

On Automorphisms of Kac–Moody Algebras and Groups

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INTRODUCTION

Let F be a field of characteristic zero, A a symmetrizable generalized Cartan matrix, $\mathfrak{g} = \mathfrak{g}(A)$ the corresponding Kac–Moody algebra over F , $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, and G the Kac–Moody group associated to \mathfrak{g}' defined over F . In this paper, we develop various aspects of the theory of Kac–Moody Lie algebras and groups related to their automorphisms as suggested by the rich literature on algebraic semisimple groups.

In a series of papers [12–16], Kac and Peterson have laid down the foundation for Kac–Moody groups (see Tits [31, 32] and references there for the earlier work). Overall in this paper we follow their approach to extend results for algebraic semisimple groups to Kac–Moody groups. In Section 1, we recall some of the basic definitions and results about Kac–Moody algebras and groups, and set the notations. We give an exposition of some of the results of [14, 15] and add a few refined results needed for our discussion.

In [7], Bruhat and Tits initiated the study of bounded subgroups for affine Tits systems. Kac and Peterson [16] extended the discussion to Kac–Moody groups. They introduced a transcendental “distance function” to measure the “size” of roots and thus established an even stronger characterization of bounded subgroups. In Section 2, we present an algebraic approach which yields also a Lie algebra analogy. Our main result is Proposition 2.8. The remaining section deals with finite dimensional algebraic subgroups of a Kac–Moody group. We rediscover the sound correspondence between Lie algebras and groups.

Abstract automorphisms of algebraic semisimple groups have been

studied by O. Schreier and van der Waerden for $PSL_n(F)$, by J. Dieudonné, L. K. Hua, C. E. Rickart, and O. T. O'Meara for classical groups, and by Borel and Tits [4] in the general case. In Section 3, we discuss abstract automorphisms of a Kac–Moody group G . Let $Aut_F(\mathfrak{g}')$ denote the group of F -automorphisms of \mathfrak{g}' and $Aut(F)$ the group of automorphisms of F . We know that $Aut_F(\mathfrak{g}') = Out(A) \rtimes Ad(\tilde{H} \rtimes G)$. Hence $Aut_F(\mathfrak{g}')$ acts on G in the obvious manner. We write $Aut(G; \mathfrak{g}')$ for the image of $Aut_F(\mathfrak{g}')$ in the group $Aut(G)$ of abstract automorphisms of G . As G is defined over \mathbb{Q} , $Aut(F)$ acts on $G = G(F)$. It was conjectured in [16] that $Aut(G) = Aut(F) \rtimes Aut(G; \mathfrak{g}')$. Let G_f be the subset of G of $Ad_{\mathfrak{g}}$ -finite elements. We show in Theorem 3.18 that if A is indecomposable and $\sigma \in Aut(G)$ with $\sigma(G_f) \subset G_f$, then σ lies in $Aut(F) \rtimes Aut(G; \mathfrak{g}')$.

Let F be algebraically closed, $\theta \in Aut_F(\mathfrak{g})$, and \mathfrak{g}^θ (resp. G^θ) the fixed point subalgebra of \mathfrak{g} (resp. subgroup of G). In Section 4, as in the finite type case, we study G^θ and some conjugacy theorems (with respect to G^θ). Following Gantmacher [8], we consider canonical decomposition of semisimple automorphisms of \mathfrak{g} of the first kind. In Theorem 4.32, we present a classification of such automorphisms in terms of orbits of the fixed point subgroup of the Weyl group in a torus. In the finite type case, this leads to a classification in terms of weighted affine Dynkin diagrams [5] generalizing the results of Kac on finite order automorphisms [10, 8.6]. For semisimple automorphisms of the second kind, we also obtain a handy decomposition through special semisimple automorphisms of the second kind. The main result is given in Proposition 4.39.

Involutions for reductive algebraic groups have been extensively studied [19, 22, 23, 25, 27]. In Section 5, we try to carry some of this to Kac–Moody groups. In Theorem 5.31, we extend the open orbit result of Vust [27] for reductive algebraic groups to Kac–Moody groups. As an application, we obtain the conjugacy theorem for split pairs (5.32). This enables us to classify involutions of the second kind in terms of admissible diagrams. A detailed discussion will be given in a separate paper. Involutions of affine Kac–Moody algebras have been studied by Levstein [18] and Bauch [1].

1. PRELIMINARIES

We recall some basic definitions and facts about Kac–Moody algebras and groups, and set the notations. This section is on the most part a review of results of [13–15]. Our main references are [10, 14, 15].

F is a field of characteristic zero with algebraic closure \bar{F} .

1.1. An $n \times n$ matrix $A = (a_{ij})$ is called a *symmetrizable generalized Cartan matrix* if it satisfies the following conditions:

- (i) $a_{ii} = 2$ for all i .
- (ii) a_{ij} are nonpositive integers for $i \neq j$.
- (iii) There exists an invertible diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ such that DA is symmetric.

The entries d_1, \dots, d_n can be chosen to be positive rational.

1.2. Kac-Moody algebras. Choose a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where \mathfrak{h} is a vector space over F of dimension $n + \text{corank}(A)$, and $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{h_1, \dots, h_n\} \subset \mathfrak{h}$ are linearly independent sets satisfying:

$$\alpha_j(h_i) = a_{ij} \quad \text{for all } i, j.$$

The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra over F generated by \mathfrak{h} and the symbols $e_i, f_i (i = 1, \dots, n)$ with the following relations:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0; & [e_i, f_j] &= \delta_{ij} h_i; \\ [h, e_i] &= \alpha_i(h) e_i; & [h, f_i] &= -\alpha_i(h) f_i \quad (h \in \mathfrak{h}); \\ (\text{ade}_i)^{1-a_{ij}}(e_j) &= (\text{adf}_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j). \end{aligned} \tag{1}$$

The derived algebra \mathfrak{g}' of \mathfrak{g} is generated by the linearly independent Chevalley generators $e_i, f_i (i = 1, \dots, n)$. The space \mathfrak{h} is canonically embedded in \mathfrak{g} . The center \mathfrak{c} of \mathfrak{g} lies in \mathfrak{h}' where

$$\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i=1}^n Fh_i. \tag{2}$$

If A is indecomposable, then every ideal of \mathfrak{g} either contains \mathfrak{g}' or is contained in \mathfrak{c} .

The Chevalley involution ω of \mathfrak{g} is the involution of \mathfrak{g} determined by

$$\omega(h) = -h \quad (h \in \mathfrak{h}), \quad \omega(e_i) = -f_i, \quad \omega(f_i) = -e_i \quad (i = 1, \dots, n). \tag{3}$$

Let \mathfrak{n}_+ (resp. \mathfrak{n}_-) denote the subalgebra of \mathfrak{g} generated by the e_i (resp. f_i) ($i = 1, \dots, n$). We have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{4}$$

Set $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$. Clearly $\omega(\mathfrak{n}_+) = \mathfrak{n}_-$ and $\omega(\mathfrak{b}_+) = \mathfrak{b}_-$.

With respect to \mathfrak{h} , we have the root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \tag{5}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ (h \in \mathfrak{h})\}$. A nonzero element $\alpha \in \mathfrak{h}^*$ with $\mathfrak{g}_\alpha \neq 0$ is called a *root* of \mathfrak{h} in \mathfrak{g} . Let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ denote the set of roots of \mathfrak{h} in \mathfrak{g} . Set

$$Q = \sum_{i=1}^n \mathbb{Z}\alpha_i; \quad Q_+ = \sum_{i=1}^n \mathbb{Z}_+ \alpha_i, \tag{6}$$

where $\mathbb{Z}_+ = \{0, 1, \dots\}$. Define a *partial ordering* \geq on \mathfrak{h}^* by setting $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. Set

$$\Delta_+ = \Delta \cap Q_+. \tag{7}$$

Then we have that $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_{\pm\alpha}$. For $\alpha = \sum_{i=1}^n k_i \alpha_i \in \Delta$, we call $ht(\alpha) = \sum_{i=1}^n k_i$ the *height* of α (with respect to Π).

Define the *fundamental reflections* $r_i \in \text{Aut}_F(\mathfrak{h})$ by

$$r_i(h) = h - \alpha_i(h) h_i \quad (h \in \mathfrak{h}, i = 1, \dots, n). \tag{8}$$

They generate the *Weyl group* W which is a Coxeter group on $\{r_1, \dots, r_n\}$ with *length functions* $w \mapsto l(w) (w \in W)$.

The root system Δ is stable under W . Set

$$\Delta^{re} = W(\Pi), \quad \Delta^{im} = \Delta - \Delta^{re}.$$

A root $\alpha \in \Delta^{re}$ (resp. Δ^{im}) is called *real* (resp. *imaginary*). If α is a real root, then $\dim \mathfrak{g}_\alpha = 1$ and $\Delta \cap \mathbb{Z}\alpha = \{\pm\alpha\}$; on the other hand if α is imaginary, then $\mathbb{Z}\alpha \subset \Delta \cup \{0\}$. Given a real root α , there exist $w \in W$ and index i such that $\alpha = w(\alpha_i)$. Then

$$r_\alpha = wr_i w^{-1} \quad \text{and} \quad \alpha^\vee = w(h_i) \tag{9}$$

depend only on α .

We choose a nondegenerate $ad(\mathfrak{g})$ -invariant symmetric F -bilinear form $(\cdot | \cdot)$ on \mathfrak{g} such that $(h_i | h_i)$ is positive rational for all i . The form is nondegenerate and W -invariant on \mathfrak{h} and hence induces an isomorphism $\nu: \mathfrak{h} \rightarrow \mathfrak{h}^*$ and a form $(\cdot | \cdot)$ on \mathfrak{h}^* .

1.3. Kac–Moody groups. We recall the construction of the group G associated to the Lie algebra \mathfrak{g}' .

A \mathfrak{g}' -module V , or (V, π) where $\pi: \mathfrak{g}' \rightarrow \text{End}_F(V)$, is called *integrable* if $\pi(x)$ is locally nilpotent for all $x \in \mathfrak{g}_\alpha$ and $\alpha \in \Delta^{re}$. Note that (\mathfrak{g}, ad) is integrable.

Let G^* be the free product of the additive groups \mathfrak{g}_α ($\alpha \in \Delta^{re}$) with canonical inclusions $i_\alpha: \mathfrak{g}_\alpha \rightarrow G^*$. For any integrable \mathfrak{g}' -module (V, π) , define a homomorphism $\pi^*: G^* \rightarrow \text{Aut}_F(V)$ by $\pi^*(i_\alpha(x)) = \exp \pi(x)$

($x \in \mathfrak{g}_\alpha, \alpha \in \Delta^{re}$). Let N^* denote the intersection of all $\ker(\pi^*)$, $G = G^*/N^*$, and $q: G^* \rightarrow G$ the canonical homomorphism. The group G is the *Kac-Moody group* associated to \mathfrak{g}' .

For $\alpha \in \Delta^{re}$ and $x \in \mathfrak{g}_\alpha$, set $\exp x = q(i_\alpha(x))$ and $U_\alpha = \exp \mathfrak{g}_\alpha$. Then U_α is an additive one parameter subgroup of G . The subgroups $U_\alpha, U_{-\alpha}$ ($\alpha \in \Pi$) generate G , and G is its own derived group. Denote by U_+ (resp. U_-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$) for $\alpha \in \Delta_+^{re}$.

Let $\alpha \in \Delta^{re}$, $e \in \mathfrak{g}_\alpha$, and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$. There is a unique homomorphism $\varphi: SL(2, F) \rightarrow G$ satisfying

$$\varphi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp te, \quad \varphi\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tf \quad (t \in F). \tag{1}$$

For each i , let φ_i denote the homomorphism of $SL(2, F)$ into G determined by α_i, e_i , and f_i . Let $G_i = \varphi_i(SL(2, F))$, $H_i = \varphi_i(\{\text{diag}(t, t^{-1}) \mid t \in F^\times\})$, and N_i be the normalizer of H_i in G_i . The φ_i are injective, and so $G_i \approx SL(2, F)$ and $H_i \approx F^\times$. Let H (resp. N) be the subgroup of G generated by the groups H_i (resp. N_i) ($i = 1, \dots, n$). Then H is an abelian normal subgroup of N and H is the direct product of the groups H_i . We have an isomorphism $\psi: W \rightarrow N/H$ such that $\psi(r_i)$ is the coset $N_iH - H$. We shall identify W with N/H using ψ . For $w \in W$, choose $n \in N$ with image w in N/H . We write

$$\begin{aligned} w(h) &= Ad(n)(h) & (h \in \mathfrak{h}), \\ wU_\pm w^{-1} &= Int(n)U_\pm, \end{aligned} \tag{2}$$

where $Int(x)y = xyx^{-1}$ ($x, y \in G$). Set $B_+ = HU_+$ and $B_- = HU_-$.

For any integrable \mathfrak{g}' -module (V, π) , we associate the homomorphism (also denoted by) $\pi: G \rightarrow Aut_F(V)$ determined by

$$\pi(\exp x) = \exp \pi(x) \quad (x \in \mathfrak{g}_\alpha, \alpha \in \Delta^{re}). \tag{3}$$

Ad is the homomorphism associated to (\mathfrak{g}, ad) . Then the kernel of Ad coincides with the center C of G , and $Ad(G)$ acts faithfully on \mathfrak{g}'/c . For any integrable \mathfrak{g}' -module (V, π) , we have that

$$\pi(Ad(g)x) = \pi(g)\pi(x)\pi(g)^{-1} \quad (g \in G, x \in \mathfrak{g}'). \tag{4}$$

1.4. Highest weight modules $L(\lambda)$. Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} (i = 1, \dots, n)\}$ and P_+ (resp. P_{++}) = $\{\lambda \in P \mid \lambda(h_i) \geq 0$ (resp. > 0) ($i = 1, \dots, n$)}. For $\lambda \in P_+$, there exists an irreducible \mathfrak{g} -module $(L(\lambda), \pi_\lambda)$, unique up to isomorphism, with a nonzero element v^+ satisfying:

$$\pi_\lambda(n_+)v^+ = 0; \quad \pi_\lambda(h)v^+ = \lambda(h)v^+ \quad (h \in \mathfrak{h}). \tag{1}$$

$L(\lambda)$ is an absolutely irreducible \mathfrak{g}' -module and $End_{\mathfrak{g}'}(L(\lambda)) = FI_{L(\lambda)}$. The module $L(\lambda)$ is called an *integrable module with highest weight λ* .

We have the weight space decomposition $L(\lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L(\lambda)_\lambda$ where $L(\lambda)_\lambda = \{v \in L(\lambda) \mid \pi_\lambda(h)v = \lambda(h)v \ (h \in \mathfrak{h})\}$. Then

$$L(\lambda)_\lambda = Fv^+. \tag{2}$$

Elements $\lambda \in \mathfrak{h}^*$ with $L(\lambda)_\lambda \neq 0$ are called *weights* of $L(\lambda)$. Let $P(\lambda)$ denote the set of weights of $L(\lambda)$. We have that $P(\lambda) \subset \lambda - Q_+$.

Consider $L(\lambda)$ as a \mathfrak{g} -module under $\pi_\lambda^* = \pi_\lambda \circ \omega$. We denote $L^*(\lambda)$ for this module and v^- for v^+ . There is a unique \mathfrak{g} -invariant F -bilinear form $\langle \cdot, \cdot \rangle$ on $L(\lambda) \times L^*(\lambda)$ such that $\langle v^+, v^- \rangle = 1$; it is nondegenerate.

1.5. LEMMA. *Let \mathfrak{p} be a subalgebra of \mathfrak{g} containing \mathfrak{b}_+ . Then there exists a subset $X \subset \Pi$ such that*

$$\mathfrak{p} = \bigoplus_{\alpha \in \Delta_X} \mathfrak{g}_\alpha + \mathfrak{b}_+,$$

where $\Delta_X = \Delta \cap \mathbb{Z}X$.

Proof. Consider the root space decomposition

$$\mathfrak{p} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{p}_\alpha$$

of \mathfrak{p} with respect to \mathfrak{h} . Let $X = \{\alpha \in \Pi \mid \mathfrak{p}_{-\alpha} \neq 0\}$. Claim: $\mathfrak{p} \cap \mathfrak{n}_-$ is generated by the $\mathfrak{g}_{-\alpha}$ ($\alpha \in X$). Suppose the assertion to be false. Choose $\gamma \in \Delta^+$ with minimal $ht(\gamma)$ such that $\mathfrak{p}_{-\gamma}$ is not contained in the subalgebra generated by the $\mathfrak{g}_{-\alpha}$ ($\alpha \in X$). By [10, Lemma 1.5], there exist $x_{-\gamma} \in \mathfrak{p}_{-\gamma}$ and e_i with $[e_i, x_{-\gamma}] \neq 0$. Clearly $[e_i, x_{-\gamma}] \in \mathfrak{p}_{-\gamma + \alpha_i}$ and by minimality of $ht(\gamma)$, we have that

$$-\gamma + \alpha_i \in \mathbb{Z}X. \tag{1}$$

There exists $x_{\gamma - \alpha_i} \in \mathfrak{g}_{\gamma - \alpha_i}$ such that $([e_i, x_{-\gamma}] \mid x_{\gamma - \alpha_i}) \neq 0$. It follows that $(e_i \mid [x_{-\gamma}, x_{\gamma - \alpha_i}]) \neq 0$ and so

$$\alpha_i \in X. \tag{2}$$

From (1) and (2), $-\gamma \in -\mathbb{Z}_+ X$. Certainly this is a contradiction. Now the assertion of the lemma is obvious.

1.6. LEMMA. (i) *If $\lambda \in P_{++}$ and $1 \neq g \in U_-$, then $\pi_\lambda(g)v^+ \notin Fv^+$.*

(ii) $U_- \cap B_+ = \{1\}$.

(iii) *If L is a normal subgroup of G contained in U_+ (resp. U_-), then $L = \{1\}$; in particular Ad is faithful on U_+ (resp. U_-).*

Proof. We establish the lemma in several steps.

(1) Let $U(\mathfrak{n}_-)$ be the enveloping algebra of \mathfrak{n}_- . Clearly \mathfrak{h} acts on $U(\mathfrak{n}_-)$ through the adjoint representation. We have the weight space decomposition $U(\mathfrak{n}_-) = \bigoplus_{\lambda \in -Q_+} U(\mathfrak{n}_-)_{\lambda}$. For any $g \in U_-$, there exist $x_i \in \mathfrak{g}_{\beta_i}$, $\beta_i \in -\Delta_+^{re}$ ($i = 1, \dots, l$) such that $g = \exp x_1 \cdots \exp x_l$. Consider the formal expansion $\exp x_1 \cdots \exp x_l = \sum (x_1^i/i_1!) \cdots \sum (x_l^i/i_l!) = \sum_{\lambda \in -Q_+} y_{\lambda}$ where $y_0 = 1$ and

$$y_{\lambda} = \sum_{i_1\beta_1 + \dots + i_l\beta_l = \lambda} \frac{x_1^{i_1} \cdots x_l^{i_l}}{i_1! \cdots i_l!} \in U(\mathfrak{n}_-)_{\lambda}.$$

(2) Let (V, π) be any integrable \mathfrak{g}' -module and $v \in V$. Then $y_{\lambda}v = 0$ for all except finitely many λ and $gv = \sum_{\lambda \in -Q_+} y_{\lambda}v$. If $g \neq 1$, there exists $0 \neq \lambda \in -Q_+$ such that $y_{\lambda} \neq 0$.

(3) For $\lambda \in P_{++}$, the line Fv^+ is not invariant under $Fe_i + Ff_i + Fh_i$ for each i . Hence by 1.5, the stabilizer of Fv^+ in \mathfrak{g} is \mathfrak{b}_+ . This yields that the map $x \mapsto \pi_{\lambda}(x)v^+$ ($x \in \mathfrak{n}_-$) is injective.

(4) Choose a basis $\{z_1, \dots, z_j, \dots\}$ of \mathfrak{n}_- . For any finite sequence $J = (j_1, \dots, j_m)$ of positive integers $j_1 \leq \dots \leq j_m$, write $z_J = z_{j_1} \cdots z_{j_m}$. The elements z_J form a basis of $U(\mathfrak{n}_-)$. Now let $1 \neq g \in U_-$. By (2), $y_{\lambda} \neq 0$ for certain $0 \neq \lambda \in -Q_+$. Write $y_{\lambda} = \sum a_J z_J$ with $a_J \in F$. Choose J with maximal length m such that $a_J \neq 0$. Consider the tensor product $V = L(\lambda) \otimes \cdots \otimes L(\lambda)$ (m copies) and the corresponding tensor product representation π of \mathfrak{g} on V . By (3), the vectors $z_J v^+$ are linearly independent. Choose a basis \mathcal{B} of $L(\lambda)$ containing v^+ and the elements $z_i v^+$. The elements $v_1 \otimes \cdots \otimes v_m$ with $v_i \in \mathcal{B}$ ($i = 1, \dots, m$) form a basis of V . One checks readily that the coefficient of $z_{j_1} v^+ \otimes \cdots \otimes z_{j_m} v^+$ in $\pi(y_{\lambda})(v^+ \otimes \cdots \otimes v^+)$ is nonzero; in particular

$$0 \neq \pi(y_{\lambda})(v^+ \otimes \cdots \otimes v^+) \in V_{m\lambda + \lambda}.$$

This implies that $\pi(g)(v^+ \otimes \cdots \otimes v^+)$ has a nontrivial component in $V_{m\lambda + \lambda}$. Now assertion (i) is obvious. Note that the line Fv^+ is invariant under $U_- \cap \mathfrak{b}_+$. Hence assertion (ii) is immediate from (i).

(5) For $x \in L$, $\pi_{\lambda}(x)$ fixes all the vectors $\pi_{\lambda}(g)v^+$ ($g \in G$). Hence $\pi_{\lambda}(x) = 1$ and so $\pi_{\lambda}^*(x) = 1$. Then by (i), $x = 1$.

1.7. Bruhat and Birkhoff decompositions. Let $\alpha, \beta \in \Delta^{re}$ with $(\alpha|\beta) \geq 0$. Then we have (see [10, 2nd edition, Ex. 5.19]):

- (i) $(\mathbb{Z}_+ \alpha + \mathbb{Z}_+ \beta) \cap \Delta = \{\alpha, \beta, \alpha + \beta\} \cap \Delta^{re}$. (1)
- (ii) $(U_{\alpha}, U_{\beta}) \subset U_{\alpha + \beta}$ if $\alpha + \beta \in \Delta^{re}$ and $= \{1\}$ otherwise.

As a consequence, one obtains

$$B_{\pm} w B_{+} r_i \subset B_{\pm} w r_i B_{+} \cup B_{\pm} w B_{+}, \tag{2}$$

hence the following theorem [15]:

THEOREM. *We have the following decompositions:*

- (i) $G = \bigcup_{w \in W} B_{+} w B_{+}$ (*Bruhat decomposition*).
- (ii) $G = \bigcup_{w \in W} B_{-} w B_{+}$ (*Birkhoff decomposition*).

1.8. Standard parabolic subgroups. For $X \subset \Pi$, let W_X denote the subgroup of W generated by the r_i with $\alpha_i \in X$. Set

$$P_X = B_{+} W_X B_{+}.$$

By 1.7(2), P_X is a subgroup of G containing B_{+} . The groups P_X and $\omega(P_X)$ are called the *standard parabolic subgroups* of G .

Suppose that P is a subgroup of G containing B_{+} . Since $G = B_{+} W B_{+}$, $P = B_{+} (W \cap P) B_{+}$. For $w \in W \cap P$, write $w = r_{i_1} \cdots r_{i_l}$ with $l = l(w)$. Then $w \alpha_{i_j} < 0$ and so $-\alpha_{i_j} \in w^{-1}(\Delta_{+}^{re})$. Hence P contains $U_{\alpha_{i_j}}$ and $U_{-\alpha_{i_j}}$. As a consequence, r_{i_j} lies in $P \cap W$. By an easy induction $r_{i_1}, \dots, r_{i_l} \in P \cap W$. We have that $P = P_X$ where $X = \{\alpha_i \in \Pi \mid r_i \in P\}$.

Let $X \subset \Pi$, U^X denote the smallest normal subgroup of U_{+} containing U_{β} ($\beta \in \Delta_{+}^{re} - \mathbb{Z}X$), U_X be the subgroup of U_{+} generated by the U_{γ} ($\gamma \in \Delta_{+}^{re} \cap \mathbb{Z}X$), G_X the subgroup of G generated by the U_{α} , $U_{-\alpha}$ ($\alpha \in X$) and $M_X = H G_X$.

LEMMA. *We have:*

- (i) $P_X = M_X \ltimes U^X$.
- (ii) $M_X \cap U_{+} = U_X$.

1.9. Large subgroup. A subgroup U'_{+} (resp. U'_{-}) of U_{+} (resp. U_{-}) is called *large* if there exist g_1, \dots, g_m in G such that $\bigcap_{i=1}^m g_i U_{\pm} g_i^{-1} \subset U'_{\pm}$.

1.10. Let $\mathfrak{a} \subset \mathfrak{gl}(r, F)$ be a Lie subalgebra consisting of nilpotent elements. Let $\mathfrak{a}_0 = \mathfrak{a}$, $\mathfrak{a}_i = [\mathfrak{a}, \mathfrak{a}_{i-1}]$ be the lower central series and $\mathfrak{a} = \bigoplus_{j=1}^m V_j$ a decomposition of \mathfrak{a} in a direct sum of vector spaces over F . Assume that $\mathfrak{a}_i = \bigoplus_{j=1}^m (\mathfrak{a}_i \cap V_j)$ ($i=0, 1, \dots$). Then we have:

- (i) $A = \exp(\mathfrak{a})$ is a unipotent subgroup of $GL(r, F)$.
- (ii) For any fixed order, the product map $\prod_j \exp(V_j) \rightarrow A$ is bijective.

This follows by an easy induction on the length of the lower central series of \mathfrak{a} .

Let $\tilde{H} = \text{Hom}(Q, F^\times)$ and let $Ad: \tilde{H} \rightarrow \text{Aut}_F(\mathfrak{g})$ be the homomorphism defined by

$$Ad(y)x = y(\alpha)x \quad (y \in \tilde{H}, s \in \mathfrak{g}_\alpha).$$

Ad induces an action of \tilde{H} on G . Define $\tilde{H} \ltimes G$ and extend Ad to $\tilde{H} \ltimes G$ in the obvious way. We also extend the action of G on $L(\Lambda)$ to $\tilde{H} \ltimes G$ by requiring \tilde{H} to fix v^+ .

1.11. PROPOSITION. *Let $S = \{w_1, \dots, w_l\}$ be a finite subset of W , $Y = \{\alpha \in \Delta_+^{re} \mid w_i \alpha < 0 \text{ for some } w_i \in S\}$ and $U_+(S)$ denote the subgroup $U_+ \cap \bigcap_{i=1}^l w_i^{-1} U_+ w_i$. Then for any \tilde{H} -invariant large subgroup U'_+ of U_+ and any order of Y , the product map*

$$\prod_{\alpha \in Y} (U_\alpha \cap U'_+) \times (U_+(S) \cap U'_+) \rightarrow U'_+$$

is bijective.

Proof. There exist $g_1, \dots, g_m \in G$ such that U'_+ contains $\bigcap_{i=1}^m g_i U_+ g_i^{-1}$. Choose $\Lambda \in P_{++}$ and a finite dimensional $\tilde{H} \ltimes B_+$ -invariant subspace V of $L(\Lambda)$ containing v^+ , $w_i^{-1} v^+$ ($i = 1, \dots, l$) and $g_j v^+$ ($j = 1, \dots, m$). Consider the representation $\pi: \tilde{H} \ltimes U_+ \rightarrow GL(V)$ (resp. $\pi: \mathfrak{b}_+ \rightarrow gl(V)$) defined by the restriction map of π_Λ to V . Set

$$\mathfrak{n}_+(S) = \mathfrak{n}_+ \cap \bigcap_{i=1}^l Ad(w_i^{-1}) \mathfrak{n}_+.$$

Then we have the decomposition

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in Y} \mathfrak{g}_\alpha \oplus \mathfrak{n}_+(S).$$

Clearly we have the following conditions:

(i) $\ker(\pi|_{\mathfrak{n}_+}) \subset \mathfrak{n}_+(S).$

(ii) There exist $\beta_1, \dots, \beta_r \in \Delta_+ - Y$ such that $\pi(\mathfrak{n}_+(S))$ has the weight decomposition (with respect to $ad(\pi(\mathfrak{h}))$)

$$\pi(\mathfrak{n}_+(S)) = \bigoplus_{i=1}^r \pi(\mathfrak{n}_+(S))_{\beta_i}.$$

The Lie algebra $\pi(\mathfrak{n}_+)$ is finite dimensional and consists of nilpotent elements. Hence it defines a unipotent group $\exp(\pi(\mathfrak{n}_+))$. Observe that $\pi(\mathfrak{g}_\alpha)$ (resp. $\pi(U_\alpha)$) ($\alpha \in \Delta_+^{re}$) generate $\pi(\mathfrak{n}_+)$ (resp. $\pi(U_+)$). We have that $\pi(U_+) = \exp(\pi(\mathfrak{n}_+))$. The decomposition $\pi(\mathfrak{n}_+) = \bigoplus_{\alpha \in Y} \pi(\mathfrak{g}_\alpha) \oplus$

$\pi(n_+(S))$ is $ad(\pi(\mathfrak{h})$)- and $Ad(\pi(\tilde{H}))$ -invariant and compatible with any term of the lower central series. Hence by 1.10,

$$\pi(U_+) = \prod_{\alpha \in Y} \pi(U_\alpha) \cdot \exp(\pi(n_+(S))).$$

Since $\ker(\pi|_{U_+}) \subset U_+(S)$, $\pi(U_+(S))$ is the subgroup of $\pi(U_+)$ consisting of the elements fixing v^+ , $w_i^{-1}v^+$ ($i = 1, \dots, l$). It follows that $\pi(U_+(S)) = \exp(\pi(n_+(S)))$.

By condition (ii), $\pi(U_+(S))$ has a $\pi(\tilde{H})$ -invariant principle series $\pi(U_+(S)) = U^{(1)} \supset \dots \supset U^{(s)} = \{1\}$ such that $U^{(i)}/U^{(i+1)}$ is isomorphic (with operators $\pi(\tilde{H})$) to some quotient of \mathfrak{g}_α with $\alpha \in \mathcal{A}_+ - Y$. As groups with operators $\pi(\tilde{H})$, $\pi(U_\alpha)$ ($\alpha \in Y$) and $\pi(U_+(S))$ have no isomorphic subquotients. Hence by [3, Lemma 3.3], the product map

$$\prod_{\alpha \in Y} (\pi(U_\alpha) \cap \pi(U'_+)) \times (\pi(U_+(S)) \cap \pi(U'_+)) \rightarrow \pi(U'_+)$$

is bijective. However, by our choice of V , $\ker(\pi|_{U_+}) \subset U'_+$ and so our desired assertion follows by pulling back the result to U'_+ .

