Perturbed Algorithms and Sensitivity Analysis for a General Class of Variational Inclusions

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In the present paper, we study a perturbed iterative method for solving a general class of variational inclusions. An existence result which generalizes some known results in this field, a convergence result, and a new iterative method are given. We also prove the continuity of the perturbed solution to a parametric variational inclusion problem. Several special cases are discussed.

1. INTRODUCTION

In a recent paper [14], the authors have studied the convergence of an iterative algorithm in order to find an approximate solution for a new class of strongly nonlinear variational inequalities which includes the models of Noor [23] and Siddiqi and Ansari [27].

The aim of this work is twofold. Firstly, we establish a convergence result for a general class of variational inclusions under some minimal hypotheses which are less demanding than the one used in [14]. Secondly, we introduce a new iterative algorithm in order to approximate a solution for this kind of problem. Then we prove convergence of this algorithm by using some fixed point theorem. We also analyze the perturbed solution of the parametric variational inclusion and prove the continuity or the Lipschitz continuity of the perturbed solution when the map $S$ is neither strongly monotone nor Lipschitz continuous. Our problem is more general than the one considered in [14, 24, 25, 28], which motivated this paper.

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Throughout this paper, unless specifically stated, we will refer to the problem

$$\text{VI}(T, A, B, S) \left\{ \begin{array}{l} \text{find } \bar{u} \in H \text{ such that} \\ 0 \in (A - B)(\bar{u}) + T(S(\bar{u})), \end{array} \right.$$ 

where

(i) $H$ is a real Hilbert space, with the associated scalar product, denoted by $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$;

(ii) $T: H \rightrightarrows H$ is a multivalued maximal monotone operator;

(iii) $S, A: H \to H$ are two mappings such that $\text{Range}(S) \cap D(T) \neq \emptyset$;

(iv) $B: H \to H$ is a nonlinear continuous mapping.

In what follows, we shall write $\to$ and $\rightharpoonup$ to denote respectively, the strong (norm) convergence and the weak convergence on $H$.

An equivalent formulation of the original problem $\text{VI}(T, A, B, S)$ is

$$\left( \mathcal{P} \right) \left\{ \begin{array}{l} \text{find } \bar{u} \in H \text{ such that} \\ \langle v^* + A(\bar{u}) - B(\bar{u}), v - S(\bar{u}) \rangle \geq 0 \text{ for all } (v, v^*) \in \text{Graph}(T). \end{array} \right.$$ 

Since $T$ is maximal monotone, $\bar{u} \in H$ is a solution of $(\mathcal{P})$ if and only if $B(\bar{u}) - A(\bar{u}) \in T(S(\bar{u}))$.

The problem $\text{VI}(T, A, B, S)$ subsumes as a particular case, variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problems.

Before we proceed any further, let us give the following examples:

(i) If $T := \partial \varphi$, where $\partial \varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi: H \to \mathbb{R} \cup \{+\infty\}$, then $\text{VI}(T, A, B, S)$ is equivalent to the problem

$$\text{Find } u \in H \text{ such that}$$

$$\langle A(u) - B(u), v - S(u) \rangle \geq \varphi(S(u)) - \varphi(v), \quad \forall v \in H. \quad (1.1)$$

Inequalities like (1.1) have been studied in [14].

(ii) Let $K$ be a nonempty closed convex subset of $H$. If in (1.1), $\varphi := \psi + I_K$, where $\psi: H \to \mathbb{R}$ is a lower semicontinuous convex function, $I_K$ is the indicator function of the set $K$, $B$ is identically null, and $S$ is the identity mapping, then $\text{VI}(T, A, B, S)$ is equivalent to the following prob-
lem studied in [8, 13]:

\[
\text{Find } u \in K \text{ such that } \langle A(u), v - u \rangle \geq \psi(u) - \psi(v), \quad \forall v \in K. \tag{1.2}
\]

(iii) Let \( K \) be as in (ii). If in (i), \( \varphi := I_{K+m(u)} \) where \( m \) is a single valued mapping on \( H \), then \( VI(T, A, B, S) \) is equivalent to the general strongly nonlinear quasivariational inequality problem given by

\[
\langle A(u) - B(u), v - S(u) \rangle \geq 0 \quad \forall v \in K(u), \tag{1.3}
\]

where \( K(u) := K + m(u) \). These kinds of problems have been studied in [24, 29], for example.

Note that for \( S(u) = u \in K(u) \) for all \( u \in H \), problem (1.3) is equivalent to the so-called strongly nonlinear quasivariational inequality problem studied in [27].

If the operator \( B \) is independent of \( u \), that is, \( B(u) = f \), \( S \) is the identity mapping, and \( m = 0 \), then (1.3) is equivalent to the problem

\[
\text{Find } u \in K \text{ such that } \langle A(u) - f, v - u \rangle \geq 0, \quad \forall v \in K. \tag{1.4}
\]

Inequalities like (1.4) are known as the classical variational inequalities and have been extensively studied in the literature (see, for example, [9, 32] among others).

(iv) Let \( K \) be as in (ii). If \( B \) is identically null and \( \varphi := I_K \), then \( VI(T, A, B, S) \) is equivalent to the problem

\[
\text{Find } u \in K \text{ such that } \langle A(u), v - S(u) \rangle \geq 0, \quad \forall v \in K, \tag{1.5}
\]

which is known as the general variational inequality studied for example in [17].

(v) Let \( K \) be a closed convex cone of \( H \), and let \( K^* \) denotes its polar, i.e.,

\[
K^* := \{ u^* \in H | \langle u^*, u \rangle \geq 0 \forall u \in K \}. 
\]

If \( \varphi := I_K \), \( B \) is identically null, and \( S(K) \subset K \), then \( VI(T, A, B, S) \) is equivalent to the problem

\[
\text{Find } u \in K \text{ such that } A(u) \in K^* \quad \text{and} \quad \langle A(u), S(u) \rangle = 0, \tag{1.6}
\]
which is known as the implicit complementarity problem, introduced and studied by Bensoussan and Lions [5]. See also Docetta and Mosco [11] and the references cited therein. Nevertheless, such problems are encountered frequently in several fields of applied mathematics such as for instance, mechanics, economic equilibrium theory, and elasticity theory.

If $S$ is the identity mapping then problem (1.6) reduces to the so-called explicit complementarity problem, introduced by Karamardian [18]. This has been extensively studied in the literature (see, for example, [15]).

