}\left(t + \frac{2\pi\tau(p)}{T(p)}\right), p\right) = \varphi\left(\tau(p) + \frac{T(p)}{2\pi}t, p\right).$$

In order to prove (c), assume that $\tau(p) > 0$ (the case $\tau(p) < 0$ follows in a similar way). By Remark 7 we know that $I_p = (-\infty, +\infty)$.

Let

$$H:(-\infty,+\infty)\to\gamma_p \quad \text{and} \quad T_\tau:(-\infty,+\infty)\to(-\infty,+\infty)$$

be defined by $H(t)=\varphi(t,p)$ and $T_\tau(t)=t+\tau(p)$.

Then $H(t)=H(s)$ implies $\varphi(t-s,p)=p$ and since γ_p is not a periodic orbit we get that $t-s=0$. On the other hand,

$$(H\circ T_\tau)(t)=H\big(t+\tau(p)\big)=\varphi\big(t+\tau(p),p\big)=\varphi\big(\tau(p),\varphi(t,p)\big)=F\big(\varphi(t,p)\big)=F\big(H(t)\big),$$

which proves the assertion. □

Proof of Theorem 2. By using Theorem 12 we know that the vector field X_μ introduced in the statement of the theorem is such that F satisfies condition X_μ. Notice also that the points $p\in\mathcal{M}^0\cup\mathcal{C}$ are critical points of X_μ. Let $p\in\mathcal{U}$ be a regular point of X_μ. It is clear that $p\notin\mathcal{C}$, that $\{x\mid V(x)=V(p)\}$ is invariant by F and has at most M connected components, one of them being γ_p. By Proposition 8 we get that for some $m\leqslant M$, the orbit γ_p is invariant by F^m. By using Theorem 1 we also know that F^m restricted to γ_p is translation-like, as we wanted to prove. □

Proof of Corollary 3. Note that for the map

$$F(x_1,x_2,\dots,x_n)=\left(x_2,x_3,\dots,x_n,\frac{R(x_2,x_3,\dots,x_n)}{x_1}\right),$$

the following equality holds

$$\mu\big(F(x)\big)=(-1)^n\det\big(DF(x)\big)\mu(x),$$

where $\mu(x)=x_1x_2\dots x_n$. Hence when n is even, F satisfies condition μ. On the other hand, when n is odd, by using Lemma 13, we know that F^2 satisfies condition μ. In both cases, by taking $p\in\mathcal{U}$ a regular point for X_μ, the fact that the first integrals of F are rational forces the finiteness of connected components of $\{x\mid V(x)=V(p)\}$. Hence by using Theorem 2 the result follows. □

Proof of Corollary 4. The proof of this result is similar to the proof of the previous corollary. The difference is that since F is a jacobian map $\mu(F(p))=\mu(p)$, so we can take as μ any first integral of F. Hence we can consider $\Phi(V_1(x),V_2(x),\dots,V_{n-1}(x))$, for any smooth function $\Phi:\mathbb{R}^{n-1}\to\mathbb{R}$. □

4. Examples

4.1. Two-dimensional examples

Example 1 (*Lyness map*). As a first example we will see that our approach can be applied to study the Lyness recurrence. Recall that its associated map is

$$F(x,y)=\left(y,\frac{a+y}{x}\right) \quad \text{with } a>0,$$

defined on $\mathcal{U} = \{(x, y) \in \mathbb{R}^2\colon\ x > 0,\ y > 0\}$. Let us check that it is in the hypotheses of Corollary 3.

It is clear that F is a diffeomorphism from $\mathcal{U}$ to $\mathcal{U}$ with inverse $F^{-1}(u, v) = ((a + u)/v, u)$. It is also known that F has the function

$$V(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy}$$

as a first integral, that the point $p_c = (x_c, x_c)$ with $x_c = \frac{1+\sqrt{1+4a}}{2}$ is a fixed point of F and that the level curves $\{V(x, y) = \alpha\} \cap \mathcal{U}$ are diffeomorphic to circles for all $\alpha > V(x_c, x_c)$. Since $n = 2$, consider the vector field given in Corollary 3, $X = xy(-\frac{\partial V}{\partial y}, \frac{\partial V}{\partial x})$. We obtain

$$X(x, y) = -(x + 1)\left(y - \frac{x + a}{y}\right)\frac{\partial}{\partial x} + (y + 1)\left(x - \frac{y + a}{x}\right)\frac{\partial}{\partial y}.$$

Of course, it coincides with the one introduced by Beukers and Cushman in [3], see (1), because as we have already said that paper was our motivating example. Since $\nabla V \neq 0$ in $\mathcal{U} \setminus \{p_c\}$ we get that all the orbits γ_p of the differential equation associated to the above vector field, except p_c of course, are periodic orbits. Thus Corollary 3 implies that F restricted to each of these orbits is conjugated to a rotation.