1.12. COROLLARY. *Let $S \subset W$, Y and $U_+(S)$ be as in 1.11. For any \tilde{H} -invariant large subgroup V of U_+ , the following conditions are equivalent:*

- (i) $V = U_+(S)$.
- (ii) *The product map $\prod_{\alpha \in Y} U_\alpha \times V \rightarrow U_+$ is bijective.*
- (iii) *V is a maximal \tilde{H} -invariant large subgroup of U_+ such that $V \cap U_\alpha = \{1\}$ for all $\alpha \in Y$.*

Proof. (i) \Rightarrow (ii) is immediate from 1.11 using U_+ for U'_+ . If U'_+ is an \tilde{H} -invariant large subgroup of U_+ such that $U'_+ \cap U_\alpha = \{1\}$ for all $\alpha \in Y$, by 1.11 we have that $U'_+ \subset U_+(S)$ and $\prod_{\alpha \in Y} U_\alpha \cap U'_+ = \{1\}$. Now the assertions (ii) \Rightarrow (iii) \Rightarrow (i) are obvious.

1.13. COROLLARY. *Let $w \in W$ and $\Phi(w) = \{\alpha \in \mathcal{A}_+^{re} \mid w\alpha < 0\}$. Then we have the following decompositions:*

- (i) *For any fixed order of $\Phi(w)$, the product map*

$$\prod_{\alpha \in \Phi(w)} U_\alpha \rightarrow U_+ \cap w^{-1}U_-w$$

is bijective.

- (ii) *The product map $(U_+ \cap w^{-1}U_-w) \times (U_+ \cap w^{-1}U_+w) \rightarrow U_+$ is bijective.*

1.14. LEMMA. *Let $w \in W$ and ${}^wU_{\pm} = wU_{\pm}w^{-1}$. Then $G = {}^wU_{\pm}NU_{\pm}$ and for every $g \in G$, its component in N is unique; moreover we have a unique decomposition $g = vnu$ such that $v \in {}^wU_{\pm} \cap nU_{\pm}n^{-1}$, $u \in U_{\pm}$, and $n \in N$.*

Proof. Suppose that $n, n' \in N$ belong to the same double coset. Write $n' = xny$ with $y \in U_{+}$ and $x \in {}^wU_{\pm} \cap nU_{\pm}n^{-1}$. Then $n^{-1}n' = x_1y$ with $x_1 = n^{-1}xn \in U_{-}$. Choose $\lambda \in P_{++}$. We have that $\pi_{\lambda}(n^{-1}n')v^{+} \in L(\lambda)_{\tau\lambda}$ where τ is the image of $n^{-1}n'$ in W . On the other hand $\pi_{\lambda}(x_1y)v^{+} \in v^{+} + \bigoplus_{\lambda < \lambda'} L(\lambda)_{\lambda}$. Hence we must have $\tau\lambda = \lambda$ and $\pi_{\lambda}(x_1)v^{+} = v^{+}$. By 1.6(i), $x_1 = 1$ and so $x = 1$. Then $y \in H \cap U_{+} = \{1\}$ and $n = n'$. The second assertion is immediate from 1.13(ii).

1.15. LEMMA. $\bigcap_{w \in W} wU_{+}w^{-1} = \{1\}$.

Proof. For $x \in \bigcap_{w \in W} wU_{+}w^{-1}$, write $x = unv$ with $n \in N, v \in U_{-}$, and $u \in U_{-} \cap nU_{+}n^{-1}$. Consider the element $vn = (n^{-1}u^{-1}n)(n^{-1}xn) \in U_{+}$. By 1.14, $n = 1, v \in U_{+} \cap U_{-} = \{1\}$, and so $x = u \in U_{+} \cap U_{-} = \{1\}$.

1.16. LEMMA. *We have:*

- (i) $N = N_G(H)$ (resp. $N = N_G(\mathfrak{h})$).
- (ii) $H = Z_G(H)$ (resp. $H = Z_G(\mathfrak{h})$).

Proof. For $g \in N_G(H)$, write $g = vnu$ with $n \in N, u \in U_{+}$, and $v \in U_{+} \cap nU_{-}n^{-1}$. Set $v' = n^{-1}vn$. Then v' lies in $U_{-} \cap nU_{+}n^{-1}$ and $v'u \in N_G(H)$. Since H normalizes U_{-} and U_{+} , $v', u \in Z_G(H)$. By 1.13(i), the centralizer of H in $U_{-} \cap n^{-1}U_{+}n$ is trivial. Hence $v' = 1$ and $N_G(H) = N \cdot N_{U_{+}}(H)$. Since $\omega(H) = H, N_G(H) = N \cdot N_{U_{-}}(H)$. Then $N_{U_{+}}(H) \subset NN_{U_{-}}(H)$ and by 1.14, $N_{U_{+}}(H) = \{1\}$. Hence (i) follows. Note that W acts faithfully on \mathfrak{h}' . Assertion (ii) follows from (i). For $g \in N_G(\mathfrak{h})$, write $g = vnu$ and $v' = n^{-1}vn$ as above. Choose $\lambda \in P_{++}$. Since $v'u \in N_G(\mathfrak{h}), \pi_{\lambda}(v'u)v^{+} \in L(\lambda)_{\lambda}$ for certain $\lambda \in \mathfrak{h}^*$. On the other hand $v' \in U_{-}$ and $u \in U_{+}$, so

$$\pi_{\lambda}(v'u)v^{+} \in v^{+} + \bigoplus_{\nu < \lambda} L(\lambda)_{\nu}.$$

Hence $\lambda = A$ and by 1.16(i), $v' = 1$. Then $N_G(\mathfrak{h}) = N \cdot N_{U_{+}}(\mathfrak{h})$ and the same argument yields the result $N_G(\mathfrak{h}) = N$.

1.17. LEMMA. *For $\lambda \in P_{+}$, let $X = \{\alpha \in \Pi \mid (\lambda|\alpha) = 0\}$.*

- (i) P_X (resp. \mathfrak{p}_X) is the stabilizer of Fv^{+} in G (resp. \mathfrak{g}).
- (ii) P_X (resp. \mathfrak{p}_X) is its own normalizer in G (resp. \mathfrak{g}).

Proof. Let P denote the stabilizer of Fv^{+} in G . Since $\lambda \in P_{+}, P \cap W = W_X$. Hence by 1.8, $P = P_X$. Note that Fv^{+} is the unique

line fixed by P_X . If $g \in N_G(P_X)$, then $\pi_A(g)Fv^+$ is stable under P_X and so $\pi_A(g)Fv^+ = Fv^+$. Hence P_X is its own normalizer. Observe that $\pi_A(f_i)v_A^+ \in Fv^+ \Leftrightarrow \pi_A(f_i)v_A^+ = 0 \Leftrightarrow \alpha_i \in X$. Hence by 1.5, \mathfrak{p}_X is the stabilizer of Fv^+ in \mathfrak{g} . The same argument yields that \mathfrak{p}_X is its own normalizer.

1.18. The flag varieties \mathcal{V}_A . For each $\alpha \in \mathcal{A} \cup \{0\}$, choose dual bases $\{x_\alpha^{(i)}\}$ of \mathfrak{g}_α and $\{y_\alpha^{(i)}\}$ of $\mathfrak{g}_{-\alpha}$. The generalized Casimir operator Ω is given by [10, 2.5]

$$\Omega = 2v^{-1}(\rho) + \sum_i y_0^{(i)}x_0^{(i)} + 2 \sum_{\alpha > 0} \sum_i y_\alpha^{(i)}x_\alpha^{(i)},$$

where ρ is an element of \mathfrak{h}^* such that $(\rho | \alpha_i) = (1/2)(\alpha_i | \alpha_i)$ ($i = 1, \dots, n$).

On $L(A)$, Ω acts as a scalar $c_A = (A + 2\rho | A)$. For $v_1 \in L(A_1)$ and $v_2 \in L(A_2)$, we have that

$$\Omega(v_1 \otimes v_2) = (c_{A_1} + c_{A_2})v_1 \otimes v_2 + 2 \sum_{\alpha \in \mathcal{A} \cup \{0\}} \sum_i x_\alpha^{(i)}v_1 \otimes y_\alpha^{(i)}v_2.$$

Thus the following conditions are equivalent:

- (i) $v_1 \otimes v_2 \in L(A_1 + A_2)$.
- (ii) $(A_1 | A_2)v_1 \otimes v_2 = \sum_{\alpha \in \mathcal{A} \cup \{0\}} \sum_i x_\alpha^{(i)}v_1 \otimes y_\alpha^{(i)}v_2$. (1)
- (iii) $\Omega(v_1 \otimes v_2) = c_{A_1 + A_2}(v_1 \otimes v_2)$.

For $A \in P_+$, let \mathcal{V}_A denote the set of nonzero elements $v \in L(A)$ satisfying

$$(A | A)v \otimes v = \sum_{\alpha \in \mathcal{A} \cup \{0\}} \sum_i x_\alpha^{(i)}v \otimes y_\alpha^{(i)}v. \quad (2)$$

Let Ω^* be the adjoint of Ω on $L^*(A) \otimes L^*(A)$ and $J_2^* = (\Omega^* - c_{2A}) \text{Sym}^2(L^*(A))$ where $\bigoplus_{k \geq 0} \text{Sym}^k(V)$ is the symmetric algebra of a vector space V . From the equivalent conditions (1), it is easy to see that \mathcal{V}_A is defined by the system of equations

$$v \in \mathcal{V}_A \Leftrightarrow 0 \neq v \in L(A), \quad f(v) = 0 \quad (f \in J_2^*). \quad (3)$$

Since Ω commutes with \mathfrak{g} (resp. G), J_2^* is \mathfrak{g} (resp. G) invariant. Hence \mathcal{V}_A is a G -invariant cone.

One of the highlights in [15] is the following theorem on flag varieties.

THEOREM. $\mathcal{V}_A = G(F^\times v^+)$.

1.19. **LEMMA.** (i) Let \mathfrak{m} be a subalgebra of \mathfrak{n}_+ of finite codimension. For $A \in P_+$, $L(A)^\mathfrak{m} = \{v \in L(A) | \pi_A(\mathfrak{m})v = \{0\}\}$ is finite dimensional.

(ii) Let \mathfrak{q} be a subspace of \mathfrak{g} containing \mathfrak{b}_+ such that $\mathfrak{q}/\mathfrak{b}_+$ is of finite dimension. Then for $\lambda \in P_+$ and $\mathfrak{m} = \mathfrak{q}^\perp = \{x \in \mathfrak{g} \mid (x|\mathfrak{q}) = \{0\}\}$, the space $L(\lambda)^\mathfrak{m}$ is finite dimensional.

Proof. (i) This is [15, Lemma 5(a)].

(ii) Let $U(\mathfrak{b}_+)$ be the enveloping algebra of \mathfrak{b}_+ and $\mathfrak{m}' = (ad(U(\mathfrak{b}_+))\mathfrak{q})^\perp$. Clearly $\mathfrak{m}' \subset \mathfrak{m}$ and it is a subalgebra of \mathfrak{n}_+ of finite codimension. Then $L(\lambda)^\mathfrak{m}$, contained in $L(\lambda)^\mathfrak{m}'$, is finite dimensional.

1.20. Let \mathfrak{p} be a Lie algebra and (V, π) a \mathfrak{p} -module, both over F . We recall the following definitions:

(i) \mathfrak{p} is π -finite (on V) if every $v \in V$ is contained in a finite dimensional \mathfrak{p} -submodule of V .

(ii) \mathfrak{p} is π -triangular if for every $v \in V$ there exist \mathfrak{p} -submodules $V_0 \subset V_1 \cdots \subset V_l$ of V such that $v \in V_l$ and $\dim(V_i) = i (i = 0, 1, \dots, l)$.

(iii) \mathfrak{p} is π -diagonalizable (resp. π -semisimple) if V is a sum of one-dimensional (resp. finite dimensional irreducible) \mathfrak{p} -submodules.

We call an element $x \in \mathfrak{p}$ π -finite (resp. π -triangular, etc.) if Fx is such.

1.21. Let \mathfrak{p} be a subalgebra of \mathfrak{g} . The following conditions are equivalent:

- (i) \mathfrak{p} is $ad_{\mathfrak{g}}$ -triangular.
- (ii) \mathfrak{p} is π_λ - and π_λ^* -triangular for all $\lambda \in P_+$.
- (iii) There exist $g \in G$ and $w \in W$ such that

$$Ad(g)\mathfrak{p} \subset \mathfrak{b}_+ \cap w(\mathfrak{b}_-).$$

As a consequence, we have the following results.

1.22. Let \mathfrak{p} be a subalgebra of \mathfrak{g} . Then \mathfrak{p} is $ad_{\mathfrak{g}/\mathfrak{c}}$ -diagonalizable (resp. -triangular) if and only if there exist $g \in G$ and $w \in W$ such that

$$Ad(g)\mathfrak{p} \subset \mathfrak{h} \quad (\text{resp. } \subset \mathfrak{b}_+ \cap w(\mathfrak{b}_-)).$$

1.23. Let \mathfrak{p} be a subalgebra of \mathfrak{g}' . Then \mathfrak{p} is $ad_{\mathfrak{g}'/\mathfrak{c}}$ -finite (resp. -triangular, -diagonalizable, or -semisimple) if and only if \mathfrak{p} is π -finite (resp. -triangular, -diagonalizable, or -semisimple) for all integrable \mathfrak{g}' -modules (V, π) .

1.24. Jordan decomposition. Let $x \in \mathfrak{g}$ be $ad_{\mathfrak{g}}$ -finite. Then there exist unique $x_s, x_n \in \mathfrak{g}$ satisfying:

(i) $x = x_s + x_n$ and $[x_s, x_n] = 0$.

(ii) x_s (resp. x_n) is π_A - and π_A^* -semisimple (resp. -locally nilpotent) for all $\lambda \in P_+$.

Furthermore, x_s (resp. x_n) is ad_g -semisimple (resp. -locally nilpotent) and $\mathfrak{g}^x = \mathfrak{g}^{x_s} \cap \mathfrak{g}^{x_n}$.

An element $x \in \mathfrak{g}$ is ad_g -triangular if and only if there exist $g \in G$ and $w \in W$ such that $\text{Ad}(g)x_s \in \mathfrak{h}$ and $\text{Ad}(g)x_n \in \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$.

1.25. Cartan subalgebras. Let \mathfrak{p} be a Lie algebra. A maximal ad_p -diagonalizable subalgebra of \mathfrak{p} is called a *split Cartan subalgebra* of \mathfrak{p} .

THEOREM [15]. *Every split Cartan subalgebra of \mathfrak{g} (resp. \mathfrak{g}' or $\mathfrak{g}'/\mathfrak{c}$) is $\text{Ad}(G)$ -conjugate to \mathfrak{h} (resp. \mathfrak{h}' or $\mathfrak{h}'/\mathfrak{c}$).*

1.26. Borel subalgebras. A subalgebra \mathfrak{a} of a Lie algebra \mathfrak{p} is called a *completely solvable subalgebra* of \mathfrak{p} if there exists a full $\text{ad}(\mathfrak{a})$ -invariant flag of the space \mathfrak{p} : $\dots, \mathfrak{a}_i \supset \mathfrak{a}_{i+1} \supset \dots (i \in \mathbb{Z})$, such that

$$(i) \quad \mathfrak{a}_0 = \mathfrak{a}, \quad (ii) \quad \bigcup_i \mathfrak{a}_i = \mathfrak{p}, \quad (iii) \quad \bigcap_i \mathfrak{a}_i = 0,$$

and

$$(iv) \quad \dim(\mathfrak{a}_i/\mathfrak{a}_{i+1}) \leq 1.$$

A maximal completely solvable subalgebra of a Lie algebra \mathfrak{p} is called a *Borel subalgebra* of \mathfrak{p} .

THEOREM [15, Theorem 3]. *Every completely solvable subalgebra of the Kac-Moody algebra \mathfrak{g} is contained in a Borel subalgebra of \mathfrak{g} . If A is indecomposable, every Borel subalgebra of \mathfrak{g} is $\text{Ad}(G)$ -conjugate to \mathfrak{b}_+ or \mathfrak{b}_- . Every Cartan subalgebra of \mathfrak{b}_+ is $\text{Ad}(U_+)$ -conjugate to \mathfrak{h} .*

1.27. Root bases. A linearly independent set $\Pi' = \{\beta_1, \dots, \beta_n\}$ of roots is called a *root basis* if for every $\alpha \in \Delta$, α or $-\alpha$ lies in $\sum_{i=1}^n \mathbb{Z}_+ \beta_i$.

PROPOSITION [10, 5.9]. *If A is indecomposable, then any root basis Π' of Δ is W -conjugate to Π or $-\Pi$.*

1.28. **LEMMA.** *Let Δ' denote the set of roots of \mathfrak{h}' in \mathfrak{g} and let $\text{res}: \Delta \rightarrow \Delta' \cup \{0\}$ be the restriction map from \mathfrak{h} to \mathfrak{h}' . Suppose that $\lambda \in \text{res}(\Delta'^e)$, $0 \neq x \in \mathfrak{g}_\lambda$, and $0 \neq y \in \mathfrak{g}_{-\lambda}$ such that $[x, y] \in \mathfrak{h}$. Then there exists $\alpha \in \Delta'^e$ with $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$.*

Proof. Let $\alpha \in \Delta'^e$ and $\beta \in \Delta$ such that $\text{res}(\beta) = l \text{res}(\alpha)$, $0 \neq l \in \mathbb{Z}$. Consider the root string $\beta - q\alpha, \dots, \beta + p\alpha$. Then $q - p = (\beta | \alpha^\vee) = 2l$. It

follows that $g_{\beta-l\alpha} \neq 0$. Suppose that A is nonaffine. Then $0 \notin \text{res}(\Delta)$ and so $\beta = l\alpha$. Since $\alpha \in \Delta^{re}$, $l = \pm 1$, and we obtain:

- (i) $\mathbb{Z} \text{res}(\alpha) \cap \Delta' = \{\pm \text{res}(\alpha)\}$.
- (ii) $\text{res}^{-1}(\text{res}(\alpha)) = \{\alpha\}$.

By (ii) of (1), the assertion of the lemma in the nonaffine case is obvious.

Now assume that A is affine. Consider $\text{supp}(x) = \{\alpha \in \Delta \text{ or } \alpha = 0 \mid x \text{ has a nonzero component in } g_\alpha\}$ (resp. $\text{supp}(y)$). There exists $\delta \in \Delta_+^{im}$ such that $\text{res}^{-1}(0) = \mathbb{Z}\delta - \{0\}$. Hence $\text{supp}(x)$ (resp. $\text{supp}(y)$) has minimal and maximal elements. Let $\alpha = \min \text{supp}(x)$ and $\beta = \min \text{supp}(y)$. Note that $(\alpha \mid \beta^\vee) = -2$ and so $[g_\alpha, g_\beta] \neq 0$. Then $\alpha + \beta = \min \text{supp}[x, y]$. Since $[x, y] \in \mathfrak{h}$, we must have that $\alpha = -\beta$. Similarly we have that $\max \text{supp}(x) = -\max \text{supp}(y)$. It follows readily that $\text{supp}(x) = \{\alpha\}$ and $\text{supp}(y) = \{-\alpha\}$.

1.29. LEMMA. *Let σ be an automorphism of g'/c over F such that $\sigma(\mathfrak{h}'/c) = \mathfrak{h}'/c$. Then there exists a basis $\{\beta_1, \dots, \beta_n\}$ of Δ with*

$$\sigma(g_{\pm\alpha_i}) = g_{\pm\beta_i} \quad (i = 1, \dots, n).$$

Proof. Clearly σ induces an automorphism of Δ' . Let $\alpha \in \Delta^{re}$.

Case 1. g is nonaffine. By 1.28(1), $\mathbb{Z} \text{res}(\alpha) \cap \Delta' = \{\pm \text{res}(\alpha)\}$ and as a consequence $\mathbb{Z} \sigma \text{res}(\alpha) \cap \Delta' = \{\pm \sigma \text{res}(\alpha)\}$. It follows that $\sigma \text{res} \alpha \in \text{res}(\Delta^{re})$. By (ii) of 1.28(1), there exists $\beta \in \Delta^{re}$ such that $\{\beta\} = \text{res}^{-1}(\sigma \text{res} \alpha)$. Then $\sigma(g_\alpha) = g_\beta$ and $\sigma(g_{-\alpha}) = g_{-\beta}$.

Case 2. g is affine. Clearly $\sigma \text{res} \alpha \neq 0$ and so $\sigma \text{res} \alpha \in \text{res}(\Delta^{re})$. By 1.28, there exists $\beta \in \Delta^{re}$ with $\sigma(g_{\pm\alpha}) = g_{\pm\beta}$.

Now let $\beta_i \in \Delta$ such that $\sigma g_{\pm\alpha_i} = g_{\pm\beta_i}$ ($i = 1, \dots, n$). Clearly $\{\beta_1, \dots, \beta_n\}$ is a basis of Δ .

1.30. COROLLARY. $N_G(\mathfrak{h}') = N$.

Proof. Let $g \in N_G(\mathfrak{h}')$. By 1.29, there exists a basis $\{\beta_1, \dots, \beta_n\}$ of Δ such that $Ad(g) g_{\pm\alpha_i} = g_{\pm\beta_i}$ ($i = 1, \dots, n$). Replacing g by an element of Ng , we may assume by 1.27 that $\beta_i \in \Delta_+$ ($i = 1, \dots, n$). In this case $g \in N_G(\mathfrak{n}_+) \cap N_G(\mathfrak{n}_-) = B_+ \cap B_- = H$.

1.31. Let \mathfrak{a} be an $ad_{g'/c}$ -diagonalizable subalgebra of g containing \mathfrak{h}' . Then $\mathfrak{a} \subset \mathfrak{h}$.

Proof. There exists $g \in G$ such that $Ad(g)\mathfrak{a} \subset \mathfrak{h}$. Then $Ad(g)\mathfrak{h}' \subset \mathfrak{h} \cap g' = \mathfrak{h}'$ and so g lies in N . Now the assertion is obvious.

1.32. Let $Aut(A)$ be the group of all permutations σ of $\{1, \dots, n\}$ such that $a_{ij} = a_{\sigma_i, \sigma_j}$. We view $Aut(A)$ as a subgroup of $Aut_F(\mathfrak{g}')$ by requiring $\sigma(e_i) = e_{\sigma_i}$ and $\sigma(f_i) = f_{\sigma_i}$ ($i = 1, \dots, n$). After reordering the indices, A decomposes into a direct sum

$$A = \text{diag}(A_1, \dots, A_l)$$

with indecomposable components A_i ($i = 1, \dots, l$). We may assume that

$$\mathfrak{g}(A) = \bigoplus_{i=1}^l \mathfrak{g}(A_i).$$

Let $I = \{i \mid A_i \text{ is not of finite type}\}$. For any subset $J \subset I$, let ω_J denote the automorphism of \mathfrak{g} such that $\omega_J|_{\mathfrak{g}(A_i)} = 1$ ($i \notin J$) and $\omega_J|_{\mathfrak{g}(A_i)}$ is the Chevalley involution ($i \in J$). Let Ω denote the set consisting of all the elements ω_J ($J \subset I$). Set

$$\text{Out}(A) = \Omega \times \text{Aut}(A).$$

The following structure theorem is immediate from 1.25, 1.27, and 1.29:

THEOREM. $Aut_F(\mathfrak{g}'/\mathfrak{c}) = \text{Out}(A) \ltimes Ad(\tilde{H} \ltimes G).$

2. BOUNDED SUBGROUPS AND SUBALGEBRAS

In this section F is an algebraically closed field of characteristic zero.

2.1. Two subspaces V_1 and V_2 of a vector space V over F are *commensurable* if $V_1 \cap V_2$ is of finite codimension in $V_1 + V_2$.

LEMMA. *Let \mathfrak{b}_1 and \mathfrak{b}_2 be commensurable Borel subalgebras of \mathfrak{g} . Then $\mathfrak{b}_1 \cap \mathfrak{b}_2$ has the same codimension in \mathfrak{b}_1 and \mathfrak{b}_2 .*

Proof. We may assume that $\mathfrak{b}_1 = \mathfrak{b}_+$ and $\mathfrak{b}_2 = Ad(g)\mathfrak{b}_+$ with $g \in G$. Write $g = unv$ such that $n \in N$ and $u, v \in U_+$. Then $\mathfrak{b}_+ \cap Ad(g)\mathfrak{b}_+ = Ad(u)(\mathfrak{b}_+ \cap w(\mathfrak{b}_+))$ where w is the image of n in W . Clearly $\mathfrak{b}_1/(\mathfrak{b}_1 \cap \mathfrak{b}_2)$, isomorphic to $\mathfrak{b}_+ / (\mathfrak{b}_+ \cap w(\mathfrak{b}_+))$, has dimension $l(w^{-1})$. Similarly $\mathfrak{b}_2/(\mathfrak{b}_1 \cap \mathfrak{b}_2)$ has dimension $l(w)$. Now the assertion follows.

2.2. A subalgebra \mathfrak{a} of \mathfrak{g} is called *large* if there exists a Borel subalgebra \mathfrak{b} of \mathfrak{g} containing \mathfrak{a} such that $\dim_F(\mathfrak{b}/\mathfrak{a}) < \infty$.

By 2.1, a large subalgebra \mathfrak{a} of \mathfrak{g} has the same codimension in any Borel subalgebra of \mathfrak{g} containing \mathfrak{a} .

2.3. LEMMA. *Let $\alpha_1 \subset \alpha_2$ be large subalgebras of n_+ . If $\alpha_1 \neq \alpha_2$, then the normalizer $n_{\alpha_2}(\alpha_1)$ of α_1 in α_2 contains α_1 properly.*

Proof. The algebra α_1 contains g_α for almost all $\alpha \in \Delta_+$. Choose a positive integer l such that $g_\alpha \subset \alpha_1$ for $ht(\alpha) > l$ and set $m = \bigoplus_{ht(\alpha) > l} g_\alpha$. Then m is an ideal of n_+ and n_+/m is a finite dimensional nilpotent Lie algebra. Since $\alpha_1/m \neq \alpha_2/m$, $\alpha_1/m \subsetneq n_{\alpha_2/m}(\alpha_1/m)$ and so our desired assertion follows.

2.4. LEMMA. *Let α be a large subalgebra of n_+ and α_1 a subalgebra of $n_g(\alpha)$ containing α . If α_1/α is solvable, then there exists a Borel subalgebra b of g containing α_1 .*

Proof. Choose $\lambda \in P_{++}$ and consider the space $V = L(\lambda)^\alpha$. Then V is finite dimensional and $n_g(\alpha)$ -stable; in particular α_1 -stable. By 1.18(3), there exists an α_1 -stable ideal of $F[V]$ which defines the image of $V \cap \mathcal{V}_\lambda$ in the projective space of the lines in V . Since $\pi_\lambda(\alpha_1)|V$ is solvable, α_1 fixes a line Fv with $v \in V \cap \mathcal{V}_\lambda$. Let b be the stabilizer of g at Fv . Then b is a Borel subalgebra of g containing α_1 .

2.5. LEMMA. *Let b_1 and b_2 be Borel subalgebras of g , $n_1 = [b_1, b_1]$, and $n_2 = [b_2, b_2]$. Then $n_1 \cap n_2 = b_1 \cap n_2 = n_1 \cap b_2$.*

Proof. We may assume that $b_1 = b_+$ and $b_2 = Ad(g)(b_\pm)$ with $g \in G$. Write $g = x_1 n x_2$ with $n \in N$, $x_1 \in U_+$, and $x_2 \in U_\pm$. Then $b_+ \cap Ad(g)(n_\pm) = Ad(x_1)(b_+ \cap Ad(n)n_\pm) \subset A(x_1)n_+ = n_+$ and so $b_1 \cap n_2 \subset n_1 \cap n_2$. The opposite inclusion is obvious and the desired equality follows.

2.6. LEMMA. *Let m be a subalgebra of g , b a Borel subalgebra of g , $n = [b, b]$, and n_1 an m -stable large subalgebra of n . Then for any large subalgebra n_2 of n_1 , there exists an m -stable large subalgebra of n contained in n_2 .*

Proof. We may assume that $b = b_+$ and $n = n_+$. Choose $\lambda \in P_{++}$ and $V = L(\lambda)^m$. Then $n_g(n_1)$ acts on V . Since the kernel of the induced map $n_g(n_1) \rightarrow gl(V)$ lies in b_+ , so $n_g(n_1)/n_1$ is finite dimensional. It follows that there exist subalgebras p_1 and p_2 of $n_g(n_1)$ such that

- (i) $p_1 + p_2 = n_g(n_1)$,
- (ii) p_1/n_1 and p_2/n_1 are solvable.

By 2.4, there exist Borel subalgebras b_1 and b_2 of g containing p_1 and p_2 , respectively. Choose Cartan subalgebras h_1 and h_2 of g contained in b_1 and

\mathfrak{b}_2 , respectively. Let Δ_1^+ (resp. Δ_2^+) denote the set of the roots of \mathfrak{h}_1 (resp. \mathfrak{h}_2) in \mathfrak{b}_1 (resp. \mathfrak{b}_2). For a positive integer k , set

$$\mathfrak{b}_1(k) = \bigoplus_{\substack{\alpha \in \Delta_1^+ \\ \text{ht}(\alpha) > k}} \mathfrak{g}_\alpha, \quad \mathfrak{b}_2(k) = \bigoplus_{\substack{\alpha \in \Delta_2^+ \\ \text{ht}(\alpha) > k}} \mathfrak{g}_\alpha,$$

where the height functions are defined with respect to the bases in Δ_1^+ and Δ_2^+ , respectively. Note that the subalgebras \mathfrak{n}_2 , \mathfrak{b}_1 , and \mathfrak{b}_2 are commensurable. There exist positive integers k and l such that $\mathfrak{b}_1(k) \subset \mathfrak{b}_2(l) \subset \mathfrak{n}_2$. We have: $U(\mathfrak{m}) \subset U(\mathfrak{p}_2)U(\mathfrak{p}_1) \subset U(\mathfrak{b}_2)U(\mathfrak{b}_1)$. It follows that $ad(U(\mathfrak{m}))\mathfrak{b}_1(k)$, contained in $\mathfrak{b}_2(l)$, generates an \mathfrak{m} -stable large subalgebra of \mathfrak{n}_2 .