(vi) If the convex cone $K$ in (v) depends upon the solution $u$, then we have the general problem

$$\text{Find } u \in K(u) \text{ such that } A(u) \in K^*(u) \text{ and } \langle A(u), S(u) \rangle = 0. \tag{1.7}$$

Such problems are called the strongly nonlinear quasicomplementarity problem. In many applications, $K(u)$ has the form

$$K(u) = K + m(u),$$

where $m$ is as in (iii).

(vii) Let $J: H \to \mathbb{R}$ be a Gâteaux differentiable convex function such that $\nabla J(u) = A(u)$ for each $u$ in $H$, $B$ is identically null, and $S$ is the identity mapping. Then (1.1) is equivalent to the convex optimization problem

$$(J + \varphi)(\bar{u}) = \inf_{v \in H} (J + \varphi)(v).$$

This problem has been studied in [21, 7] among others.

According to a fixed point theorem [6], we obtain an approximate solution for $VI(T, A, B, S)$. More specifically, we study iterative algorithms in which an elementary step amounts to solving the generalized variational inclusion $VI(T, A, B, S)$, which consists in coupling an iterative method with a data perturbation. This is close to the approach found in [14], except that these authors prove the convergence by assuming that the map $S$ is strongly monotone and Lipschitz continuous. The assumption of Lipschitz continuous and strong monotonicity of the operator $A$ with respect to $S$ remains the key assumption in the convergence proofs presented hereafter.

A strong motivation for the study of this type of problem is its deep connections with nonlinear analysis, variational convergence, and its applications in areas such as optimization theory, mechanics, variational calculus, etc. For more details we refer to Duvaut and Lions [12], Mosco [20], Panagiotopoulos [30], and Isac [15] for the complementarity problems.
The paper is organized as follows: In Section 2 we recall some basic concepts. In Section 3 we give an existence result which includes, as a special case, some known result for the complementarity problems and we study a perturbed algorithm for solving $VI(T, A, B, S)$, obtained by coupling an iterative method with a data perturbation. Then we prove a convergence result. Section 4 is devoted to a new perturbed algorithm by using an equivalent formulation of problem $VI(T, A, B, S)$ as a fixed point problem. Sensitivity analysis is the purpose of Section 5.

2. PRELIMINARIES AND NOTATIONS

We recall some definitions of variational convergence and some results about maximal monotone operators theory drawn from Attouch [2] and Brezis [4].

For a multivalued operator $T: H \rightrightarrows H$, we denote by

$$D(T) := \{u \in H \mid T(u) \neq \emptyset\},$$

the domain of $T$,

$$R(T) := \bigcup_{u \in H} T(u),$$

the range of $T$,

$$\text{Graph}(T) := \{(u, u^*) \in H \times H \mid u \in D(T) \text{ and } u^* \in T(u)\},$$

the graph of $T$. Throughout the paper we identify operators with their graphs.

We recall that $T$ is monotone if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$ we have

$$\langle v^* - u^*, v - u \rangle \geq 0.$$

$T$ is maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

$T^{-1}$ is the operator defined by

$$v \in T^{-1}(u) \Leftrightarrow u \in T(v).$$

We recall that, for any $\lambda > 0$ the resolvent of index $\lambda$ of $T$ is defined by

$$J^T_\lambda := (I + \lambda T)^{-1},$$
if $T$ is maximal monotone then for any $\lambda > 0$, $J^T_{\lambda}$ is nonexpansive, i.e.,

$$|J^T_{\lambda}(u) - J^T_{\lambda}(v)| \leq |u - v|.$$  

The Yosida approximate of index $\lambda$ of $T$ is defined by

$$T_{\lambda} := \frac{1}{\lambda}(I - J^T_{\lambda}),$$

which is everywhere defined and Lipschitz continuous with Lipschitz constant $1/\lambda$.

Let $\{T^n | n \in \mathbb{N}\}$, $T$ be a sequence of maximal monotone operators. The sequence $\{T^n | n \in \mathbb{N}\}$ is said to be graph-convergent to $T$, and we write $T^n \xrightarrow{G} T$, if the following property holds: for every $(x, y) \in \text{Graph}(T)$, there exists a sequence $(x_n, y_n) \in \text{Graph}(T^n)$ such that $x_n \to x$ and $y_n \to y$ in $H$.

We recall that the following statements are equivalent [2, Proposition 3.60]:

1. $T^n \xrightarrow{G} T$;
2. $J^{T^n}_{\lambda}(x) \to J^T_{\lambda}(x)$ in $H$ for every $x \in H$ and $\lambda > 0$;
3. $J^{T^n}_{\lambda_0}(x) \to J^T_{\lambda_0}(x)$ in $H$ for every $x \in H$ and for some $\lambda_0 > 0$.

Let $f$ be a mapping from $H$ to $H$. $f$ is said to be

- monotone if
  $$\langle f(x) - f(y), x - y \rangle \geq 0 \quad \text{for all } x, y \in D(f),$$
- strictly monotone if $f$ is monotone and strict inequality holds for $x \neq y$,
- strongly monotone, if there exists a constant $\alpha > 0$ such that
  $$\langle f(x) - f(y), x - y \rangle \geq \alpha|x - y|^2 \quad \text{for all } x, y \in D(f),$$
- Lipschitz continuous, if there exists a constant $k > 0$ such that
  $$|f(x) - f(y)| \leq k|x - y|.$$

We recall that a strongly monotone operator is strictly monotone.

Considering the mappings $A, S: H \to H$ and $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$, we say that:

1. $A$ is a $\phi$-Lipschitz mapping with respect to $S$ if
   $$|A(u) - A(v)| \leq |S(u) - S(v)|\phi(|S(u) - S(v)|) \quad \forall u, v \in H,$$
(ii) \( A \) is a \( \psi \)-strongly monotone mapping with respect to \( S \) if

\[
\langle A(u) - A(v), S(u) - S(v) \rangle \geq |S(u) - S(v)|^2 \psi(|S(u) - S(v)|)
\]

\( \forall u, v \in H. \)

If in this definition, \( \phi \) and \( \psi \) are constant (\( \phi \equiv k > 0, \psi \equiv \alpha > 0 \)) then we say that \( A \) is a \( k \)-Lipschitz (respectively \( \alpha \)-strongly monotone) with respect to \( S \). Obviously if \( S(u) = u, \forall u \in H \), we obtain the classical definitions of a Lipschitz and a strongly monotone mapping.