Example 2 (*Gumovski–Mira-type maps*). Consider the family of jacobian maps

$$F_{\{A,B,C\}}(x, y) = \left(y, -x + \frac{By + C}{y^2 + A}\right),$$

where A, B and C are real constants. It is easy to check that the family of functions

$$V_{\{A,B,C\}}(x, y) = x^2y^2 + A\left(x^2 + y^2\right) - Bxy - C(x + y)$$

is invariant under the action of $F_{\{A,B,C\}}$. By Corollary 4, we can consider the vector field

$$X_{\{A,B,C\}}(x, y) = V_{\{A,B,C\}}(x, y)\left(-\frac{\partial V_{\{A,B,C\}}}{\partial y}, \frac{\partial V_{\{A,B,C\}}}{\partial x}\right).$$

A particular subfamily of the above type of maps, are the Gumovski–Mira maps [9]:

$$F_{\{1,\beta,\alpha\}}(x, y) = \left(y, -x + \frac{\alpha + \beta y}{1 + y^2}\right).$$

It is easy to check that for all α, β these maps are diffeomorphisms from $\mathbb{R}^2$ to $\mathbb{R}^2$.

From [2] and [9] we know when $\alpha = 0$ and $\beta \in (0, 2)$ the level curves of $V_{\{1,\beta,0\}}$ in $\mathbb{R}^2$ are closed curves surrounding the origin, which is the unique fixed point of $F_{\{1,\beta,0\}}$, see Fig. 1. These closed curves are periodic orbits of $X_{\{1,\beta,0\}}$. It is also easy to prove that the only critical point of $X_{\{1,\beta,0\}}$ is the origin and that each level set has at most one connected component. Thus by using Corollary 4, we know that on each of these closed curves $F_{\{1,\beta,0\}}$, with $\beta \in (0, 2)$, is conjugated to a rotation. A detailed study of the rotation number can be found in [2].

When $\beta > 2$ the phase portrait of the vector field $X_{\{1,\beta,0\}}$ is given by two centers surrounded by a polycycle composed by a saddle with a couple of homoclinic trajectories surrounding both centers. The rest of the orbits are given by closed curves surrounding the polycycle, see Fig. 2.

Both centers and the saddle point are fixed points of F. By studying the number of connected components of the level sets of $F_{\{1,\beta,0\}}$ and by continuity arguments we have that all the orbits of $X_{\{1,\beta,0\}}$ are invariant by $F_{\{1,\beta,0\}}$, so again from Corollary 4 we know that $F_{\{1,\beta,0\}}$ is conjugated to a rotation in any of the above mentioned closed curves, and conjugated to a translation in any of the separatrices of the polycycle.

A different situation appears for this other map studied by Gumovski and Mira in [9]:

$$F_{\{-a^2,-2,0\}}(x,y) = \left(y, -x + \frac{-2y}{y^2 - a^2}\right) \quad \text{with } a \neq 1.$$

In this case the map is not defined in the whole $\mathbb{R}^2$. We are going to apply again Corollary 4, but this time to a given open invariant subset of $\mathbb{R}^2$, where the restriction of F is a diffeomorphism. The level curves of the invariant $V_{\{-a^2,-2,0\}}$ have been studied in [9]. The results are showed in Fig. 3. We can see that the origin is a center such that its period annulus is the open bounded set delimited by two heteroclinic orbits which joint the two saddles.

Let U be the period annulus of the origin, i.e.,

$$U = \left\{(x,y) \in \mathbb{R}^2\colon -\sqrt{a^2-1} < x < \sqrt{a^2-1},\ \frac{-ax + a^2 - 1}{x - a} < y < \frac{ax + a^2 - 1}{x - a}\right\}.$$

Observe $F_{\{-a^2,-2,0\}}$ is bijective on U (thus a diffeomorphism). This is because it is globally injective and on U it has a well-defined inverse, $F^{-1}_{\{-a^2,-2,0\}}(u,v) = (\frac{-2u+a^2v-u^2v}{u^2-a^2}, u)$. However $F_{\{-a^2,-2,0\}}$ is not a diffeomorphism on $\mathbb{R}^2 \setminus U$.

The origin is again a fixed point of $F_{\{-a^2,-2,0\}}$, and by continuity arguments it can be seen that the connected components of the closed level sets of $V_{\{-a^2,-2,0\}}$ which contain the closed curves are invariant by $F_{\{-a^2,-2,0\}}$. So by Corollary 4, on each closed curve on the whole period annulus U, $F_{\{-a^2,-2,0\}}$ is conjugated to a rotation of the circle.

