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Vertices contained in all or in no minimum total dominating set of a tree

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. We characterize the set of vertices of a tree that are contained in all, or in no, minimum total dominating sets of the tree.

Keywords: Total domination; Total domination number; Tree

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with $|V|=n$. The *open neighborhood* of $v\in V$ is $N(v)=\{u\in V \mid uv\in E\}$. A set $S\subseteq V$ is a *dominating set* of G if every vertex in $V-S$ is adjacent to a vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S\subseteq V$ is a *total dominating set*, abbreviated *TDS*, of G if every vertex in V is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S=V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set.

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Total domination was introduced by Cockayne et al. [2] and is studied, for example, in [3,7]. For a survey of domination and its variations, see the books by Haynes et al. [5,6].

Mynhardt [8] characterized the set of vertices that are contained in all, or in no, minimum dominating sets of a tree. Hammer et al. [4] investigated vertices belonging to all or to no maximum stable sets of a graph. In this paper we characterize the set of vertices of a tree T that are contained in all, or in no, $\gamma_t(T)$-sets.

1.1. Notation

For notation and graph theory terminology we in general follow Chartrand and Lesniak [1]. In particular, we denote the distance between two vertices u and v by $d(u,v)$. The degree of a vertex v is a graph G is denoted by $\deg_G v$, or simply by $\deg v$ if the graph G is clear from context. The diameter of a graph G is denoted by $\mathrm{diam}(G)$. A path on n vertices is denoted by P_n.

We define the sets $\mathscr{A}_t(G)$ and $\mathscr{N}_t(G)$ of a graph G by

$$\mathscr{A}_t(G) = \{v \in V(G) \mid v \text{ is in every } \gamma_t(G)\text{-set}\}, \text{ and}$$

$$\mathscr{N}_t(G) = \{v \in V(G) \mid v \text{ is in no } \gamma_t(G)\text{-set}\}.$$

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v. The maximal subtree at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. The set of leaves in T is denoted by $L(T)$ and the set of support vertices by $S(T)$. A *strong support vertex* is adjacent to at least two leaves. The set of leaves in T_v distinct from v we denote by $L(v)$, i.e., $L(v) = D(v) \cap L(T)$. We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of T is denoted by $B(T)$. For $j = 0,1,2,3$, we define

$$L^j(v) = \{u \in L(v) \mid d(u,v) \equiv j \pmod 4\}.$$

We sometimes write $L_T^j(v)$ to emphasize the tree (or subtree) concerned.

2. Tree pruning

We next describe a technique called *tree pruning*, which will allow us to characterize the sets $\mathscr{A}_t(T)$ and $\mathscr{N}_t(T)$ for an arbitrary tree T. Given a vertex u of T, we say we *attach* a path of length q to u if we join u to a leaf of the path P_q.

Let v be a vertex of T that is not a support vertex. The pruning of T is performed with respect to the root. Hence suppose T is rooted at v, i.e., $T = T_v$. If $\deg u \leqslant 2$ for each $u \in V(T_v) - \{v\}$, then let $\bar{T}_v = T$. Otherwise, let u be a branch vertex at maximum distance from v; note that $|C(u)| \geqslant 2$ and $\deg x \leqslant 2$ for each $x \in D(u)$. We now apply the following pruning process:

- If $|L^2(u)| \geqslant 1$, then delete $D(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \geqslant 1$, $L^2(u) = \emptyset$ and $|L^3(u)| \geqslant 1$, then delete $D(u)$ and attach a path of length 2 to u.