We end this section by recalling a basic result due to Boyd and Wong [6] which will be useful:

**Theorem 2.1** (Boyd and Wong). Let \((X, d)\) be a complete metrically convex metric space. If for the mapping \( T: X \to X \) there is a mapping \( \varphi: P \to \mathbb{R}^+ \) (where \( P := \{d(x, y) \mid x, y \in X\} \)) satisfying:

1. \( d(T(x), T(y)) \leq \varphi(d(x, y)), \forall x, y \in X; \)
2. \( \varphi(t) < t \) for all \( t \in P \setminus \{0\}. \)

Then \( T \) has a unique fixed point.

### 3. AN EXISTENCE RESULT FOR THE GENERAL CLASS OF VARIATIONAL INCLUSIONS

In this section, we will extend ideas developed by G. Isac for the complementarity problems [16] to more general problems.

A natural connection between \( VI(T, A, B, S) \) and the fixed point theory will be used to prove some existence results. The following lemma is in this sense:

**Lemma 3.1.** \( \bar{u} \) is a solution of \( VI(T, A, B, S) \) if and only if \( \bar{u} \) is a fixed point of the map

\[
F(u) := u - S(u) + J^T_\lambda \left( S(u) - \lambda (A - B)(u) \right).
\]

**Proof.** Let \( \bar{u} \) be a fixed point of the map \( F \). Then

\[
S(\bar{u}) = J^T_\lambda \left( S(\bar{u}) - \lambda (A - B)(\bar{u}) \right).
\]

From the definition of the proximal mapping \( J^T_\lambda \), one has

\[
B(\bar{u}) - A(\bar{u}) \in T(S(\bar{u})).
\]

Thus \( \bar{u} \) is a solution of \( VI(T, A, B, S) \).
Remark 3.1. We note that when $T := \partial \varphi$, Lemma 3.1 is similar to Lemma 2.1 in [14].

Remark 3.2. If we consider the mappings $f, h : H \rightarrow H$ defined by

\[ f(x) := A(u), \quad \text{where } u \text{ is an arbitrary element of } S^{-1}(x), \]

and

\[ h(x) := B(u), \quad \text{where } u \text{ is an arbitrary element of } S^{-1}(x), \]

then problem $VI(T, A, B, S)$ is equivalent to the following one, denoted by $VI(T, f, h)$:

\[
\begin{cases}
\text{find } \overline{u} \in H \text{ such that } \\
0 \in f(\overline{u}) - h(\overline{u}) + T(\overline{u}).
\end{cases}
\]

We note that if $A$ is a $\phi$-Lipschitz (respectively a $\psi$-strongly monotone) mapping with respect to $S$, then $f$ is a $\phi$-Lipschitz (respectively a $\psi$-strongly monotone). Moreover, problem $VI(T, f, h)$ has a solution if and only if the mapping

\[ g(u) := J^T_\lambda \left( u + \lambda h(u) - \lambda f(u) \right) \]

has a fixed point.

Theorem 3.2. Suppose the following assumptions are satisfied:

(i) $A$ is a $\phi_1$-Lipschitz and a $\psi$-strongly monotone mapping with respect to $S$;

(ii) $B$ is a $\phi_2$-Lipschitz mapping with respect to $S$;

(iii) there exists $\lambda > 0$ such that

\[ \lambda^2 \left( \phi_1^2(t) + \phi_2^2(t) \right) < 2 \lambda \psi(t) < 1 + \lambda^2 \left( \phi_1^2(t) + \phi_2^2(t) \right), \quad \forall t \in \mathbb{R}^+. \]

Then the problem $VI(T, A, B, S)$ has at least one solution, which is unique when $S$ is one to one.

Proof. According to Remark 3.2, problem $VI(T, A, B, S)$ is equivalent to problem $VI(T, f, h)$. The proof will be complete if we show that the mapping $g$ has a fixed point. We have

\[
\begin{align*}
|g(u) - g(v)|^2 &= |J^T_\lambda (u - \lambda(f - h)(u)) - J^T_\lambda (v - \lambda(f - h)(v))|^2 \\
&\leq |u - \lambda(f - h)(u) - v + \lambda(f - h)(v)|^2 \\
&\leq |u - v - \lambda(f(u) - f(v))|^2 + \lambda^2|h(u) - h(v)|^2 \\
&\leq |(u - v) - \lambda(f(u) - f(v))|^2 + \lambda^2\phi_2^2(|u - v|)|u - v|^2.
\end{align*}
\]
Hence,
\[(u - v) - \lambda(f(u) - f(v))|^2 = |u - v|^2 - 2\lambda\langle f(u) - f(v), u - v \rangle + \lambda^2|f(u) - f(v)|^2.
\]
By assumptions (i) and Remark 3.1, we obtain
\[\|(u - v) - \lambda(f(u) - f(v))\|^2 \leq (1 - 2\lambda\psi(|u - v|))|u - v|^2.
\]
Let \(\varphi(t) := t\sqrt{1 - 2\lambda\psi(t) + \lambda^2(\phi_1^2(t) + \phi_2^2(t))}, \ t \geq 0.\) Using assumption (iii) we have \(\varphi(t) < t.\)

All assumptions of the Boyd and Wong Theorem being satisfied, we obtain the existence of an unique fixed point of \(g,\) which is a solution of problem \(VI(T, A, B, S)\) via Remark 3.2 and Lemma 3.1.

**Corollary 3.3.** Suppose that the following assumptions are satisfied:

(i) \(A\) is \(k\)-Lipschitz and \(\alpha\)-strongly monotone with respect to \(S;\)
(ii) \(B\) is \(\gamma\)-Lipschitz with respect to \(S.\)

Then the problem \(VI(T, A, B, S)\) has at least one solution, which is unique when \(S\) is one to one.

**Proof.** We have to check condition (iii) of Theorem 3.2. If we use a constant \(\alpha_1\) satisfying \(0 < \alpha_1 < \alpha,\) we may find a real number \(\lambda > 0\) such that \(0 < \lambda < 2\alpha_1/(k^2 + \gamma^2).\) Noticing that \(A\) is \(\alpha_1\)-strongly monotone with respect to \(S,\) so there exists \(\lambda > 0\) satisfying
\[\lambda^2k^2 + \lambda^2\gamma^2 < 2\alpha_1\lambda < \min(1 + \lambda^2k^2 + \lambda^2\gamma^2, 2\alpha\lambda).
\]
The result follows from Theorem 3.2.

