Coloring the complements of intersection graphs of geometric figures

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Abstract

Let $G$ be the complement of the intersection graph $G$ of a family of translations of a compact convex figure in $\mathbb{R}^n$. When $n = 2$, we show that $\chi(G) \leq \min\{3\omega(G) - 2, 6\gamma(G)\}$, where $\omega(G)$ is the size of the minimum dominating set of $G$. The bound on $\chi(G) \leq 6\gamma(G)$ is sharp. For higher dimension we show that $\chi(G) \leq \lceil \frac{2(n^2 - n + 1)^{1/2}n - 1}{\gamma(G) - 1} \rceil$, for $n \geq 3$. We also study the chromatic number of the complement of the intersection graph of homothetic copies of a fixed convex body in $\mathbb{R}^n$.

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1. Introduction

The intersection graph $G$ of a family $\mathcal{F}$ of sets is the graph with vertex set $\mathcal{F}$ in which two members of $\mathcal{F}$ are adjacent if and only if they have nonempty intersection. Asplund and Grünbaum [1] and Gyárfás and Lehel [5] studied many interesting problems on the chromatic numbers of intersection graphs of convex figures in the plane. Many problems of this type can be stated as follows. For a class $\mathcal{G}$ of intersection graphs and for a positive integer $k$, find $\chi(\mathcal{G}, k)$, where $\chi(\mathcal{G}, k)$ is the maximum chromatic number of a graph in $\mathcal{G}$ with the clique number at most $k$.

Recent papers on intersection graphs of translations of a plane figure include [2,4,7,9,10,13]. Let $\mathcal{T}_n$ be the family of intersection graphs of translations of a fixed compact convex figure in $\mathbb{R}^n$. In connection with the channel assignment problem in broadcast networks, Clark et al. [2] and Gräf et al. [4] considered colorings of graphs in the class $\mathcal{U}$ of intersection graphs of unit disks in the plane. Perepelitsa [13] considered the more general family $\mathcal{T}_2$ of intersection graphs of translations of a fixed compact convex figure in the plane. She proved that $\chi(\mathcal{T}_2, k) \leq 8k - 7$. Kim et al. [7] improved Perepelitsa’s bound further. They proved that every graph in $\mathcal{T}_2$ is $(3k - 3)$-degenerate, which implies that $\chi(\mathcal{T}_2, k) \leq 3k - 2$. The result that every graph in $\mathcal{T}_2$ is $(3k - 3)$-degenerate is sharp; there is a construction of intersection graph of unit disks that is not $(3k - 4)$-degenerate. In [9], Kostochka obtained upper bounds on $\chi(\mathcal{T}_n, k)$ in higher dimensions $\mathbb{R}^n$.

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Another problem is finding the maximum chromatic number of the complement of an intersection graph of convex sets. The chromatic number of the complement of an intersection graph is the minimum number of cliques that suffices to cover the vertices of the graphs. Several results about chromatic numbers of complements of intersection graphs of boxes appear in [3, 5, 6]. Let \( B_n \) be the family of complements of the intersection graphs of boxes in \( \mathbb{R}^n \). Gyárfás and Lehel [5] proved that \( \chi(B_n, k) \leq k(k - 1)/2 \), and Károlyi [6] proved that \( \chi(B_n, k) \leq (1 + o(1))k(\log_2 k)^{n-1} \). Fon-Der-Flaass and Kostochka [3] gave a simple proof of a slight refinement of the Károlyi’s bound:

\[
\chi(B_n, k) \leq k(\log_2 k)^{n-1} + n - 0.5k(\log_2 k)^{n-2} \quad \text{for } n \geq 2.
\]  

They also obtained a lower bound for a special case: \( \chi(B_n, 2) \geq \lfloor 5n/3 \rfloor \). One of the main questions is whether \( \chi(B_n, k) \) is superlinear.

In this paper, we consider several types of convex sets. As in [7], we consider a family of convex bodies obtained by translations of a fixed convex body in \( \mathbb{R}^n \). We study the chromatic numbers of the complements of intersection graphs of convex bodies obtained by translations of a fixed convex body in \( \mathbb{R}^n \).