We like to point out that since in $\mathbb{R}^2 \setminus U$ the map is no more a diffeomorphism, we cannot apply our results in order to characterize the dynamics of the map. Moreover, under the light of Example 4 below, there is no reason to believe that each one of the connected components of the level sets of $V_{\{-a^2,-2,0\}}$, formed by two solutions of the differential equation associated to $X_{\{-a^2,-2,0\}}$, is invariant by F^m for any $m \geqslant 1$.

Example 3. Consider the recurrence

$$x_{k+2} = \frac{ax_{k+1} + b}{(cx_{k+1} + d)x_k} \quad \text{with } a, b, c, d > 0,$$

studied in [13]. It has an associated map

$$F(x,y) = \left(y, \frac{ay + b}{(cy + d)x}\right),$$

which is a diffeomorphism of the first quadrant $\mathcal{U} := \{x > 0, y > 0\}$.

From [13] we know that it has in $\mathcal{U}$ the first integral

$$V(x,y)=\big[(d+cx)(dx+a)y^2+\big(a^2+bd+x^2\big(ac+d^2\big)\big)y+(dx+a)(ax+b)\big]/(xy),$$

and that its level curves in $\mathcal{U}$ are closed curves surrounding the unique fix point of F in $\mathcal{U}$. By using again Corollary 4 we obtain that F restricted to each level set of V in $\mathcal{U}$ is conjugated to a rotation.

Example 4. Consider the map

$$\tilde{F}(y,z)=\left(-\frac{1+y+z}{z},\frac{1+y+(2-a)z}{z(a+y+z)}\right),$$

with $a\in\mathbb{R}$. It is easy to check that $H(y,z)=\frac{(1+y+z)(y+a-1)}{z}$ is a first integral of $\tilde{F}$, see [4]. Indeed this map is constructed in that paper by noticing that $F^4(-1,y,z)=(-1,\tilde{F}(y,z))$, where F is given in formula (8) of Example 5. It is easy to verify that $\tilde{F}$ satisfies condition X where $X(y,z)=X_1(y,z)\partial/\partial y+X_2(y,z)\partial/\partial z$, being

$$X_1(y,z):=-z(1+z)\frac{\partial H(y,z)}{\partial z}=(y+1)(z+1)(a-1+y)/z,$$
$$X_2(y,z):=z(1+z)\frac{\partial H(y,z)}{\partial y}=(z+1)(a+2y+z).$$

The existence of this vector field is in fact a consequence of the study of Example 5.

On the other hand, it is proved in [4] that most level sets $H_h:=\{(y,z)\mid H(y,z)=h\}$ are hyperbolas and that there are many values of h and $a>0$ for which the orbit of $\tilde{F}$, which of course lies in a fix hyperbola, is dense over it. In fact, in these cases, there is an appropriate compactification, so that $\tilde{F}$ extends to a map which is conjugated to an irrational rotation of the circle (see Proposition 15 and the proof of Proposition 17 of [4] for more details). Therefore, although the union of the two branches of the set H_h is invariant by $\tilde{F}$, there is no any $m\geqslant 1$ such that one of the branches is invariant by $\tilde{F}^m$.

This example shows the importance of the hypothesis that F is a diffeomorphism in Theorems 1 and 2.

4.2. *Three-dimensional examples*

Example 5 (*Todd recurrence*). Consider the third order Lyness recurrence

$$x_{k+3}=\frac{a+x_{k+1}+x_{k+2}}{x_k}\quad\text{with } a>0,\ x_1>0,\ x_2>0,\ x_3>0,$$

also called Todd recurrence, see [4,6,8,12]. It has the associated map

$$F(x,y,z)=\left(y,z,\frac{a+y+z}{x}\right),\tag{8}$$

defined in $\mathcal{U} = \{(x, y, z) \in \mathbb{R}^3\colon\ x > 0,\ y > 0,\ z > 0\}$. This map is a diffeomorphism from $\mathcal{U}$ into itself. From [6,7], it is known that it has a couple of functionally independent first integrals of F, given by

$$V_1(x, y, z) = \frac{(x+1)(y+1)(z+1)(a+x+y+z)}{xyz},$$
$$V_2(x, y, z) = \frac{(1+y+z)(1+x+y)(a+x+y+z+xz)}{xyz}.$$