3.1. A Perturbed Iterative Algorithm. In order to obtain an approximate solution of \(VI(T, A, B, S),\) we consider sequences \(\{u_n \mid n \in \mathbb{N}\}\) constructed according to the formula
\[
\begin{aligned}
\text{(A)} & \quad \text{Given } u_0 \in H \\
& \quad u_{n+1} := J_{\lambda}^{T_n}(u_n + \lambda h(u_n) - \lambda f(u_n)) + e_n,
\end{aligned}
\]
where \(\{e_n \mid n \in \mathbb{N}\}\) is a sequence of the element of \(H\) introduced to take into account a possible inexact computation, and \(\{T_n \mid n \in \mathbb{N}\}\) a sequence of maximal monotone operators approximating \(T\) in a specific sense.
Theorem 3.4. Suppose that the following statements are satisfied:

(i) $T^n$ graph-converge to $T$ ($T^n \xrightarrow{G} T$);
(ii) $A$ is $k$-Lipschitz and $\alpha$-strongly monotone with respect to $S$;
(iii) $B$ is a $\gamma$-Lipschitz continuous with respect to $S$;
(iv) $\lim_{n \to \infty} |e_n| = 0$.

Then the sequence $\{u_n\}_{n \in \mathbb{N}}$ generated by $(\mathcal{A})$ converges strongly to a solution $u$ of $VI(T, A, B, S)$.

Proof. According to Corollary 3.3, there exists a unique solution $u$ of $VI(T, f, h)$ which is a solution of $VI(T, A, S, H)$ and is unique when $S$ is one to one.

By setting $F(u) := u + \lambda h(u) - \lambda f(u)$, we have

$$|u_{n+1} - u| = |J_{\lambda^n}^T(F(u_n)) - J_{\lambda^n}^T(F(u)) + e_n|$$

$$\leq |J_{\lambda^n}^T(F(u_n)) - J_{\lambda^n}^T(F(u))| + |J_{\lambda^n}^T(F(u)) - J_{\lambda}^T(F(u))| + |e_n|$$

$$\leq |F(u_n) - F(u)| + |e_n|,$$

where $e_n := |J_{\lambda}^T(F(u)) - J_{\lambda}^T(F(u_n))| + |e_n|$ converges to 0 as $n \to \infty$. We have

$$|F(u_n) - F(u)| \leq |u_n - u - \lambda(f(u_n) - f(u))| + \lambda|h(u_n) - h(u)|.$$

Since $f$ is $k$-Lipschitz and $\alpha$-strongly monotone, we get

$$|u_n - u - \lambda(f(u_n) - f(u))| \leq \sqrt{1 - 2 \alpha \lambda + \lambda^2 k^2} |u_n - u|.$$

Using the Lipschitz continuity of $h$, we obtain

$$|u_{n+1} - u| \leq \left(\sqrt{1 - 2 \alpha \lambda + \lambda^2 k^2} + \lambda \gamma\right)|u_n - u| + e_n.$$  

By setting $\theta := \sqrt{1 - 2 \alpha \lambda + \lambda^2 k^2} + \lambda \gamma$, we finally obtain

$$|u_{n+1} - u| \leq \theta |u_n - u| + e_n.$$

Hence

$$|u_{n+1} - u| \leq \theta^{n+1} |u_0 - u| + \sum_{j=1}^{n} \theta^j e_{n+1-j},$$

$\theta < 1$ for $0 < \lambda < 2(\alpha - \gamma)/(k^2 - \gamma^2)$.

The result follows from Ortega and Rheinboldt [26], since $e_n \to 0$ as $n \to \infty$. \qed
Remark 3.3. We note that for problem (1.1), the iterative scheme (3.8) reduces to

\[ u_{n+1} = P_{K_n(u)} \left( u_n + \lambda h(u_n) - f(u_n) \right) + e_n, \]

where \( P \) stands for the projection operator of \( H \) into \( K_n(u) \) and \( \{K_n(u) \mid n \in \mathbb{N}\} \) is a sequence of closed convex subset of \( H \) approximating \( K(u) \).

If \( K_n(u) = K(u) = K + m(u) \), then (3.8) is equivalent to the algorithm

\[ u_{n+1} = m(u_n) + P_K \left( u_n - \lambda (f(u_n) - h(u_n)) - m(u_n) \right), \]

which has been studied in [27].

One can see that our general algorithm includes many previously known iterative methods as special cases.

Remark 3.4. We have proved the convergence of the iterative algorithm to a solution of the problem \( VI(T, A, B, S) \) without any condition on the map \( S \). However, if we want the uniqueness of the solution, we have to make some assumptions on \( S \), for example, \( S \) can be supposed strongly monotone and Lipschitz continuous (see [14, 24]).

Remark 3.5. If we consider problem (1.1), then the assumption \( T^n \rightarrow T \) is satisfied when the sequence \( \{\varphi_n \mid n \in \mathbb{N}\} \) converges in the sense of Mosco to \( \varphi \), that is:

- \( \forall u \in H, \forall \{u_n \mid n \in \mathbb{N}\} \) such that \( u_n \rightharpoonup u \), then \( \varphi(u) \leq \liminf_{n \to +\infty} \varphi_n(u_n) \),
- \( \forall u \in H, \exists \{u_n \mid n \in \mathbb{N}\} \) such that \( u_n \rightharpoonup u \) and \( \varphi(u) \geq \limsup_{n \to +\infty} \varphi_n(u_n) \).