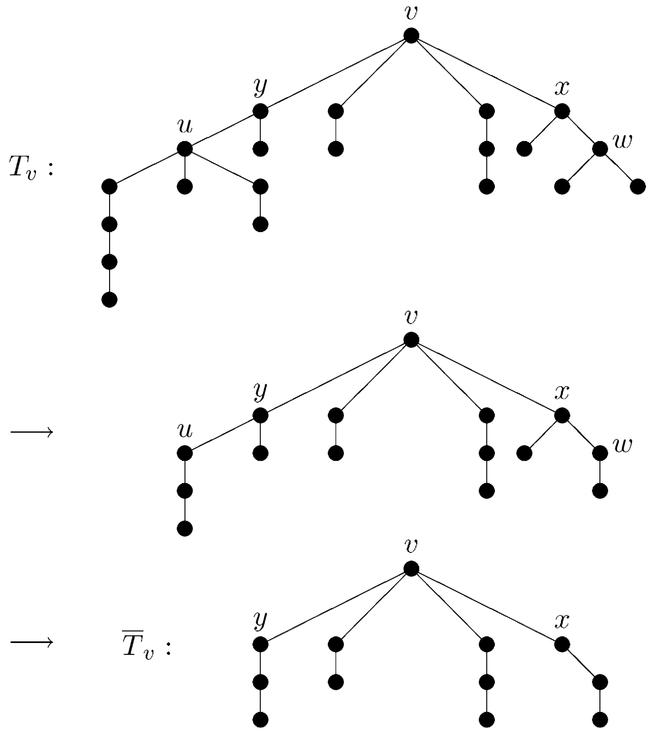

Fig. 1. The pruning $\bar{T}_v$ of the tree T_v.

- If $|L^1(u)| \geqslant 1$ and $L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 1 to u.
- If $L^1(u) = L^2(u) = \emptyset$ and $|L^3(u)| \geqslant 1$, then delete $D(u)$ and attach a path of length 3 to u.
- If $L^1(u) = L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 4 to u.

This step of the pruning process, where all the descendants of u are deleted and a path of length 1, 2, 3, or 4 is attached to u to give a tree in which u has degree 2, is called a *pruning of* T_v *at* u. Repeat the above process until a tree $\bar{T}_v$ is obtained with $\deg u \leqslant 2$ for each $u \in V(\bar{T}_v) - \{v\}$. The tree $\bar{T}_v$ is unique and is called the *pruning* of T_v. To simplify notation, we write $\bar{L}^j(v)$ instead of $L^j_{\bar{T}_v}(v)$.

To illustrate the pruning process, consider the tree T in Fig. 1. The vertices u and w are branch vertices at maximum distance 2 from v. Since $|L^2(u)| = 1$, we delete $D(u)$ and attach a path of length 2 to u. Since $|L^1(w)| = 2$ and $L^2(w) = L^3(w) = \emptyset$, we delete $D(w)$ and attach a path of length 1 to w. This pruning of T_v at u and w produces the intermediate tree as shown in Fig. 1. In this tree, the vertices x and y are branch vertices at maximum distance 1 from v. Since $|L^2(x)| = 1$, we delete $D(x)$ and attach a path of length 2 to x. Since $|L^1(y)| = 1$, $L^2(y) = \emptyset$ and $|L^3(y)| = 1$, we delete $D(y)$ and attach a path of length 2 to y. This pruning of T_v at x and y produces the pruning $\bar{T}_v$ of T_v.

We shall prove:

Theorem 1. *Let T be a tree rooted at a vertex v such that* $\deg u \leqslant 2$ *for each* $u \in V(T) - \{v\}$. *Then*

(a) $v \in \mathscr{A}_t(T)$ *if and only if v is a support vertex or* $|L^1(v) \cup L^2(v)| \geqslant 2$;
(b) $v \in \mathscr{N}_t(T)$ *if and only if* $L^1(v) \cup L^2(v) = \emptyset$.

Theorem 2. *Let v be a vertex of a tree T. Then*

(a) $v \in \mathscr{A}_t(T)$ *if and only if v is a support vertex or* $|\bar{L}^1(v) \cup \bar{L}^2(v)| \geqslant 2$;
(b) $v \in \mathscr{N}_t(T)$ *if and only if* $\bar{L}^1(v) \cup \bar{L}^2(v) = \emptyset$.