2.7. LEMMA. *Let \mathfrak{m} be a subalgebra of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} , $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and \mathfrak{n}_1 an \mathfrak{m} -stable large subalgebra of \mathfrak{n} . Then for any \mathfrak{m} -stable large subalgebra \mathfrak{a} of \mathfrak{b} , $\mathfrak{a} \cap \mathfrak{n}$ is \mathfrak{m} -stable.*

Proof. By 2.6, there exists an \mathfrak{m} -stable large subalgebra \mathfrak{a}_1 of \mathfrak{b} contained in $\mathfrak{a} \cap \mathfrak{n}$. Let \mathfrak{a}_2 be the ideal of \mathfrak{a} generated by \mathfrak{a}_1 . Clearly \mathfrak{a}_2 is \mathfrak{m} -stable and $\mathfrak{a}_2 \subset \mathfrak{a} \cap \mathfrak{n}$. For $x \in \mathfrak{m}$, $Fx + \mathfrak{a} \subset \mathfrak{n}_\mathfrak{g}(\mathfrak{a}_2)$ and $(Fx + \mathfrak{a})/\mathfrak{a}_2$ is solvable. Hence by 2.4, there exists a Borel subalgebra \mathfrak{b}_1 containing $Fx + \mathfrak{a}$. Note that $[x, \mathfrak{a}] \subset \mathfrak{a} \cap [\mathfrak{b}_1, \mathfrak{b}_1]$. By 2.5, $[x, \mathfrak{a}] \subset \mathfrak{a} \cap \mathfrak{n}(x \in \mathfrak{m})$. Thus $\mathfrak{a} \cap \mathfrak{n}$ is \mathfrak{m} -stable.

2.8. PROPOSITION. *Let M be a subgroup of $Aut_F(\mathfrak{g})$ and \mathfrak{m} a subalgebra of \mathfrak{g} . Assume that M normalizes \mathfrak{m} . The following conditions are equivalent:*

- (i) *Let \mathfrak{p} be the smallest M -stable (resp. \mathfrak{m} -stable) subspace of \mathfrak{g} containing \mathfrak{b}_+ . Then \mathfrak{b}_+ is of finite codimension in \mathfrak{p} .*
- (ii) *There exists an M -stable (resp. \mathfrak{m} -stable) large subalgebra of \mathfrak{n}_+ .*
- (iii) *There exists an M - and \mathfrak{m} -stable large subalgebra of \mathfrak{n}_+ .*
- (iv) *There exist $X \subset \Pi$ of finite type and $g \in G$ such that $Ad(g)\mathfrak{p}_X$ is M - and \mathfrak{m} -stable.*

Proof. (i) \Rightarrow (ii). Let $(\cdot | \cdot)$ be a nondegenerate symmetric $ad_\mathfrak{g}$ -invariant F -bilinear form of \mathfrak{g} . Let \mathfrak{a} denote the orthogonal complement of $\mathfrak{g}' \cap \mathfrak{p}$ in \mathfrak{g}' (resp. \mathfrak{p} in \mathfrak{g}). Then \mathfrak{a} is M -stable (resp. \mathfrak{m} -stable) and $\mathfrak{a} \subset \mathfrak{b}_+$ (resp. $\mathfrak{a} \subset \mathfrak{n}_+$). Let \mathfrak{a}_1 be the smallest subalgebra of \mathfrak{g} containing \mathfrak{a} . Clearly \mathfrak{a}_1 is an M -stable large subalgebra of \mathfrak{b}_+ (resp. \mathfrak{m} -stable large subalgebra of \mathfrak{n}_+). By 2.5, $\mathfrak{a}_1 \cap \mathfrak{n}_+$ is M -stable. Now $\mathfrak{n}_1 = \mathfrak{a}_1 \cap \mathfrak{n}_+$ is an M -stable (resp. \mathfrak{m} -stable) large subalgebra of \mathfrak{n}_+ .

(ii) \Rightarrow (iii). Let \mathfrak{n}_1 (resp. \mathfrak{n}_2) be an \mathfrak{m} -stable (resp. M -stable) large subalgebra of \mathfrak{n}_+ . By 2.6, there exists an \mathfrak{m} -stable large subalgebra \mathfrak{a} of \mathfrak{n}_+ contained in \mathfrak{n}_2 . Let \mathfrak{n}_3 denote the subalgebra generated by $x(\mathfrak{a})(x \in M)$.

Since M normalizes m and a is m -stable, n_3 is an M - and m -stable large subalgebra of n_+ .

(iii) \Rightarrow (iv). We prove this in several steps.

(1) Let a be a maximal M - and m -stable large subalgebra of g commensurable to b_+ and let b be a Borel subalgebra of g containing a . Set $n = [b, b]$ and $a_1 = n \cap a$. By 2.5 and 2.7, a_1 is M - and m -stable. Consider the algebra $p = n_g(a_1)$. Then p/a_1 is finite dimensional. Choose subalgebras a_2, a_3, p_1 , and p_2 of p such that

(i) a_2/a_1 is the radical of p/a_1 ,

(ii) $a_2 \subset a_3 = p_1 \cap p_2$, and p_1/a_2 and p_2/a_2 are opposite Borel subalgebras of p/a_2 .

(2) Clearly $a \subset p$, p is M - and m -stable and so is a_2 . Note that $(a_2 + a)/a_1$ is solvable. By 2.4, $a_2 + a$ is an M - and m -stable large subalgebra of g . Hence the maximality condition of a yields that $a_2 \subset a$.

(3) Let b_1 and b_2 be Borel subalgebra of g containing p_1 and p_2 , respectively. Set $n_1 = [b_1, b_1]$ and $n_2 = [b_2, b_2]$. Then $a_1 = n_1 \cap n_2$. Observe that $(b_1 \cap p)/a_1$ and $(b_2 \cap p)/a_1$ are solvable. Hence by the condition (ii) of (1), $b_1 \cap p = p_1$ and $b_2 \cap p = p_2$. Consider the algebra $l = n_1 \cap n_2 \cap p$. Since $l \subset a_3$ and a_3/a_2 is a Cartan subalgebra of p/a_2 , l is ad_{p/a_2} -diagonalizable. On the other hand, l is ad_{g/a_2} -locally nilpotent. It follows that $l \subset a_2$. By (2) and 2.5, $l \subset a \cap n = a_1$. By 2.5, $a_1 \subset n_1 \cap n_2$. Thus $a_1 = n_{n_1 \cap n_2}(a_1)$ and by 2.3, $a_1 = n_1 \cap n_2$.

(4) Replacing m and M by $Ad(G)$ -conjugates, we may assume that $b_1 = b_+$, $b_2 = w(b_+)$ with $w \in W$ and $h \subset a_3$. Claim: $p_2 = b_2$. Suppose the assertion to be false. There exists an index i such that $w(g_{\alpha_i}) \notin p_2$. Since $h \subset a_3 \subset p_2$, p_2 is h -stable and so $wr_i(b_+) \supset p_2$. By (3), we have that

$$a_1 = n_+ \cap w(n_+) = n_+ \cap wr_i(n_+).$$

Obviously this is a contradiction. Hence $p_2 = b_2$ and similarly $p_1 = b_+$. By 1.5, $n_g(a_1) = p_X$ for a certain $X \subset \Pi$ of finite type. Since $n_g(a_1)$ is M - and m -stable, assertion (iv) follows.

(iv) \Rightarrow (i). Let $X, Y, \subset \Pi$ and $w \in W$ such that $w(p_X) = p_Y$. Replacing w by an element of wW_X , we may assume that $w(\Delta^+) = \Delta^+$. Then $w = 1$ and $p_X = p_Y$. Hence p_X and p_Y are $Ad(G)$ -conjugate if and only if $X = Y$. Let $Aut(A)_X = \{\sigma \in Aut(A) \mid \sigma X = X\}$ and $Aut_F(g; g') = \{\sigma \in Aut_F(g) \mid \sigma|_{g'} = 1\}$. Then M is contained in the group $Ad(g)(Aut(A)_X \ltimes Ad(\tilde{H} \ltimes P_X)) Ad(g^{-1}) \cdot Aut_F(g; g')$. Clearly $m \subset Ad(g) p_X$. Now assertion (i) is immediate.

2.9. For finite type $X \subset \Pi$, the group P_X (resp. the subalgebra p_X of g) and its conjugates are called *finite type parabolic subgroups* of G (resp. *finite type parabolic subalgebras* of g).

THEOREM. *Let M (resp. \mathfrak{m}) be a subgroup of G (resp. subalgebra of \mathfrak{g}). The following conditions are equivalent:*

- (i) M (resp. \mathfrak{m}) is contained in a finite type parabolic subgroup of G (resp. subalgebra of \mathfrak{g}).
- (ii) M is contained in a finite number of double cosets $B_+ w B_+$ ($w \in W$) (resp. there exist finitely many Borel subalgebras $\mathfrak{b}_1, \dots, \mathfrak{b}_l$ commensurable to \mathfrak{b}_+ such that $U(\mathfrak{m}) \subset U(\mathfrak{b}_1) \cdots U(\mathfrak{b}_l)$).
- (iii) M (resp. \mathfrak{m}) is π_Λ -finite for all $\Lambda \in P_+$.
- (iv) There exists $\Lambda \in P_{++}$ such that M (resp. \mathfrak{m}) is π_Λ -finite.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are immediate.

(iv) \Rightarrow (i). Choose a finite dimensional M -submodule (resp. \mathfrak{m} -submodule) V of $L(\Lambda)$ containing v^+ and a finite dimensional \mathfrak{b}_+ -submodule V_1 of $L(\Lambda)$ containing V . Let $\pi: \mathfrak{b}_+ \rightarrow \mathfrak{gl}(V_1)$ denote the homomorphism defined by the restriction map of π_Λ . Set $\mathfrak{p} = \{x \in \mathfrak{g} \mid \pi_\Lambda(x)V = 0\}$. Clearly \mathfrak{p} is M - (resp. \mathfrak{m} -) stable and $\ker(\pi) \subset \mathfrak{p} \subset \mathfrak{b}_+$. By 2.5, $\mathfrak{p} \cap \mathfrak{n}_+$ is an M -stable large subalgebra of \mathfrak{n}_+ . From 2.8, (i) follows for M . Since \mathfrak{p} is a large subalgebra of \mathfrak{b}_+ , $L(\Lambda)^\mathfrak{p}$ is finite dimensional. Clearly $v^+ \in V \subset L(\Lambda)^\mathfrak{p}$. Then $n_{\mathfrak{g}}(\mathfrak{p})/\mathfrak{p}$ acts faithfully on $L(\Lambda)^\mathfrak{p}$. Choose subalgebras \mathfrak{p}_1 and \mathfrak{p}_2 of $n_{\mathfrak{g}}(\mathfrak{p})$ such that

$$n_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}_1 + \mathfrak{p}_2, \tag{1}$$

$\mathfrak{p}_1/\mathfrak{p}$ and $\mathfrak{p}_2/\mathfrak{p}$ are solvable.

The same argument as 2.4 implies that \mathfrak{p}_1 and \mathfrak{p}_2 are large subalgebras of \mathfrak{g} . Choose Borel subalgebras \mathfrak{b}_1 and \mathfrak{b}_2 containing \mathfrak{p}_1 and \mathfrak{p}_2 , respectively. We have that $U(\mathfrak{m}) \subset U(n_{\mathfrak{g}}(\mathfrak{p})) = U(\mathfrak{p}_1) U(\mathfrak{p}_2) \subset U(\mathfrak{b}_1) U(\mathfrak{b}_2)$. Note that $\mathfrak{b}_1, \mathfrak{b}_2$, and \mathfrak{b}_+ are commensurable. Then condition 2.8(i) holds for \mathfrak{m} . By 2.8, (i) follows for \mathfrak{m} .

2.10. For $X \subset \Pi$, set $\mathfrak{p}_X^- = \omega(\mathfrak{p}_X)$ and $P_X^- = \omega(P_X)$ where ω is the Chevalley involution of \mathfrak{g} . The following conjugacy theorem, due to Kac and Peterson [16], is immediate from 2.9.

THEOREM. *Let M (resp. \mathfrak{m}) be a subgroup of G (resp. a subalgebra of \mathfrak{g}). The following conditions are equivalent:*

- (i) M (resp. \mathfrak{m}) is $Ad_{\mathfrak{g}}$ -finite (resp. $ad_{\mathfrak{g}}$ -finite).
- (ii) There exist finite type $X, Y \subset \Pi$ and $w \in W$ such that M (resp. \mathfrak{m}) can be conjugated into $P_X \cap w P_Y^- w^{-1}$ (resp. $\mathfrak{p}_X \cap w(\mathfrak{p}_Y^-)$).

2.11. Algebraic subalgebras and subgroups. A subalgebra \mathfrak{a} of \mathfrak{g}' is called an *algebraic subalgebra* if for every integrable representation (V, π) of \mathfrak{g}' , \mathfrak{a} satisfies the following conditions:

- (i) \mathfrak{a} is π -finite.
- (ii) For any \mathfrak{a} -stable finite dimensional subspace W of V , the algebra $\pi(\mathfrak{a})|W$ is algebraic. (1)

Given an algebraic subalgebra \mathfrak{a} of \mathfrak{g}' , a subgroup A of G is called a *connected algebraic subgroup* of G with Lie algebra \mathfrak{a} if for every finite dimensional \mathfrak{a} -stable subspace W of any integrable \mathfrak{g}' -module (V, π) , W is A -stable and $\pi(A)|W$ is the connected algebraic subgroup of $GL(W)$ with Lie algebra $\pi(\mathfrak{a})|W$.

LEMMA. Let $\mathfrak{a}_i (i=1, \dots, l)$ be algebraic subalgebras of \mathfrak{g}' and A_i connected algebraic subgroups of G with Lie algebras \mathfrak{a}_i , respectively. Suppose that $\mathfrak{a}_1, \dots, \mathfrak{a}_l$ generate an $ad_{\mathfrak{g}'}$ -finite subalgebra \mathfrak{p} of \mathfrak{g}' . Then we have:

- (i) \mathfrak{p} is an algebraic subalgebra of \mathfrak{g}' .
- (ii) The subgroup generated by A_1, \dots, A_l is a connected algebraic subgroup of G with Lie algebra \mathfrak{p} .

Proof. Let m be a positive integer. A subalgebra of $gl(m, F)$ (resp. subgroup of $GL(m, F)$) generated by algebraic subalgebras (resp. connected algebraic subgroups) is algebraic (resp. connected and algebraic). Now the assertion is immediate from the definition.

2.12. For $\alpha \in \Delta^{re}$ and any integrable \mathfrak{g}' -module (V, π) , \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$ are π -locally nilpotent. By 2.11(i), the algebras $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, F\alpha^\vee$, and $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + F\alpha^\vee$ are algebraic subalgebras of \mathfrak{g}' . For $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$, there is a unique homomorphism $\varphi: SL(2, F) \rightarrow G$ satisfying:

$$\varphi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp te, \quad \varphi\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tf, \quad (t \in F).$$

Let $G_{(\alpha)} = \varphi(SL(2, F))$ and $H_\alpha = \varphi(\{\text{diag}(t, t^{-1}) | t \in F^\times\})$. Then $G_{(\alpha)}$ and H_α are independent of the choice of e and f . By 2.11(ii), the groups $U_\alpha, U_{-\alpha}, H_\alpha$, and $G_{(\alpha)}$ are connected algebraic subgroups of G with Lie algebras $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, F\alpha^\vee$, and $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + F\alpha^\vee$, respectively.

For $X \in \Pi$, set $\Delta_X = \Delta \cap \mathbb{Z}X$ and $\mathfrak{p}'_X = \mathfrak{g}' \cap \mathfrak{p}_X$. Let U_{-X} denote the subgroup generated by the $U_\alpha (\alpha \in \Delta^{re} \cap (-\mathbb{Z}_+ X))$ and U^{-X} the smallest normal subgroup of U_- containing all the U_β with $\beta \in -\Delta_+^{re}$ and $\beta \notin \mathbb{Z}X$.

PROPOSITION. For finite type $X, Y \subset \Pi$ and $w \in W$, the group $P_X \cap wP_Y^- w^{-1}$ is a connected algebraic subgroup of G with Lie algebra $\mathfrak{p}'_X \cap w(\omega(\mathfrak{p}'_Y))$.

Proof. We establish the assertion in several steps.

(1) Let \mathfrak{p} denote the algebra $\mathfrak{p}'_X \cap w(\omega(\mathfrak{p}'_Y))$. Then $\mathfrak{p} = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \psi} \mathfrak{g}_\alpha$ where $\psi = (\Delta^+ \cup \Delta_X) \cap w(-\Delta^+ \cup \Delta_Y)$. Note that ψ is a finite subset of Δ^{re} . Hence by 2.11(i), \mathfrak{p} is an algebraic subalgebra of \mathfrak{g}' .

(2) For $\alpha \in \Delta^{re}$, as a group with operators \tilde{H} , U_α is simple. For $\alpha, \beta \in \Delta^{re}$, the groups U_α and U_β are isomorphic with operators \tilde{H} if and only if $\alpha = \beta$.

(3) The group $W_X \cap wW_Y w^{-1}$ is generated by the reflections $r_\alpha (\alpha \in \Delta^{re} \cap ZX \cap ZWY)$. Consider the Tits cone C . Choose $v_1, v_2 \in C$ such that $\Delta \cap ZX = \{\alpha \in \Delta \mid \langle v_1, \alpha \rangle = 0\}$ and $\Delta \cap ZWY = \{\alpha \in \Delta \mid \langle v_2, \alpha \rangle = 0\}$. Let $[v_1, v_2]$ denote the line segment connecting v_1 and v_2 . For $\alpha \in \Delta$, α vanishes on $[v_1, v_2]$ if and only if $\alpha \in ZX \cap ZWY$. It follows that there exists $v \in [v_1, v_2]$ such that $\Delta \cap ZX \cap ZWY = \{\alpha \in \Delta \mid \langle v, \alpha \rangle = 0\}$. Let W_1 be the group generated by the $r_\alpha (\alpha \in \Delta^{re} \cap ZX \cap ZWY)$. Clearly $W_1 \subset W_X \cap wW_Y w^{-1}$. On the other hand, $W_X \cap wW_Y w^{-1}$ fixes v and so the opposite inclusion follows.

(4) Let $\psi_1 = \Delta^+ \cap w(-(\Delta^+ - ZY))$ and $\psi_2 = w(\Delta_Y) \cap \Delta^+$. Note that $wU^{-Y} w^{-1}$ is an \tilde{H} -invariant large subgroup of $wU_- w^{-1}$. By 1.8 and 1.11, we have that

$$wP_Y^- w^{-1} = U_2^+ wW_Y w^{-1} U_2^- (wU^{-Y} w^{-1} \cap U_-) U_1,$$

where $U_1 = \prod_{\alpha \in \psi_1} U_\alpha$ and $U_2^\pm = \prod_{\alpha \in \pm \psi_2} U_\alpha$. It follows readily that $P_X \cap wP_Y^- w^{-1}$ is generated by the groups $U_\alpha (\alpha \in \psi_1 \cup \psi_2)$, $U_{-X} \cap (U_2^- \ltimes (wU^{-Y} w^{-1} \cap U_-))$ and $W_X \cap wW_Y w^{-1}$. The group $U_{-X} \cap (U_2^- \ltimes (wU^{-Y} w^{-1} \cap U_-))$ is \tilde{H} -invariant. As a consequence of (2), it is generated by the U_α such that U_α is contained either in $U_{-X} \cap U_2^-$ or $U_{-X} \cap wU^{-Y} w^{-1}$. Note that for $\alpha \in \Delta^{re}$, r_α is contained in the group generated by H , U_α , and $U_{-\alpha}$. Now by (3), it follows that $P_X \cap wP_Y^- w^{-1}$ is generated by the groups H and $U_\alpha (\alpha \in \psi)$. Our desired assertion is immediate from 2.11 and (1).

2.13. LEMMA. Let \mathfrak{a} be an algebraic subalgebra of \mathfrak{g}' , A a connected algebraic subgroup of G with Lie algebra \mathfrak{a} , and W a finite dimensional \mathfrak{a} -submodule of a fixed integrable \mathfrak{g}' -module (V, π) . Suppose that A and \mathfrak{a} act on W faithfully. Then we have:

(i) Let P (resp. \mathfrak{p}) be a subgroup (resp. subalgebra) of A (resp. \mathfrak{a}) such that $\pi(P)|W$ is the connected algebraic group with Lie algebra $\pi(\mathfrak{p})|W$. Then P is a connected algebraic subgroup of G with Lie algebra \mathfrak{p} .

(ii) If \mathfrak{a}_1 and \mathfrak{a}_2 are algebraic subalgebras of \mathfrak{g}' contained in \mathfrak{a} , then $\mathfrak{a}_1 \cap \mathfrak{a}_2$ is an algebraic subalgebra of \mathfrak{g}' .

Proof. Let W' be a \mathfrak{p} -stable finite dimensional subspace of an integrable \mathfrak{g}' -module (V', π') . We show that $\pi'(P)|W'$ is the connected algebraic subgroup of $GL(W')$ with Lie algebra $\pi'(\mathfrak{p})|W'$. Replacing W' by the \mathfrak{a} -submodule generated by W' , we may assume that W' is \mathfrak{a} -stable. Consider the \mathfrak{g}' -module $(V \oplus V', \pi \oplus \pi')$. Let $A_W, A_{W'}$, and $A_{W \oplus W'}$ denote the images of A in $GL(W), GL(W')$, and $GL(W \oplus W')$, respectively. We have the following commutative diagram

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow \alpha & \downarrow \beta & \searrow \gamma & \\
 A_{W'} & \xleftarrow{\sigma} & A_{W \oplus W'} & \xrightarrow{\tau} & A_{W'}
 \end{array} \tag{1}$$

with maps defined in the obvious manner. Since γ is bijective, so is τ . Note that σ and τ , defined by restriction maps, are rational homomorphisms. It follows that τ is an isomorphism and $\sigma \circ \tau^{-1}$ is a rational homomorphism. Clearly $\pi'(P)|W'$ and $\pi'(\mathfrak{p})|W'$ are the images of $\pi(P)|W$ and $\pi(\mathfrak{p})|W$, respectively, under $\sigma \circ \tau^{-1}$. Hence $\pi'(P)|W'$ is the connected algebraic subgroup of $GL(W')$ with Lie algebra $\pi'(\mathfrak{p})|W'$. By definition, (i) follows.

Note that $\pi(\mathfrak{a}_1 \cap \mathfrak{a}_2)|W = (\pi(\mathfrak{a}_1)|W) \cap (\pi(\mathfrak{a}_2)|W)$ is algebraic. From (i), $\mathfrak{a}_1 \cap \mathfrak{a}_2$ is an algebraic subalgebra of \mathfrak{g}' .

2.14. PROPOSITION. (i) If \mathfrak{a} is an algebraic subalgebra of \mathfrak{g}' , then there is a unique connected algebraic subgroup A of G with Lie algebra \mathfrak{a} ; moreover there exists an integrable \mathfrak{g}' -module (V, π) with a finite dimensional \mathfrak{a} -submodule M on which \mathfrak{a} and A act faithfully.

(ii) Let \mathfrak{a}_1 and \mathfrak{a}_2 be algebraic subalgebras of \mathfrak{g}' . If A is a connected algebraic subgroup of G with Lie algebra \mathfrak{a}_1 (resp. \mathfrak{a}_2), then $\mathfrak{a}_1 = \mathfrak{a}_2$.

Proof. We prove the assertion in several steps.

(1) If A is a connected algebraic subgroup of G with Lie algebra \mathfrak{a} , then for any $g \in G, gAg^{-1}$ is a connected algebraic subgroup of G with Lie algebra $Ad(g)\mathfrak{a}$.

(2) Let A be a connected algebraic subgroup of G with Lie algebra \mathfrak{a} and (V, π) any integrable \mathfrak{g}' -module. For $v \in V, v$ is fixed by \mathfrak{a} if and only if it is fixed by A . For a finite dimensional subspace W of V, W is \mathfrak{a} -stable if and only if it is A -stable.

(3) By definition, \mathfrak{a} is $ad_{\mathfrak{g}}$ -finite. By 2.10 and (1), we may assume that there exist finite type $X, Y \subset \Pi$ and $w \in W$ such that

$$\mathfrak{a} \subset \mathfrak{p}_X \cap w(\mathfrak{p}_Y^-).$$

Now choose an integrable \mathfrak{g}' -module (V, π) such that there exist $v_1, v_2 \in V$ and a finite dimensional $(\mathfrak{p}_X \cap w(\mathfrak{p}_Y^-) \cap \mathfrak{g}')$ -submodule M of V satisfying:

- (a) The stabilizer of \mathfrak{g}' (resp. G) at Fv_1 is $\mathfrak{p}_X \cap \mathfrak{g}'$ (resp. P_X).
- (b) The stabilizer of \mathfrak{g}' (resp. G) at Fv_2 is $\mathfrak{g}' \cap w(\mathfrak{p}_Y^-)$ (resp. $wP_Y^-w^{-1}$).
- (c) $P_X \cap wP_Y^-w^{-1}$ and $\mathfrak{p}_X \cap w(\mathfrak{p}_Y^-) \cap \mathfrak{g}'$ act faithfully on M .

Let A be the subgroup of $P_X \cap wP_Y^-w^{-1}$ such that $\pi(A)|M$ is the connected algebraic subgroup of $GL(M)$ with Lie algebra $\pi(\mathfrak{a})|M$. Then by 2.13(i), A is a connected algebraic subgroup of G with Lie algebra \mathfrak{a} . Clearly such A in $P_X \cap wP_Y^-w^{-1}$ is unique. By (2) and conditions (a) and (b), any connected algebraic subgroup of G with Lie algebra \mathfrak{a} lies in $P_X \cap wP_Y^-w^{-1}$. Thus the uniqueness of A follows. By (2), \mathfrak{a}_1 normalizes \mathfrak{a}_2 . Hence $\mathfrak{a}_1 + \mathfrak{a}_2$ is a subalgebra of \mathfrak{g}' . Since $U(\mathfrak{a}_1 + \mathfrak{a}_2) = U(\mathfrak{a}_1)U(\mathfrak{a}_2)$, $\mathfrak{a}_1 + \mathfrak{a}_2$ is $ad_{\mathfrak{g}'}$ -finite. By 2.11, A is a connected algebraic subgroup of G with Lie algebra $\mathfrak{a}_1 + \mathfrak{a}_2$. A simple dimension argument yields $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathfrak{a}_1 + \mathfrak{a}_2$.

The following corollary is immediate from 2.13 and 2.14.

2.15. COROLLARY. (i) *Let A be a connected algebraic subgroup of G with Lie algebra \mathfrak{a} . Then $N_G(\mathfrak{a}) = N_G(A)$.*

(ii) *Let A_i be a connected algebraic subgroup of G with Lie algebra $\mathfrak{a}_i (i=1, 2)$. Then $A_1 \subset A_2$ if and only if $\mathfrak{a}_1 \subset \mathfrak{a}_2$.*

2.16. Let A be a connected algebraic subgroup of G with Lie algebra \mathfrak{a} . The group A is called a *subtorus* of G if \mathfrak{a} is $ad_{\mathfrak{g}'}$ -diagonalizable; A is called *unipotent* if \mathfrak{a} is $ad_{\mathfrak{g}'}$ -locally nilpotent.

LEMMA. *Let A be a connected algebraic subgroup of G with Lie algebra \mathfrak{a} . Then A is generated by subtori and unipotent subgroups.*

Proof. Let (V, π) be an integrable \mathfrak{g}' -module such that there exists a finite dimensional \mathfrak{a} -submodule W on which A and \mathfrak{a} acts faithfully. The assertion is true for $\pi(A)|W$. Note that in the commutative diagram 2.13(1), $\sigma \circ \tau^{-1}$ carries tori (resp. unipotent subgroups) to groups of the same kind. The lemma now is immediate from 2.13.

2.17. COROLLARY. *Let A be a connected algebraic subgroup of G with Lie algebra \mathfrak{a} . Then $Z_G(\mathfrak{a}) = Z_G(A)$.*

Proof. By 2.16 and 2.15(ii), we may assume that A is a torus or A is unipotent.

Case 1. A is a torus. Let (V, π) be any integrable \mathfrak{g}' -module. Let P (resp. P') be the set of weights of \mathfrak{a} (resp. A) in V . Note that $v \in V$ (resp. a line Fv in V) is fixed by \mathfrak{a} if and only if it is fixed by A . Hence there is a bijection $\tau: P \rightarrow P'$ such that $V_\lambda = V_{\tau(\lambda)}$ ($\lambda \in P$). Observe that $g \in Z_G(\mathfrak{a})$ (resp. $Z_G(A)$) if and only if g leaves invariant V_λ for all $\lambda \in P$ (resp. P') and all integrable \mathfrak{g}' -module (V, π) . Now the assertion is obvious.

Case 2. A is unipotent.

Let $X \in \mathfrak{g}'$ be $ad_{\mathfrak{g}'}$ -locally nilpotent and $g \in G$. Then $g \exp Xg^{-1} = \exp X$ if and only if $Ad(g)X = X$. Hence the assertion in this case follows easily.

2.18. A subgroup M of G is called an algebraic subgroup of G with Lie algebra \mathfrak{m} if the following conditions hold:

- (i) \mathfrak{m} is an algebraic subalgebra of \mathfrak{g}' .
- (ii) The connected algebraic subgroup M^0 of G with Lie algebra \mathfrak{m} is a subgroup of M of finite index.

We write $L(M)$ for the Lie algebra of M .

LEMMA. *Let M and N be algebraic subgroups of G . Then $M \cap N$ is an algebraic subgroup of G with Lie algebra $L(M) \cap L(N)$.*

Proof. We prove the lemma in several steps.

(1) Let A be an algebraic subgroup of G with Lie algebra \mathfrak{a} and (V, π) any integrable \mathfrak{g}' -module. Let l be a line in V and A_l (resp. \mathfrak{a}_l) the stabilizer of A (resp. \mathfrak{a}) at l . Then A_l is an algebraic subgroup of G with Lie algebra \mathfrak{a}_l .

Enlarging V if necessary, we may assume that there exists a finite dimensional A -submodule W , containing l , on which A and \mathfrak{a} act faithfully on W . Clearly $(\pi(A)|W)$, is an algebraic subgroup of $GL(W)$ with Lie algebra $(\pi(\mathfrak{a})|W)_l$. Then by 2.13(i), the assertion is immediate.

(2) Since M is $ad_{\mathfrak{g}'}$ -finite, we may assume that there exist finite type $X, Y \in \Pi$ and $w \in W$ such that

$$M \subset P_X \cap wP_Y^- w^{-1}, \quad L(M) \subset \mathfrak{p}_X \cap w(\mathfrak{p}_Y^-).$$

Choose an integrable \mathfrak{g}' -module (V, π) such that there exist $v_1, v_2 \in V$ satisfying the conditions:

- (a) The stabilizer of \mathfrak{g}' (resp. G) at Fv_1 is $\mathfrak{p}_X \cap \mathfrak{g}'$ (resp. P_X).
- (b) The stabilizer of \mathfrak{g}' (resp. G) at Fv_2 is $\mathfrak{g}' \cap w(\mathfrak{p}_Y^-)$ (resp. $wP_Y^- w^{-1}$).

Then by (1), $N \cap P_X \cap wP_Y^- w^{-1}$ is an algebraic subgroup of G with Lie algebra $L(N) \cap \mathfrak{p}_X \cap w(\mathfrak{p}_Y^-)$.