4. A NEW ITERATIVE ALGORITHM

In this section we study the problem

\[ VI(T, f, h) \left\{ \begin{array}{l} \text{find } \bar{u} \in H \text{ such that} \\ 0 \in f(\bar{u}) - h(\bar{u}) + T(\bar{u}). \end{array} \right. \]

We will transform \( VI(T, f, h) \) into a fixed point problem. The following lemma is in this sense:

Lemma 4.1. If there exists a fixed point \( \bar{u} \in H \) of the map \( G \) defined by

\[ G(u) := J^T_A(u) + h(J^T_A(u)) - \lambda f(J^T_A(u)), \]

then \( J^T_A(\bar{u}) \) is a solution of \( VI(T, f, h) \).
Proof. If we write $u^* := J_\lambda^T(\bar{u})$, we have

$$G(\bar{u}) = u^* + \lambda h(u^*) - \lambda f(u^*).$$

Hence

$$G(\bar{u}) = \bar{u} - u^* = \lambda h(u^*) - \lambda f(u^*).$$

We have

$$u^* = J_\lambda^T(\bar{u}) \iff u^* + \lambda T(u^*) \supseteq \bar{u}$$

$$\iff \bar{u} - u^* \in \lambda T(u^*)$$

$$\Rightarrow \lambda h(u^*) - \lambda f(u^*) \in \lambda T(u^*)$$

$$\Rightarrow h(u^*) - f(u^*) \in T(u^*).$$

Thus $u^*$ is a solution of $VI(T, f, h)$.

Proof.

**Theorem 4.2.** If $f$ is $k$-Lipschitz and $\alpha$-strongly monotone and $h$ is $\gamma$-Lipschitz, then $G$ has a unique fixed point $\bar{u}$, and $J_\lambda^T(\bar{u})$ is the solution of $VI(T, f, h)$.

**Proof.** We have shown that there exists a real number $\lambda > 0$ such that

$$\lambda^2(k^2 + \gamma^2) < 2\lambda \alpha_1 < \min(1 + \lambda^2 k^2 + \lambda^2 \gamma^2, 2\lambda \alpha),$$

where $0 < \alpha_1 < \alpha$. We have

$$|G(x) - G(y)|^2 = |J_\lambda^T(x) + \lambda h(J_\lambda^T(x)) - \lambda f(J_\lambda^T(x)) - J_\lambda^T(y) - \lambda h(J_\lambda^T(y)) + \lambda f(J_\lambda^T(y))|^2.$$

By setting $x^* := J_\lambda^T(x)$ and $y^* := J_\lambda^T(y)$, we can write

$$|G(x) - G(y)|^2 \leq |x^* - y^* - \lambda(f(x^*) - f(y^*))|^2 + \lambda^2|h(x^*) - h(y^*)|^2$$

$$= |x^* - y^*|^2 - 2\lambda\langle x^* - y^*, f(x^*) - f(y^*)\rangle$$

$$+ \lambda^2[f(x^*) - f(y^*)]^2 + \lambda^2[h(x^*) - h(y^*)]^2.$$

Hence $|G(x) - G(y)|^2 \leq (1 - 2\lambda \alpha_1 + \lambda^2(k^2 + \gamma^2))|x^* - y^*|^2$.

We finally obtain, since $J_\lambda^T$ is nonexpansive

$$|G(x) - G(y)| \leq \theta|x - y|,$$

where $\theta := \sqrt{1 - 2\lambda \alpha_1 + \lambda^2(k^2 + \gamma^2)}$, belongs to $]0, 1[.$
The result follows from Theorem 2.1.

In order to obtain an approximate solution of $VI(T, f, h)$, we can apply a successive approximation perturbed method to the problem of solving $G(w) = u$. The result procedure is

$$
\begin{align*}
\text{(B)} \quad & \text{Given } u_0 \in H \\
& u_{n+1} = J^T_T(u_n) + \lambda h(J^T_T(u_n)) - \lambda f(J^T_T(u_n)) + e_n,
\end{align*}
$$

where the sequence $\{T^n \mid n \in \mathbb{N}\}$ approximates $T$ and $e_n$ are satisfied:

**Theorem 4.3.** Assume that the following conditions are satisfied:

(i) $T^n$ graph-converges to $T$ ($T^n \to T$);

(ii) $f$ is $k$-Lipschitz and $\alpha$-strongly monotone;

(iii) $h$ is $\gamma$-Lipschitz;

(iv) $e_n \to 0$ as $n \to +\infty$.

Then the sequence $\{u_n \mid n \in \mathbb{N}\}$ generated by (B) converges strongly to the unique fixed point $\bar{u}$ of $G$, and $J^T_T(\bar{u})$ solves $VI(T, f, h)$.

**Proof.** According to Theorem 4.2, $G$ has a unique fixed point $\bar{u}$. We have

$$
|u_{n+1} - \bar{u}| = |J^T_T(u_n) + \lambda h(J^T_T(u_n)) - \lambda f(J^T_T(u_n)) - J^T_T(\bar{u})|
$$

$$
= |\lambda h(J^T_T(u_n)) - \lambda f(J^T_T(u_n)) + e_n|
$$

$$
\leq |J^T_T(u_n) - J^T_T(\bar{u})| - \lambda |f(J^T_T(u_n)) - f(J^T_T(\bar{u}))| + \gamma |h(J^T_T(u_n)) - h(J^T_T(\bar{u}))| + |e_n|
$$

$$
\leq \left( \sqrt{1 - 2\lambda \alpha + \lambda^2 k^2} + \lambda \gamma \right) |J^T_T(u_n) - J^T_T(\bar{u})|
$$

$$
+ (1 + \lambda \gamma + \lambda k) |J^T_T(\bar{u}) - J^T_T(\bar{u})| + |e_n|
$$

$$
\leq \theta |J^T_T(u_n) - J^T_T(\bar{u})| + \varepsilon_n,
$$

where $\theta := \sqrt{1 - 2\lambda \alpha + \lambda^2 k^2} + \lambda \gamma$, and

$$
\varepsilon_n := (1 + \lambda \gamma + \lambda k) |J^T_T(\bar{u}) - J^T_T(\bar{u})| + |e_n|,
$$

which converges to 0 as $n \to \infty$.

Hence,

$$
|u_{n+1} - \bar{u}| \leq \theta |J^T_T(u_n) - J^T_T(\bar{u})| + \varepsilon_n.
$$
Since \( J_\lambda^{T^\nu} \) is nonexpansive, we get
\[
|u_{n+1} - \bar{u}| \leq \theta |u_n - \bar{u}| + \varepsilon_n.
\]
By a similar argument used in the proof of Theorem 3.4, the result follows.

\[\square\]

Remark 4.1. We note that for \( T := \partial I_K \), where \( I_K \) is the indicator function of a nonempty convex set \( K \), the iterative scheme \((\mathscr{A})\) reduces to
\[
\begin{align*}
u_{n+1} := P_K(u_n) + \lambda h(P_K(u_n)) - \lambda f(P_K(u_n)) + \varepsilon_n,
\end{align*}
\]
where \( P_K \) stands for the projection operator of \( H \) into \( K \).