To illustrate Theorem 2, note that in the pruning $\bar{T}_v$ of the tree T in Fig. 1, $|\bar{L}^0(v)| = |\bar{L}^1(v)| = 0$, $|\bar{L}^2(v)| = 1$ and $|\bar{L}^3(v)| = 3$; that is, $|\bar{L}^1(v) \cup \bar{L}^2(v)| = 1$. Hence, by Theorem 2, $v \notin \mathscr{A}_t(T) \cup \mathscr{N}_t(T)$.

3. Preliminary results

We begin with an elementary fact about total dominating sets of the path P_n.

Fact 3. *For* $n \geqslant 2$, $\gamma_t(P_n) = n/2$ *if* $n \equiv 0 \pmod 4$, $\gamma_t(P_n) = (n+2)/2$ *if* $n \equiv 2 \pmod 4$, *and* $\gamma_t(P_n) = (n+1)/2$ *otherwise. Furthermore, there is a* $\gamma_t(P_n)$*-set that contains one of its end-vertices if and only if* $n \equiv 2$ *or* $3 \pmod 4$.

We also need the following result.

Fact 4. *Any TDS of a tree T contains every support vertex of T. Moreover, if* $\operatorname{diam}(T) \geqslant 3$, *there is a minimum TDS which contains no leaf.*

Proof. Every leaf is uniquely dominated by the support vertex adjacent to it, and so any TDS contains every support vertex. Suppose $\operatorname{diam}(T) \geqslant 3$ and let D be a $\gamma_t(T)$-set that contains as few leaves as possible. Suppose D contains a leaf w. Let v be the support vertex adjacent to w and let u be a neighbor of v of degree at least 2. Then $(D - \{w\}) \cup \{u\}$ is a $\gamma_t(T)$-set containing fewer leaves than D, a contradiction. Hence D contains no leaf. □

Lemma 5. *Let T' be a tree with $v \in V(T')$. Let u' be a vertex of T' that has a neighbor distinct from v (possibly, $u' = v$). Let T be the tree obtained from T' by attaching a path of length* 4 *to u'. Then*

(a) $\gamma_t(T) = \gamma_t(T') + 2$;
(b) $v \in \mathscr{A}_t(T')$ *if and only if* $v \in \mathscr{A}_t(T)$;
(c) $v \in \mathscr{N}_t(T')$ *if and only if* $v \in \mathscr{N}_t(T)$.

Proof. Suppose T is obtained from T' by adding the path u, w, x, y and the edge uu'.

(a) Any $\gamma_t(T')$-set can be extended to a TDS of T by adding the set $\{w, x\}$, and so $\gamma_t(T) \leqslant \gamma_t(T') + 2$. Now let D be a $\gamma_t(T)$-set and let $D' = D \cap V(T')$. By Fact 4, we can choose D to contain no leaf, and so $\{w, x\} \subset D$. Furthermore, we may assume that $u \notin D$, for otherwise we may replace it by a vertex in $N(u') - \{u\}$ (since D is a TDS of T, such a vertex is necessarily adjacent to a vertex of D). Hence D' is a TDS of T', and so $\gamma_t(T') \leqslant |D'| = \gamma_t(T) - 2$.

(b) Suppose that $v \notin \mathscr{A}_t(T')$. Let S' be a $\gamma_t(T')$-set that does not contain v. Then $S = S' \cup \{w, x\}$ is a TDS of T of cardinality $|S'| + 2 = \gamma_t(T)$, and so S is a $\gamma_t(T)$-set that does not contain v. Hence $v \notin \mathscr{A}_t(T)$. Conversely, suppose that $v \in \mathscr{A}_t(T')$. Let D be an arbitrary $\gamma_t(T)$-set and let $D' = D \cap V(T')$. By Fact 4, $x \in D$. The set D contains exactly one of w and y. If $u \notin D$, then D' is a TDS of T' of cardinality $\gamma_t(T) - 2 = \gamma_t(T')$, and so D' is a $\gamma_t(T')$-set. Since $v \in \mathscr{A}_t(T')$, $v \in D' \subset D$. If $u \in D$, then $|D'| = |D| - 3$. By our choice of u', u' has a neighbor u^* in T' distinct from v. Now, $D' \cup \{u^*\}$ is a TDS of T' and $|D'| + 1 \geqslant |D' \cup \{u^*\}| \geqslant \gamma(T') = |D| - 2 = |D'| + 1$, hence $D' \cup \{u^*\}$ is a $\gamma_t(T')$-set. Since $v \in \mathscr{A}_t(T')$ and $v \neq u^*$, $v \in D' \subset D$. Hence $v \in \mathscr{A}_t(T)$.