(3) By (2), we may assume that there exists a connected algebraic subgroup A of G with Lie algebra \mathfrak{a} such that M and N lie in A . Choose an integrable \mathfrak{g}' -module (V, π) such that there exists a finite dimensional \mathfrak{a} -submodule W on which A and \mathfrak{a} act faithfully. Clearly $\pi(M \cap N)|W$ is an algebraic subgroup of $GL(W)$ with Lie algebra $\pi(L(M) \cap L(N))|W$. By 2.13, the lemma follows.

2.19. COROLLARY. *Let $M_i (i \in I)$ be a family of algebraic subgroups of G . Then $\bigcap_{i \in I} M_i$ is an algebraic subgroup of G with Lie algebra $\bigcap_{i \in I} L(M_i)$.*

Proof. For any finite subset J of I , $M_J = \bigcap_{j \in J} M_j$ is an algebraic subgroup of G . Choose J such that $L(M_J)$ has the least dimension and the index $[M_J : M_J^0]$ is minimal. Then $\bigcap_{i \in I} M_i = M_J$ and $L(\bigcap_{i \in I} M_i) = \bigcap_{i \in I} L(M_i)$.

2.20. Let M be a subgroup of G . Assume that M is $Ad_{\mathfrak{g}}$ -finite. By 2.10, 2.12, and 2.19, there exists a unique minimal algebraic subgroup of G containing M . We write $Cl(M)$ for the minimal algebraic subgroup of G containing M .

LEMMA. *Let M be an algebraic subgroup of G and M_1 a subgroup of M such that $M = Cl(M_1)$. Then we have that $Z_G(M_1) = Z_G(M)$ and $N_G(M_1) \subset N_G(M)$.*

Proof. Let (V, π) be any integrable \mathfrak{g}' -module. For $g \in Z_G(M_1)$, consider the \mathfrak{g}' -module $(V \oplus V, \pi \oplus \pi^g)$ where

$$\pi^g(x)v = \pi(gxg^{-1})v \quad \{x \in G, v \in V\}.$$

Let W denote the diagonal subspace $\{(v, v) | v \in V\}$ of $V \oplus V$. Clearly W is M_1 -stable. Since $M = Cl(M_1)$, W is M -stable. Then $\pi^g|W = \pi|W$ and so $g \in Z_G(M)$. Clearly $N_G(M_1) \subset N_G(M)$ follows from the uniqueness of $Cl(M_1)$.

3. ABSTRACT AUTOMORPHISMS OF G

In this section, F is a field of characteristic zero with algebraic closure \bar{F} .

3.1. Extension of base fields. Let $k \subset K$ be fields of characteristic zero and A be a symmetrizable generalized Cartan matrix. Let $\mathfrak{g}(k)$ (resp. $G(k)$, $L(A)(k)$, $L^*(A)(k)$, etc.) denote the Kac–Moody algebra $\mathfrak{g}(A)$ (resp. the

corresponding objects) over k . Then $\mathfrak{g}(K) = \mathfrak{g}(k) \otimes_k K$, and for $A \in P_+ \cap \mathfrak{h}^*(k)$, $L(A)(K) = L(A)(k) \otimes_k K$ and $L^*(A)(K) = L^*(A)(k) \otimes_k K$. For any $\tau \in \text{Aut}_k(K)$, τ induces naturally an automorphism of $\mathfrak{g}(K)$ (resp. $G(K)$, $L(A)(K)$, $L^*(A)(K)$, etc.), also denoted by τ , satisfying the following conditions:

- (i) For $\lambda \in K$ and $x \in \mathfrak{g}(K)$ (resp. $L(A)(K)$, $L^*(A)(K)$), $\tau(\lambda x) = \tau(\lambda) \tau(x)$.
- (ii) For $x \in \mathfrak{g}(K)$ (resp. $G(K)$) and $v \in L(A)(K)$, $\tau(\pi_A(x) v) = \pi_A(\tau(x)) \tau(v)$.
- (iii) For $x \in \mathfrak{g}(K)$ (resp. $G(K)$) and $v \in L^*(A)(K)$, $\tau(\pi_A^*(x) v) = \pi_A^*(\tau(x)) \tau(v)$.
- (iv) For $g \in G(K)$ and $x \in \mathfrak{g}(K)$, $\tau(\text{Ad}(g) x) = \text{Ad}(\tau(g)) \tau(x)$.
- (v) For $x_i \in \mathfrak{g}_{\beta_i}(k)$, $\lambda_i \in K$, and $\beta_i \in \Delta^{re}(i = 1, \dots, l)$, $\tau(\exp \lambda_1 x_1 \cdots \exp \lambda_l x_l) = \exp \tau(\lambda_1) x_1 \cdots \exp \tau(\lambda_l) x_l$.

3.2. LEMMA. *Let $\text{Gal}(\bar{F}/F)$ denote the Galois group of F -automorphisms of \bar{F} . Then the fixed point group*

$$G(\bar{F})^{\text{Gal}(\bar{F}/F)} = \{g \in G(\bar{F}) \mid \tau(g) = g \text{ for all } \tau \in \text{Gal}(\bar{F}/F)\}$$

coincides with $G(F)$.

Proof. Let $g \in G(\bar{F})^{\text{Gal}(\bar{F}/F)}$. Write $g = xny$ with $n \in N$, $y \in U_+$, and $x \in U_- \cap nU_-n^{-1}$. Note that

$$N(\bar{F}) = N(F) H(\bar{F}). \tag{1}$$

Since U_{\pm} are $\text{Gal}(\bar{F}/F)$ -stable, we have that $x, y, n \in G(\bar{F})^{\text{Gal}(\bar{F}/F)}$. Choose an element $A \in P_{++} \cap \mathfrak{h}^*(F)$ and $0 \neq v^+ \in L(A)_+(F)$. Consider the element $\pi_A(x) v^+$. Clearly it is fixed by $\text{Gal}(\bar{F}/F)$ and so lies in $\mathcal{V}(A)_-(F)$ (for definition of $\mathcal{V}(A)_-$, see [15]). By [15, (v) of Theorem 1], there exist $\lambda \in F^\times$ and $x_1 \in U_-(F)$ such that $\pi_A(x) v^+ = \lambda \pi_A(x_1) v^+$. By 1.6(i), $\lambda = 1$ and $x = x_1 \in U_-(F)$. Observe that ω commutes with $\text{Gal}(\bar{F}/F)$. We also have that $y \in U_+(F)$. To show that $n \in N(F)$, by (1) we may assume that $n \in H(\bar{F})$. However, $H(\bar{F})$ is $\text{Gal}(\bar{F}/F)$ -equivariantly isomorphic to $(\bar{F}^\times)^n$ and so the assertion is obvious in this case.

3.3. PROPOSITION. *For $g \in G(\bar{F})$, the following conditions are equivalent:*

- (i) *There exist $x \in G(\bar{F})$ and $w \in W$ such that*

$$xgx^{-1} \in B_+(\bar{F}) \cap wB_-(\bar{F})w^{-1}.$$

- (ii) For any integrable representation (V, π) of $\mathfrak{g}'(\bar{F})$, g is π -finite.
- (iii) For any $\lambda \in P_+$, g is π_λ - and π_λ^* -finite.
- (iv) There exists $\lambda \in P_{++}$ such that g is π_λ - and π_λ^* -finite.
- (v) g is $Ad_{\mathfrak{g}'(\bar{F})}$ -finite.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (v). By 2.9, there exist finite type parabolic subgroups P_1 and P_2 of G such that $g \in P_1 \cap \omega(P_2)$. By 2.10, assertion (v) follows.

(v) \Rightarrow (i). Let L be a connected algebraic group defined over \bar{F} and τ a rational endomorphism of L . A result of Steinberg [26] yields that there exist τ -stable Borel subgroups of L . From 2.10, there exist $g_1, g_2 \in G$ such that $g_1 B_+ g_1^{-1}$ and $g_2 B_- g_2^{-1}$ are normalized by g . Hence $g \in g_1 B_+ g_1^{-1} \cap g_2 B_- g_2^{-1}$. Write $g_1^{-1} g_2 = unv$ with $u \in B_+$, $v \in B_-$ and $n \in N$. Set $x = (g_1 u)^{-1}$ and w the image of n in W . Then xgx^{-1} lies in $B_+(\bar{F}) \cap wB_-(\bar{F})w^{-1}$.

3.4. An element $g \in G(\bar{F})$ is called *unipotent* (resp. *semisimple*) if for every integrable representation (V, π) of $\mathfrak{g}'(\bar{F})$, g is π -locally unipotent (resp. V is a sum of one-dimensional $\pi(g)$ -submodules).

An element $g \in G(\bar{F})$ is said to have a *Jordan decomposition* if there exist semisimple s and unipotent u in $G(\bar{F})$ such that $g = su$ and $su = us$. The Jordan decomposition if it exists is unique.

LEMMA. *We have:*

- (i) If $g \in G(\bar{F})$ has a Jordan decomposition, then g is $Ad_{\mathfrak{g}'(\bar{F})}$ -finite.
- (ii) If $g \in G(F)$ is $Ad_{\mathfrak{g}'}$ -finite, there exist $s, u \in G(F)$ and $x \in \mathfrak{g}'(F)$ such that
 - (a) s is semisimple, u is unipotent, and $g = su = us$,
 - (b) x is $ad_{\mathfrak{g}'}$ -locally nilpotent and $u = \exp x$.

Proof. Assertion (i) is immediate from the definitions. First we establish assertion (ii) for $F = \bar{F}$. By 3.3, we may assume that $g \in B_+ \cap wB_- w^{-1}$ with $w \in W$. Set $B_w = B_+ \cap wB_- w^{-1}$ and $U_w = U_+ \cap wU_- w^{-1}$. Clearly $B_w = H \times U_w$ is a connected algebraic solvable subgroup of G with Lie algebra $\mathfrak{b}_+ \cap \omega(\mathfrak{b}_-)$. Note that H (resp. U_w) is $Ad_{\mathfrak{g}'}$ -diagonalizable (resp. $Ad_{\mathfrak{g}'}$ -locally unipotent). It follows that g has a Jordan decomposition $g = su$ in B_w . Since $U_w = \exp(\mathfrak{n}_+ \cap \omega(\mathfrak{n}_-))$, there exists $x \in \mathfrak{n}_+ \cap \omega(\mathfrak{n}_-)$ such that $u = \exp x$. Now return to the general case $F \subset \bar{F}$. By the uniqueness of Jordan decomposition, we have that $\tau(s) = s$ and $\tau(u) = u$ for $\tau \in \text{Gal}(\bar{F}/F)$. By 3.2, $s, u \in G(F)$. Then $\exp x = \exp \tau(x)$ for $\tau \in \text{Gal}(\bar{F}/F)$ and so $x \in \mathfrak{g}'(F)$.

3.5. LEMMA. *Let V be a finite dimensional vector space over F and $\pi: SL(2, \mathbb{Q}) \rightarrow GL(V)$ a nontrivial homomorphism. Then π can be extended to a rational homomorphism, denoted again by π , from $SL(2, F)$ into $GL(V)$; moreover there exist nilpotent elements X, Y and diagonalizable Z in $gl(V)$ satisfying the following conditions:*

- (i) $[X, Y] = Z, [Z, X] = 2X, [Z, Y] = -2Y.$
- (ii) $\pi\left(\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}\right) = \exp tX, \pi\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tY (t \in F).$

(iii) *The Zariski closure of the images of the diagonal matrices is a one-dimensional F -split torus with Lie algebra $\bar{F}Z$.*

Proof. Let U_+ (resp. U_-) denote the subgroup of $SL(2, \mathbb{Q})$ of unipotent upper (resp. lower) triangular matrices and H be the subgroup of $SL(2, \mathbb{Q})$ of diagonal matrices. First we show that $\pi(U_+)$ and $\pi(U_-)$ are unipotent. Suppose the assertion to be false; say $\pi(U_+)$ is not unipotent. Set $\bar{V} = V \otimes_F \bar{F}$. Since U_+ is commutative, there exists a nontrivial homomorphism $\lambda: U_+ \rightarrow \bar{F}^\times$ such that

$$\bar{V}_\lambda = \{v \in \bar{V} \mid \pi(x)v = \lambda(x)v, x \in U_+\} \neq 0.$$

For $t \in H, \pi(t)\bar{V}_\lambda = \bar{V}_{t\lambda}$ where $(t\lambda)(x) = \lambda(t^{-1}xt)$ ($x \in U_+$). Since V is finite dimensional, there exist only finitely many such \bar{V}_λ . Hence the stabilizer $H^\lambda = \{t \in H \mid t\lambda = \lambda\}$ is a subgroup of H of finite index. Now choose $t \in H^\lambda$ with $t^2 \neq 1$. Then $U_+ = [t, U_+] = \{txt^{-1}x^{-1} \mid x \in U_+\}$. This implies that for $x \in U_+, \lambda(x) = 1$. This contradicts our choice of λ . Hence $\pi(U_+)$ and $\pi(U_-)$ are unipotent. Then there exist nilpotent $X, Y \in gl(V)$ such that

$$\pi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp tX, \quad \pi\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tY (t \in \mathbb{Q}).$$

Note that $\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix} (t \in \mathbb{Q}^\times)$. It follows that

$$\exp((\exp t\text{ad}(X))(-t^{-1}Y)) = \exp((\exp t^{-1}\text{ad}(Y))(tX)).$$

Since \exp is injective on nilpotent elements, we have that

$$(\exp t\text{ad}(X))(-t^{-1}Y) = (\exp t^{-1}\text{ad}(Y))(tX). \tag{1}$$

From (1), we have the relation

$$\begin{aligned} -t^{-1}Y - [X, Y] - \frac{t}{2}[X, [X, Y]] - \dots \\ = tX + [Y, X] + \frac{t^{-1}}{2}[Y, [Y, X]] + \dots \quad (t \in \mathbb{Q}^\times). \end{aligned} \tag{2}$$

Note the summations in (2) are finite. Set $Z = [X, Y]$. It follows easily that

$$[X, Y] = Z, \quad [Z, X] = 2X, \quad [Z, Y] = -2Y.$$

Then we have a homomorphism $sl(2, F) \rightarrow gl(V)$ sending $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to X , Y , and Z , respectively. Clearly the corresponding homomorphism of $SL(2, F)$ into $GL(V)$ is the desired extension of π . Let $d\pi$ be the differential of π . Since Z is the image of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ under $d\pi$, condition (iii) is immediate.

3.6. COROLLARY. *Let B_+ (resp. B_-) be the subgroup of $SL(2, \mathbb{Q})$ of upper triangular matrices (resp. lower triangular matrices), $U_+ = [B_+, B_+]$, and $U_- = [B_-, B_-]$. Let V be a vector space over F and $\pi: SL(2, \mathbb{Q}) \rightarrow GL(V)$ a homomorphism. The following conditions are equivalent:*

- (i) $SL(2, \mathbb{Q})$ is π -finite.
- (ii) The group U_+ (resp. U_-) is π -finite.
- (iii) Every $x \in SL(2, \mathbb{Q})$ is π -finite.
- (iv) Every $x \in B_+$ (resp. B_-) is π -finite.

In this case, there exist locally nilpotent elements X , Y , and diagonalizable element Z of $gl(V)$ satisfying the following conditions:

(a) $[X, Y] = Z, [Z, X] = 2X, [Z, Y] = -2Y.$

(b) $\pi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp tX, \pi\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tY (t \in \mathbb{Q}).$

(c) π extends to a locally rational homomorphism of $SL(2, F)$ into $GL(V)$.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Since U_- is a conjugate of U_+ , U_- is also π -finite. Note that $SL(2, \mathbb{Q}) = U_+ U_- U_+ U_-$. Clearly (iii) follows easily.

(iii) \Rightarrow (iv) is obvious.

(iv) \Rightarrow (i). Note that U_- is a conjugate of U_+ and $SL(2, \mathbb{Q}) = U_+ U_- U_+ U_-$. It suffices to show that U_+ is π -finite. Let $x \in U_+$ and $\langle x \rangle$ the cyclic group generated by x . For any positive integer $m > 1$, let $D = \langle \text{diag}(m, m^{-1}) \rangle$ and $U = \{y \in U_+ \mid y^{m^l} \in \langle x \rangle \text{ for some } l \in \mathbb{Z}_+\}$. Clearly D normalizes U and $D \rtimes U = D \langle x \rangle D$. This implies that $D \rtimes U$ is π -finite. Now let W be any $D \rtimes U$ -stable finite dimensional subspace of V . For any homomorphism $\lambda: U \rightarrow \bar{F}^\times$, set

$$\bar{W}_\lambda = \{v \in W \otimes_F \bar{F} \mid \pi(y)v = \lambda(y)v (y \in U)\}.$$

Let $\lambda_1, \dots, \lambda_l$ be all the homomorphisms such that $\bar{W}_{\lambda_i} \neq 0$. There exists a subgroup D_1 of D of finite index such that $\pi(D_1) \bar{W}_{\lambda_i} = \bar{W}_{\lambda_i} (i=1, \dots, l)$. Choose $1 \neq t \in D_1$. Then $[t, U] \subset \bigcap_{i=1}^l \ker(\lambda_i)$. It follows that there exists a positive integer r prime to m such that $\pi(y^r)|W$ is unipotent for $y \in U$. Now let W_1 be any finite dimensional $\langle x \rangle$ -stable subspace of V . There exists a finite dimensional $D \ltimes U$ -stable subspace W of V containing W_1 . Hence for any positive integer $m > 1$, there exists a positive integer r prime to m such that $\pi(x^r)|W_1$ is unipotent. This yields readily that $\pi(x)|W_1$ is unipotent. Hence for $x \in U_+$, $\pi(x)$ is locally unipotent. Note that $U_+ \cong \mathbb{Q}$. It follows that U_+ is π -locally unipotent and in particular π -finite.

The other assertions are immediate from 3.5.

In the following, for simplicity of notations, \mathfrak{g} , \mathfrak{g}' , and G stand for $\mathfrak{g}(F)$, $\mathfrak{g}'(F)$, and $G(F)$, respectively.

3.7. LEMMA. Let $\beta_i \in \mathcal{A}^{re}$, $e'_i \in \mathfrak{g}_{\beta_i}$, $f'_i \in \mathfrak{g}_{-\beta_i} (i=1, \dots, n)$, and σ the endomorphism of \mathfrak{h}' given by $\sigma(h_i) = \beta_i^\vee (i=1, \dots, n)$. Assume that

- (i) $(\beta_i^\vee | \beta_j) = (h_i | \alpha_j)$ and $[e'_i, f'_j] = \delta_{ij} \beta_i^\vee$ for all i, j ;
- (ii) \mathfrak{g}_{β_i} and $\mathfrak{g}_{-\beta_i} (i=1, \dots, n)$ generate the Lie algebra \mathfrak{g}' .

Then σ extends to an automorphism of \mathfrak{g}' , also denoted by σ , such that

$$\sigma(e_i) = e'_i, \quad \sigma(f_i) = f'_i \quad (i=1, \dots, n).$$

Proof. By (ii), $\Delta \subset \sum_{i=1}^n \mathbb{Z} \beta_i$ and as a consequence $h_i \in \sum_{i=1}^n F \beta_i^\vee (i=1, \dots, n)$. Hence σ is an automorphism of \mathfrak{h}' over F . Now the assertion is immediate from the characterization of the Lie algebra \mathfrak{g}' [10, 1.5].

3.8. Let G_f be the subset of G defined by

$$G_f = \{ g \in G \mid g \text{ is } Ad_{\mathfrak{g}}\text{-finite} \}. \tag{1}$$

Let $Aut_f(G)$ denote the set of abstract automorphisms σ of G satisfying the condition

$$\sigma(G_f) \subset G_f. \tag{2}$$

In the following, we discuss such automorphisms of G . Recall the embeddings of $SL(2, F)$ in G associated to real roots. For $\alpha \in \mathcal{A}^{re}$, $e \in \mathfrak{g}_\alpha$, and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$, there exists a homomorphism $\varphi_\alpha: SL(2, F) \rightarrow G$ determined by

$$\varphi_\alpha \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \exp xe, \quad \varphi_\alpha \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) = \exp xf \quad (x \in F). \tag{3}$$

Let λ_α denote the one parameter subgroup

$$\lambda_\alpha(t) = \varphi_\alpha(\text{diag}(t, t^{-1})) \quad (t \in F^\times). \tag{4}$$

Then λ_α is independent of the choice of e and f . Let (V, π) be any integrable \mathfrak{g}' -module. We have that

$$\pi(\lambda_\alpha(t)) v = t^{\langle \alpha^\vee | \tau \rangle} v \quad (t \in F^\times, v \in V_\tau, \tau \in \mathfrak{h}^*). \tag{5}$$

3.9. LEMMA. *Let $\sigma \in \text{Aut}_f(G)$, $\alpha \in \Delta^{re}$, $e \in \mathfrak{g}_\alpha$, and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$. Then we have the following conditions:*

- (i) *There exist $\text{ad}_{\mathfrak{g}}$ -locally nilpotent elements $X, Y \in \mathfrak{g}'$ such that $\sigma(\exp te) = \exp tX$, $\sigma(\exp tf) = \exp tY$ ($t \in \mathbb{Q}$).*
- (ii) *Set $Z = [X, Y]$. Then Z is $\text{ad}_{\mathfrak{g}}$ -diagonalizable, $[Z, X] = 2X$, and $[Z, Y] = -2Y$.*
- (iii) *$\sigma(\lambda_\alpha(\mathbb{Q}^\times))$ is $\text{Ad}_{\mathfrak{g}}$ -diagonalizable and the minimal algebraic subgroup of $G(\bar{F})$ containing $\sigma(\lambda_\alpha(\mathbb{Q}^\times))$ is a one-dimensional subtorus of $G(\bar{F})$ with Lie algebra $\bar{F}Z$.*

Proof. Note that $\sigma\varphi_\alpha(SL(2, \mathbb{Q})) \subset G_f$. Our assertions are immediate from 3.6 and (b) of 3.4(ii).

3.10. \mathfrak{g}' is defined over \mathbb{Q} and $\mathfrak{g}' = \mathfrak{g}'_{\mathbb{Q}} \otimes_{\mathbb{Q}} F$. We assume that $h_i, e_i, f_i \in \mathfrak{g}'_{\mathbb{Q}}$ ($i = 1, \dots, n$). Let M be a subgroup of G such that M is $\text{Ad}_{\mathfrak{g}}$ -finite. By 2.20, there is a unique minimal algebraic subgroup $Cl(M)$ of $G(\bar{F})$ containing M . We say that $Cl(M)$ is the closure of M in $G(\bar{F})$ and M is dense in $Cl(M)$.

LEMMA. *If $\sigma \in \text{Aut}_f(G)$, then there exists $g \in G$ such that $\sigma(H) = gHg^{-1}$.*

Proof. Apply 3.9 to $\alpha_1, \dots, \alpha_n$. There exist $\text{ad}_{\mathfrak{g}}$ -diagonalizable elements $Z_1, \dots, Z_n \in \mathfrak{g}'$ such that $T_i = Cl(\sigma(H_i(\mathbb{Q})))$ is a one-dimensional subtorus of $G(\bar{F})$ with Lie algebra $\bar{F}Z_i$ ($i = 1, \dots, n$). By 2.20, T_i centralizes T_j and by 2.17, T_i centralizes Z_j . Hence Z_i centralizes Z_j and $\sum_{i=1}^n \bar{F}Z_i$ is $\text{ad}_{\mathfrak{g}}$ -diagonalizable. There exists a split Cartan subalgebra $\text{Ad}(g)\mathfrak{h}'$ with $g \in G$ containing the elements Z_i ($i = 1, \dots, n$). Clearly gHg^{-1} centralizes Z_i and so by 2.17, gHg^{-1} centralizes T_i . By 1.16(ii), $\sigma(H(\mathbb{Q})) \subset gHg^{-1}$. Since $H(\mathbb{Q})$ is dense in $H(\bar{F})$, by 2.20, H centralizes $\sigma^{-1}(gHg^{-1})$. By 1.16(ii), H and $\sigma^{-1}(gHg^{-1})$ are their own centralizer. Hence $H = \sigma^{-1}(gHg^{-1})$. Now our desired assertion follows.

3.11. LEMMA. *Let $\sigma \in \text{Aut}_f(G)$ such that $\sigma(H) = H$. Then $\sigma(H(\mathbb{Q}))$ is dense in $H(\bar{F})$.*

Proof. Note that $H(\mathbb{Q}) = \prod_{i=1}^n H_i(\mathbb{Q})$. From 3.9(iii), $Cl(\sigma(H(\mathbb{Q})))$ is subtorus of $G(\bar{F})$. Let \mathfrak{a} denote its Lie algebra. By 2.18(ii), $\mathfrak{a} \subset \mathfrak{h}' \otimes \bar{F}$. Since $H(\mathbb{Q})$ is W -invariant, so is $\sigma(H(\mathbb{Q}))$ and by 2.20 and 2.15(i), \mathfrak{a} is W -invariant. Claim: $\alpha | \mathfrak{a} \neq 0$ for all $\alpha \in \Delta^{re}$. Suppose that $\alpha \in \Delta^{re}$ such that $\alpha | \mathfrak{a} = 0$. Then \mathfrak{a} centralizes $F\alpha^\vee + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ and so does $\sigma(H(\mathbb{Q}))$. Let $G_{(\alpha)}$ denote the subgroup of G generated by U_α and $U_{-\alpha}$. Since $G_{(\alpha)}(\bar{F})$ is the connected algebraic subgroup of $G(\bar{F})$ with Lie algebra $\bar{F}\alpha^\vee + \mathfrak{g}_\alpha(\bar{F}) + \mathfrak{g}_{-\alpha}(\bar{F})$, by 2.17, $\sigma(H(\mathbb{Q}))$ centralizes $G_{(\alpha)}$. It follows that $H(\mathbb{Q})$ centralizes $\sigma^{-1}(G_{(\alpha)})$. Since $H(\mathbb{Q})$ is dense in $H(\bar{F})$, by 2.20, H centralizes $\sigma^{-1}(G_{(\alpha)})$ and so $\sigma^{-1}(G_{(\alpha)}) \subset H$. However, $G_{(\alpha)} \approx SL(2, F)$ is not commutative. Certainly we are led to a contradiction. So, for $\alpha \in \Delta^{re}$, there exists $x \in \mathfrak{a}$ with $\alpha(x) \neq 0$. Then $\alpha^\vee = \alpha(x)^{-1}(x - r_\alpha(x))$ lies in \mathfrak{a} . Hence $\mathfrak{a} = \mathfrak{h}' \otimes_F \bar{F}$ and as a consequence $\sigma(H(\mathbb{Q}))$ is dense in $H(\bar{F})$.

3.12. LEMMA. *Let $\sigma \in Aut_f(G)$ such that $\sigma(H) = H$. Then there exists a map, also denoted by σ , $\sigma: \Delta^{re} \rightarrow \Delta^{re}$ such that for every $\alpha \in \Delta^{re}$, $e \in \mathfrak{g}_\alpha$, and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$, there exist $X \in \mathfrak{g}_{\sigma\alpha}$, $Y \in \mathfrak{g}_{-\sigma\alpha}$ satisfying the following conditions:*

- (i) $[X, Y] = (\sigma\alpha)^\vee$.
- (ii) $\sigma(\exp te) = \exp tX$, $\sigma(\exp tf) = \exp tY$ ($t \in \mathbb{Q}$).
- (iii) $\sigma(\lambda_\alpha(t)) = \lambda_{\sigma\alpha}(t)$ ($t \in \mathbb{Q}^\times$).

Proof. By 3.9, there exist $ad_{\mathfrak{g}}$ -locally nilpotent elements $X, Y \in \mathfrak{g}'$ and $ad_{\mathfrak{g}}$ -diagonalizable $Z \in \mathfrak{g}'$ satisfying the following conditions:

- (a) $[X, Y] = Z$, $[Z, X] = 2X$, $[Z, Y] = -2Y$.
- (b) $\sigma(\exp te) = \exp tX$, $\sigma(\exp tf) = \exp tY$ ($t \in \mathbb{Q}$).

(c) $Cl(\lambda_\alpha(\mathbb{Q}^\times))$ is a one-dimensional subtorus of $G(\bar{F})$ with Lie algebra $\bar{F}Z$.

It suffices to show that there exists $\sigma\alpha \in \Delta^{re}$ with $X \in \mathfrak{g}_{\sigma\alpha}$ and $Y \in \mathfrak{g}_{-\sigma\alpha}$. Since $\lambda_\alpha(\mathbb{Q}^\times)$ is independent of the choice of e and f , so is $\sigma\alpha$.

First we show that FX and FY are \mathfrak{h}' -invariant. Note that $\exp(\mathbb{Q}e)$ is $H(\mathbb{Q})$ -invariant. From (b), $\mathbb{Q}X$ is $Ad(\sigma(H(\mathbb{Q})))$ -invariant. By 3.11, $\sigma(H(\mathbb{Q}))$ is dense in $H(\bar{F})$ and so FX is $Ad(H)$ -invariant. Consider the Lie subalgebra $\mathfrak{p} = \mathfrak{h}' + FX$. Clearly it is $ad_{\mathfrak{g}}$ -triangular. Thus by [15, Lemma 6], there exist $g \in G$ and $w \in W$ such that

$$Ad(g)(\mathfrak{p}) \subset \mathfrak{b}_+ \cap w(\mathfrak{b}_-).$$

We may assume that $Ad(g)\mathfrak{h}' = \mathfrak{h}'$ and by 1.30, $g \in N$. Then $FX \subset w_1(\mathfrak{n}_+) \cap w_2(\mathfrak{n}_-)$ for some $w_1, w_2 \in W$. As a consequence, there exist $\beta_1, \dots, \beta_l \in \Delta^{re}$ such that $X \in \bigoplus_{i=1}^l \mathfrak{g}_{\beta_i}$. Note that FX is \mathfrak{h}' -invariant. Hence

there exists $\lambda \in (\mathfrak{h}')^* \cap \text{res}(\Delta^{re})$ such that $X \in \mathfrak{g}_\lambda$. Since $\sigma(H) = H$, $Cl(\sigma(\lambda_\alpha(\mathbb{Q}^\times))) \subset H(\bar{F})$ and by 2.15(ii), $Z \in \mathfrak{h}'$. Similarly there exists $\tau \in (\mathfrak{h}')^* \cap \text{res}(\Delta^{re})$ such that $Y \in \mathfrak{g}_\tau$. However, $[X, Y] = Z \in \mathfrak{h}'$ and $Z \neq 0$. It follows that $\tau = -\lambda$. Then our assertion follows from 1.28.