Let \( \{K_n \mid n \in \mathbb{N}\} \) be an increasing sequence of closed convex subset of \( H \) such that \( K = \bigcup_{n \in \mathbb{N}} K_n \), then assumption (i) of Theorem 4.3 is satisfied since \( I_{K_n} \) converges in the Mosco sense to \( I_K \).

4.1. The Case Where the Parameter \( \lambda > 0 \) Is Perturbed. To begin with, let us review some standard results from operator theory and convex analysis.

For any subset \( C \) of \( H \) and \( x \) in \( H \), \( d(x, C) := \inf_{y \in C} |x - y| \) is the distance from \( x \) to \( C \) (we set \( d(x, \emptyset) = +\infty \)).

For any \( \rho \geq 0 \), \( \rho B \) denotes the closed ball of radius \( \rho \), and \( C_\rho := C \cap \rho B \). For \( C, D \subset H \), the excess function of \( C \) on \( D \) is defined as
\[
e(C, D) := \sup_{x \in C} d(x, C),
\]
with the natural convention that \( e(\emptyset, D) = 0 \).

For any \( \rho \geq 0 \), the \( \rho \)-Hausdorff distance between \( C \) and \( D \) is given by
\[
\text{haus}_\rho(C, D) := \max\{e(C_\rho, D); e(D_\rho, C)\}.
\]
For more details about the \( \rho \)-Hausdorff distance, we refer to [3].

A convergence notion is naturally attached to the notion of \( \rho \)-Hausdorff distance. A sequence \( \{T^n \mid n \in \mathbb{N}\} \) of maximal monotone operators is convergent to an operator \( T \), if

for each \( \rho \geq 0 \),
\[
\lim_{n \to \infty} \text{haus}_\rho(\text{Graph}(T^n), \text{Graph}(T)) = 0.
\]

Another type of distance between maximal monotone operators is given by
\[
d_{\lambda, \rho}(T, T') := \sup_{|x| \leq \rho} |J_\lambda^T x - J_\lambda^T' x| = \lambda \sup_{|x| \leq \rho} |T_\lambda x - T'_\lambda x|.
\]
In the sequel, unless specifically stated, we suppose that the sequence of non-negative real numbers \( \{\lambda_n \mid n \in \mathbb{N}\} \) is bounded as follows:

\[ 0 < \lambda \leq \lambda_n \leq \bar{\lambda} < +\infty. \]

**Lemma 4.4 [22].** Let \( \{T^n \mid n \in \mathbb{N}\} \), \( T \) be a sequence of maximal monotone operators, for \( \rho \geq 0 \), the following estimate holds,

\[ d_{\lambda_n}(T, T^n) \leq (2 + \bar{\lambda}) \text{baru}_{\rho}(T, T^n), \]

where \( \rho' := \max (\rho + \bar{\lambda}^{-1}J^*_A(0) ; \bar{\lambda}^{-1}(\rho + \bar{\lambda}^{-1}J^*_A(0))) \).

This section is concerned with the perturbed iterative method for solving the problem \( \text{VI}(T, f, H) \)

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0 \in H \\
u_{n+1} := J^{T^n}_{\lambda_n}(u_n) + \lambda_n h(J^{T^n}_{\lambda_n}(u_n)) - \lambda_n f(J^{T^n}_{\lambda_n}(u_n)) + e_n.
\end{array} \right.
\end{align*}
\]

The case \( f \) and \( h \) identically null, the iterative procedure above is nothing else but an approximation method combining a data perturbation by variational convergence with the proximal point algorithm, which has been already treated in the literature (see Refs. [22, 33, 21] among others.

**Theorem 4.5.** Assume that the following statements are satisfied:

(i) \( f \) is \( k \)-Lipschitz and \( \alpha \)-strongly monotone;

(ii) \( h \) is \( \gamma \)-Lipschitz;

(iii) \( \forall \rho \geq 0, \lim_{n \to +\infty} \text{baru}_{\rho}(T, T^n) = 0; \)

(iv) \( \lim_{n \to +\infty} |e_n| = 0; \)

then the sequence \( \{u_n \mid n \in \mathbb{N}\} \) generated by (\( \Theta \)) converges strongly to the unique fixed point of \( G \).

**Proof.** By setting \( u^*_n := J^{T^n}_{\lambda_n}(u_n), \hat{u}_n := J^{T^n}_A(u), \) and \( \bar{u}_n := J^{T^n}_{\lambda_n}(u) \), we have

\[
|u_{n+1} - u| = |u^*_n + \lambda_n h(u^*_n) - \lambda_n f(u^*_n) - \bar{u}_n - \lambda_n h(\bar{u}_n) + \lambda_n f(\bar{u}_n) + e_n|
\]

\[
\leq |u^*_n - \hat{u}_n| + \lambda_n |f(u^*_n) - f(\hat{u}_n)| + \lambda_n |h(u^*_n) - h(\hat{u}_n)|
\]

\[
+ |\hat{u}_n - \bar{u}_n| + \lambda_n |h(\hat{u}_n) - h(\bar{u}_n)|
\]

\[
\leq \sigma (\lambda_n) |u^*_n - \bar{u}_n| + (1 + \lambda_n \gamma + \lambda_n k) \left| J^{T^n}_{\lambda_n}(u) - J^{T^n}_{\lambda_n}(u) \right| + |e_n|. \]
where \( \sigma(\lambda) := (1 - 2\alpha\lambda + \lambda^2)k^2 + \lambda\gamma \). Since \( J_{\lambda_n}^{T^n} \) is nonexpansive, we obtain
\[
|u_{n+1} - u| \leq \sigma(\lambda_n)|u_n - u| + (1 + \tilde{\lambda}(\gamma + k))(J_{\lambda_n}^{T^n}(u) - J_{\lambda_n}(u)) + |e_n|.
\]
We have
\[
\left| J_{\lambda_n}^{T^n}(u) - J_{\lambda_n}(u) \right| \leq d_{\lambda_n,\rho}(T^n, T), \quad \text{with} \ \rho := |u|,
\]
Using Lemma 4.4 we get
\[
|u_{n+1} - u| \leq \theta|u_n - u| + (1 + \tilde{\lambda}(\gamma + k))(2 + \tilde{\lambda})\text{haus}_\rho(T, T^n) + |e_n|,
\]
where \( \theta := \max(\sigma(\lambda), \sigma(\tilde{\lambda})) \).
By setting \( e_n := (1 + \tilde{\lambda}(\gamma + k))(2 + \tilde{\lambda})\text{haus}_\rho(T, T^n) + |e_n| \), we can write
\[
|u_{n+1} - u| \leq \theta^{n+1}|u_0 - u| + \sum_{j=1}^n \theta^j|e_{n+1-j}|
\]
The assumption \( \theta < 1 \) is obtained if
\[
\tilde{\lambda} < \frac{2(\alpha - \gamma)}{k^2 - \gamma^2}.
\]
Assumptions (iii) and (iv) imply that \( e_j \to 0 \) as \( j \to +\infty \).
The result follows from Ortega and Rheinboldt [26].