(c) Suppose $v \notin \mathscr{N}_t(T')$. Let S' be a $\gamma_t(T')$-set that contains v. Then $S = S' \cup \{w, x\}$ is a $\gamma_t(T)$-set that contains v, and so $v \notin \mathscr{N}_t(T)$. Conversely, suppose $v \in \mathscr{N}_t(T')$. Let D be an arbitrary $\gamma_t(T)$-set and let $D' = D \cap V(T')$. Using the notation introduced above, D' or $D' \cup \{u^*\}$ is a $\gamma_t(T')$-set. Since $v \in \mathscr{N}_t(T')$, $v \notin D'$ and so $v \notin D$. □

4. Proof of Theorem 1

If v is a support vertex, then Theorem 1 holds by Fact 4. Hence we may assume that v is not a support vertex of T.

Suppose $L(v) = L^0(v)$ and T' is a path P_5 containing v as a leaf. Then $L^1(v) \cup L^2(v) = \emptyset$. By repeated applications of Lemma 5 it follows that $v \in \mathscr{A}_t(T)$ ($\mathscr{N}_t(T)$, respectively), if and only if $v \in \mathscr{A}_t(T')$ ($\mathscr{N}_t(T')$, respectively). The result now follows from Fact 3. Hence we may assume that $L(v) \neq L^0(v)$.

For each $w \in L(v)$, let T^* be the tree obtained by replacing the v–w path in T with a v–w path of length j, $j = 0, 5, 2, 3$ if $w \in L^i(v)$, $i = 0, 1, 2, 3$, respectively. By repeated applications of Lemma 5 it now follows that $v \in \mathscr{A}_t(T)$ ($\mathscr{N}_t(T)$, respectively) if and only if $v \in \mathscr{A}_t(T^*)$ ($\mathscr{N}_t(T^*)$, respectively). If v is a leaf of T^*, then T^* is a path of order $n \geqslant 3$, so $|L^1(v) \cup L^2(v)| \leqslant 1$ and the result follows from Facts 3 and 4. Hence we may assume that v is not a leaf.

To prove Theorem 1 we may therefore assume without loss of generality that $\deg v \geqslant 2$ and every leaf of T is at distance 2, 3, or 5 from v. We consider the following cases.

Case 1: $|L^2(v)| = 1$, $L^1(v) = \emptyset$.

Let u be the leaf at distance 2 from v. Then every leaf distinct from u is at distance 3 from v, i.e., $L^3(v) = L(v) - \{u\}$. Since any $\gamma_t(T)$-set contains every support vertex and at least one neighbor of every support vertex, $\gamma_t(T) \geqslant 2|L(v)|$. On the other hand, $D^* = S(T) \cup L(T)$ is a TDS of T of cardinality $2|L(v)|$, and so $\gamma_t(T) = 2|L(v)|$. Moreover, D^* and $(D^* - \{u\}) \cup \{v\}$ are $\gamma_t(T)$-sets that show that $v \notin \mathscr{A}_t(T) \cup \mathscr{N}_t(T)$.

Case 2: $|L^1(v)|=1$, $L^2(v)=\emptyset$.