For \( n = 2 \) in (1), \( \chi(B_2, k) \leq k\log_2 k + 2 - 0.5k \). In this paper, we will show that \( \chi(\mathcal{F}_2, k) \) is linear, where \( \mathcal{F}_2 \) is the family of the complements of the intersection graphs in \( \mathcal{F}_2 \). We will prove the following results.

**Theorem 1.** Let \( \overline{G} \) be the complement of an intersection graph \( G \) of translations of a fixed compact convex set in the plane. Then \( \chi(\overline{G}) \leq \min\{3\gamma(G) - 2, 6\gamma(G)\} \) where \( \gamma(G) \) is the size of the minimum dominating set of \( G \). The bound \( \chi(\overline{G}) \leq 6\gamma(G) \) is sharp.

Theorem 1 implies that \( \chi(\overline{G}) \leq 3\gamma(G) - 2 \). By Construction 2 in Section 2 the maximum possible chromatic number of the intersection graph \( G \) defined in Theorem 1 is \( \frac{3}{2} \gamma(G) \leq \chi(\overline{G}) \).

**Theorem 2.** Let \( \mathcal{F} \) be a family of translates of a convex body \( M \) in \( \mathbb{R}^n \) for \( n \geq 3 \). If \( G \) is the intersection graph of \( \mathcal{F} \), then \( \chi(\overline{G}) \leq \lfloor 2(n^2 - n + 1)^{1/2}(n^2 - n + 1)^{1/2}(k - 1) + 1 \rfloor \), where \( k = \gamma(G) \).

We also study a more general case. We denote \( x + \lambda D = \{x + \lambda w : w \in D\} \) where \( x \) is a point in \( \mathbb{R}^n \) and \( \lambda \) a real number. Two convex bodies \( K \) and \( D \) in \( \mathbb{R}^n \) are called homothetic if \( K = x + \lambda D \) for a point \( x \) in \( \mathbb{R}^n \) and some \( \lambda > 0 \). We consider intersection graphs of families of homothetic copies of a fixed compact convex body. An intersection graph of a family of disks of various sizes is a special case. Let \( \mathcal{H}_2 \) be a family of intersection graphs of homothetic copies of a fixed convex set in the plane and \( \mathcal{H}_2 \) be the family of complements of intersection graphs in \( \mathcal{H}_2 \). We show that \( \chi(\mathcal{H}_2, k) \) is also linear.

**Theorem 3.** Let \( \mathcal{D}_2 \) be a family of homothetic copies of a fixed compact convex set \( D \) in the plane. If \( \overline{G} \) is the complement of the intersection graph \( G \) of \( \mathcal{D}_2 \), then \( \chi(\overline{G}) \leq 6\gamma(G) - 5 \).

In Section 2 we will prove Theorem 1; in Section 3 we will prove Theorem 2. In Section 4 we will prove Theorem 3. Here we review basic terms and notation. A graph \( G \) is \( d \)-degenerate if every subgraph \( G' \) of \( G \) has a vertex whose degree in \( G' \) is at most \( d \). The neighborhood of a vertex set \( A \) in \( G \) is \( N_G(A) \); the closed neighborhood \( N_G[A] \) of \( A \) is \( N_G[A] = A \cup N_G(A) \). The maximum size of an independent set in \( G \) is denoted by \( \alpha(G) \). A set \( S \) is called a dominating set of a graph \( G \) if every vertex \( v \) outside \( S \) has a neighbor in \( S \). \( G[M] \) denotes the subgraph induced by \( M \) in \( G \).

2. Coloring the complement of an intersection graph in the plane

In this section, \( G \) is an intersection graph of convex sets obtained by translations of a fixed convex set in the plane and \( \overline{G} \) is its complement. We define \( x + K = \{x + y : y \) is a point in \( K\} \) where \( x \) is a point and \( K \) is a convex set in the plane. That is, \( x + K \) is a translation of \( K \). Our first tool is an old result of Minkowski [12].