5. SENSITIVITY ANALYSIS FOR VARIATIONAL INCLUSIONS

We now consider the general parametric variational inclusion,
\[
\text{Find } u \in H \text{ such that } 0 \in A(u, \omega) + T(S(u), \omega), \quad (5.1)
\]
where \( \omega \in \Omega \) is an open subset of \( H \), and \( A(u, \omega) \) and \( T(u, \omega) \) are given operators defined on \( H \times \Omega \) which take values in \( H \).
If \( T := \partial \varphi \), where \( \varphi \) is a convex lower semicontinuous function defined on \( H \times \Omega \), then problem (5.1) is equivalent to finding \( u \in H \) such that
\[
\langle A(u, \omega), v - S(u) \rangle \geq \varphi(S(u), \omega) - \varphi(v, \omega) \quad \forall (v, \omega) \in H \times \Omega.
\]
If \( \varphi(\cdot, \omega) := I_{K_\omega} \), where \( \{K_\omega \mid \omega \in \Omega \} \) is a family of closed convex subsets of \( H \), then problem (5.2) is equivalent to finding \( u \in K_\omega \) such that

\[
\langle A(u, \omega), v - S(u) \rangle \geq 0 \quad \forall v \in K_\omega.
\] (5.3)

We note that for \( S := 1d \), problems like (5.3) has been studied by Dafermos [10].

Sensitivity analysis of variational inequalities is now the object of great attention for many engineers and mathematicians. Several useful theories have now been established by Tobin [31], Kyparisis [19], and Dafermos [10], among others. We mainly follow the ideas and technique of Dafermos, extended by Noor [25] for a class of quasivariational inequalities. The purpose of this section is the analysis of the perturbed solution of the parametric variational inclusion (5.1).

We assume that for some \( \bar{\omega} \in \Omega \), there exists a solution \( \bar{u} \). We want to investigate conditions which imply uniqueness of the perturbed solution \( u(\omega) \) near \( \bar{u} \), for each \( \omega \) in a neighborhood of \( \bar{\omega} \) and those conditions under which the function \( u(\omega) \) is continuous or Lipschitz continuous. We assume that \( \mathcal{B} \) is the closure of a ball in \( H \) centered at \( \bar{u} \). Here we are only interested in the case that the solutions \( u(\omega) \) lie in the interior of \( \mathcal{B} \). For this we suppose that the mappings \( A(\cdot, \omega) \) and \( T(\cdot, \omega) \) are defined on \( \mathcal{B} \times \Omega \). We note \( T_{\omega}(\cdot) \) for \( T(\cdot, \omega) \).

Via Remark 3.2, \( u(\omega) \) is a solution of (5.1) iff \( u(\omega) \) is a fixed point of the map

\[
K(u, \omega) := J_{t_\omega}(u - \lambda f(u, \omega)).
\]

To obtain a fixed point of the map \( K(u, \omega) \), we can apply Theorem 3.2. However, we do not have the uniqueness of the solution \( u(\omega) \) for each \( \omega \). Therefore we have to make some supplementary conditions on the map \( S \). As proposed by Noor, when \( S \) is Lipschitz continuous and strongly monotone and when \( \lambda \) satisfies some conditions (see Ref. [25]) we have the uniqueness. If these hypotheses are not verified, the following results are in this sense.

The following concept will be useful: An operator \( A(\cdot, \omega) \) defined on \( \mathcal{B} \times \Omega \) is said to be:

(a) a locally \( \psi \)-strongly monotone mapping with respect to \( S \) if

\[
\langle A(u, \omega) - A(v, \omega), S(u) - S(v) \rangle \geq |S(u) - S(v)|^2 \psi(|S(u) - S(v)|)
\]

\( \forall \omega \in \Omega, \ u, v \in \mathcal{B} \);
LEMMA 5.1. Assume that the following assumptions are satisfied:

(i) $A(u, \omega)$ defined on $\mathcal{B} \times \Omega$ is a locally $\psi$-strongly monotone mapping with respect to $S$;

(ii) $S$ is an expansive mapping, i.e.,
$$\exists \rho \geq 1: |S(u) - S(v)| \geq \rho |u - v| \quad \forall u, v \in \mathcal{B};$$

(iii) $|A(u, \omega) - A(v, \omega)| \leq |u - v| \psi(|S(u) - S(v)|);

(iv) There exists $\lambda > 0$ such that
$$\lambda^2 \psi^2(t) < 2 \lambda \psi(t) < 1 + \lambda^2 \psi^2(t) \quad \forall t \in \mathbb{R}^+.$$ 

Then problem (5.1) has a unique solution, denoted by $u(\omega)$. Consequently, $u(\overline{\omega}) = \overline{u} = K(u(\overline{\omega}), \overline{\omega})$.

Remark 5.1. If $\phi$ and $\psi$ are constant, i.e., $\phi = k > 0$, $\psi = \gamma > 0$, then condition (iv) is always satisfied.

We recall that a mapping $h: H \to H$ is said to be accretive iff
$$|x - y| \leq |(x - y) + \rho (h(x) - h(y))|, \quad \forall x, y \in H, \rho \geq 0.$$ 

Also, $U: H \to H$ is said to be pseudo-contractive iff
$$|x - y| \leq |(1 + \rho)(x - y) - (U(x) - U(y))|, \quad \forall x, y \in H, \rho > 0.$$ 

We recall this result due to Kato and Browder (see Ref. [16]): if $g := \text{Id} - U$, then the mapping $U$ is pseudo-contractive iff $g$ is accretive.