Let u be the leaf at distance 5 from v in T, and let $v, u_1, u_2, \ldots, u_4, u$ be the v–u path. Every leaf distinct from u is at distance 3 from v. Any TDS in T contains at least three vertices on the v–u path and at least two vertices distinct from v on any other path from v to a leaf. Hence $\gamma_t(T)\geqslant 1+2|L(v)|$. On the other hand, $D^*=S(T)\cup(C(v)-\{u_1\})\cup\{u_2,u_3\}$ is a TDS of T of cardinality $1+2|L(v)|$, and so $\gamma_t(T)=1+2|L(v)|$. Moreover, D^* and $(D^*-\{u_2\})\cup\{v\}$ are $\gamma_t(T)$-sets that show that $v\notin\mathscr{A}_t(T)\cup\mathscr{N}_t(T)$.

Case 3: $|L^1(v)|=|L^2(v)|=0$.

Then every leaf is at distance 3 from v. It is easy to see that any TDS in T contains exactly two vertices distinct from v on every path from v to a leaf, and that no $\gamma_t(T)$-set contains v. Hence $v\in\mathscr{N}_t(T)$ (and obviously $v\notin\mathscr{A}_t(T)$).

In what follows, let D denote an arbitrary $\gamma_t(T)$-set.

Case 4: $|L^2(v)|\geqslant 2$.

Let u and w be two leaves at distance 2 from v in T, with u' and w' the support vertices adjacent to u and w, respectively. If $v\notin D$, then $\{u,u',w,w'\}\subseteq D$ and so $(D-\{u,w\})\cup\{v\}$ is a TDS of T of cardinality $\gamma_t(T)-1$, which is impossible. Hence $v\in D$ and thus $v\in\mathscr{A}_t(T)$.

Case 5: $|L^2(v)|=1$, $|L^1(v)|\geqslant 1$.

Let u with $N(u)=\{u'\}$ be the leaf at distance 2 from v. Let w be a leaf at distance 5 from v with $v, w_1, w_2, \ldots, w_4, w$ the v–w path. If $v\notin D$, then $\{u,u',w_2,w_3,w_4\}\subseteq D$, and so $(D-\{u,w_2\})\cup\{v\}$ is a TDS of T of cardinality $\gamma_t(T)-1$, which is impossible. Hence $v\in D$ and thus $v\in\mathscr{A}_t(T)$.

Case 6: $|L^2(v)|=0$, $|L^1(v)|\geqslant 2$.

Let u and w be two leaves at distance 5 from v in T with $P_u: v,u_1,u_2,\ldots,u_4,u$ and $P_w: v,w_1,w_2,\ldots,w_4,w$ the v–u and v–w paths, respectively. For $i\in\{u,w\}$, define $D_i=V(P_i)\cap D$. If $v\notin D$, then $\{u_2,w_2\}\subseteq D$ to dominate $\{u_1,w_1\}$. Hence $|D_i|\geqslant 3$. But then (since D dominates v) $D'=(D-(D_1\cup D_2))\cup\{v,u_3,u_4,w_3,w_4\}$ is a TDS with $|D'|<\gamma_t(T)$, a contradiction. Thus $v\in\mathscr{A}_t(T)$.

Since these cases exhaust all possibilities, the proof of Theorem 1 is complete. □

5. Proof of Theorem 2

Theorem 2 is an immediate consequence of Theorem 1, Lemma 5 and the following lemma.

Lemma 6. *Consider a path P in a tree T_1* with $L(P)\subseteq L(T_1)$, $w\in V(P)\cup B(T_1)$ *and* $v\in V(T_1)-V(P)$. *For a set (to be defined) $X\subset V(P)-\{w\}$, let $T_2=T_1-X$. If*