3.13. LEMMA. *Let $\sigma \in \text{Aut}_f(G)$ be such that $\sigma(H) = H$, and let $\sigma: \Delta^{re} \rightarrow \Delta^{re}$ be the induced map as in 3.12. Let $e'_i \in \mathfrak{g}_{\sigma\alpha_i}$ and $f'_i \in \mathfrak{g}_{-\sigma\alpha_i}$ be such that $\sigma(\exp e_i) = \exp e'_i$ and $\sigma(\exp f_i) = \exp f'_i$ ($i = 1, \dots, n$). Then we have:*

- (i) $[e'_i, f'_j] = \delta_{ij}(\sigma\alpha_i)^\vee$ for all i, j .
- (ii) $((\sigma\alpha_i)^\vee | \sigma\alpha_j) = (h_i | \alpha_j)$ for all i, j .
- (iii) $\mathfrak{g}_{\sigma\alpha_i}, \mathfrak{g}_{-\sigma\alpha_i}$ ($i = 1, \dots, n$) generate \mathfrak{g}' .

Proof. Let $\beta_i = \sigma\alpha_i$ ($i = 1, \dots, n$). By 3.12(i),

$$[e'_i, f'_i] = \beta_i^\vee \quad (i = 1, \dots, n).$$

Assume that $i \neq j$. Consider the relation

$$\exp te_i \exp f_j \exp(-te_i) = \exp f_j \quad (t \in \mathbb{Q}).$$

We have that $\exp((\exp tade'_i)f'_i) = \exp f'_i$ ($t \in \mathbb{Q}$). Since $(\exp tade'_i)f'_i$ and f'_i are $ad_{\mathfrak{g}}$ -locally nilpotent, so $(\exp tade'_i)f'_i$. It follows that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} (ade'_i)^n f'_i = 0 \quad (t \in \mathbb{Q}). \tag{1}$$

Since the summation in (1) is a finite sum, (1) yields that $(ade'_i)^n f'_i = 0$ for all $n \geq 1$; in particular $[e'_i, f'_j] = 0$. Thus (i) is established. By 3.12(iii), we have that

$$\sigma(\lambda_{\alpha_i}(t)) = \lambda_{\beta_i}(t) \quad (t \in \mathbb{Q}^\times).$$

Note that

$$\lambda_{\alpha_i}(t) \exp e_j \lambda_{\alpha_i}(t)^{-1} = \exp t^{\langle \alpha_i^\vee | \alpha_j \rangle} e_j \quad (t \in \mathbb{Q}^\times). \tag{2}$$

Applying σ to (2), it yields that

$$\exp t^{\langle \alpha_i^\vee | \alpha_j \rangle} e'_j = \exp t^{\langle \beta_i^\vee | \beta_j \rangle} e'_j \quad (t \in \mathbb{Q}).$$

Hence (ii) follows.

Let \mathfrak{p} denote the subalgebra of \mathfrak{g}' generated by $\mathfrak{g}_{\beta_i}, \mathfrak{g}_{-\beta_i}$ ($i = 1, \dots, n$). Since $\sigma(U_{\pm\alpha_i}) \subset U_{\pm\beta_i}$, so $U_{\pm\beta_i}$ ($i = 1, \dots, n$) generate G . It follows that \mathfrak{p} is G -invariant and so is an ideal of \mathfrak{g}' . Without losing any generality, we may assume that \mathfrak{g} is indecomposable. Now it is easy to see that $\mathfrak{p} = \mathfrak{g}'$.

3.14. LEMMA. *If $\sigma \in \text{Aut}_F(G)$, then there exists $\tau \in \text{Aut}_F(\mathfrak{g}')$ such that $\tau \circ \sigma(H) = H$ and $\tau \circ \sigma(U_\alpha) = U_\alpha$ for all $\alpha \in \Delta^{re}$.*

Proof. By 3.10, we may assume that $\sigma(H) = H$. By 3.13 and 3.7, there exists $\tau \in \text{Aut}_F(\mathfrak{g}')$ such that $\tau \circ \sigma(H) = H$ and $\tau \circ \sigma(U_{+\alpha_i}) \subset U_{\pm\alpha_i}$ ($i = 1, \dots, n$). Note that $\tau \circ \sigma(r_{\alpha_i}) = r_{\alpha_i}$ ($i = 1, \dots, n$) and so $\tau \circ \sigma(w) = w$ for every $w \in W$. Given $\alpha \in \Delta^{re}$, choose α_i and $w \in W$ with $\alpha = w\alpha_i$. Then $U_\alpha = wU_{\alpha_i}w^{-1}$ and so

$$\tau \circ \sigma(U_\alpha) \subset U_\alpha \quad (\alpha \in \Delta^{re}).$$

Now consider the Bruhat decomposition $G = \bigcup_{w \in W} B_+ w B_+$. We have that $\tau \circ \sigma(B_+ w B_+) = B_+ w B_+$ for all $w \in W$. Given $w \in W$, the product map $(U_+ \cap wU_- w^{-1}) \times wB_+ \rightarrow B_+ w B_+$ is bijective. It follows that $\tau \circ \sigma(U_+ \cap wU_- w^{-1}) = U_+ \cap wU_- w^{-1}$ for all $w \in W$. Now the assertion $\tau \circ \sigma(U_\alpha) = U_\alpha$ ($\alpha \in \Delta^{re}$) is immediate.

3.15. LEMMA. *Let H (resp. U_+, U_-) be the subgroup of $SL(2, F)$ of diagonal matrices (resp. unipotent upper triangular, unipotent lower triangular matrices). If σ is an automorphism of $SL(2, F)$ such that $\sigma(H) = H$ and $\sigma(U_\pm) = U_\pm$, then there exists an automorphism θ of F and $0 \neq c \in F$ satisfying the following conditions:*

- (i) $\sigma(\text{diag}(t, t^{-1})) = \text{diag}(\theta(t), \theta(t^{-1}))$ ($t \in F^\times$).
- (ii) $\sigma\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \theta(x)c \\ 0 & 1 \end{pmatrix}$, $\sigma\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ \theta(x)^{-1}c^{-1} & 1 \end{pmatrix}$, ($x \in F$).

Proof. Set $\lambda(t) = \text{diag}(t, t^{-1})$ ($t \in F^\times$) and $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $v(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ ($x \in F$). Let $\theta: F^\times \rightarrow F^\times$ (resp. $\psi: F \rightarrow F$) be the function determined by the condition

$$\begin{aligned} \sigma(\lambda(t)) &= \lambda(\theta(t)) & (t \in F^\times), \\ \sigma(u(x)) &= u(\psi(x)) & (x \in F). \end{aligned}$$

We have the relation that

$$u(x)v(-x^{-1})u(x) = \lambda(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (x \in F^\times). \tag{1}$$

Applying σ to (1), we obtain that

$$u(\psi(x))v(y)u(\psi(x)) = \lambda(\theta(x)) \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \tag{2}$$

for some $y \in F$ and $c \in F^\times$. Then we have that

$$y = -\psi(x)^{-1}, \quad \psi(x) = c\theta(x) \quad (x \in F^\times). \tag{3}$$

The function $\theta = c^{-1}\psi$, being additive and multiplicative, is an automorphism of F . From (3), $\sigma(v(x)) = v(\theta(x) c^{-1})$ ($x \in F$).

3.16. LEMMA. *Let m be a positive integer. Then the subfield of F generated by x^m ($x \in F$) coincides with F .*

Proof. Let F_1 be the subfield of F generated by x^m ($x \in F$) over \mathbb{Q} . For $a \in F$ and $\lambda \in \mathbb{Q}$,

$$a^m + \binom{m}{1} \lambda a^{m-1} + \dots + \lambda^m = (a + \lambda)^m \in F_1. \tag{1}$$

Choose $\lambda_1, \dots, \lambda_{m+1}$ distinct rational numbers. Since the corresponding Vandemonde determinant is nonzero, conditions (1) for $\lambda_1, \dots, \lambda_{m+1}$ yield that $a, \dots, a^m \in F_1$. Hence F_1 coincides with F .

3.17. LEMMA. *Assume that the generalized Cartan matrix A is symmetrizable and indecomposable. Let $\sigma \in \text{Aut}_f(G)$ such that $\sigma(H) = H$ and $\sigma(U_\alpha) = U_\alpha$ ($\alpha \in \Delta^{re}$). Then there exist $c_1, \dots, c_n \in F^\times$ and an automorphism θ of F satisfying the conditions*

$$\begin{aligned} \sigma(\exp x e_i) &= \exp(\theta(x) c_i e_i), \\ \sigma(\exp x f_i) &= \exp(\theta(x) c_i^{-1} f_i) \quad (x \in F, i = 1, \dots, n). \end{aligned}$$

Proof. For $\alpha \in \Delta^{re}$, let $G_{(\alpha)}$ denote the subgroup of G generated by U_α and $U_{-\alpha}$. Clearly $G_{(\alpha)}$ is σ -invariant. Choose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ with $[e, f] = \alpha^\vee$. By 3.15, there exist an automorphism θ_α of F and $c_\alpha \in F^\times$ such that

$$\begin{aligned} \sigma(\lambda_\alpha(t)) &= \lambda_\alpha(\theta_\alpha(t)) \quad (t \in F^\times), \\ \sigma(\exp x e) &= \exp(\theta_\alpha(x) c_\alpha e), \quad \sigma(\exp x f) = \exp(\theta_\alpha(x) c_\alpha^{-1} f) \quad (t \in F). \end{aligned} \tag{1}$$

Hence it suffices to show that $\theta_{\alpha_1} = \dots = \theta_{\alpha_n}$. For $\alpha \in \Delta^{re}$, write $\alpha^\vee = m_1 \alpha_1^\vee + \dots + m_n \alpha_n^\vee$. Then

$$\lambda_\alpha(t) = \lambda_{\alpha_1}(t)^{m_1} \dots \lambda_{\alpha_n}(t)^{m_n} \quad (t \in F^\times). \tag{2}$$

Since $H = H_1 \times \dots \times H_n$ is a direct product,

$$\theta_\alpha(t^{m_i}) = \theta_{\alpha_i}(t^{m_i}) \quad (t \in F^\times, i = 1, \dots, n).$$

By 3.16, if $m_i \neq 0$, then $\theta_{\alpha_i} = \theta_\alpha$. Since A is indecomposable, the assertion $\theta_{\alpha_1} = \dots = \theta_{\alpha_n}$ follows readily.

3.18. Recall that \mathfrak{g}' has a \mathbb{Q} -structure $\mathfrak{g}'(\mathbb{Q})$ by the construction of Kac-Moody algebras over \mathbb{Q} . For $\alpha \in \Delta^{re}$, choose $e_\alpha \in \mathfrak{g}_\alpha(\mathbb{Q})$. Then $Aut(F)$ acts on G defined by

$$\sigma(\exp x e_\alpha) = \exp(\sigma(x) e_\alpha) \quad (\sigma \in Aut(F), x \in F, \alpha \in \Delta^{re}). \tag{1}$$

By 1.32, $Aut_F(\mathfrak{g}') = Out(A) \rtimes Ad(\tilde{H} \rtimes G)$. Then $Aut_F(\mathfrak{g}')$ acts on G in the obvious manner. Let $Aut(G; \mathfrak{g}')$ denote the image of $Aut_F(\mathfrak{g}')$ in $Aut(G)$. Note that $Aut(F)$ normalizes $Aut_F(\mathfrak{g}')$. Hence $Aut(F)$ normalizes $Aut(G; \mathfrak{g}')$.

Now we are ready to present our main result on abstract automorphisms of G .

THEOREM. *Assume that the generalized Cartan matrix A is symmetrizable and indecomposable. Let G be the Kac-Moody group associated to $\mathfrak{g}'(A)$ over F , $G_f = \{g \in G \mid g \text{ is } Ad_{\mathfrak{g}}\text{-finite}\}$, and $Aut_f(G) = \{\sigma \in Aut(G) \mid \sigma(G_f) \subset G_f\}$. Then $Aut_f(G) = Aut(F) \rtimes Aut(G; \mathfrak{g}')$.*

Proof. Let $\tau \in Aut(F) \cap Aut(G; \mathfrak{g}')$. By (1), $\tau(\exp e_i) = \exp e_i$, $\tau(\exp f_i) = \exp f_i$ ($i = 1, \dots, n$). Since $\tau \in Aut(G; \mathfrak{g}')$, $\tau = 1$ follows easily. Clearly $Aut(F) \rtimes Aut(G; \mathfrak{g}') \subset Aut_f(G)$. Now let $\sigma \in Aut_f(G)$. By 3.14, we may assume that $\sigma(H) = H$ and $\sigma(U_\alpha) = U_\alpha$ ($\alpha \in \Delta^{re}$). Then by 3.17, σ lies in $Aut(F) \rtimes Aut(G; \mathfrak{g}')$.

3.19. **COROLLARY.** *Let $G_u = \{g \in G \mid g \text{ is } Ad_{\mathfrak{g}}\text{-locally unipotent}\}$ and $\sigma \in Aut(G)$ such that $\sigma(G_u) \subset G_f$. Then σ lies in $Aut(F) \rtimes Aut(G; \mathfrak{g}')$.*

Proof. First we show that for $\alpha \in \Delta^{re}$ and $x \in \mathfrak{g}_\alpha$, the group $\sigma(\exp \mathbb{Q}x)$ is $Ad_{\mathfrak{g}}$ -finite. For $t \in \mathbb{Q}$, let $\sigma(\exp tx) = s(t) u(t)$ denote the Jordan decomposition. Then there exists an $ad_{\mathfrak{g}}$ -locally nilpotent $X \in \mathfrak{g}'$ such that $u(t) = \exp tX$ ($t \in \mathbb{Q}$). Let $M = \{s(t) \mid t \in \mathbb{Q}\}$. The group $H(\mathbb{Q})$ acts on $\mathbb{Q}x$ (resp. M) through Ad (resp. $\text{Int} \circ \sigma$). The map $\beta: \mathbb{Q}x \rightarrow M$, defined by $\beta(tx) = s(t)$ ($t \in \mathbb{Q}$), is an $H(\mathbb{Q})$ -homomorphism, and $\ker(\beta)$ is $H(\mathbb{Q})$ -invariant. As an $H(\mathbb{Q})$ -group, $\mathbb{Q}x$ is simple. Hence β is either trivial or is an isomorphism. We may assume that $\ker(\beta) = \{0\}$. For $t_1, t_2 \in \mathbb{Q}^\times$, consider the cyclic groups $\langle s(t_1) \rangle$ and $\langle s(t_2) \rangle$ generated by $s(t_1)$ and $s(t_2)$, respectively. Clearly the intersection $\langle s(t_1) \rangle \cap \langle s(t_2) \rangle$ is of finite index in $\langle s(t_1) \rangle$ and $\langle s(t_2) \rangle$. For $t \in \mathbb{Q}^\times$, the identity component S of $Cl(\langle s(t) \rangle)$ is independent of t . It follows that S is $\sigma(H(\mathbb{Q}))$ -invariant. Note that $\beta^{-1}(M \cap S)$ is a nontrivial $H(\mathbb{Q})$ -invariant subgroup of $\mathbb{Q}x$. Hence $\mathbb{Q}x = \beta^{-1}(M \cap S)$ and so

$$\sigma(\exp \mathbb{Q}x) \subset S \exp \mathbb{Q}X. \tag{1}$$

Since S is a subtorus of $G(\bar{F})$, (1) implies easily that $\sigma(\exp \mathbb{Q}x)$ is $Ad_{\mathfrak{g}}$ -finite. Now set $Aut_f(G)' = \{\sigma \in Aut(G) \mid \sigma(G_u) \subset G_f\}$. By 3.6, we can replace $Aut_f(G)$ by $Aut_f(G)'$ in all the above discussion. Thus our assertion follows.

3.20. COROLLARY. *For $\sigma \in Aut(G)$, $\sigma \in Aut_f(G)$ if and only if σ carries Borel subgroups of G to Borel subgroups of G .*

Proof. (\Rightarrow) This is immediate from 3.18.

(\Leftarrow) For $x \in G_u$, there exist $g \in G$ and $w \in W$ such that $gxg^{-1} \in B_+ \cap wB_-w^{-1}$. The assumption yields that $\sigma(x)$ satisfies a similar condition. Hence $\sigma(x) \in G_f$ and by 3.19, our assertion follows.

4. AUTOMORPHISMS OF KAC-MOODY ALGEBRAS

In this section, F is an algebraically closed field of characteristic zero.

4.1. Let σ be an automorphism of a Lie algebra \mathfrak{p} (resp. a group P). The *fixed point subalgebra* \mathfrak{p}^σ (resp. *fixed point subgroup* P^σ) is defined by $\mathfrak{p}^\sigma = \{x \in \mathfrak{p} \mid \sigma(x) = x\}$ (resp. $P^\sigma = \{x \in P \mid \sigma(x) = x\}$).

4.2. Let G be a connected algebraic subgroup of $GL(m, F)$ and $x \in GL(m, F)$ an element normalizing G . For a connected solvable algebraic subgroup B_1 of $Z_G(x)$, there exists an x -stable Borel subgroup of G containing B_1 [26].

4.3. LEMMA. *Let \mathfrak{p} be a finite dimensional Lie algebra over F and $\sigma \in Aut_F(\mathfrak{p})$. If \mathfrak{b}_1 is a solvable subalgebra of \mathfrak{p}^σ , then there exists a σ -stable Borel subalgebra of \mathfrak{p} containing \mathfrak{b}_1 .*

Proof. Let \mathfrak{r} be the radical of \mathfrak{p} . Passing over to $\mathfrak{p}/\mathfrak{r}$, we may assume that \mathfrak{p} is semisimple. Then $ad(\mathfrak{p})$ is algebraic and σ normalizes the connected algebraic group with Lie algebra $ad(\mathfrak{p})$. Our assertion is now immediate from 4.2.

4.4. Let G be a connected algebraic solvable subgroup of $GL(m, F)$ and M a solvable subgroup of $GL(m, F)$ normalizing G . If M consists of semi-simple elements, then we have the following:

(i) M leaves invariant a maximal torus of G .

(ii) M -stable maximal tori of G are conjugate by elements of $Z_{C^\infty(G)}(M)$ where $C^\infty(G) = \bigcap_{i=0}^\infty C^i(G)$ with $C^0(G) = G$ and $C^{i+1}(G) = [G, C^i(G)]$.

These results were established in [5, 3.2 and 3.4]. One deduces easily the following results on Lie algebras.

4.5. LEMMA. *Let \mathfrak{p} be a finite dimensional solvable Lie algebra over F and $\sigma \in \text{Aut}_F(\mathfrak{p})$. If σ is semisimple, then we have the following conditions:*

- (i) σ leaves invariant a Cartan subalgebra of \mathfrak{p} .
- (ii) σ -stable Cartan subalgebras of \mathfrak{p} are conjugate by elements of $\exp(\text{ad}([\mathfrak{p}, \mathfrak{p}]^\sigma))$.

In the following, A is a symmetrizable generalized Cartan matrix, $\mathfrak{g} = \mathfrak{g}(A)$ the Kac-Moody algebra, and G is the Kac-Moody group associated to \mathfrak{g}' over F .

4.6. Let σ be an automorphism of \mathfrak{g} over F . By 1.26, if A is indecomposable, $\sigma(\mathfrak{b}_+)$ is $\text{Ad}(G)$ -conjugate to \mathfrak{b}_+ or \mathfrak{b}_- . We say that σ is of the first kind (resp. of the second kind) if $\sigma(\mathfrak{b}_+)$ is commensurable to \mathfrak{b}_+ (resp. \mathfrak{b}_-). Clearly for indecomposable A and $\sigma \in \text{Aut}_F(\mathfrak{g})$, the following conditions are equivalent:

- (i) σ is of the first kind (resp. the second kind).
- (ii) $\sigma(\mathfrak{b}_+)$ is $\text{Ad}(G)$ -conjugate to \mathfrak{b}_+ (resp. \mathfrak{b}_-).
- (iii) For any Borel subalgebra \mathfrak{b} of \mathfrak{g} , $\sigma(\mathfrak{b})$ and \mathfrak{b} are commensurable (resp. $\sigma(\mathfrak{b}) \cap \mathfrak{b}$ is finite dimensional).

An element $\sigma \in \text{Aut}_F(\mathfrak{g})$ is called $\text{Ad}_{\mathfrak{g}}$ -finite if $\sigma|_{\mathfrak{g}'}$ is locally finite.

4.7. THEOREM. *Let $\sigma \in \text{Aut}_F(\mathfrak{g})$ be of the first kind and $\mathfrak{a} \subset \mathfrak{g}^\sigma$ be an $\text{ad}_{\mathfrak{g}}$ -triangular subalgebra. If σ is $\text{Ad}_{\mathfrak{g}}$ -finite, then there exists $g \in G$ such that $\text{Ad}(g)\mathfrak{b}_+$ (resp. $\text{Ad}(g)\mathfrak{b}_-$) is a σ -stable Borel subalgebra of \mathfrak{g} containing \mathfrak{a} .*

Proof. By 2.8, there exist a finite type $X \subset \Pi$ and $x \in G$ such that the algebra $\mathfrak{p} = \text{Ad}(x)(\mathfrak{p}_X)$ (resp. $\text{Ad}(x)(\mathfrak{p}_{\bar{X}})$) is σ - and \mathfrak{a} -stable. By 1.17(ii), $\mathfrak{a} \subset \mathfrak{p}$. Let $\mathfrak{n}_+^X = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ ($\alpha \notin \mathbb{Z}X$). Note that $\mathfrak{p}/\text{Ad}(x)(\mathfrak{n}_+^X)$ is a finite dimensional reductive Lie algebra and the image of $\sigma(\text{Ad}(x)(\mathfrak{n}_+^X))$ in $\mathfrak{p}/\text{Ad}(x)(\mathfrak{n}_+^X)$ is a nilpotent ideal. Let $\mathfrak{q} = \text{Ad}(x)(\mathfrak{h}_1 \oplus \mathfrak{n}_+^X)$ where $\mathfrak{h}_1 = \bigcap_{\alpha_i \in X} \ker(\alpha_i)$. It follows that \mathfrak{q} is σ - and \mathfrak{a} -stable. Applying 4.3 to $\mathfrak{p}/\mathfrak{q}$, it follows readily that there exists a σ -stable Borel subalgebra \mathfrak{b} , commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-), of \mathfrak{g} containing \mathfrak{a} . Now the assertion is obvious.

4.8. COROLLARY. *Let $\sigma \in \text{Aut}_F(\mathfrak{g})$ be semisimple of the first kind. Then we have the following conditions:*

(i) *There exists a σ -stable Borel subalgebra of \mathfrak{g} commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-).*

(ii) *For any σ -stable Borel subalgebra \mathfrak{b} of \mathfrak{g} , there exists a σ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} contained in \mathfrak{b} .*

Proof. (i) is immediate from 4.7.

(ii) We may assume that \mathfrak{b} is \mathfrak{b}_+ . Choose a σ -stable Borel subalgebra \mathfrak{b}_1 which is $Ad(G)$ -conjugate to \mathfrak{b}_- . Note that Cartan subalgebras of $\mathfrak{b} \cap \mathfrak{b}_1$ are Cartan subalgebras of \mathfrak{g} . Now our assertion is immediate from 4.5(i).

4.9. A pair $(\mathfrak{t}, \mathfrak{b})$ of subalgebras of \mathfrak{g} is called a *standard pair* if \mathfrak{b} is a Borel subalgebra of \mathfrak{g} and \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} contained in \mathfrak{b} . Given a standard pair $(\mathfrak{t}, \mathfrak{b})$, let \mathfrak{b}^- denote the Borel subalgebra of \mathfrak{g} such that $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{t}$. Set $B = N_G(\mathfrak{b})$, $B^- = N_G(\mathfrak{b}^-)$, $U = [B, B]$, $U^- = [B^-, B^-]$, $N = N_G(\mathfrak{t})$, and $T = Z_G(\mathfrak{t})$.

An element $\sigma \in Aut_F(\mathfrak{g})$ is called *quasi-semisimple of the first kind* if there exists a σ -stable standard pair $(\mathfrak{t}, \mathfrak{b})$. Given $\sigma \in Aut_F(\mathfrak{g})$, σ induces an automorphism, also denoted by σ , of G . We write G^σ or G_σ for the fixed point subgroup of G .

LEMMA. *Let $\sigma \in Aut_F(\mathfrak{g})$ be quasi-semisimple of the first kind and $(\mathfrak{t}, \mathfrak{b})$ be a σ -stable standard pair. Then we have the following conditions:*

(i) $G^\sigma = U_\sigma^\pm N_\sigma U_\sigma$.

(ii) *The subgroup $G_0^\sigma = \langle U_\sigma^-, U_\sigma \rangle$ of G^σ generated by U_σ^- and U_σ is a normal subgroup of G^σ .*

Proof. Consider the decomposition $G = U^\pm NU$. For $g \in G^\sigma$, write $g = vnu$ such that $n \in N$, $u \in U$ and $v \in U^\pm \cap nU^-n^{-1}$. Note that the groups U , U^- , and N are σ -stable. By 1.14, $u, v, n \in G^\sigma$. Hence (i) follows.

Given $n \in N$, $nU^\pm n^{-1} = (nU^\pm n^{-1} \cap U)(nU^\pm n^{-1} \cap U^-)$. It follows that for $n \in N_\sigma$, $nU_\sigma^\pm n^{-1} \subset U_\sigma U_\sigma^-$. Clearly G_0^σ is a normal subgroup of G^σ .

4.10. LEMMA. *Let $\sigma \in Aut_F(\mathfrak{g})$ be semisimple. If $(\mathfrak{t}, \mathfrak{b})$ is a σ -stable standard pair, then $z_g(\mathfrak{t}^\sigma) = \mathfrak{t}$.*

Proof. We may assume that $\mathfrak{t} = \mathfrak{h}$ and $\mathfrak{b} = \mathfrak{b}_+$. It follows that Δ_+ is σ -stable and so is Π . Let \mathfrak{h}_1 be a σ -stable subspace of \mathfrak{h} with $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{h}_1$. Since Π is σ -stable, $\sigma^m | \Pi = 1$ for some positive integer m and so $\sigma^m | \mathfrak{h}_1 = 1$. Choose $X \in \mathfrak{h}_1$ such that $\alpha_i(X) = 1$ ($i = 1, \dots, n$) and set $Y = \sum_{i=1}^m \sigma^i X$. Clearly $Y \in \mathfrak{h}^\sigma$ and $z_g(Y) = \mathfrak{h}$. Our assertion is now obvious.

4.11. LEMMA. *Let $\sigma \in Aut_F(\mathfrak{g})$ and let $(\mathfrak{t}_1, \mathfrak{b})$, $(\mathfrak{t}_2, \mathfrak{b})$ be σ -stable standard pairs. Then there exists $u \in U^\sigma$ such that $\mathfrak{t}_2 = Ad(u) \mathfrak{t}_1$.*

Proof. By 1.26, t_1 and t_2 are $Ad(U)$ -conjugate. Choose $u \in U$ with $t_2 = Ad(u)t_1$. Since t_2 and t_1 are σ -stable, $u^{-1}\sigma(u) \in N_U(t_1) = \{1\}$ and so $u \in U^\sigma$.

4.12. LEMMA. *Let $\sigma \in Aut_F(\mathfrak{g})$ be semisimple of the first kind and $\mathfrak{p} \subset \mathfrak{g}^\sigma$. The following conditions are equivalent:*

- (i) \mathfrak{p} is a maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of \mathfrak{g}^σ .
- (ii) There exists a σ -stable standard pair (t, \mathfrak{b}) such that $\mathfrak{p} = t^\sigma$.

Proof. (i) \Rightarrow (ii). By 4.7, there exist σ -stable Borel subalgebras \mathfrak{b}_1 and \mathfrak{b}_2 , containing \mathfrak{p} , of \mathfrak{g} commensurable with \mathfrak{b}_+ and \mathfrak{b}_- , respectively. Consider the subalgebra $z_{\mathfrak{b}_1 \cap \mathfrak{b}_2}(\mathfrak{p})$. It is a finite dimensional σ -stable solvable subalgebra of $\mathfrak{b}_1 \cap \mathfrak{b}_2$. By 4.5(i), \mathfrak{p} is contained in a σ -stable Cartan subalgebra \mathfrak{t} of $\mathfrak{b}_1 \cap \mathfrak{b}_2$. The maximality condition of \mathfrak{p} implies easily that $t^\sigma = \mathfrak{p}$. Clearly the standard pair (t, \mathfrak{b}_1) has the desired condition in (ii).

(ii) \Rightarrow (i). By 4.10, $t = z_{\mathfrak{g}}(t^\sigma)$ and so condition (i) is obvious.

4.13. THEOREM. *Let $\sigma \in Aut_F(\mathfrak{g})$ be semisimple of the first kind. Then we have the following conditions:*

- (i) Maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebras of \mathfrak{g}^σ are $Ad(G_0^\sigma)$ -conjugate.
- (ii) The group G_0^σ , defined in 4.9, is independent of the choice of a σ -stable standard pair.

Proof. We prove the assertions in several steps.

(1) Let \mathfrak{p}_1 and \mathfrak{p}_2 be maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebras of \mathfrak{g}^σ . There exist σ -stable Borel subalgebras \mathfrak{b}_1 and \mathfrak{b}_2 of \mathfrak{g} , and $x_1 \in U_1^\sigma, x_2 \in U_2^\sigma$ such that $\mathfrak{p}_1 = Ad(x_1 x_2^{-1}) \mathfrak{p}_2$. By 4.7 and 4.12, there exist σ -stable standard pairs (t_1, \mathfrak{b}_1) and (t_2, \mathfrak{b}_2) such that $\mathfrak{p}_1 = t_1^\sigma, \mathfrak{p}_2 = t_2^\sigma$, and \mathfrak{b}_1 and \mathfrak{b}_2 are commensurable with \mathfrak{b}_+ and \mathfrak{b}_- , respectively. Then $\mathfrak{b}_1 \cap \mathfrak{b}_2$ is a finite dimensional σ -stable solvable subalgebra. By 4.5(i), there exists a σ -stable Cartan subalgebra \mathfrak{t}_3 of $\mathfrak{b}_1 \cap \mathfrak{b}_2$. By 4.11, there exist $x_1 \in U_1^\sigma$ and $x_2 \in U_2^\sigma$ such that $t_1 = Ad(x_1) t_3$ and $t_2 = Ad(x_2) t_3$. Then $t_1 = Ad(x_1 x_2^{-1}) t_2$ and so $\mathfrak{p}_1 = Ad(x_1 x_2^{-1}) \mathfrak{p}_2$.

(2) Let (t, \mathfrak{b}) and (t_1, \mathfrak{b}_1) be σ -stable standard pairs. Then $\langle (U^-)^\sigma, U^\sigma \rangle = \langle (U_1^-)^\sigma, U_1^\sigma \rangle$. Indeed, by 4.10, 4.12, and (1), there exists $x \in G^\sigma$ such that $Ad(x)t_1 = t$. Since the groups are normal in G^σ , we may assume that $t_1 = t$. Then by 1.13(ii), $U_1^\pm = (U_1^\pm \cap U)(U_1^\pm \cap U^-)$ and as a consequence $(U_1^\pm)^\sigma \subset U^\sigma(U^-)^\sigma$. Thus $\langle (U_1^-)^\sigma, U_1^\sigma \rangle$ is contained in $\langle (U^-)^\sigma, U^\sigma \rangle$. Similarly we have the opposite inclusion relation.