LEMMA 5.2. Assume that (i) and (iii) in Lemma 5.1 are satisfied. If in addition:

(a) $S - \rho \text{Id}$ is accretive for some $\rho > 1$,

(b) $\exists \lambda > 0: (\lambda^2/\rho) \psi^2(t) < 2 \lambda \psi(t) < \rho + \lambda^2/\rho$.

Then the problem (5.1) has a unique solution $u(\omega)$ and $u(\overline{\omega}) = \overline{u} = K(u(\overline{\omega}), \overline{\omega})$.

Proof. This follows the one given in [16] for the implicit complementarity problems.
**Lemma 5.3.** If $A(\bar{u}, \omega)$ and the map $\omega \mapsto J^T_k(\bar{u} - \lambda f(\bar{u}, \omega))$ are continuous (resp. Lipschitz continuous) in $\omega$ at $\bar{\omega}$, then the function $u(\omega)$ solution of (5.1) is continuous (resp. Lipschitz continuous) at $\omega = \bar{\omega}$.

**Proof.** For $\omega \in \Omega$, using Lemma 5.1 or Lemma 5.2 and the triangle inequality, we have

$$|u(\omega) - u(\bar{\omega})| = |K(u, \omega) - K(\bar{u}, \bar{\omega})|$$

$$\leq |K(u(\omega), \omega) - K(u(\bar{\omega}), \omega)|$$

$$+ |K(u(\bar{\omega}), \omega) - K(\bar{u}, \bar{\omega})|$$

$$\leq \sigma(|u - \bar{u}|) |u(\omega) - u(\bar{\omega})|$$

$$+ |J^T_k(\bar{u} - \lambda f(\bar{u}, \omega)) - J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega}))|$$

$$+ |J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega})) - J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega}))|$$

$$\leq \sigma(|u - \bar{u}|) |u(\omega) - u(\bar{\omega})| + \lambda |f(\bar{u}, \omega) - f(\bar{u}, \bar{\omega})|$$

$$+ |J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega})) - J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega}))|,$$

where $\sigma(t) := \sqrt{1 - 2\lambda \psi(t)} + \lambda \phi^2(t)$.

Hence,

$$|u(\omega) - u(\bar{\omega})| \leq \frac{1}{1 - \sigma(|u - \bar{u}|)}$$

$$\times |J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega})) - J^T_k(\bar{u} - \lambda f(\bar{u}, \bar{\omega}))|$$

$$+ \frac{\lambda}{1 - \sigma(|u - \bar{u}|)} |f(\bar{u}, \omega) - f(\bar{u}, \bar{\omega})|.$$

We assume $\lambda$ sufficiently small so that $\bar{u} - \lambda f(\bar{u}, \bar{\omega}) \in \mathcal{B}$. Since $A(\bar{u}, \omega)$ is continuous (resp. Lipschitz continuous) in $\omega$ at $\bar{\omega}$ then $f(\bar{u}, \omega)$ is continuous (resp. Lipschitz continuous) in $\omega$ at $\bar{\omega}$. The required result follows from the last inequality.

**Remark 5.2.** Consider the case when $T_\omega = \partial \varphi_\omega$, where $\{\varphi_\omega : \omega \in \Omega\}$ is a sequence of lower semicontinuous convex and proper functions, then it would be desirable to replace the continuity assumption of the proximal mapping, defined by

$$\omega \mapsto J^T_k(\bar{\tau})$$ for a fixed $\bar{\tau}$ in $\mathcal{B}$,

in Lemma 5.3 by conditions bearing directly on $\{\varphi_\omega : \omega \in \Omega\}$. The next result is in that direction.
The following definition will be useful.

**Definition 5.1.** A parametrized family \( \{ \varphi_\omega \mid \omega \in \Omega \} \) is said to be epi-continuous at \( \omega = \bar{\omega} \), if for every converging sequence \( \omega_n \to \bar{\omega} \), the following properties are satisfied for every \( \nu \in H \):

(i) there exists a sequence \( \{ v_n \mid n \in \mathbb{N} \} \) converging to \( \nu \) such that

\[
\varphi_\omega(v) \geq \limsup_{n \to +\infty} \varphi_{\omega_n}(v_n);
\]

(ii) for every sequence \( \{ v_n \mid n \in \mathbb{N} \} \) converging to \( \nu \)

\[
\varphi_{\omega_n}(v) \leq \liminf_{n \to +\infty} \varphi_{\omega_n}(v_n).
\]

**Proposition 5.4.** Assume that the family \( \{ \varphi_\omega \mid \omega \in \Omega \} \) is epi-continuous at \( \omega = \bar{\omega} \), then the map \( \omega \mapsto J_\lambda^\nu(v) \) is continuous at \( \omega = \bar{\omega} \).

**Proof.** This follows by applying Theorem 3.26 in [2].

**Lemma 5.5.** If the assumptions of Lemma 5.1 or 5.2 and Lemma 5.3 hold, then there exists a neighborhood \( \mathcal{N} \subset \Omega \) of \( \bar{\omega} \) such that for \( \omega \in \mathcal{N} \), \( u(\omega) \) is the unique solution of the parametric variational inclusion (5.1) in the interior of \( \mathcal{B} \).

**Proof.** This is similar to the one of Lemma 2.5 in [10].

We now state the main result of this section:

**Theorem 5.6.** Let \( \bar{u} \) be the solution of the parametric variational inclusion (5.1) at \( \bar{\omega} \). Assume that assumptions of Lemma 5.1 or those of Lemma 5.2 are satisfied. If the operator \( A(\bar{u}, \omega) \) and the map \( \omega \mapsto J_\lambda^\nu(\bar{u} - \lambda f(\bar{u}, \bar{\omega})) \) are continuous (resp. Lipschitz continuous) at \( \omega = \bar{\omega} \), then there exists a neighborhood \( \mathcal{N} \subset \Omega \) of \( \bar{\omega} \) such that for every \( \omega \in \mathcal{N} \), the parametric variational inclusion (5.1) has a unique solution in the interior of \( \mathcal{B} \) denoted by \( u(\omega) \), \( u(\bar{\omega}) = \bar{u} \), and \( u(\omega) \) is continuous (resp. Lipschitz continuous) at \( \omega = \bar{\omega} \).

**References**