In order to use Corollary 3 we have to consider the vector field

$$X(x, y, z) = xyz \cdot \big(\nabla V_1(x, y, z) \times \nabla V_2(x, y, z)\big) = \sum_{i=1}^{3} X_i(x, y, z) \frac{\partial}{\partial x_i},$$

defined by

$$X_1(x, y, z) := (x+1)(1+y+z)(yz-x-y-a)G(x, y, z)/\big(xy^2z^2\big),$$
$$X_2(x, y, z) := (y+1)(z-x)(a+x+y+z+xz)G(x, y, z)/\big(x^2yz^2\big),$$
$$X_3(x, y, z) := (z+1)(1+x+y)(y+z+a-xy)G(x, y, z)/\big(x^2y^2z\big),$$

where $G(x, y, z) = -y^3 - (x+a+1+z)y^2 - (a+x+z)y + x^2z^2 + xz + x^2z + xz^2$. It is not difficult to check that their critical points in $\mathcal{U}$ are either the points of the surface $\mathcal{G} := \{(x, y, z) \mid G(x, y, z) = 0\}$ or the points of the line $\mathcal{L} := \{(x, (x+a)/(x-1), x) \mid x > 1\}$. The line $\mathcal{L}$ is formed, precisely, by the fix point of F and the 2-periodic points of F. Moreover in [6] it is proved that the level curves of (V_1, V_2) in $\mathcal{U} \setminus (\mathcal{L} \cup \mathcal{G})$ are formed by two disjoints circles. Thus if for each $p \in \mathcal{U} \setminus (\mathcal{L} \cup \mathcal{G})$ we consider the connected component of the level set of (V_1, V_2) which passes through it, which is diffeomorphic to a circle, we obtain that F^2 restricted to it is conjugated to a rotation. Further discussion on the rotation numbers of F^2 and its periods can be found in [6].

As we have seen in the proof of Corollary 3, the map F^2 satisfies condition μ with $\mu(x, y, z) = xyz$. In [6] it has been proved that F satisfies as well condition $\hat{\mu}$ being $\hat{\mu}(x, y, z) = G(x, y, z)$ the function given above. By using Lemma 13 we obtain that F^2 satisfies condition $\tilde{\mu}$ with $\tilde{\mu} = (x^2y^2z^2)/G(x, y, z)$. This is precisely the factor for $\nabla V_1 \times \nabla V_2$ used in [6] to study the dynamics given by F. Notice that with this new $\tilde{\mu}$ the vector field is $\tilde{X} = \tilde{\mu} \cdot (\nabla V_1 \times \nabla V_2) = \sum_{i=1}^{3} \tilde{X}_i \partial/\partial x_i$, where

$$\tilde{X}_1(x, y, z) := (x+1)(1+y+z)(a+x+y-yz)/(yz),$$
$$\tilde{X}_2(x, y, z) := (y+1)(x-z)(a+x+y+z+xz)/(xz),$$
$$\tilde{X}_3(x, y, z) := (z+1)(1+x+y)(xy-y-a-z)/(xy).$$

Its only critical points in $\mathcal{U}$ are precisely the points of the line $\mathcal{L}$ and it allows to study the dynamics given by F in the whole $\mathcal{U}$ also including the surface $\mathcal{G}$. Hence F^2, and indeed F, is conjugated to a rotation of the circle on the periodic orbits of $\tilde{X}$ that foliate $\mathcal{G}$.

Example 6. In [11], nine third-order difference equations, with a couple of two functionally independent first integrals, have been introduced. In this example we consider the diffeomorphism

$$F(x,y,z)=\left(y,z,\frac{(y+1)(z+1)}{x(1+y+z)}\right),$$

defined in $\mathcal{U}=\{(x,y,z)\in\mathbb{R}^3\colon\ x>0,\ y>0,\ z>0\}$. It corresponds to a very particular case of equation (Y_1) of [11]. From [15], we know that it has the following 2-first integrals (that is first integrals of F^2), given by

$$I_1(x,y,z)=\frac{1+x+y+z+xy+yz+xyz}{xz}\quad\text{and}\quad I_2(x,y,z)=\frac{1+x+z+xy+xz+yz}{y}.$$

Consider the new two first integrals of F,

$$V_1=I_1+I_2\quad\text{and}\quad V_2=I_1I_2,$$

see also [15]. It is not difficult to check that the vector field that appears in Corollary 4, $X(x,y,z)=xyz\cdot(\nabla V_1(x,y,z)\times\nabla V_2(x,y,z))$, has the surface

$$\mathcal{G}=\bigl\{G(x,y,z):=-(1+x)(1+z)y^2+\bigl[x^2z+\bigl(z^2-1\bigr)x-(1+z)\bigr]y\\ +xz(x+1)(z+1)=0\bigr\}$$

and the line $\mathcal{L}=\{x=z;\,xy^2+(x^2-1)-1-x=0\}$, full of critical points. Moreover, as in the previous example, the line $\mathcal{L}$ is filled by the fix points of F^2, and F^2 satisfies condition $\hat{\mu}$ with $\hat{\mu}(x,y,z)=G(x,y,z)$. At this point we can argue similarly than in the previous case to describe the dynamics of F^2.