(i) $P=u,w,x$, $X=\{x\}$ *and v is not a leaf adjacent to w*,
(ii) $P=u,w,x,y$, *where* $\deg_{T_1}x=2$, *and* $X=\{u\}$,
(iii) $P=t,u,w,x,y$, *where* $\deg_{T_1}u=\deg_{T_1}x=2$, *and* $X=\{t,u\}$,
(iv) $P=t,u,w,x,y,z$, *where* $\deg_{T_1}u=\deg_{T_1}x=\deg_{T_1}y=2$, *and* $X=\{x,y,z\}$,
(v) $P=u,w,x,y,z$, *where* $\deg_{T_1}x=\deg_{T_1}y=2$, *and* $X=\{u,z\}$,
(vi) $P=s,t,u,w,x,y,z$, *where* $\deg_{T_1}i=2$ *for* $i\in\{t,u,x,y\}$, *and* $X=\{x,y,z\}$,

then

(a) $\gamma_t(T_2) = \begin{cases} \gamma_t(T_1) & \textit{for (i) and (ii)}, \\ \gamma_t(T_1) - 1 & \textit{for (iii) and (v)}, \\ \gamma_t(T_1) - 2 & \textit{for (iv) and (vi)}. \end{cases}$

(b) $v \in \mathscr{A}_t(T_1)$ *if and only if* $v \in \mathscr{A}_t(T_2)$.

(c) $v \in \mathscr{N}_t(T_1)$ *if and only if* $v \in \mathscr{N}_t(T_2)$.

Proof. In each case the proof of (c) is similar to the proof of (b) and hence omitted.

(i) Note that any minimal TDS of T_1 contains at most one of u and x. By Fact 4, $w \in \mathscr{A}_t(T_1) \cap \mathscr{A}_t(T_2)$. Statements (a) and (b) now follow from the fact that any TDS of T_2 is a TDS of T_1, and any minimal TDS of T_1 corresponds to a minimal TDS of T_2, where we replace x with u if necessary.

(ii) (a) Let D_1 be a $\gamma_t(T_1)$-set. By Fact 4, $\{w, x\} \subseteq D_1$. Hence $u \notin D_1$ and D_1 is a TDS of T_2. Now let D_2 be a $\gamma_t(T_2)$-set. By Fact 4, we can choose D_2 so that $\{w, x\} \subseteq D_2$. Hence D_2 is a TDS of T_1 and the result follows.

(b) It is clear from the proof of (a) that $\mathscr{A}_t(T_2) \subseteq \mathscr{A}_t(T_1)$. Suppose that $v \in \mathscr{A}_t(T_1)$ and let S_2 be an arbitrary $\gamma_t(T_2)$-set. By Fact 4, $x \in S_2$. Since S_2 contains exactly one of w and y, $S_1 = (S_2 - \{y\}) \cup \{w\}$ is a TDS of T_1 of cardinality $|S_2| = \gamma_t(T_2) = \gamma_t(T_1)$, i.e. a $\gamma_t(T_1)$-set. Thus $v \in S_1$. Since $v \neq w$, $v \in S_2$ as required.

(iii) (a) It is easy to see that $\{u, w, x\} \subseteq D_1$ for any $\gamma_t(T_1)$-set D_1; hence $D_1 - \{u\}$ is a TDS of T_2. On the other hand, since any $\gamma_t(T_2)$-set D_2 contains exactly one of w and y, it is obvious that $(D_2 - \{y\}) \cup \{w, u\}$ is a TDS of T_1 and the result follows.

(b) Suppose that $v \in \mathscr{A}_t(T_2)$. Let D_1 be a $\gamma_t(T_1)$-set; as above $D_1 - \{u\}$ is $\gamma_t(T_2)$-set. Since $v \neq u$, $v \in D_1$ and it follows that $v \in \mathscr{A}_t(T_1)$. Conversely, suppose $v \in \mathscr{A}_t(T_1)$ and let S_2 be any $\gamma_t(T_2)$-set. By Fact 4, $x \in S_2$. Since S_2 contains exactly one of w and y, $S_1 = (S_2 - \{y\}) \cup \{u, w\}$ is a $\gamma_t(T_1)$-set, hence $v \in S_1$. Now $v \notin \{u, w\}$, therefore $v \in S_2$ and so $v \in \mathscr{A}_t(T_2)$.