(3) The assertion (ii) is immediate from (2). Now we return to (i). By (2), the groups U_1^σ and U_2^σ are contained in G_0^σ . Then by (1), \mathfrak{p}_1 and \mathfrak{p}_2 are $Ad(G_0^\sigma)$ -conjugate.

4.14. COROLLARY. *Let σ be a semisimple automorphism of \mathfrak{g} of the first kind. Then maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebras of $(\mathfrak{g}')^\sigma$ are $Ad(G_0^\sigma)$ -conjugate.*

4.15. Let σ be a semisimple automorphism of \mathfrak{g} of the first kind. For simplicity of description, we may assume that $(\mathfrak{h}, \mathfrak{b}_+)$ is a σ -stable standard pair. The generalized Cartan matrix A is symmetrizable. Write

$$A = DB,$$

where $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ and B is symmetric. Since σ is semisimple, there exists σ -stable complementary subspace \mathfrak{h}'' to \mathfrak{h}' in \mathfrak{h} . We define a symmetric bilinear F -valued form $(\cdot | \cdot)$ on \mathfrak{h} by the conditions

$$(h_i | h) = \varepsilon_i \langle \alpha_i, h \rangle \quad \text{for } h \in \mathfrak{h}, i = 1, \dots, n;$$

$$(x | y) = 0 \quad \text{for } x, y \in \mathfrak{h}''.$$

σ induces a permutation τ of $\{1, \dots, n\}$ such that $\sigma\alpha_i = \alpha_{\tau(i)}$ and $\sigma h_i = h_{\tau(i)}$ ($i = 1, \dots, n$). Clearly we have that

$$a_{ij} = a_{\tau(i)\tau(j)} \quad (i, j = 1, \dots, n). \quad (1)$$

Let J be the permutative matrix corresponding to τ . Then (1) implies that $JAJ^{-1} = A$. Suppose that A is indecomposable. By [10, 2.3], D is unique up to a constant factor. Note that $\varepsilon_i/\varepsilon_j$ ($i, j = 1, \dots, n$) are positive rational numbers. It follows that $D = JDJ^{-1}$ and so

$$\varepsilon_i = \varepsilon_{\tau(i)} \quad (i = 1, \dots, n). \quad (2)$$

From (2), $(h_i | h) = (\sigma h_i | \sigma h)$ for $h \in \mathfrak{h}$ and $i = 1, \dots, n$. Hence the form $(\cdot | \cdot)$ on \mathfrak{h} is σ -invariant. By [10, Theorem 2.2], $(\cdot | \cdot)$ extends uniquely to a σ -invariant and $ad_{\mathfrak{g}}$ -invariant bilinear nondegenerate symmetric form on \mathfrak{g} .

LEMMA. *Let σ be a semisimple automorphism of \mathfrak{g} . Then there exists a nondegenerate symmetric σ -invariant and $ad_{\mathfrak{g}}$ -invariant bilinear form $(\cdot | \cdot)$ on \mathfrak{g} .*

Proof. By a simple reduction argument, we may assume that A is indecomposable.

Case 1. σ is of the first kind. The assertion is established in the above discussion.

Case 2. σ is of the second kind. Then σ^2 is of the first kind. We may assume that $(\mathfrak{h}, \mathfrak{b}_+)$ is σ^2 -stable. Consider the automorphism $\omega \circ \sigma = \tau$. Clearly $\tau|\mathfrak{h} = -\sigma|\mathfrak{h}$ and there exists a positive integer l such that $\tau^l g_{\pm\alpha_i} = g_{\pm\alpha_i}$ ($i = 1, \dots, n$). It follows that τ is semisimple. Note that a nondegenerate symmetric $ad_{\mathfrak{g}}$ -invariant form $(\cdot|\cdot)$ on \mathfrak{g} is ω -invariant. Hence $(\cdot|\cdot)$ is σ -invariant if and only if it is τ -invariant. Now the assertion follows from that in Case 1.

4.16. Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} and $\Phi(\mathfrak{t}, \mathfrak{g})$ the set of roots of \mathfrak{t} in \mathfrak{g} . A subset Φ^+ of $\Phi(\mathfrak{t}, \mathfrak{g})$ is called a *positive system* if there exists a standard pair $(\mathfrak{t}, \mathfrak{b})$ such that $\Phi^+ = \Phi(\mathfrak{t}, \mathfrak{b})$. Given a positive system Φ^+ , it determines a subset $\Pi = \{\beta_1, \dots, \beta_n\}$, called the set of simple roots, such that every $\alpha \in \Phi(\mathfrak{t}, \mathfrak{g})$ is an integral combination of Π with all coefficients in \mathbb{Z}_+ or $-\mathbb{Z}_+$.

Let σ be a semisimple automorphism of \mathfrak{g} of the first kind. A σ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} is called *fundamental* (for σ) if there exists a σ -stable Borel subalgebra of \mathfrak{g} containing \mathfrak{t} .

PROPOSITION. *Let $\sigma \in Aut_F(\mathfrak{g})$ be semisimple of the first kind and \mathfrak{t} a σ -stable Cartan subalgebra of \mathfrak{g} . The following conditions are equivalent:*

- (i) \mathfrak{t} is fundamental.
- (ii) There exists a σ -stable positive system of $\Phi(\mathfrak{t}, \mathfrak{g})$.
- (iii) \mathfrak{t}^σ is a maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of \mathfrak{g}^σ .
- (iv) $\mathfrak{t}^\sigma \cap \mathfrak{g}'$ is a maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of $(\mathfrak{g}')^\sigma$.

Proof. (i) \Leftrightarrow (ii) is obvious from the definitions.

(i) \Leftrightarrow (iii) is immediate from 4.12.

(iii) \Rightarrow (iv). Let \mathfrak{a}' be a maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of $(\mathfrak{g}')^\sigma$. Choose any maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra \mathfrak{a} of \mathfrak{g}^σ containing \mathfrak{a}' . By 4.13, \mathfrak{t}^σ and \mathfrak{a} are $Ad(G^\sigma)$ -conjugate. This implies that $\mathfrak{t}^\sigma \cap \mathfrak{g}'$ and \mathfrak{a}' are also $Ad(G^\sigma)$ -conjugate. Thus (iv) follows.

(iv) \Rightarrow (iii). Since $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}'$, so $\mathfrak{g}^\sigma = \mathfrak{t}^\sigma + (\mathfrak{g}')^\sigma$. Let \mathfrak{p} be any maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of \mathfrak{g}^σ containing \mathfrak{t}^σ . Then $\mathfrak{p} = \mathfrak{t}^\sigma + (\mathfrak{p} \cap \mathfrak{g}')$. By (iv), $\mathfrak{p} \cap \mathfrak{g}'$ must coincide with $\mathfrak{t}^\sigma \cap \mathfrak{g}'$ and so $\mathfrak{t}^\sigma = \mathfrak{p}$ is a maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebra of \mathfrak{g}^σ .

4.17. **LEMMA.** *Let σ be an automorphism of \mathfrak{g} and \mathfrak{t} a σ -stable Cartan subalgebra. If α is a real root of \mathfrak{t} with $\mathfrak{g}_\alpha \subset \mathfrak{g}^\sigma$, then $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}^\sigma$.*

Proof. Consider the subalgebra $\mathfrak{m} = F\alpha^\vee + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$. It is σ -stable and isomorphic to $sl(2, F)$. Since $\sigma\alpha = \alpha$, $\sigma|F\alpha^\vee = 1$ and σ is trivial on $F\alpha^\vee + \mathfrak{g}_\alpha$. This yields readily that σ is trivial on \mathfrak{m} .

4.18. Let σ be a quasi-semisimple automorphism of \mathfrak{g} of the first kind, $(\mathfrak{t}, \mathfrak{b})$ a σ -stable standard pair, and Π the set of simple roots of \mathfrak{t} in \mathfrak{b} . We call σ a *diagram automorphism* (with respect to $(\mathfrak{t}, \mathfrak{b})$) if for every $(\alpha, i) \in \Pi \times \mathbb{Z}$ with $\sigma^i\alpha = \alpha$, we have $\sigma^i|_{\mathfrak{g}_\alpha} = 1$. It is easy to see that the above condition is equivalent to the following condition:

(i) There exist $0 \neq e_\alpha \in \mathfrak{g}_\alpha (\alpha \in \Pi)$ such that $\sigma e_\alpha = e_{\sigma\alpha} (\alpha \in \Pi)$.

4.19. Let $A = (a_{ij})$ be the generalized Cartan matrix such that $\mathfrak{g} = \mathfrak{g}(A)$. Recall that $Aut(A)$ is the group of permutations σ of $\{1, \dots, n\}$ such that

$$a_{ij} = a_{\sigma i, \sigma j} \quad (i, j = 1, \dots, n). \tag{1}$$

Let Q denote the vector space $\sum_{i=1}^n F\alpha_i$. Then $\mathfrak{h}/\mathfrak{c}$ is identified with the dual space Q^* of Q . Given $\sigma \in Aut(A)$, σ defines an automorphism of Q by $\sigma\alpha_i = \alpha_{\sigma i} (i = 1, \dots, n)$ and so an automorphism of $\mathfrak{h}/\mathfrak{c}$. From (1), $\mathfrak{h}'/\mathfrak{c}$ is $Aut(A)$ -stable. Since $Aut(A)$ is finite, there exists a subspace \mathfrak{h}'' of \mathfrak{h} such that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ and $(\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}$ is $Aut(A)$ -stable. Now define an action of $Aut(A)$ on \mathfrak{h} by the following conditions. For $\sigma \in Aut(A)$,

$$\begin{aligned} \sigma h_i &= h_{\sigma i} \quad (i = 1, \dots, n), \\ \sigma|_{\mathfrak{h}''} &\text{ is the pull back of } \sigma|_{(\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}}. \end{aligned} \tag{2}$$

It follows that

$$\langle \sigma h, \alpha_{\sigma i} \rangle = \langle h, \alpha_i \rangle \quad (h \in \mathfrak{h}, i = 1, \dots, n).$$

From [10, 1.4], σ extends to an automorphism of \mathfrak{g} by

$$\begin{aligned} \sigma e_i &= e_{\sigma i}, \quad \sigma f_i = f_{\sigma i} \quad (i = 1, \dots, n), \\ \sigma|_{\mathfrak{h}} &\text{ is given in (2).} \end{aligned} \tag{3}$$

Clearly elements of $Aut(A)$ are diagram automorphisms of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$ and ω centralizes $Aut(A)$.

4.20. Let $Aut_F(\mathfrak{g}; \mathfrak{g}')$ denote the group of automorphisms of \mathfrak{g} which are trivial on \mathfrak{g}' . Clearly $Aut_F(\mathfrak{g}; \mathfrak{g}')$ is a normal subgroup of $Aut_F(\mathfrak{g})$. For $\sigma \in Aut_F(\mathfrak{g}; \mathfrak{g}')$, \mathfrak{h}' is σ -stable and by 1.31, \mathfrak{h} is σ -stable. Since $\sigma \mathfrak{g}_{\alpha_i} = \mathfrak{g}_{\alpha_i} (i = 1, \dots, n)$, σ acts trivially on $\mathfrak{h}/\mathfrak{c}$. Hence there is $f \in Hom_F(\mathfrak{h}'', \mathfrak{c})$ such that

$$\sigma x = x + f(x), \quad x \in \mathfrak{h}''.$$

Conversely given any $f \in \text{Hom}_F(\mathfrak{h}'', \mathfrak{c})$, let σ_f be the linear map defined by

$$\sigma_f|_{\mathfrak{g}'} = 1 \quad \text{and} \quad \sigma_f(x) = x + f(x) \quad (x \in \mathfrak{h}'').$$

One checks easily that $\sigma_f \in \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$. Hence $f \mapsto \sigma_f (f \in \text{Hom}_F(\mathfrak{h}'', \mathfrak{c}))$ defines an isomorphism between $\text{Hom}_F(\mathfrak{h}'', \mathfrak{c})$ and $\text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$.

4.21. LEMMA. *Let G be a connected unipotent algebraic group, τ a semi-simple automorphism of G , and $C(\tau, G) = \{\tau(g)g^{-1} \mid g \in G\}$. We have the following:*

(i) *The product map $C(\tau, G) \times G^{\tau} \rightarrow G$ is an isomorphism of varieties.*

(ii) *Assume that M is an algebraic group containing G and $t \in M$ is a semisimple element normalizing G . For $g \in G$, the element tg is semisimple if and only if $g \in \{t^{-1}xtx^{-1} \mid x \in G\}$.*

Proof. (i) is immediate from [3, 11.1] and (ii) is proved in [5, 1.7].

4.22. PROPOSITION. *Let σ be a diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{t}, \mathfrak{b})$. Assume that A is indecomposable. Then the following conditions are equivalent:*

(i) *σ is of finite order.*

(ii) *σ is semisimple.*

(iii) *Let $\text{Aut}_F^0(\mathfrak{g}) = \text{Aut}_F(\mathfrak{g}; \mathfrak{g}') \times \text{Ad}(\tilde{H} \ltimes G)$. The automorphism σ is $\text{Aut}_F^0(\mathfrak{g})$ -conjugate to an element of $\text{Aut}(A)$.*

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Let \mathfrak{b}^- denote the Borel subalgebra of \mathfrak{g} such that $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{t}$. By 4.17, σ is also a diagram automorphism with respect to $(\mathfrak{t}, \mathfrak{b}^-)$. Replacing \mathfrak{b} by \mathfrak{b}^- if necessary, we may assume that \mathfrak{b} is commensurable with \mathfrak{b}_+ . Then choose an element $g \in G$ satisfying the conditions

$$\text{Ad}(g)\mathfrak{t} = \mathfrak{h}, \quad \text{Ad}(g)\mathfrak{b} = \mathfrak{b}_+.$$

Replacing σ by $\text{Ad}(g) \circ \sigma \circ \text{Ad}(g^{-1})$, we may assume that $\mathfrak{t} = \mathfrak{h}$ and $\mathfrak{b} = \mathfrak{b}_+$. There exists a permutation τ of $\{1, \dots, n\}$ such that

$$\sigma \mathfrak{g}_{\alpha_i} = \mathfrak{g}_{\alpha_{\tau(i)}} \quad (i = 1, \dots, n).$$

By assumption, σ is a diagram automorphism and so there exist $0 \neq e'_i \in \mathfrak{g}_{\alpha_i}$ with $\sigma e'_i = e'_{\tau(i)}$ ($i = 1, \dots, n$). Choose $x \in \tilde{H}$ such that $\text{Ad}(x)e_i = e'_i$ ($i = 1, \dots, n$). Replacing σ by $\text{Ad}(x^{-1}) \circ \sigma \circ \text{Ad}(x)$, we may assume that $\sigma e_i = e_{\tau(i)}$ ($i = 1, \dots, n$). Consider the diagram automorphism τ , also denoted by τ , corresponding to the permutation τ as in 4.19. Then $\sigma|_{\mathfrak{g}'} = \tau|_{\mathfrak{g}'}$ and so

$\sigma = \tau \circ \gamma$ with $\gamma \in \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$. Let M denote the image of $\text{Aut}(A) \rtimes \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$ in $GL(\mathfrak{h})$. Note that the group $\text{Aut}_F(\mathfrak{g}; \mathfrak{g}') | \mathfrak{h}$ is a connected unipotent algebraic group. From 4.21(ii), σ is $\text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$ -conjugate to τ . Hence assertion (iii) follows.

(iii) \Rightarrow (i) is obvious.

4.23. Recall that $\tilde{H} = \text{Hom}(Q, F^\times)$ where $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$. For $x \in \tilde{H}$, $Ad(x)$ is the automorphism of \mathfrak{g} determined by $Ad(x)v = x(\alpha)v (\alpha \in \Delta, x \in \mathfrak{g}_\alpha)$. For $\sigma \in \text{Aut}_F(\mathfrak{g})$ with $\sigma(\mathfrak{h}) = \mathfrak{h}$, σ determines an automorphism of Q , denoted also by σ , such that $\sigma\mathfrak{g}_\alpha = \mathfrak{g}_{\sigma\alpha} (\alpha \in \Delta)$. Then σ acts on \tilde{H} by $(\sigma x)(\alpha) = x(\sigma^{-1}\alpha) (x \in \tilde{H}, \alpha \in Q)$. One checks readily that $\sigma Ad(x)\sigma^{-1} = Ad(\sigma x) (x \in \tilde{H})$. In particular σ normalizes $Ad(\tilde{H})$. Now let τ be any automorphism of \mathfrak{g} . There exists $g \in G$ such that $Ad(g)\mathfrak{h} = \tau(\mathfrak{h})$. Then $\tau Ad(\tilde{H})\tau^{-1} = Ad(g) Ad(\tilde{H}) Ad(g^{-1})$ and so $Ad(\tilde{H} \rtimes G)$ is a normal subgroup of $\text{Aut}_F(\mathfrak{g})$.

PROPOSITION. $\text{Aut}_F(\mathfrak{g}) = \text{Out}(A) \rtimes (\text{Aut}_F(\mathfrak{g}; \mathfrak{g}') \times Ad(\tilde{H} \rtimes G))$.

Proof. Let $\tau \in \text{Aut}_F(\mathfrak{g})$. Replacing τ by an element in $\tau \text{Out}(A)$ if necessary, we may assume that τ is of the first kind. Choose $g \in G$ with $\tau(\mathfrak{h}) = Ad(g)\mathfrak{h}$ and $\tau(\mathfrak{b}_+) = Ad(g)\mathfrak{b}_+$. Replacing τ by $Ad(g^{-1}) \circ \tau$, we may assume that $(\mathfrak{h}, \mathfrak{b}_+)$ is τ -stable. Replacing τ by an element of the form $Ad(x) \circ \tau (x \in \tilde{H})$, we may assume that τ is a diagram automorphism. Then $\tau \in \text{Aut}(A) \rtimes \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$. Now our assertion is obvious.

4.24. LEMMA. *Let G be a connected semisimple algebraic group, T a maximal torus of G , and B a Borel subgroup of G containing T . Let τ be a diagram automorphism of G with respect to (T, B) and $S = T_0^\tau$ the identity component of T^τ . Then we have the following:*

(i) G_τ is semisimple.

(ii) $N_G(S) = N_{G_0^\tau}(S) \cdot T$.

(iii) *Let $w \in W(T)$ be the element such that $B \cap wBw^{-1} = T$. Then there exists $n \in N_{G_0^\tau}(T)$ with image w in W .*

Proof. (i) and (ii) were established in [5, 9.9]. Note that $w \in W^\tau$ and so S is w -stable. Now assertion (iii) is immediate from (ii).

4.25. Let τ be a diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$. For $X \subset \Pi$, G_X is the subgroup of G generated by $U_\alpha, U_{-\alpha} (\alpha \in X)$. If X is a τ -orbit of finite type, then G_X is a connected semisimple τ -stable algebraic subgroup of G . From 4.24(iii), there exists an element $n_X \in G_X$ satisfying:

$n_X \in N \cap G_0^\tau$ and the image of n_X in W_X is the element of longest length. (1)

LEMMA. Let N_0^τ denote the subgroup of N generated by the elements n_X (X τ -orbit in Π of finite type). We have the following conditions:

- (i) W^τ coincides with the image of N_0^τ in W .
- (ii) $N^\tau = H^\tau N_0^\tau$.

Proof. Let $w \in W^\tau$ and $\Phi(w) = \{\alpha \in \Delta_+ \mid w\alpha < 0\}$. Write $w = r_{i_1} \cdots r_{i_l}$ with $l = l(w)$. We show that w lies in the image $Im(N_0^\tau)$ of N_0^τ in W by induction on l . Observe that $\Phi(w)$ is τ -stable and α_{i_l} lies in $\Phi(w)$. Consider the τ -orbit X in Π containing α_{i_l} . Since $X \subset \Phi(w)$ and $\Phi(w)$ is finite, X is of finite type. Choose n_X as in (1) and let r_X be the image of n_X in W . Clearly $w r_X \in W^\tau$ and $l(w r_X) = l(w) - Card(\Delta_X^+)$ where $\Delta_X^+ = \Delta^+ \cap \mathbb{Z}X$. Thus $w r_X \in Im(N_0^\tau)$ and so $w = (w r_X) r_X^{-1}$ lies in $Im(N_0^\tau)$. Hence (i) follows. Note that (ii) is immediate from (i).

4.26. PROPOSITION. Let τ be a diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$. Then we have the following conditions:

- (i) $G^\tau = H^\tau \langle U_+^\tau, U_-^\tau \rangle$.
- (ii) If τ is semisimple, then τ is a diagram automorphism of \mathfrak{g} with respect to any τ -stable standard pair $(\mathfrak{t}, \mathfrak{b})$.

Proof. (i) is immediate from 4.9 and 4.25(ii). By 4.12, \mathfrak{h}^τ and \mathfrak{t}^τ are maximal $ad_{\mathfrak{g}}$ -diagonalizable subalgebras of \mathfrak{g}^τ . By 4.13(i), \mathfrak{h}^τ and \mathfrak{t}^τ are $Ad(G^\tau)$ -conjugate. By 4.10, $\mathfrak{h} = z_{\mathfrak{g}}(\mathfrak{h}^\tau)$ and $\mathfrak{t} = z_{\mathfrak{g}}(\mathfrak{t}^\tau)$. Hence \mathfrak{h} and \mathfrak{t} are $Ad(G^\tau)$ -conjugate. We may assume that $\mathfrak{t} = \mathfrak{h}$. There exists a decomposition $A = A_1 + A_2$ with corresponding decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that \mathfrak{g}_1 and \mathfrak{g}_2 are τ -stable, and $\mathfrak{b} \cap \mathfrak{g}_1$ (resp. $\mathfrak{b} \cap \mathfrak{g}_2$) is commensurable with $\mathfrak{b}_+ \cap \mathfrak{g}_1$ (resp. $\mathfrak{b}_- \cap \mathfrak{g}_2$). It suffices to show (ii) under the condition that \mathfrak{b} is commensurable with \mathfrak{b}_\pm . Replacing \mathfrak{b} by \mathfrak{b}^- if necessary, we may assume that \mathfrak{b} and \mathfrak{b}_+ are commensurable. It follows that there exists $w \in W$ with $\mathfrak{b} = w(\mathfrak{b}_+)$. Since \mathfrak{b} and \mathfrak{b}_+ are τ -stable, so $w \in W^\tau$. Now the assertion is immediate from 4.25(i).

4.27. LEMMA. Let τ and τ' be diagram automorphisms of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$. Then $\tau|_{\mathfrak{h}} = \tau'|_{\mathfrak{h}}$ if and only if τ and τ' are $Ad(\tilde{H})$ -conjugate.

Proof. (\Rightarrow) By definition, there exist nonzero elements $x_i, y_i \in \mathfrak{g}_{\alpha_i}$ ($i = 1, \dots, n$) such that $\tau(x_i) = x_j, \tau'(y_i) = y_j$ if $\tau(\alpha_i) = \alpha_j$. Choose $t \in \tilde{H}$ with $Ad(t)x_i = y_i$ ($i = 1, \dots, n$). Then τ and $Ad(t^{-1}) \circ \tau' \circ Ad(t)$ coincide on \mathfrak{h} and \mathfrak{g}_{α_i} ($i = 1, \dots, n$) and by 4.17, also on $\mathfrak{g}_{-\alpha_i}$ ($i = 1, \dots, n$). This implies that $\tau = Ad(t^{-1}) \circ \tau' \circ Ad(t)$.

(\Leftarrow) Note that $Ad(\tilde{H})$ acts trivially on \mathfrak{h} . Hence the assertion is obvious.

4.28. LEMMA. *Let T be an algebraic torus, τ an automorphism of T of finite order, $C(\tau, T) = \{\tau(t) t^{-1} \mid t \in T\}$, and T_0^τ the identity component of T^τ . Then $T = C(\tau, T) \cdot T_0^\tau$ is an almost direct product.*

Proof. Let $X_*(T)$ denote the group of one parameter subgroups of T . Consider the action of τ on $X_*(T) \otimes \mathbb{Q}$. Since τ is of finite order, $X_*(T) \otimes \mathbb{Q}$ is completely reducible for the action of τ . It follows that there exists a τ -stable subtorus T_1 of T such that $T = T_1 \cdot T_0^\tau$ is an almost direct product. One checks easily that $T_1 = C(\tau, T)$.

4.29. PROPOSITION. *Let σ be a semisimple automorphism of \mathfrak{g} of the first kind, $(\mathfrak{t}, \mathfrak{b})$ a σ -stable standard pair, Π the set of simple roots of \mathfrak{t} in \mathfrak{b} , $\tilde{T} = \text{Hom}(\mathbb{Z}\Pi, F^\times)$, and \tilde{T}_0^σ the identity component of \tilde{T}^σ . Then we have a decomposition*

$$\sigma = Ad(t) \circ \tau,$$

where τ is a semisimple diagram automorphism with respect to $(\mathfrak{t}, \mathfrak{b})$ and $t \in \tilde{T}_0^\sigma$; moreover τ is unique modulo $Ad(C(\sigma, \tilde{T})) \cap Ad(\tilde{T}_0^\sigma)$.

Proof. Without loss of generality, we may assume that $\mathfrak{t} = \mathfrak{h}$, $\mathfrak{b} = \mathfrak{b}_+$, and $\tilde{T} = \tilde{H}$. There exists a permutation of $\{1, \dots, n\}$, also denoted by σ , such that

$$\sigma \mathfrak{g}_{\alpha_i} = \mathfrak{g}_{\alpha_{\sigma(i)}} \quad (i = 1, \dots, n).$$

Choose $t \in \tilde{H}$ satisfying the conditions

$$(Ad(t) \circ \sigma) e_i = e_{\sigma(i)} \quad (i = 1, \dots, n). \tag{1}$$

Set $\sigma_1 = Ad(t) \circ \sigma$. Then by (1), σ_1 is a diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$. Note that $\sigma_1|_{\mathfrak{g}'}$ is of finite order and $\sigma_1|_{\mathfrak{h}} = \sigma|_{\mathfrak{h}}$. Hence σ_1 is semisimple. Clearly σ_1 acts on $\mathbb{Z}\Pi$ through a permutation of Π . Hence σ_1 acts on \tilde{H} through a finite group. By 4.28, we can write

$$t^{-1} = y(x^{-1}\sigma_1(x)),$$

with $y \in \tilde{H}_0^{\sigma_1}$ and $x \in \tilde{H}$. Observe that

$$Ad(x^{-1}\sigma_1(x)) = Ad(x^{-1}) \circ \sigma_1 \circ Ad(x) \circ \sigma_1^{-1}. \tag{2}$$

Set $\tau = Ad(x^{-1}) \circ \sigma_1 \circ Ad(x)$. Then τ is a semisimple diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$, and by (2),

$$\sigma = Ad(y) \circ \tau. \tag{3}$$

Since $Ad(\tilde{H})$ acts trivially on \mathfrak{h} , $\sigma|_{\mathfrak{h}} = \sigma_1|_{\mathfrak{h}} = \tau|_{\mathfrak{h}}$ and so $\tilde{H}^\sigma = \tilde{H}^{\sigma_1} = \tilde{H}^\tau$. Hence (3) is a desired decomposition of σ . The uniqueness of τ is immediate from 4.27.

4.30. Let σ be a semisimple automorphism of \mathfrak{g} of the first kind and $(\mathfrak{t}, \mathfrak{b})$ a σ -stable standard pair. A decomposition of σ as in 4.29 will be called a *canonical decomposition* of σ with respect to the standard pair $(\mathfrak{t}, \mathfrak{b})$. For semisimple Lie algebras, the notion of a canonical decomposition of semisimple automorphisms is due to Gantmacher [8].

4.31. LEMMA. *Let σ be a semisimple automorphism of \mathfrak{g} of the first kind. Then there exists $y \in Aut_F(\mathfrak{g}; \mathfrak{g}') \times Ad(\tilde{H} \ltimes G)$ such that $y\sigma y^{-1}$ has a canonical decomposition $y\sigma y^{-1} = Ad(x) \circ \tau$ with $\tau \in Aut(A)$ and $x \in \tilde{H}_0^\tau$.*

Proof. There exists a σ -stable standard pair $(\mathfrak{t}, \mathfrak{b})$ such that \mathfrak{b} and \mathfrak{b}_+ are commensurable. Choose $g \in G$ with $Ad(g)\mathfrak{t} = \mathfrak{h}$ and $Ad(g)\mathfrak{b} = \mathfrak{b}_+$. Replacing σ by $Ad(g) \circ \sigma \circ Ad(g^{-1})$, we may assume that $(\mathfrak{h}, \mathfrak{b}_+)$ is σ -stable. By 4.29, we have a canonical decomposition $\sigma = Ad(x) \circ \tau_1$ where τ_1 is a semisimple diagram automorphism of \mathfrak{g} with respect to $(\mathfrak{h}, \mathfrak{b}_+)$ and $x \in \tilde{H}_0^{\tau_1}$. There exists $\tau \in Aut(A)$ such that $\tau|_{\mathfrak{g}'}$ and $\tau_1|_{\mathfrak{g}'}$ are $Ad(\tilde{H})$ -conjugate. Replacing σ by an $Ad(\tilde{H})$ -conjugate, we may assume that $\tau_1|_{\mathfrak{g}'} = \tau|_{\mathfrak{g}'}$. By 4.21(ii), τ_1 and τ are $Aut_F(\mathfrak{g}; \mathfrak{g}')$ -conjugate. Note that $Aut_F(\mathfrak{g}; \mathfrak{g}')$ centralizes $Ad(\tilde{H})$. Now the assertion is immediate.

4.32. Let $\tau_i \in Aut(A)$, $x_i \in \tilde{H}_0^{\tau_i}$ ($i = 1, 2$). Now we discuss the condition when $Ad(x_1) \circ \tau_1$ and $Ad(x_2) \circ \tau_2$ are conjugate to each other in $Aut_F(\mathfrak{g})$. From 4.23,

$$Aut_F(\mathfrak{g}) = Out(A) \ltimes (Aut_F(\mathfrak{g}; \mathfrak{g}') \times Ad(\tilde{H} \ltimes G)). \tag{1}$$

Hence τ_1 and τ_2 are conjugate in $Out(A)$.