**Lemma 4 (Minkowski [12]).** If \( K \) is a convex body in \( \mathbb{R}^n \) for \( n \geq 2 \), then \( \{x + K \cap (y + K) \neq \emptyset \) if and only if \( \{x + \frac{1}{2}(K + (-K)) \} \cap (y + \frac{1}{2}(K + (-K))) \neq \emptyset \).
Note that the set $\frac{1}{2}[K + (-K)]$ is centrally symmetric for every $K$. By Lemma 4, it suffices to prove Theorems 1 and 2 for centrally symmetric convex bodies. A convex body $A$ is called highest in a family of convex bodies if it has a point that has the biggest $n$th-coordinate in $\mathbb{R}^n$.

**Lemma 5.** If $A$ is a highest convex set in $\mathcal{V}(G)$, then $\chi(\mathcal{G}[N_G(A)])$ decomposes into at most three independent sets. That is, $\chi(\mathcal{G}[N_G(A)]) \leq 3$.

**Proof.** Let $A$ be a highest member in $\mathcal{V}(G)$. For convenience, we assume that the center of $A$ is the origin $O = (0, 0)$. Let $z$ be the rightmost point on the horizontal axis that belongs to $A$. If $z = (0, 0)$, then $A$ is an interval with the center $O$ and $G$ is an intersection graph of intervals on the plane. So, we assume $z \neq (0, 0)$. Let $B = A - 2z$ and $C = A + 2z$. Since $A$ is convex and centrally symmetric, both $B$ and $C$ intersect $A$, but either has no common interior points with $A$. Note that $B$ and $C$ may or may not belong to $\mathcal{V}(G)$.

In $G$, $N_G(A)$ decomposes into three subsets $N_1, N_2$, and $N_3$ such that

$N_1 = \{U : U \text{ intersects } A \text{ and } B\}$,

$N_2 = \{U : U \text{ intersects } A \text{ and } C, \text{ but not } B\}$,

$N_3 = \{U : U \text{ intersects } A, \text{ but not } B \text{ or } C\}$.

In Theorem 8 in [7], the following three claims are proved.

Claim 1: $U \cap V \neq \emptyset$ for any convex sets $U$ and $V$ in $N_1$.

Claim 2: $U \cap V \neq \emptyset$ for any convex sets $U$ and $V$ in $N_2$.

Claim 3: $U \cap V \neq \emptyset$ for any convex sets $U$ and $V$ in $N_3$.

It implies that $G[N_1], G[N_2]$, and $G[N_3]$ are complete subgraphs of $G$. Hence $N_G(A)$ is the union of three independent sets in $\mathcal{G}$. □

**Lemma 6.** For any convex set $A$, $\chi(\mathcal{G}[N_G(A)]) \leq 6$.

**Proof.** Let $M^+$ be the set of neighbors of $A$ whose center is above the horizontal line passing the center of $A$ and $M^- = N[A] - M^+$. From Lemma 5, $\chi(\mathcal{G}[M^-]) \leq 3$ and $\chi(\mathcal{G}[M^+]) \leq 3$. Hence $\chi(\mathcal{G}[N_G(A)]) \leq 6$, since $N_G[A] = M^+ \cup M^-$. □

Now we will prove the main theorem.

**Theorem 1.** Let $\mathcal{G}$ be the complement of an intersection graph $G$ of translations of a fixed compact convex set in the plane. Then $\chi(\mathcal{G}) \leq \min\{\gamma(G) - 2, 6\gamma(G)\}$ where $\gamma(G)$ is the size of the minimum dominating set of $G$. The bound $\chi(\mathcal{G}) \leq 6\gamma(G)$ is sharp.