(iv) (a) Let D_1 be a $\gamma_t(T_1)$-set. By Fact 4, we can choose D_1 with $\{u, w, x, y\} \subseteq D_1$, in which case $D_1 - \{x, y\}$ is a TDS of T_2. On the other hand, if D_2 is a $\gamma_t(T_2)$-set, we can choose D_2 so that $\{u, w\} \subseteq D_2$. Then $D_2 \cup \{x, y\}$ is a TDS of T_1 and the result follows.

(b) Suppose $v \in \mathscr{A}_t(T_2)$. By Fact 4, $y \in D_1$ for any $\gamma_t(T_1)$-set D_1; note that D_1 also contains exactly one of x and z. Hence $D_2 = D_1 - \{x, y, z\}$ is a TDS of T_2 of cardinality $|D_1| - 2 = \gamma_t(T_2)$ and thus a $\gamma_t(T_2)$-set. Therefore $v \in D_2$ and since $v \notin \{x, y, z\}$, $v \in D_1$. On the other hand, if $v \notin \mathscr{A}_t(T_2)$ and S_2 is a $\gamma_t(T_2)$-set not containing v, then $S_1 = S_2 \cup \{x, y\}$ is a $\gamma_t(T_1)$-set not containing v. Hence $v \notin \mathscr{A}_t(T_1)$.

(v) (a) By Fact 4 we can choose a $\gamma_t(T_1)$-set D_1 with $\{w, x, y\} \subseteq D_1$. Then $D_1 - \{y\}$ is a TDS of T_2. Similarly, we can choose a $\gamma_t(T_2)$-set D_2 with $\{x, w\} \subseteq D_2$. Hence $D_2 \cup \{y\}$ is a TDS of T_1.

(b) Suppose $v \in \mathscr{A}_t(T_2)$. Any $\gamma_t(T_1)$-set D_1 contains $\{w, y\}$ and exactly one of x and z. Hence $D_2 = (D_1 - \{y, z\}) \cup \{x\}$ is a TDS of T_2 of cardinality $|D_1| - 1 = \gamma_t(T_2)$ and hence a $\gamma_t(T_2)$-set. Therefore $v \in D_2$ and since $v \neq x$, $v \in D_1$. Now suppose $v \in \mathscr{A}_t(T_1)$. Any $\gamma_t(T_2)$-set S_2 contains x and exactly one of y and w. Thus $S_1 = S_2 \cup \{w, y\}$ is a

TDS of T_1 of cardinality $|S_2| + 1 = \gamma_t(T_1)$, i.e., a $\gamma_t(T_1)$-set. Since $v \in \mathscr{A}_t(T_1)$, $v \in S_1$. Since $v \notin \{w, y\}$, $v \in S_2$ as required.

(vi) (a) We can choose a $\gamma_t(T_1)$-set D_1 such that $\{t, u, x, y\} \subseteq D_1$ and $D_1 - \{x, y\}$ is a TDS of T_2. Furthermore, any $\gamma_t(T_2)$-set can be extended to a TDS of T_1 by adding the set $\{x, y\}$.

(b) Suppose $v \in \mathscr{A}_t(T_2)$. Any $\gamma_t(T_1)$-set D_1 contains $\{t, y\}$, exactly one of x and z and exactly one of u and s, so $D_2 = (D_1 - \{s, x, y, z\}) \cup \{u\}$ is a TDS of T_2 of cardinality $|D_1| - 2 = \gamma_t(T_2)$, i.e., a $\gamma_t(T_2)$-set. Thus $v \in D_2$. Since $v \neq u$, $v \in D_1$. On the other hand, if $v \notin \mathscr{A}_t(T_2)$ and S_2 is a $\gamma_t(T_2)$-set not containing v, then $S_1 = S_2 \cup \{x, y\}$ is a $\gamma_t(T_1)$-set not containing v. Hence $v \notin \mathscr{A}_t(T_1)$. □

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