Suppose that $\tau_1 = \tau_2 = \tau$ and $y \in Aut_F(\mathfrak{g}; \mathfrak{g}') \times Ad(\tilde{H} \ltimes G)$ such that

$$y(Ad(x_1) \circ \tau) y^{-1} = Ad(x_2) \circ \tau. \tag{2}$$

Set $\sigma = Ad(x_2) \circ \tau$. From (2), $(y(\mathfrak{h}), y(\mathfrak{b}_+))$ is a σ -stable standard pair. From 4.10 and 4.13(i), there exists $g \in G^\sigma$ such that $gy(\mathfrak{h}) = \mathfrak{h}$. Replacing y by gy , we may assume that

$$y(\mathfrak{h}) = \mathfrak{h}.$$

Choose $w \in W$ such that $y(\mathfrak{b}_+) = w(\mathfrak{b}_+)$. Since \mathfrak{b}_+ and $w(\mathfrak{b}_+)$ are τ -stable, so $w \in W^\tau$ and by 4.25(i), w has a preimage $n \in N^\tau$. Set $y_1 = Ad(n^{-1})y$. Then $(\mathfrak{h}, \mathfrak{b}_+)$ is a y_1 -stable standard pair. It follows that

$y_1 \in \text{Aut}_F(\mathfrak{g}; \mathfrak{g}') \times \text{Ad}(\tilde{H})$. From (2) and $n \in N^\tau$, $y_1 \tau y_1^{-1} | \mathfrak{h} = \tau | \mathfrak{h}$ and by 4.27, $y_1 \in \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')^\tau \times \text{Ad}(\tilde{H})$. Hence we may assume that

$$y \in \text{Ad}(N^\tau) \text{Aut}_F(\mathfrak{g}; \mathfrak{g}')^\tau \text{Ad}(\tilde{H}). \tag{3}$$

Now we are ready to present our main result on classification of semi-simple automorphisms of \mathfrak{g} of the first kind.

THEOREM. (i) For $\tau \in \text{Aut}(A)$, let $W^\tau = \{w \in W \mid \tau w = w\tau \text{ on } \mathfrak{h}\}$. Then W^τ coincides with the image of N^τ in W .

(ii) Let $\{\tau_1, \dots, \tau_l\}$ be a complete set of representatives of the conjugacy classes in $\text{Out}(A)$ containing automorphisms of \mathfrak{g} of the first kind. For τ_i , set $C(\tau_i, \tilde{H}) = \{\tau_i(t) t^{-1} \mid t \in \tilde{H}\}$ and $S_i = \tilde{H}/C(\tau_i, \tilde{H})$. Then every semi-simple automorphism of \mathfrak{g} of the first kind is $\text{Aut}_F(\mathfrak{g})$ -conjugate to an element of the form $\text{Ad}(x) \circ \tau_i$ with $x \in \tilde{H}_0^{\tau_i}$; moreover $\text{Ad}(x) \circ \tau_i$ and $\text{Ad}(y) \circ \tau_j$ with $x \in \tilde{H}^{\tau_i}$, $y \in \tilde{H}^{\tau_j}$ are conjugate to each other if and only if $i = j$ and x, y have the same image in the orbit space $S_i/\text{Out}(A)^{\tau_i} \times W^{\tau_i}$.

Proof. (i) This is immediate from 4.25(i).

(ii) Note that $\text{Aut}_F(\mathfrak{g}; \mathfrak{g}')$ centralizes $\text{Ad}(\tilde{H})$. From the above discussion, $\text{Ad}(x) \circ \tau_i$ and $\text{Ad}(y) \circ \tau_j$ are conjugate if and only if $i = j$ and they are $\text{Out}(A)^{\tau_i} \text{Ad}(N^{\tau_i} \tilde{H})$ -conjugate. Since $\text{Ad} | \tilde{H}$ is injective, our desired conclusion follows.

In the following, we discuss semisimple automorphisms of \mathfrak{g} of the second kind. First we need some definitions.

4.33. Let $X \subset \mathfrak{H}$, $\mathfrak{p} = \text{Ad}(g) \mathfrak{p}_X$ (resp. $\text{Ad}(g) \mathfrak{p}_X^-$) with $g \in G$ and \mathfrak{m} a subalgebra of \mathfrak{p} . We call \mathfrak{m} a *Levi-subalgebra* of \mathfrak{p} if \mathfrak{m} is $\text{Ad}(N_G(\mathfrak{p}))$ -conjugate to $\text{Ad}(g)(\mathfrak{h} + \bigoplus_{\alpha \in \Delta_X} \mathfrak{g}_\alpha)$ where $\Delta_X = \Delta \cap \mathbb{Z}X$.

For $\sigma \in \text{Aut}_F(\mathfrak{g})$, a subalgebra \mathfrak{a} of \mathfrak{g} is called a σ -split toral subalgebra of \mathfrak{g} if $\sigma | \mathfrak{a} = -1$ and \mathfrak{a} is $\text{ad}_{\mathfrak{g}}$ -diagonalizable. A parabolic subalgebra \mathfrak{p} of \mathfrak{g} is called σ -split if $\mathfrak{p} \cap \sigma(\mathfrak{p})$ is a Levi-subalgebra of \mathfrak{p} and $\sigma(\mathfrak{p})$.

4.34. **LEMMA.** Let \mathfrak{m} be a finite dimensional semisimple Lie algebra over F and σ a semisimple automorphism of \mathfrak{m} . If there exists a nontrivial σ -split toral subalgebra \mathfrak{a} of \mathfrak{m} , then there exists a proper σ -split parabolic subalgebra \mathfrak{p} of \mathfrak{m} such that \mathfrak{p} is σ^2 -stable.

Proof. By 4.8, there exists a σ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{m} contained in $z_{\mathfrak{m}}(\mathfrak{a})$. We may assume that \mathfrak{a} is algebraic. There exists $0 \neq x \in \mathfrak{a}$ such that $\alpha(x) \in \mathbb{Z}(\alpha \in \Phi(\mathfrak{t}, \mathfrak{m}))$. Set $\varphi = \{\alpha \in \Phi(\mathfrak{t}, \mathfrak{m}) \mid \alpha(x) \geq 0\}$ and $\mathfrak{p} = \mathfrak{t} + \bigoplus_{\alpha \in \varphi} \mathfrak{m}_\alpha$. Since $\sigma(x) = -x$, \mathfrak{p} is a σ -split parabolic subalgebra of \mathfrak{m} and is σ^2 -stable.

4.35. PROPOSITION. *Let σ be a semisimple automorphism of \mathfrak{g} of the second kind. Then we have the following conditions:*

(i) *There exists a σ -split finite type parabolic subalgebra \mathfrak{q} of \mathfrak{g} such that \mathfrak{q} is σ^2 -stable.*

(ii) *Let \mathfrak{p} be a minimal σ -split finite type parabolic subalgebra such that \mathfrak{p} is σ^2 -stable. If \mathfrak{t} is a σ -stable Cartan subalgebra of \mathfrak{g} contained in $\mathfrak{p} \cap \sigma(\mathfrak{p})$, then $\mathfrak{t}^{-\sigma}$ is a maximal σ -split toral subalgebra of \mathfrak{g} and $z_{\mathfrak{g}}(\mathfrak{t}^{-\sigma}) = \mathfrak{p} \cap \sigma(\mathfrak{p})$.*

Proof. Set $\tau = \sigma^2$. Then τ is a semisimple automorphism of \mathfrak{g} of the first kind. By 4.8, there exists a τ -stable Borel subalgebra \mathfrak{b} of \mathfrak{g} commensurable with \mathfrak{b}_+ . By 4.5, there exists a σ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} contained in $\mathfrak{b} \cap \sigma(\mathfrak{b})$. For clarity of description, we may assume that $\mathfrak{t} = \mathfrak{h}$ and $\mathfrak{b} = \mathfrak{b}_+$. As in the proof of 4.10, there exists $X \in \mathfrak{h}^{\tau}$ such that $\langle X, \alpha_i \rangle = 1 (i = 1, \dots, n)$. Set $Y = X - \sigma X$. Clearly we have that

$$\sigma Y = -Y. \tag{1}$$

Since σ is of the second kind, $\langle \sigma X, \alpha \rangle < 0$ for almost all $\alpha \in \Delta_+$, and so

$$\langle Y, \alpha \rangle > 0 \quad \text{for almost all } \alpha \in \Delta_+. \tag{2}$$

Let $\psi_s = \{\alpha \in \Delta \mid \alpha(Y) = 0\}$, $\psi_u = \{\alpha \in \Delta \mid \alpha(Y) > 0\}$, and $\psi = \psi_s \cup \psi_u$. By (1), we have that

$$\Delta_+ \cap \sigma(-\Delta_+) \subset \psi_u. \tag{3}$$

By (2), the algebra $\mathfrak{q} = \mathfrak{h} + \bigoplus_{\alpha \in \psi} \mathfrak{g}_{\alpha}$ is a finite type parabolic subalgebra of \mathfrak{g} . From (1), $\sigma\psi_s = \psi_s$, $\sigma\psi_u = -\psi_u$, and $\tau\psi = \psi$. Hence \mathfrak{q} is σ -split and τ -stable. Thus (i) follows.

Now let \mathfrak{p} and \mathfrak{t} be as in (ii). We may assume that $\mathfrak{p} = \mathfrak{p}_X$, $\sigma(\mathfrak{p}) = \mathfrak{p}_{\bar{X}}$, \mathfrak{h} is σ -stable, and \mathfrak{b}_+ is τ -stable. Set $\mathfrak{a} = \mathfrak{h}^{-\sigma}$ and $\mathfrak{l} = \mathfrak{p}_X \cap \mathfrak{p}_{\bar{X}}$.

The minimality of \mathfrak{p} yields that \mathfrak{l} has no proper σ -split and τ -stable parabolic subalgebras. By 4.34, we have that

$$\sigma\text{-split toral subalgebras of } \mathfrak{l} \text{ are central in } \mathfrak{l}. \tag{4}$$

From (4), we have $\mathfrak{l} \subset z_{\mathfrak{g}}(\mathfrak{a})$. Choose $Y \in \mathfrak{h}^{-\sigma}$ as above. By (3), $\Delta_+ - \mathbb{Z}X$ lies in ψ_u and so $z_{\mathfrak{g}}(\mathfrak{a}) \subset z_{\mathfrak{g}}(Y) \subset \mathfrak{l}$. Hence $z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{l}$ and by (4), \mathfrak{a} is a maximal σ -split toral subalgebra of \mathfrak{g} contained in the center of \mathfrak{l} . This implies that $\mathfrak{a} \subset \mathfrak{t}$ and so $\mathfrak{a} \subset \mathfrak{t}^{-\sigma}$. Now the assertion is obvious.

4.36. LEMMA. *Let $G = SL(2, F)$ (resp. $Ad(SL(2, F))$), T a maximal torus of G , $N = N_G(T)$, and σ an automorphism of G such that $\sigma|T = -1$. Then $N^{\sigma} \not\subset T$.*

Proof. We may assume that T is the subgroup of G of diagonal matrices. Let $x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\sigma = \text{Int}(t) \circ \text{Int}(x)$ with $t \in T$. Choose $s \in T$ with $s^2 = t$. We have that $\sigma = \text{Int}(s) \circ \text{Int}(\sigma) \circ \text{Int}(s^{-1})$. Clearly $sxs^{-1} \in N^\sigma$ and $sxs^{-1} \notin T$.

4.37. LEMMA. *Let G be a connected semisimple algebraic group defined over F , T a maximal torus of G , $N = N_G(T)$, B a Borel subgroup of G containing T , τ a diagram automorphism of G with respect to (T, B) , and $S = T_0^\tau$ the identity component of T^τ . Suppose that λ is an automorphism of G such that $\lambda|_T = -1$ and $\tau\lambda = \lambda\tau$. Then we have that $(N^\lambda \cap N^\tau)T = N_G(S)$.*

Proof. Let $\Phi(S, G)$ be the set of roots of S in G and Π the set of simple roots of T in B . Consider the disjoint union $\Pi = \Pi_1 \cup \dots \cup \Pi_l$ of τ -orbits. For each i , choose $\alpha_i \in \Pi_i$ and set $\beta_i = \alpha_i|_S$. Then by [5, 9.8], $\Phi(S, G)$ is a root system with Weyl group $N_G(S)/T$ and $\{\beta_1, \dots, \beta_l\}$ is a basis of $\Phi(S, G)$. Let G_i be the group generated by the $U_\alpha(\alpha \in \Phi(T, G) \cap \mathbb{Z}\Pi_i)$, $T_i = (T \cap G_i)^\circ$, $S_i = (T_i^\tau)^\circ$, and $M_i = (G_i^\tau)^\circ$. Then M_i is a connected semisimple group of rank 1, M_i is λ -stable and $\lambda|_{S_i} = -1$. By 4.36, there exists $n_i \in N_{M_i}(S_i)$ such that $n_i \in M_i^\lambda$ and $n_i \notin S_i$. Note that $S = Z_S(M_i)S_i$ and so $n_i \in N_G(S)$. As in [5, 8.7], the image of n_i in $N_G(S)/T$ is the reflection r_{β_i} . Clearly by our choice, $n_i \in N^\lambda \cap N^\tau$ ($i = 1, \dots, l$) and the assertion follows.

4.38. LEMMA. *Let $X \subset \Pi$ be a finite type subset and $\text{Aut}(A; X) = \{\tau \in \text{Aut}(A) \mid \tau X = X\}$. Then we have the following conditions:*

- (i) *Given $\tau \in \text{Aut}(A; X)$, there exists $n_X \in N^\tau \cap N^\omega \cap G_X$ such that its image in W_X is the longest element.*
- (ii) *Set $\xi = \tau\omega \text{Ad}(n_X)$. The parabolic subalgebra \mathfrak{p}_X is ξ -split, ξ^2 -stable, and ξ is of finite order.*
- (iii) *There are only finitely many such ξ .*
- (iv) *\mathfrak{h} , H , and \tilde{H} are ξ -stable.*

Proof. Since $\tau\omega = \omega\tau$, (i) is immediate from 4.37. Note that $H^\omega = (\mathbb{Z}/2\mathbb{Z})^n$. It follows that $n_X^4 = 1$ and there are only finitely many such choices of n_X . This implies (iii) and that ξ is of finite order. The conditions on \mathfrak{p}_X and (iv) are obvious due to our construction of ξ .

4.39. The automorphisms ξ of \mathfrak{g} constructed in 4.38 are called *special semisimple automorphisms of \mathfrak{g} of the second kind*.

PROPOSITION. *Let σ be a semisimple automorphism of \mathfrak{g} of the second kind. Then σ is $\text{Aut}_F(\mathfrak{g}; \mathfrak{g}') \times (\text{Ad}(\tilde{H}) \rtimes \text{Ad}(G))$ -conjugate to an element of the form $\text{Ad}(s) \circ \xi$ where ξ is a special semisimple automorphism of \mathfrak{g} of the second kind and $s \in \tilde{H}^\xi$.*

Proof. Replacing σ by an $Ad(G)$ -conjugate, by 4.35, we may assume that there exists a finite type $X \subset \Pi$ such that the following conditions hold:

$$\sigma(\mathfrak{p}_X) = \mathfrak{p}_X^-, \quad \mathfrak{p}_X \text{ is } \sigma^2\text{-stable,} \quad \text{and} \quad (\mathfrak{h}, \mathfrak{p}_X \cap \mathfrak{b}_+) \text{ is } \sigma\text{-stable.} \quad (1)$$

Choose $n_X \in N \cap G_X$ such that its image in W_X is the longest element. Then $(\mathfrak{h}, \mathfrak{b}_+)$ is $\omega Ad(n_X) \sigma^{-1}$ -stable. There exists $\tau \in Aut(A; X)$ such that $\tau \omega Ad(n_X) \sigma^{-1}$ acts trivially on Δ . By 4.37, we may assume that $n_X \in N^\tau \cap N^\omega \cap G_X$. Set $\xi = \tau \omega Ad(n_X)$. Choose $s \in \tilde{H}$ such that $Ad(s) \circ \xi | \mathfrak{g}' = \sigma | \mathfrak{g}'$. Replacing σ by an $Ad(\tilde{H})$ -conjugate, we may assume by 4.28 that $s \in \tilde{H}^\xi$. Thus $Ad(s) \circ \xi$ is semisimple. From 4.21, σ is $Aut_F(\mathfrak{g}; \mathfrak{g}')$ -conjugate to $Ad(s) \circ \xi$.

5. INVOLUTIONS OF KAC-MOODY ALGEBRAS

In this section, F is an algebraically closed field of characteristic zero.

5.1. An *involution* θ of \mathfrak{g} is an automorphism of \mathfrak{g} over F of order two. Given an involution θ of \mathfrak{g} , it induces an involution, denoted also by θ , of G . We write \mathfrak{g}^θ (resp. G^θ) for the fixed point subalgebra of \mathfrak{g} (resp. subgroup of G).

LEMMA. *Let θ_1 and θ_2 be involutions of \mathfrak{g} . If $\mathfrak{g}^{\theta_1} = \mathfrak{g}^{\theta_2}$, then θ_1 and θ_2 are $Aut_F(\mathfrak{g}; \mathfrak{g}')$ -conjugate.*

Proof. We may assume that A is indecomposable. Consider the non-degenerate bilinear symmetric form $(\cdot | \cdot)$ on \mathfrak{g}'/c . It is θ_1 - and θ_2 -invariant. This implies that θ_1 and θ_2 have the same eigenspaces in \mathfrak{g}'/c . Hence $\theta_1 \theta_2$ induces the trivial automorphism on \mathfrak{g}'/c . In particular, \mathfrak{h}' is $\theta_1 \theta_2$ -stable and by 1.31, so is \mathfrak{h} . Hence \mathfrak{g}_α is $\theta_1 \theta_2$ -stable for every $\alpha \in \Delta$. As a consequence $\theta_1 \theta_2$ is trivial on \mathfrak{g}' . We have that $\theta_2 = x \circ \theta_1$ with $x \in Aut_F(\mathfrak{g}; \mathfrak{g}')$. Now the assertion follows readily from 4.21(ii).

5.2. **LEMMA.** *Let θ be an involution of \mathfrak{g} and $x \in G$. Then the following conditions are equivalent:*

- (i) $x\theta(x)^{-1} \in Z(G)$.
- (ii) $x\theta(x)^{-1} \in Z_G(\mathfrak{g}^\theta)$.
- (iii) $x \in N_G(\mathfrak{g}^\theta)$.

Proof. (i) \Rightarrow (ii). This is obvious.

(ii) \Rightarrow (iii). Set $z = x\theta(x)^{-1}$. For $v \in \mathfrak{g}^\theta$, $\theta(Ad(x^{-1})v) = Ad(x^{-1})Ad(z)v = Ad(x^{-1})v$. Hence $x \in N_G(\mathfrak{g}^\theta)$.

(iii) \Rightarrow (i). Consider the involution $\tau = Ad(x^{-1}) \circ \theta \circ Ad(x)$. Then $g^\tau = Ad(x^{-1}) g^\theta = g^\theta$. By 4.15, there exists a θ - and ad_{g^θ} -invariant non-degenerate bilinear symmetric form $(\cdot | \cdot)$ on \mathfrak{g} . The form is also τ -invariant. Hence θ and τ have the same eigenspaces in \mathfrak{g} . This implies that $\theta = \tau$. Now condition (i) is obvious.

5.3. LEMMA. For $x \in G$, $Ad(x) \circ \theta$ is an involution if and only if $x\theta(x) \in Z(G)$.

Proof. Note that $(Ad(x) \circ \theta)^2 = Ad(x\theta(x))$. Thus the assertion is obvious.

The following results are immediate from our study of semisimple automorphisms of \mathfrak{g} of the first kind.

5.4. PROPOSITION. Let θ be an involution of \mathfrak{g} of the first kind. Then we have the following conditions:

- (i) There exists a θ -stable standard pair $(\mathfrak{t}, \mathfrak{b})$.
- (ii) A θ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} is contained in a θ -stable Borel subalgebra of \mathfrak{g} if and only if \mathfrak{t}^θ is a maximal ad_{g^θ} -diagonalizable subalgebra of \mathfrak{g}^θ .
- (iii) Let \mathfrak{t} be as in (ii). Then $\mathfrak{t} = z_{\mathfrak{g}}(\mathfrak{t}^\theta)$.
- (iv) Maximal ad_{g^θ} -diagonalizable subalgebras of \mathfrak{g}^θ are $Ad(G^\theta)$ -conjugate.

5.5. PROPOSITION. Let θ be an involution of \mathfrak{g} of the first kind and X denote the set of $Ad(G^\theta)$ -conjugate classes of θ -stable Borel subalgebras of \mathfrak{g} commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-). Assume that $(\mathfrak{h}, \mathfrak{b}_+)$ is θ -stable, $W^\theta = \{w \in W \mid \theta w = w\theta\}$ and W_0^θ is the image of N^θ in W . Then $X \approx W_0^\theta \backslash W^\theta$.

Proof. Let \mathfrak{b} be a θ -stable Borel subalgebra of \mathfrak{g} commensurable with \mathfrak{b}_+ . By 4.8(ii), there exists a θ -stable Cartan subalgebra \mathfrak{t} contained in \mathfrak{b} . By (iii) and (iv) of 5.4, there exists $g \in G^\theta$ such that $Ad(g)(\mathfrak{b})$ contains \mathfrak{h} . It follows that \mathfrak{b} is $Ad(G^\theta)$ -conjugate to $w(\mathfrak{b}_+)$ with $w \in W^\theta$. For $w_1, w_2 \in W^\theta$, from 4.11, $w_1(\mathfrak{b}_+)$ and $w_2(\mathfrak{b}_+)$ are $Ad(G^\theta)$ -conjugate if and only if there exists $n \in N^\theta$ such that $Ad(n)(w_1(\mathfrak{b}_+)) = w_2(\mathfrak{b}_+)$. Now the assertion follows.

5.6. LEMMA. Let θ be an involution of \mathfrak{g} , \mathfrak{t} a θ -stable Cartan subalgebra of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} , and let $g \in G$ be such that $g\theta(g) \in Z(G)$. The following two conditions are equivalent:

(i) For $\tau = Ad(g) \circ \theta$, there exists a τ -stable Cartan subalgebra contained in \mathfrak{b} .

(ii) Let $B = N_G(\mathfrak{b})$, $U = [B, B]$, and $N = N_G(t)$. There exists $x \in U$ such that $xg\theta(x)^{-1}$ lies in N .

Proof. For any $y \in G$, we have

$$Ad(yg\theta(y)^{-1})(t) = Ad(y) \tau Ad(y)^{-1}(t). \tag{1}$$

(i) \Rightarrow (ii). Choose $x \in U$ such that $Ad(x^{-1})(t)$ is τ -stable. Then by (1), $xg\theta(x)^{-1}$ lies in N .

(ii) \Rightarrow (i). From (1), $Ad(x^{-1})(t)$ is a τ -stable Cartan subalgebra of \mathfrak{g} contained in \mathfrak{b} .

5.7. LEMMA. Let θ be an involution of \mathfrak{g} of the second kind. Then every Borel subalgebra \mathfrak{b} of \mathfrak{g} commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-) contains a θ -stable Cartan subalgebra of \mathfrak{g} .

Proof. Consider the algebra $\mathfrak{b} \cap \theta(\mathfrak{b})$. It is a finite dimensional θ -stable solvable Lie algebra. Note that Cartan subalgebras of $\mathfrak{b} \cap \theta(\mathfrak{b})$ are Cartan subalgebras of \mathfrak{g} . Now our assertion is immediate from 4.5.

5.8. COROLLARY. Let θ be an involution of \mathfrak{g} and $Q = \{g\theta(g)^{-1} \mid g \in G\}$. The following conditions are equivalent:

(i) Every Borel subalgebra of \mathfrak{g} contains a θ -stable Cartan subalgebra of \mathfrak{g} .

(ii) Assume that A is indecomposable. Every Borel subalgebra of \mathfrak{g} commensurable to \mathfrak{b}_+ contains a θ -stable Cartan subalgebra of \mathfrak{g} .

(iii) Let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} containing a θ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} . For every $g \in Q$, there exists $u \in U$ such that $ug\theta(u)^{-1}$ lies in N .

Proof. (i) \Rightarrow (ii). This is obvious.

(ii) \Rightarrow (i). By 5.7, we may assume that θ is of the first kind. Without loss of generality, by 4.31, we may assume that $(\mathfrak{b}, \mathfrak{b}_+)$ is θ -stable and θ has a canonical decomposition

$$\theta = Ad(t) \circ \tau,$$

where $\tau \in Aut(A)$ and $t \in \tilde{H}^r$. Since $\theta^2 = 1$, we have $t^2 = 1$. For any $s \in \tilde{H}$, $\omega Ad(s) \omega^{-1} = Ad(s^{-1})$ and so ω commutes with $Ad(t)$. Note that ω centralizes $Aut(A)$. Hence ω commutes with θ . Now let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} commensurable with \mathfrak{b}_- . Then $\omega(\mathfrak{b})$ is a Borel subalgebra

of \mathfrak{g} commensurable with \mathfrak{b}_+ and by assumption, there exists a θ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{g} contained in $\omega(\mathfrak{b})$. Since $\omega\theta = \theta\omega$, $\omega(\mathfrak{t})$ is a θ -stable Cartan subalgebra of \mathfrak{g} contained in \mathfrak{b} .

(i) \Leftrightarrow (iii). For $x \in G$, set $\tau = Ad(x\theta(x)^{-1}) \circ \theta$. Then we have that

$$\theta(Ad(x^{-1})y) = Ad(x^{-1})(\tau(y)) \quad (y \in \mathfrak{g}). \tag{1}$$

From (1) and 5.6, the following conditions are equivalent:

- (a) $Ad(x^{-1})\mathfrak{b}$ contains a θ -stable Cartan subalgebra of \mathfrak{g} .
- (b) \mathfrak{b} contains a τ -stable Cartan subalgebra of \mathfrak{g} .
- (c) There exists $u \in U$ such that $u(x\theta(x^{-1}))\theta(u^{-1})$ lies in N .

Now the assertion (i) \Leftrightarrow (iii) is immediate.

5.9. COROLLARY. *Let θ be an involution of \mathfrak{g} of the first kind and $(\mathfrak{t}, \mathfrak{b})$ a θ -stable standard pair. Let $B = N_G(\mathfrak{b})$, $U = [B, B]$, $N = N_G(\mathfrak{t})$, and $v \in U$ such that $\theta(v) = v^{-1}$. Then there exists $u \in U$ satisfying $v = u\theta(u)^{-1}$.*

Proof. Set $\tau = Ad(v) \circ \theta$. By 5.3, τ is an involution of \mathfrak{g} . Clearly \mathfrak{b} is τ -stable. By 4.8(ii), \mathfrak{b} contains a τ -stable Cartan subalgebra of \mathfrak{g} . From 5.6, there exists $u \in U$ such that $u^{-1}v\theta(u) \in N$. However, $N \cap U = \{1\}$ and so $v = u\theta(u)^{-1}$.

5.10. COROLLARY. *Let θ be an involution of \mathfrak{g} of the second kind, \mathfrak{t} a θ -stable Cartan subalgebra of \mathfrak{g} , and \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-). For every $g \in G$ with $g\theta(g) \in Z(G)$, there exists $x \in U$ such that $xg\theta(x)^{-1}$ lies in N .*

Proof. Set $\tau = Ad(g) \circ \theta$. Then τ is an involution of \mathfrak{g} of the second kind. Now our assertion is immediate from 5.6 and 5.7.

Remark. The results of 5.10 were first obtained by Springer [25] for reductive algebraic groups and θ -stable standard pairs (T, B) using Bruhat decomposition. In the following, we present an example that 5.10 is not true in general for involutions of the first kind.

5.11. EXAMPLE. Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\Pi = \{\alpha_1, \alpha_2\}$. Then G is a central extension by F^\times of $SL(2, F[z, z^{-1}])$ and U_+ is isomorphic to the free product of the two groups $\begin{pmatrix} 1 & F[z] \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} F[z] & \\ 0 & 1 \end{pmatrix}$. Order α_1 and α_2 so that

$$U_{\alpha_1} = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}, \quad U_{\alpha_2} = \begin{pmatrix} 1 & 0 \\ Fz & 1 \end{pmatrix}.$$

Consider the involution $\tau \in \text{Aut}(A)$ with $\tau\alpha_1 = \alpha_2$ and $\tau\alpha_2 = \alpha_1$. On $SL(2, F[z, z^{-1}])$, the action of τ is given by

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & z^{-1}c \\ zb & a \end{pmatrix}. \tag{1}$$

Then we have the following conditions:

(2) The group $V = U_+ \cap r_{\alpha_1}U_+ r_{\alpha_1} \cap r_{\alpha_2}U_+ r_{\alpha_2}$ is a normal subgroup of U_+ , $U_+ = U_{\alpha_1}U_{\alpha_2}V$ and U_+/V is abelian.

(3) $G^\tau = H^\tau$.

(4) Set $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$. The equation $u = x\tau(x)^{-1}$ has no solution for $x \in r_{\alpha_1}U_+ r_{\alpha_1}$.

Condition (2) is immediate from 1.7(2) and 1.11. From 4.25, $N^\tau = H^\tau$ and by 4.9(i), $G^\tau = H^\tau U_+^\tau = H^\tau U_-^\tau$ and so $G^\tau = H^\tau$. Set

$$y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

By (1), $u = y\tau(y)^{-1}$ and by (3), the set of the x satisfying $u = x\tau(x)^{-1}$ coincides with yH^τ . Note that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U_{\alpha_1}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in U_{\alpha_2}$. Thus (4) follows.

The image of N in $SL(2, F[z, z^{-1}])$ consists of the elements

$$\begin{pmatrix} \lambda z^m & 0 \\ 0 & \lambda^{-1} z^{-m} \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda^{-1} z^{-m} \\ -\lambda z^m & 0 \end{pmatrix} \quad (\lambda \in F^\times, m \in \mathbb{Z}).$$

One checks readily the following

(5) The equation $v\tau(v)^{-1} \in U_+$ has no nontrivial solutions for $v \in U_{-\alpha_1}$ and $n \in N$.

From (4) and (5), the equation $xu\tau(x)^{-1} \in N$ has no solution for $x \in r_{\alpha_1}U_+ r_{\alpha_1}$.

Now set $\mathfrak{b} = r_{\alpha_1}(\mathfrak{b}_+)$. Then 5.10 is not true for τ and $(\mathfrak{h}, \mathfrak{b})$. By the proof of (i) \Leftrightarrow (iii) in 5.8, the Borel subalgebra $Ad(y^{-1})r_{\alpha_1}(\mathfrak{b}_+)$ of \mathfrak{g} contains no τ -stable Cartan subalgebra of \mathfrak{g} .

Note that V is τ -stable, $u \in V$, and $\tau(u) = u^{-1}$. Clearly from (4), the equation $x\tau(x)^{-1} = u$ has no solutions in V .

5.12. Let θ be an involution of \mathfrak{g} , \mathfrak{t} a θ -stable Cartan subalgebra of \mathfrak{g} , and \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} . Assume that every Borel subalgebra of \mathfrak{g} commensurable with \mathfrak{b} contains a θ -stable Cartan subalgebra of \mathfrak{g} . Set

$$B = N_G(\mathfrak{b}), \quad U = [B, B], \quad T = Z_G(\mathfrak{t}), \quad N = N_G(\mathfrak{t}).$$

In the following, we discuss the double coset decomposition $G^\theta \backslash G/B$ of G .