**Proof.** We pick a highest convex set in $\mathcal{V}(G)$ and call it $A_1$ and put $M_1 = N_G[A_1]$. By Lemma 5, $\chi(\mathcal{G}[M_1]) \leq 3$. Delete $M_1$ from $\mathcal{V}(G)$ and pick a highest convex set $A_2$ among $V_1 = \mathcal{V}(G) - M_1$ and put $M_2 = A_2 \cup (N_G(A_2) \cap V_1)$. Again, $\chi(\mathcal{G}[M_2]) \leq 3$. Delete $M_2$ from $V_1$ and continue this process with formula $M_j = A_j \cup (N_G(A_j) \cap V_{j-1})$ and $V_{j-1} = V_{j-2} - M_{j-1}$ until no vertices remain. This process ends within $\gamma(G)$ steps, since $\{A_1, A_2, \ldots \}$ is an independent set in $G$. Furthermore, $V(G) = \bigcup_{i=1}^{l} M_i$ where $l \leq \gamma(G)$. Note that $\chi(\mathcal{G}[M_i]) \leq 3$ for all $i (1 \leq i \leq l)$. If $l = \gamma(G)$, then $\mathcal{G}[M_{\gamma(G)}]$ must be an independent set. Otherwise, we have an independent set whose size is greater than $\gamma(G)$. Hence $\chi(\mathcal{G}[M_{\gamma(G)}]) \leq 1$. Therefore

$$\chi(\mathcal{G}) \leq \sum_{i=1}^{l} \chi(\mathcal{G}[M_i]) \leq 3(\gamma(G) - 1) + 1 = 3\gamma(G) - 2.$$ 

Now we will prove the second bound. Let $D = \{v_1, v_2, \ldots, v_{\gamma(G)}\}$ be a minimum-sized dominating set of $G$. Hence $V(G) = \bigcup_{i=1}^{\gamma(G)} N_G[v_i]$. Using Lemma 6, we have

$$\chi(\mathcal{G}) \leq \sum_{i=1}^{\gamma(G)} \chi(\mathcal{G}[N_G[v_i]]) \leq 6\gamma(G).$$

Construction 1 below shows that the bound $\chi(\mathcal{G}) \leq 6\gamma(G)$ is sharp. □
Construction 1. Let \( P = \{u_0, u_1, u_2, \ldots, u_{6k-7}\} \) where \( k \geq 7 \) be points in the plane such that \( u_0 \) is the origin and \( \{u_1, u_2, \ldots, u_{6k-7}\} \) are equally spaced on the circle \( C_2 \) with radius 2 whose center is the origin. Let \( W_F \) be the family of unit disks in the plane whose centers are \( P \). Let \( W \) be the intersection graph of unit disks in \( W_F \).

We will show that \( \chi(W) = 6 \). Because \( \omega(W) = k \), \( \chi(W) \geq \omega(W) = (6k - 6)/k > 5 \). Hence \( \chi(W) \geq 6 \). On the other hand, it is easily checked using Lemma 6 that \( \chi(W) \leq 6 \). Hence \( \chi(W) = 6 \). Now let \( G \) be an intersection graph consisting of \( t \) disjoint copies of \( W \). Then \( \gamma(G) = t \) and \( \chi(G) = 6t = 6\gamma(G) \).

Construction 2. Let \( F_5 \) be a family of five convex sets such that the intersection graph of \( F_5 \) is \( C_5 \) that is the cycle of length 5. Let \( F \) be the family of \( k \) disjoint copies of \( F_5 \). If \( G \) is the intersection graph of \( F \), then \( \chi(G) = 2k \). Since \( \chi(C_5) = 3 \), \( \chi(G) = 3k = \frac{3}{2} \alpha(G) \).

3. Convex figures in higher dimensions

In this section, \( H \) is an intersection graph of convex bodies in \( \mathbb{R}^n \), and \( \overline{H} \) is the complement of \( H \).

Lemma 7. Let \( A \) be a centrally symmetric convex body in \( \mathbb{R}^n \) and let \( B \) be a translation of \( A \). If the center of \( B \) is in \( A \), then \( B \) contains the center of \( A \).