5. Figures

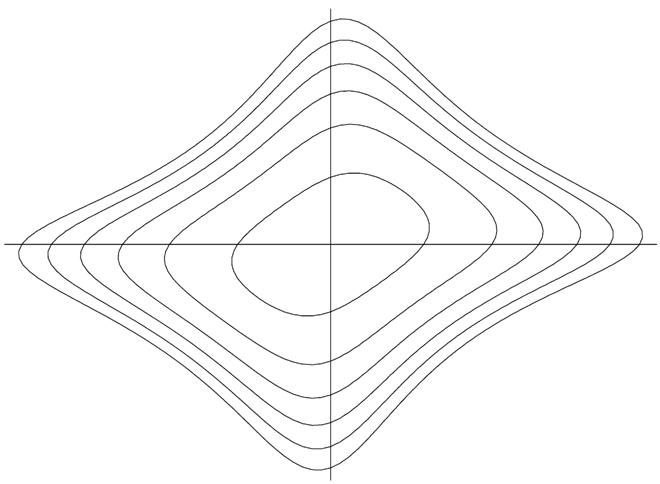

Fig. 1. Level curves of the invariant $V_{\{1,\beta,0\}}$ associated to the Gumovski–Mira map $F_{\{1,\beta,0\}}$ for $\beta\in(0,2)$.

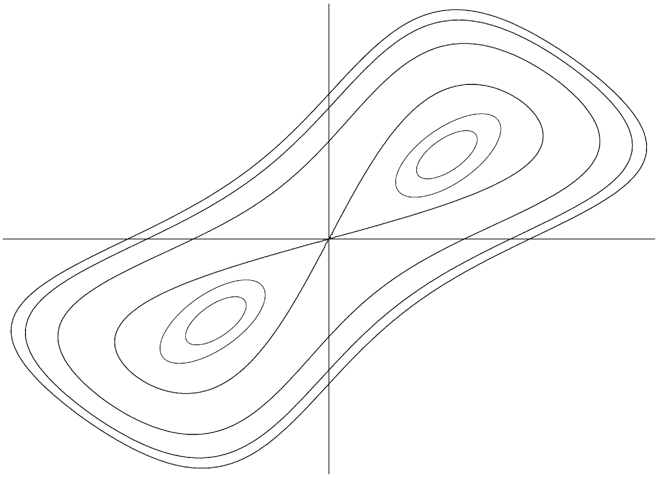

Fig. 2. Level curves of the invariant $V_{\{1,\beta,0\}}$ associated to the Gumovski–Mira map $F_{\{1,\beta,0\}}$ for $\beta > 2$.

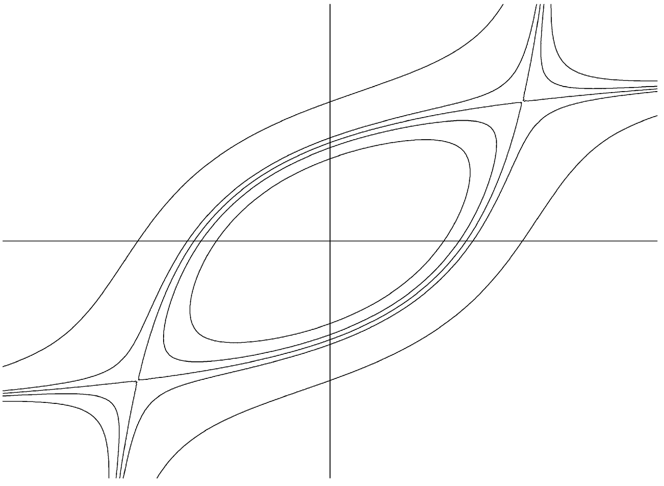

Fig. 3. Level curves of the invariant $V_{\{-a^2,-2,0\}}$ associated to the Gumovski–Mira map $F_{\{-a^2,-2,0\}}$ for $|a| > 1$.

Acknowledgments

The authors are supported by MEC through grants MTM2005-06098-C02-01 (first and second authors) and DPI2005-08-668-C03-1 (third author). Both GSD–UAB and CoDALab groups are supported by the Government of Catalonia through the SGR program.

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