Let $\tau: G \rightarrow G$ denote the map given by

$$\tau(x) = x^{-1}\theta(x) \quad (x \in G).$$

Set $\tau^{-1}N = \{g \in G \mid g^{-1}\theta(g) \in N\}$. Then the group $G^\theta \times T$ acts on $\tau^{-1}N$ by

$$(x, y)z = xzy^{-1} \quad ((x, y) \in G^\theta \times T, z \in \tau^{-1}N).$$

PROPOSITION. *Let V be a set of representatives of $G^\theta \times T$ -orbits in $\tau^{-1}N$. Then G is a disjoint union of the double cosets $G^\theta vB$ ($v \in V$).*

Proof. From the proof of (i) \Leftrightarrow (iii) in 5.8, we have that $G = (\tau^{-1}N)U$. Hence it remains to show that the cosets $G^\theta vB$ ($v \in V$) are disjoint. Let $g_1, g_2 \in \tau^{-1}N$ such that $g_2 \in G^\theta g_1 B$. Write $g_2 = xg_1 y$ with $x \in G^\theta$ and $y \in B$. Note that $B = TU$. Write $y = zu$ with $z \in T$ and $u \in U$. Then

$$g_2^{-1}\theta(g_2) = u^{-1}(g_1 z)^{-1}\theta(g_1 z)\theta(u) \in UN\theta(U).$$

By 1.14, the N -component is unique and so

$$g_2^{-1}\theta(g_2) = (g_1 z)^{-1}\theta(g_1 z).$$

Hence $g_2 \in G^\theta g_1 T$ and our assertion follows.

Remark. The above description of $G^\theta \backslash G/B$ is due to Springer [25] for reductive algebraic groups. In the following, we present a different characterization which is due to Rossmann [23] and Matsuki [19] both in the finite type case.

5.13. LEMMA. *Let θ be an involution of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} , and let $\mathfrak{t}_1, \mathfrak{t}_2$ be θ -stable Cartan subalgebras of \mathfrak{g} contained in \mathfrak{b} . Then \mathfrak{t}_1 and \mathfrak{t}_2 are $Ad(G^\theta \cap U)$ -conjugate.*

Proof. Choose $x \in U$ such that $\mathfrak{t}_2 = Ad(x)\mathfrak{t}_1$. Since \mathfrak{t}_1 and \mathfrak{t}_2 are θ -stable, so $x^{-1}\theta(x) \in N = N_G(\mathfrak{t}_1)$. However, by 1.14, the N -component in the decomposition $U \backslash G/\theta(U)$ is unique. This implies that $x \in G^\theta \cap U$.

5.14. Let θ, \mathfrak{t} , and \mathfrak{b} be as in 5.12. Let A denote the set of θ -stable Cartan subalgebras of \mathfrak{g} . Then G^θ acts on A on the left by

$$x \cdot \mathfrak{a} = Ad(x)\mathfrak{a} \quad (x \in G^\theta, \mathfrak{a} \in A).$$

Consider the map

$$\pi: G^\theta \backslash G/B \rightarrow G^\theta \backslash A$$

sending $G^\theta gB$ to the $Ad(G^\theta)$ -conjugacy class $[\mathfrak{a}]$ where \mathfrak{a} is a θ -stable Cartan subalgebra of \mathfrak{g} contained in $Ad(g) \mathfrak{b}$. By 5.13, the map π is well defined. For a θ -stable Cartan subalgebra \mathfrak{a} of \mathfrak{g} , choose $g \in G$ with $\mathfrak{a} = Ad(g) \mathfrak{t}$. Note that the Borel subalgebras of \mathfrak{g} which contain \mathfrak{a} and are commensurable with \mathfrak{b} are $N_G(\mathfrak{a})$ -conjugate. Thus the fiber $\pi^{-1}([\mathfrak{a}])$ is given by

$$\pi^{-1}([\mathfrak{a}]) = G^\theta \backslash G^\theta N_G(\mathfrak{a}) gB/B. \tag{1}$$

Let $n, n_1 \in N_G(\mathfrak{a})$ such that $n_1 g \in G^\theta ngB$. Write

$$n_1 g = xngtu,$$

with $x \in G^\theta$, $t \in T$, and $u \in U$. Then $Ad(ng) \mathfrak{t}$ and $Ad(ngtu) \mathfrak{t}$ are θ -stable Cartan subalgebras of \mathfrak{g} contained in $Ad(ng) \mathfrak{b}$. By 5.13, there exists $v \in U$ such that

$$ngv(ng)^{-1} \in G^\theta, \quad Ad(ngv) \mathfrak{t} = Ad(ngtu) \mathfrak{t}. \tag{2}$$

From (2), $tu^{-1}t^{-1}v \in N_G(\mathfrak{t}) \cap U = \{1\}$, so $tut^{-1} = v$ and as a consequence

$$n_1 = xngtug^{-1} = xngv(ng)^{-1} n(gtg^{-1}) \in G^\theta nZ_G(\mathfrak{a}). \tag{3}$$

Now set $W_G(\mathfrak{a}) = N_G(\mathfrak{a})/Z_G(\mathfrak{a})$, $K = G^\theta$, and $W_K(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. From (3), we have that

$$\pi^{-1}([\mathfrak{a}]) \simeq W_K(\mathfrak{a}) \backslash W_G(\mathfrak{a}).$$

This gives the following description of the double coset decomposition $G^\theta \backslash G/B$ of G .

PROPOSITION. *Let $\{\mathfrak{a}_i | i \in I\}$ be a set of representatives of the $Ad(G^\theta)$ -conjugate classes of θ -stable Cartan subalgebras of \mathfrak{g} . Then*

$$G^\theta \backslash G/B \simeq \bigcup_{i \in I} W_{G^\theta}(\mathfrak{a}_i) \backslash W_G(\mathfrak{a}_i).$$

5.15. Let θ be an involution of \mathfrak{g} . A toral subalgebra \mathfrak{a} of \mathfrak{g} is θ -split if $\theta|_{\mathfrak{a}} = -1$. A parabolic subalgebra \mathfrak{p} of \mathfrak{g} is θ -split if $\mathfrak{p} \cap \theta(\mathfrak{p})$ is a Levi-subalgebra of \mathfrak{p} and $\theta(\mathfrak{p})$.

In the following, θ is an involution of \mathfrak{g} of the second kind. Let \mathfrak{t} be a θ -stable Cartan subalgebra of \mathfrak{g} and \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} . Let Δ (resp. Δ^+) denote the set of roots of \mathfrak{t} in \mathfrak{g} (resp. \mathfrak{b}) and Π the set of simple roots of \mathfrak{t} in \mathfrak{b} .

The pair $(\mathfrak{t}, \mathfrak{b})$ is called a *split pair* if there exists a subset X of Π satisfying the following conditions:

- (i) $\Delta^+ \cap \theta(\Delta^+) = \Delta_X^+$ where $\Delta_X^+ = \Delta^+ \cap \mathbb{Z}X$.
- (ii) $\theta|_{\mathfrak{g}_\alpha} = 1$ for all $\alpha \in \Delta_X^+$.

5.16. LEMMA. *Let $(\mathfrak{t}, \mathfrak{b})$ be a split pair, $X \subset \Pi$ as in the definition, and \mathfrak{g}_X the subalgebra of \mathfrak{g} generated by $\mathfrak{g}_{\pm\alpha} (\alpha \in X)$. Then we have the following:*

- (i) \mathfrak{g}_X is a finite dimensional semisimple Lie algebra and $\theta|_{\mathfrak{g}_X} = 1$.
- (ii) The algebra $\mathfrak{p}_X = \mathfrak{b} + \mathfrak{g}_X$ is a minimal θ -split parabolic subalgebra of \mathfrak{g} .
- (iii) Set $\mathfrak{a} = \mathfrak{t}^{-\theta}$. Then \mathfrak{a} is a maximal θ -split toral subalgebra of \mathfrak{g} and $z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{p}_X \cap \theta(\mathfrak{p}_X) = \mathfrak{t} + \mathfrak{g}_X$.

Proof. Since θ is of the second kind, the set $\Delta_X^+ = \Delta^+ \cap \theta(\Delta^+)$ is finite and so \mathfrak{g}_X is a finite dimensional semisimple Lie algebra. By 4.17, $\theta|_{\mathfrak{g}_{-\alpha}} = 1 (\alpha \in X)$ and so $\theta|_{\mathfrak{g}_X} = 1$.

Note that \mathfrak{p}_X is θ -split, $\mathfrak{t} + \mathfrak{g}_X$ is a Levi-subalgebra of \mathfrak{p}_X , and $\mathfrak{t} + \mathfrak{g}_X$ contains no proper θ -split parabolic subalgebras. Hence condition (ii) is obvious. Assertion (iii) follows from (ii) and 4.35(ii).

5.17. PROPOSITION. *Let \mathfrak{t} be a θ -stable Cartan subalgebra of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} commensurable with \mathfrak{b}_+ (resp., \mathfrak{b}_-), $\psi = \Delta^+ \cap \theta(\Delta^+)$, and $\xi = \Delta^+ \cap \theta(-\Delta^+)$. Then the following conditions are equivalent:*

- (i) $(\mathfrak{t}, \mathfrak{b})$ is a split pair.
- (ii) $\theta|_{\mathfrak{g}_\alpha} = 1$ for every $\alpha \in \psi$.
- (iii) $\theta|_{\mathfrak{g}_\alpha} = 1$ for every $\alpha \in \Pi \cap \psi$.
- (iv) $\mathfrak{b} + \mathfrak{g}^\theta = \mathfrak{g}$.
- (v) $\mathfrak{t}^{-\theta}$ is a maximal θ -split toral subalgebra of \mathfrak{g} and $\mathfrak{b} \cap \theta(\mathfrak{b})$ has the least dimension among Borel subalgebras of \mathfrak{g} containing \mathfrak{t} commensurable with \mathfrak{b}_+ (resp. \mathfrak{b}_-).

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (i). Observe that ξ and ψ satisfy the following closedness condition:

$$\text{If } \alpha, \beta \in \xi \text{ (resp. } \psi) \text{ and } \alpha + \beta \in \Delta, \text{ then } \alpha + \beta \in \xi \text{ (resp. } \psi). \tag{1}$$

Let $X = \Pi \cap \psi$ and $\Delta_X^+ = \Delta^+ \cap \mathbb{Z}X$. Identify the space $\mathbb{Z}\Pi/\mathbb{Z}X$ with $\bigoplus_{\alpha \in \Pi - X} \mathbb{Z}\alpha$. Clearly θ induces an automorphism $\bar{\theta}$ of $\mathbb{Z}\Pi/\mathbb{Z}X$. If $\alpha \in \Pi - X$, then $\alpha \in \xi$ and so $\bar{\theta}\alpha$ has nonpositive coefficients with respect to the basis $\Pi - X$. It follows that if $\alpha \in \Delta^+$ and $\alpha \notin \Delta_X^+$, then $\theta\alpha \in -\Delta^+$.

Hence $\psi \subset \mathcal{A}_X^+$ and from (1) we have the opposite inclusion relation. This yields that $\psi = \mathcal{A}_X^+$. By 4.17, $\theta|_{\mathfrak{g}_{\pm\alpha}} = 1 (\alpha \in X)$ and as a consequence $\theta|_{\mathfrak{g}_\beta} = 1 (\beta \in \mathcal{A} \cap ZX)$. Thus (i) follows.

(i) \Rightarrow (iv). Clearly we have that $\mathfrak{p}_X + \theta(\mathfrak{p}_X) \subset \mathfrak{p}_X + \mathfrak{g}^\theta$. Note that $\theta|_{\mathfrak{g}_X} = 1$ and so $\mathfrak{p}_X + \mathfrak{g}^\theta = \mathfrak{b} + \mathfrak{g}^\theta$. Since \mathfrak{p}_X is θ -split, $\mathfrak{p}_X + \theta(\mathfrak{p}_X) = \mathfrak{g}$. Now we have the desired condition $\mathfrak{g} = \mathfrak{b} + \mathfrak{g}^\theta$.

(iv) \Rightarrow (ii). Let \mathfrak{q} denote the subspace of \mathfrak{g} given by $\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\psi} \mathfrak{g}_\alpha$. Clearly \mathfrak{q} is θ -stable and contains \mathfrak{b} . By (iv), θ acts trivially on $\mathfrak{g}/\mathfrak{q}$. However, $\bigoplus_{\alpha \in -\psi} \mathfrak{g}_\alpha \simeq \mathfrak{g}/\mathfrak{q}$ with operator θ and so $\mathfrak{g}_\alpha \subset \mathfrak{g}^\theta$ for all $\alpha \in -\psi$. By 4.17, (ii) follows.

(i) \Rightarrow (v). By 5.16(iii), $\mathfrak{t}^{-\theta}$ is a maximal θ -split toral subalgebra of \mathfrak{g} and by 5.16(i), $\mathfrak{g}_X \subset \mathfrak{g}^\theta$. Let \mathfrak{p} be any Borel subalgebra of \mathfrak{g} containing \mathfrak{t} . Then $\mathfrak{p} \cap \theta(\mathfrak{p})$ contains $\mathfrak{t} + (\mathfrak{p} \cap \mathfrak{g}_X)$. Now both $\mathfrak{t} + (\mathfrak{b} \cap \mathfrak{g}_X)$ and $\mathfrak{t} + (\mathfrak{p} \cap \mathfrak{g}_X)$ are Borel subalgebras of $\mathfrak{t} + \mathfrak{g}_X$; in particular they have the same dimension. Thus the minimality condition on $\dim(\mathfrak{b} \cap \theta(\mathfrak{b}))$ follows.

(v) \Rightarrow (ii). We establish the assertion in several steps.

Step 1. Choose $Y \in \mathfrak{t}$ such that $\alpha(Y) > 0$ for every $\alpha \in \Pi$. Set $Z = Y - \theta(Y)$, $\phi_s = \{\alpha \in \mathcal{A} \mid \alpha(Z) = 0\}$, $\phi_u = \{\alpha \in \mathcal{A} \mid \alpha(Z) > 0\}$, $\phi = \phi_s \cup \phi_u$, and $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \phi} \mathfrak{g}_\alpha$. Clearly we have that

$$\xi \subset \phi_u, \quad \mathcal{A} = \phi \cup -\phi, \quad \text{and} \quad \phi \text{ is a closed subset of } \mathcal{A}. \quad (2)$$

From (2), the algebra \mathfrak{p} is a θ -split finite type parabolic subalgebra of \mathfrak{g} containing \mathfrak{t} .

Step 2. Set $\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \phi_s} \mathfrak{g}_\alpha$. Let \mathfrak{b}_1 be a Borel subalgebra of \mathfrak{m} containing \mathfrak{t} . Then $\mathfrak{q} = \mathfrak{b}_1 \oplus \bigoplus_{\alpha \in \phi_u} \mathfrak{g}_\alpha$ is a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} . Clearly $\mathfrak{q} \cap \theta(\mathfrak{q}) \subset \mathfrak{b}_1$ and $\xi \subset \phi_u$. By the minimality condition on $\dim(\mathfrak{b} \cap \theta(\mathfrak{b}))$, we have that

$$\xi = \phi_u, \text{ and } \mathfrak{m} \text{ contains no proper } \theta\text{-split parabolic subalgebras containing } \mathfrak{t}. \quad (3)$$

Step 3. By 4.34, $\mathfrak{t}^{-\theta}$ is central in \mathfrak{m} . If $\theta|_{[\mathfrak{m}, \mathfrak{m}]} \neq 1$, by a result of Vust [27], there exists nontrivial θ -split toral subalgebra of $[\mathfrak{m}, \mathfrak{m}]$. However, $\mathfrak{t}^{-\theta}$ is a maximal θ -split toral subalgebra of \mathfrak{g} . Hence we must have that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}^\theta$. Since $\xi = \phi_u$, so $\psi \subset \phi_s$ and as a consequence $\mathfrak{g}_\alpha \subset \mathfrak{g}^\theta$ for every $\alpha \in \psi$.

5.18. COROLLARY. *Let θ be an involution of \mathfrak{g} of the second kind, \mathfrak{t} a θ -stable Cartan subalgebra of \mathfrak{g} , and \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing \mathfrak{t} . Assume that $\mathfrak{b} \cap \theta(\mathfrak{b})$ has the least dimension among all Borel subalgebras of \mathfrak{g} . Then $(\mathfrak{t}, \mathfrak{b})$ is a split pair.*

Proof. Let $\psi = \Delta^+ \cap \theta(\Delta^+)$ and $X = \Pi \cap \psi$. Suppose that $\alpha \in X$ and $\theta\alpha \neq \alpha$. One checks readily that $r_\alpha(\Delta^+) \cap \theta(r_\alpha(\Delta^+)) = \psi - \{\alpha, \theta\alpha\}$. Then we have that $\dim(r_\alpha(\mathfrak{b}) \cap \theta(r_\alpha(\mathfrak{b}))) < \dim(\mathfrak{b} \cap \theta(\mathfrak{b}))$ which is a contradiction. Hence $\theta|_X = 1$. As in (iii) \Rightarrow (i) of 5.17, we derive also the condition $\psi = \Delta^+ \cap \mathbb{Z}X$. Let \mathfrak{g}_X be the subalgebra of \mathfrak{g} generated by $\mathfrak{g}_{\pm\alpha}$ ($\alpha \in X$). Then \mathfrak{g}_X is a finite dimensional semisimple Lie algebra. The minimality condition implies that $\mathfrak{t} + \mathfrak{g}_X$ contains no proper θ -split parabolic subalgebras. By a result of Vust [27], $\mathfrak{g}_X \subset \mathfrak{g}^\theta$. By 5.17(ii), $(\mathfrak{t}, \mathfrak{b})$ is a split pair.

5.19. COROLLARY. *Let θ be an involution of \mathfrak{g} of the second kind. There exists a split pair $(\mathfrak{t}, \mathfrak{b})$ for θ .*

In the following, we discuss the conjugacy theorem for split pairs. In the finite case, the result is due to Vust [27] which is a simple consequence of 5.17(iv). For the general case, we need a proper topology on G to establish the result. We recall some of the basic results on regular functions on G by Kac and Peterson [13].

5.20. For every $\alpha \in \Delta^{re}$, we fix a nonzero element $e_\alpha \in \mathfrak{g}_\alpha$ and coordinatize U_α by putting $x_\alpha(t) = \exp(te_\alpha)$ ($t \in F$). Given $\beta = (\beta_1, \dots, \beta_k) \in (\Delta^{re})^k$, set $x_\beta: F^k \rightarrow G$ by

$$x_\beta(t_1, \dots, t_k) = x_{\beta_1}(t_1) \cdots x_{\beta_k}(t_k),$$

and denote the image of x_β by U_β .

A function $f: G \rightarrow F$ is called *weakly regular* if $f \circ x_\beta: F^k \rightarrow F$ is a polynomial function for all $\beta \in (\Delta^{re})^k$ and $k \in \mathbb{Z}_+$. Denote by $F[G]_{w.r.}$ the algebra of all weakly regular functions. Let V be a \mathfrak{g}' -module. Given $v^* \in V^*$ and $v \in V_{fin}$, the matrix coefficient $f_{v^*,v}(g) = \langle gv, v^* \rangle$ ($g \in G$) is a weakly regular function of G .

A weakly regular function f of G is called *strongly regular* if there exist large subgroups U'_\pm of U_\pm such that $f(u_- gu_+) = f(g)$ for all $g \in G$ and $u_\pm \in U'_\pm$. For $\lambda \in P_+$, the matrix coefficients $f_{v^*,v}$ with $v \in L(\lambda)$ and $v^* \in L^*(\lambda)$ are strongly regular functions of G . We denote by $F[G]$ the algebra of all strongly regular functions on G . Then $F[G]$ is a $G \times G$ -module under π where

$$(\pi(g_1, g_2)f)(g) = f(g_1^{-1}gg_2). \tag{1}$$

5.21. See [13, Theorem 1]. The linear map $\phi: \bigoplus_{\lambda \in P_+} L^*(\lambda) \otimes L(\lambda) \rightarrow F[G]$, defined by $\phi(v^*v) = f_{v^*,v}$, is an isomorphism of $G \times G$ -modules.

5.22. If G is of finite type, then $F[G] = F[G]_{w.r.}$ is the ordinary ring of regular functions of the algebraic group G .

5.23. For any subgroup M of G , we denote by $F[M]$ the restriction of $F[G]$ to M .

For $X \subset \Pi$, \mathfrak{g}_X (resp. G_X) is the subalgebra (resp. subgroup) generated by $\mathfrak{g}_{\pm\alpha_i}$ (resp. $U_{\pm\alpha_i}$) with $\alpha_i \in X$, and $\mathfrak{h}'_X = \mathfrak{h} \cap \mathfrak{g}_X$. Observe that by [10, 10.4], the cyclic \mathfrak{g}_X -submodule generated by v_A^+ (resp. v_A^-) coincides with $L(A|\mathfrak{h}'_X)$ (resp. $L^*(A|\mathfrak{h}'_X)$). By 5.21, the algebra $F[G_X]_{s.r.}$ coincides with $F[G_X]$ defined by restriction.

5.24. For $A \in P_+$, write θ_A for $f_{v_A^-, v_A^+}$ and set $S = \{\theta_A | A \in P_+\} \subset F[G]$. By [13, Lemma 4.4], the map $\phi: U_- \times H \times U_+ \rightarrow G$, defined by multiplication induces an isomorphism $\phi^*: S^{-1}F[G] \simeq F[U_-] \otimes F[H] \otimes F[U_+]$.

5.25. Let $w \in W$, $U_1 = U_+ \cap wU_-w^{-1}$ and $U_2 = U_+ \cap wU_+w^{-1}$. By [13, Lemma 4.5], the product map $\psi: U_1 \times U_2 \rightarrow U_+$ induces an isomorphism $\psi^*: F[U_+] \simeq F[U_1] \otimes F[U_2]$. Similar assertion holds for U_- .

5.26. Let $X \subset \Pi$ be a finite type subset. Then G_X is a finite dimensional semisimple algebraic group. Let $w \in W_X$ be the longest element. Set

$$\begin{aligned} U^X &= U_+ \cap wU_+w^{-1} & U_X &= U_+ \cap wU_-w^{-1}, \\ U^{-X} &= U_- \cap wU_-w^{-1}, & U_{-X} &= U_- \cap wU_+w^{-1}. \end{aligned}$$

Then from 5.24 and 5.25, the product map

$$\phi: U^{-X} \times U_{-X} \times H \times U_X \times U^X \rightarrow G$$

induces an isomorphism

$$\phi^*: S^{-1}F[G] \simeq F[U^{-X}] \otimes F[U_{-X}] \otimes F[H] \otimes F[U_X] \otimes F[U^X].$$

HG_X is an algebraic subgroup of G and $F[HG_X]$ coincides with the algebra of regular functions of the algebraic group HG_X ; moreover the product map

$$\psi: U_{-X} \times H \times U_X \rightarrow HG_X$$

induces an isomorphism $\psi^*: S^{-1}F[HG_X] \simeq F[U_{-X}] \otimes F[H] \otimes F[U_X]$.

5.27. We introduce the Zariski topology (resp. w -Zariski topology) on G defined by the strongly regular functions (resp. weakly regular functions), i.e., a closed set is the set of zeros of an ideal of $F[G]$ (resp. $F[G]_{w.r.}$). Note that $F[G] \subset F[G]_{w.r.}$. Hence a Zariski-closed (resp. Zariski-open) subset of G is w -Zariski-closed (resp. w -Zariski-open) in G .

5.28. LEMMA. (i) *If Y is Zariski-open in HG_X , then $U^{-X}YU^X$ is Zariski-open in G .*

(ii) *If Y is Zariski-closed in HG_X , then $U^{-X}YU^X$ is Zariski-closed in $U^{-X}HG_XU^X$.*

Proof. Choose $\lambda \in P_{++}$. Then U_-HU_+ coincides with the set $\{g \in G \mid \theta_\lambda(g) \neq 0\}$ and so U_-HU_+ is Zariski-open in G . If Y is Zariski-open in $U_{-X}HU_X$, by 5.26, $U^{-X}YU^X$ is Zariski-open in U_-HU_+ and so in G . Let Y be a Zariski-open subset of HG_X . There exist $x_1, \dots, x_m \in HG_X$ and Zariski-open subsets Y_1, \dots, Y_m of $U_{-X}HU_X$ such that $Y = \bigcup_{i=1}^m Y_i x_i$. By 1.8(i), HG_X normalizes U^X and so

$$U^{-X}YU^X = \bigcup_{i=1}^m U^{-X}Y_i U^X x_i. \tag{1}$$

Note that $F[G]$ is a $G \times G$ -module under π (5.20(1)). It follows that right translations of G are homeomorphisms with respect to the Zariski topology of G . From (1), $U^{-X}YU^X$ is a finite union of Zariski-open subsets of G . Hence (i) follows; (ii) is obvious from (i).

5.29. LEMMA. *G is irreducible with respect to w -Zariski topology.*

Proof. Set $V = U_{x_1} \times U_{-x_1} \times \dots \times U_{x_n} \times U_{-x_n}$. Suppose that $G = X \cup Y$ where X and Y are w -Zariski-closed subsets of G . Let $V^{(l)} = V \times \dots \times V$ (l copies) and $m_l: V^{(l)} \rightarrow G$ the map defined by multiplication. $V^{(l)}$ is endowed with the Zariski-topology in the obvious manner. From the definition of weakly regular functions, m_l is continuous. Since $V^{(l)}$ is irreducible, $V^{(l)} \subset m_l^{-1}(X)$ or $V^{(l)} \subset m_l^{-1}(Y)$. Assume that $G \neq X$. Note that V generates G . Then $m_l(V^{(l)}) \not\subset X$ for some l . It follows that $m_j(V^{(l)}) \subset Y$ for all $j \geq l$. Hence $G = Y$.

5.30. LEMMA. *Let G be a reductive algebraic group, θ an involution of G , and $Q' = \{g \in G \mid \theta(g) = g^{-1}\}$. Let the (left) twisted action of G be given by $x * y = xy\theta(x)^{-1}$ ($x, y \in G$). Then Q' is a finite union of closed $G*$ -orbits.*

Proof. Let G° denote the identity component of G and $G = \bigcup_{i=1}^l G^\circ x_i$. Assume that $Q' \cap Gx_i \neq \emptyset$. We may assume that $x_i \in Q'$. Set $\tau_i = \text{Int}(x_i) \circ \theta$. For $x \in G^\circ$, $xx_i \in Q'$ if and only if $\tau_i(x) = x^{-1}$. Observe that

$$y(xx_i)\theta(y)^{-1} = (yx\tau_i(y)^{-1})x_i.$$

Hence we may assume that G is connected. Then the assertion is a result of Richardson [21, Sect. 9].

5.31. THEOREM. *Let $(\mathfrak{t}, \mathfrak{b})$ be a split pair for an involution θ of \mathfrak{g} of the second kind. Then the set BG^θ is w -Zariski-open in G .*

Proof. Without loss of generality, we may assume that $\mathfrak{t} = \mathfrak{h}$ and $\mathfrak{b} = \mathfrak{b}_+$. Let $X \in \Pi$ such that

$$\Delta_+ \cap \theta(\Delta_+) = \Delta_+ \cap \mathbb{Z}X.$$

Set $Q = \{x\theta(x)^{-1} \mid x \in G\}$ and $Q'_X = \{g \in HG_X \mid \theta(g) = g^{-1}\}$. By 5.16, we have

$$\theta|_{G_X} = 1, \tag{1}$$

$$\theta(U^X) = U^{-X}. \tag{2}$$

Set $M_X = HG_X$. Consider the P_{X^*} -orbits in $Q \cap U^X M_X U^{-X}$. Note that $U^X M_X U^{-X} \cap N = M_X \cap N$. From 5.10, we have

(3) Every P_{X^*} -orbit in $Q \cap U^X M_X U^{-X}$ meets Q'_X . From (2) and (3), the following follows:

(4) Every P_{X^*} -orbit in $Q \cap U^X M_X U^{-X}$ is of the form $Q \cap U^X Y U^{-X}$ where Y is an M_{X^*} -orbit in Q'_X .

From (4), 5.28, and 5.30, there are finitely many P_{X^*} -orbits in $Q \cap U^X M_X U^{-X}$ and every such orbit is closed, hence also open, in $Q \cap U^X M_X U^{-X}$. By 5.28, $U^X M_X U^{-X}$ is open in G . It follows that every P_{X^*} -orbit in $Q \cap U^X M_X U^{-X}$ is open in Q . Note that the map τ , defined by $\tau(x) = x\theta(x)^{-1}$, is continuous and $Q = \tau(G)$. Then the set $P_X G^\theta$, being the inverse image of an open P_{X^*} -orbit in Q , is w -Zariski open in G . However, by (1), $B_+ G^\theta = P_X G^\theta$ and the desired assertion follows.

5.32. COROLLARY. *Let θ be an involution of \mathfrak{g} of the second kind. Assume that $(\mathfrak{t}, \mathfrak{b})$ and $(\mathfrak{t}_1, \mathfrak{b}_1)$ be split pairs for θ such that \mathfrak{b} and \mathfrak{b}_1 are commensurable. Then there exists $g \in G^\theta$ such that $Ad(g)\mathfrak{t} = \mathfrak{t}_1$ and $Ad(g)\mathfrak{b} = \mathfrak{b}_1$.*

Proof. Choose $x \in G$ with $\mathfrak{b}_1 = Ad(x)\mathfrak{b}$. By 5.31, BG^θ and $(xBx^{-1})G^\theta$ are w -Zariski-open in G . By 5.29, G is irreducible and so $BG^\theta = Bx^{-1}G^\theta$, both being open double cosets. Hence we may write $x = gy$ with $g \in G^\theta$ and $y \in B$. Then $\mathfrak{b}_1 = Ad(g)\mathfrak{b}$ and by 5.13, we may choose $g \in G^\theta$ satisfying $\mathfrak{t}_1 = Ad(g)\mathfrak{t}$.

5.33. COROLLARY. *Maximal θ -split toral subalgebras of \mathfrak{g} for an involution θ of the second kind are $Ad(G^\theta)$ -conjugate.*

Proof. Let \mathfrak{a} be a maximal θ -split toral subalgebra of \mathfrak{g} . Choose a Borel subalgebra \mathfrak{p} of \mathfrak{g} containing \mathfrak{a} commensurable with \mathfrak{b}_+ . Then there exists

a θ -stable Cartan subalgebra t of \mathfrak{g} such that $\mathfrak{a} \subset t$ and $t \subset \mathfrak{p} \cap \theta(\mathfrak{p})$. By 5.17(v), there exists a split pair (t, b) . Now our assertion is immediate from 5.32.

Remark. From 5.19 and 5.32, a satisfactory classification of involutions of the second kind, in terms of the Dynkin diagram, is possible. A detailed discussion will be presented in a separate paper.

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