Proof. Without loss of generality, we may assume that the center of \( A \) is the origin and the center of \( B \) is the point \( x \). If \( x \in A \), then \( -x \in A \) because \( A \) is centrally symmetric. Hence \( 0 = x + (-x) \in x + A = B \). \( \square \)

Corollary 8. Let \( A \) be a centrally symmetric convex body in \( \mathbb{R}^n \). Let \( B_1, \ldots, B_s \) be translations of \( A \) by \( x_1, \ldots, x_s \), respectively. If for each \( i \) (\( 1 \leq i \leq s \)), the center \( x_i \) of \( B_i \) is in \( A \), then the center of \( A \) is a common point of \( A, B_1, B_2, \ldots, B_s \). Consequently, \( \{A, B_1, B_2, \ldots, B_s\} \) induces an independent set in \( \overline{H} \).

Now we will prove Theorem 2 by adapting an idea from [8].

Theorem 2. Let \( F \) be a family of translates of a convex body \( M \) in \( \mathbb{R}^n \). If \( G \) is the intersection graph of \( F \), then \( \chi(G) \leq [2(n^2 - n + 1)^{1/2}]^{n-1} [2(n^2 - n + 1)^{1/2}] (k - 1) + 1 \), where \( k = \alpha(G) \).

Proof. From Lemma 4, we may assume that the convex body \( M \) is centrally symmetric. Lassak [11] proved that for every centrally symmetric convex body \( M \in \mathbb{R}^n \) there are parallelotopes \( R \) and \( S \) such that \( R \subseteq M \subseteq S \) and \( R \) and \( S \) are homothetic with homothetic ratio \( (n^2 - n + 1)^{1/2} \). From now on, we denote \( \lambda = (n^2 - n + 1)^{1/2} \) for simple notation.

Let \( A \) be the highest convex set in \( F \). Let \( R_A \) and \( S_A \) be translates of \( R \) and \( S \) that are parallelotopes such that \( R_A \subseteq A \subseteq S_A \) and \( R_A \) and \( S_A \) are homothetic with homothetic ratio \( \lambda \). Since \( A \) is centrally symmetric, all convex bodies in \( N_G[A] \) have their centers in the bottom half of the double-sized convex set \( 2A \) of \( A \). Hence there exists a parallelotope \( S_{2A} \) such that \( R_{2A} \subseteq A \subseteq S_{2A} \) and \( R_A \) and \( S_{2A} \) are homothetic with homothetic ratio \( 2\lambda \). Hence the bottom half of \( S_{2A} \) is covered by \( [2\lambda]^{n-1} [\lambda] \) copies of \( R_A \). Also, since a set of convex bodies whose centers are in \( R_A \) have a common point by Corollary 8, we have \( \chi(G|N_G[A]) \leq [2\lambda]^{n-1} [\lambda] \).

As in Theorem 1 and using the above argument, we have a decomposition

\[
V(G) = \bigcup_{i=1}^{k} M_i \quad \text{where} \quad \chi(G|M_{i}) \leq [2\lambda]^{n-1} [\lambda] \quad \text{for} \quad 1 \leq i \leq k.
\]

Note that \( M_k \) is an intersecting family in the decomposition. Hence, we have

\[
\chi(G) \leq \sum_{i=1}^{k} \chi(G|M_{i}) = 1 + \sum_{i=1}^{k-1} \chi(G|M_{i}) \leq 1 + \sum_{i=1}^{k-1} [2\lambda]^{n-1} [\lambda] = [2\lambda]^{n-1} [\lambda](k - 1) + 1. \quad \square
\]

Remark 9. When \( F \) is a family of translates of a convex body \( M \) in \( \mathbb{R}^n \) and \( G \) is the intersection graph of \( F \), we can show that \( \chi(G) \leq [2(n^2 - n + 1)^{1/2}]^{n-1} [(n^2 - n + 1)^{1/2}] (\omega(G) - 1) + 1 \), where \( \omega(G) \) is the clique number of \( G \), by using the argument in Theorem 2.
4. Convex figures with different sizes

In this section, we will consider a more general setting: shrinking and expanding of the convex body are allowed. We consider intersection graphs of families of homothetic copies of a fixed compact convex set. An intersection graph of a family of disks of various sizes is a special case.

For \(0 \leq \lambda \leq 1\), we define \(w(u, v, \lambda) = (1-\lambda)v + \lambda u\) and \(W(U, v, \lambda) = \{w(u, v, \lambda) : u \in U\}\). The following observation appears in [7].

**Lemma 10.** Given a convex set \(U\), for each \(v \in U\) and \(0 \leq \lambda \leq 1\), the set \(W(U, v, \lambda)\) is contained in \(U\) and contains \(v\).

**Proof.** Since \(w(u, v, 0) = v \in U\) and \(w(u, v, 1) = u \in U\), we have \(w(u, v, \lambda) \in U\) for every \(0 \leq \lambda \leq 1\). On the other hand, \(v = w(v, v, \lambda) \in W(U, v, \lambda)\) for every \(0 \leq \lambda \leq 1\). Hence \(W(U, v, \lambda) \subseteq U\) and \(v \in W(U, v, \lambda)\). \(\square\)

Now we will prove Theorem 3.

**Theorem 3.** Let \(D_2\) be a family of homothetic copies of a fixed compact convex set \(D\) in the plane. If \(\overline{G}\) is the complement of the intersection graph \(G\) of \(D_2\), then \(\chi(\overline{G}) \leq 6\chi(G) - 5\).

**Proof.** Let \(Z_1\) be a smallest homothetic copy of \(D\) in \(D_2\), and put \(M_1 = NG[Z_1]\). We will show that \(\chi(\overline{G}[M_1]) \leq 6\). For every \(U \in M_1\), let \(\lambda\) be the positive real number such that \(Z_1 = u + \lambda U\) for some \(u\). That is, \(U\) and \(Z_1\) are homothetic. For every \(U \in M_1\), choose a point \(z \in Z_1 \cap U\) and let \(U^* = W(U, z, \lambda)\). Note that \(U^*\) is a translate of \(Z_1\) and \(U^* = (1-\lambda)z + \lambda U\). By Lemma 10, the intersection graph \(G_{Z_1}\) of the family \(Z_1 = \{Z_1 \cup \{U^* : U \in M_1\}\}\) is a subgraph of \(G[M_1]\). Hence the minimum number of cliques covering \(G[M_1]\) is at most the minimum number of cliques covering \(G_{Z_1}\). From Lemma 6, \(\chi(\overline{G}[M_1]) \leq \chi(\overline{G}_{Z_1}) \leq 6\).

Now put \(V_1 = V(G) - NG[Z_1]\) and let \(Z_2\) be a smallest convex set in \(V_1\). Put \(M_2 = G[Z_2] \cap V_1\). With the same argument as above, \(\chi(\overline{G}[M_2]) \leq 6\). We continue this procedure with formula \(M_i = NG[Z_i] \cap V_{i-1}\) and \(V_{i-1} = V_{i-2} - M_{i-1}\) until all convex sets are gone. Clearly the procedure ends within \(\chi(G)\) steps. Hence we can write \(V(G) = \bigcup_{i=1}^l M_i\) where \(l \leq \chi(G)\). If \(l = \chi(G)\), then \(\overline{G}[M_{\chi(G)}]\) is an independent set. Therefore

\[
\chi(\overline{G}) \leq \sum_{i=1}^l \chi(\overline{G}[M_i]) \leq 6(\chi(G) - 1) + 1 = 6\chi(G) - 5.
\]

Using an argument similar to those in Theorems 3 and 2, we have the following Theorem.

**Theorem 11.** If \(\overline{H}\) is the complement of the intersection graph \(H\) of homothetic copies of a fixed compact convex body in \(\mathbb{R}^d\) with \(n \geq 3\), then \(\chi(\overline{H}) \leq \lceil 2(n^2 - n + 1)^{1/2} \rceil (\chi(H) - 1) + 1\).

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