JOURNAL OF FUNCTIONAL ANALYSIS **139**, 261–300 (1996)
ARTICLE NO. 0086

Lipschitz Algebras and Derivations of von Neumann Algebras

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Received October 12, 1994

Let $\mathscr{M}$ be an abelian von Neumann algebra and E an abelian operator $\mathscr{M}$-bimodule. Then the domain of any ultraweakly closed derivation from $\mathscr{M}$ into E is a Lipschitz algebra. Conversely, every Lipschitz algebra can be realized as such

This paper is the result of an attempt to understand the role of Lipschitz algebras in Alain Connes' theory of noncommutative metric spaces (see [22], Chapter VI). We found that the class of Lipschitz algebras is identical to the class of domains of unbounded derivations of abelian von Neumann algebras (Theorems 9 and 16). The exact statement of this result requires the notion of measurable metric space (see Section I) and a definition of unbounded derivations which is appropriate to the von Neumann algebra context (see Section II).

The relation to noncommutative metric theory arises in two ways. First, Lipschitz algebras are dual to metric spaces in much the same way that abelian C^*-algebras are dual to topological spaces ([77]; see Section I below), thus creating an analogy with noncommutative topology. Second, unbounded derivations also arise in Connes' theory, although our approach is slightly different (see Section V). In Section V we discuss the implications of our results for the theory of noncommutative metric spaces.

We also prove some miscellaneous other results about domains of derivations of von Neumann algebras, in Section II. Although there is a large literature on unbounded derivations of C^*-algebras (e.g. see the monographs [9] and [68], the survey [65], and the several survey papers in [56] and [57]), relatively little attention has been paid to the von Neumann algebra case (but see [10], [11], [13], [14]). Presumably this is because the main examples of unbounded derivations involve

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0022-1236/96 $18.00

C^*-algebras. Nonetheless, it may be possible to reformulate some of these examples in a von Neumann algebra setting. We discuss this issue in Section VI and we use our conclusions to analyze the noncommutative torus as a noncommutative metric space.

Briefly, the contents are as follows. Section I—review of Lipschitz algebras. Section II—basic definitions and general results on domains of unbounded derivations of von Neumann algebras. Section III—exhibition of Lipschitz algebras as domains. Section IV—characterization of domains as Lipschitz algebras. Section V—implications for noncommutative metric theory. Section VI—C^*-algebra versus von Neumann algebra examples; illustration with the noncommutative torus.

I owe several intellectual debts. My postgraduate advisor, Charles Akemann, read several versions of this manuscript and offered invaluable advice; in particular, the C*-algebra versus von Neumann algebra issue discussed in Section VI is largely his contribution. My graduate advisor, William Bade, provided much-needed support and encouragement of my prior work on Lipschitz spaces on which this paper is based. Jeremy Kahn directed me to [22] and Marc Rieffel directed me to [63], [64], and [59]; these (especially [22] and [64]) were the starting point for the work described here. Marc Rieffel first suggested to me that there might be a noncommutative generalization of Lipschitz algebras, and he also commented on a very early version of this manuscript. Finally, Thomas Wolff sent me a clever proof of a theorem about the Fourier series of Lipschitz functions on the torus, which is used in Section VI.

I. Lipschitz Algebras

Lipschitz algebras play a major role in this paper. A reasonably rich literature exists on this subject; see [1], [2], [8], [19], [24], [26], [27], [32], [33], [35], [37], [38], [39], [40], [41], [42], [43], [50], [52], [53], [54], [60], [66], [67], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78].

A general survey of Lipschitz spaces is forthcoming [79]. In the meantime, we present here a brief description of some of the main results of the subject.

Lip *and* Lip_0

We begin by defining the *Lipschitz algebras* $\mathrm{Lip}(X)$ and $\mathrm{Lip}_0(X)$. For any metric space X, $\mathrm{Lip}(X)$ is the space of all bounded complex-valued Lipschitz functions on X, with norm

$$\|f\|_L = \max(L(f), \|f\|_\infty).$$

Here $L(f)$ denotes the Lipschitz number of f,

$$L(f) = \sup\{|f(x) - f(y)|/\rho(x, y) : x, y \in X, x \neq y\}.$$

If X has a distinguished "base point" e and the diameter of X is finite, then $\mathrm{Lip}_0(X)$ is defined to be the space of all complex-valued Lipschitz functions on X which vanish at e, with norm $L(\cdot)$.

$\mathrm{Lip}(X)$ and $\mathrm{Lip}_0(X)$ are both Banach algebras in the weak sense that there exists $C \geqslant 1$ such that $\|fg\| \leqslant C \|f\| \|g\|$ for all f, g. The norms can be modified so that $C = 1$, but doing so has some unwanted consequences and no real benefits.

Every $\mathrm{Lip}(X)$ is isometrically isomorphic to some $\mathrm{Lip}_0(Y)$, whereas $\mathrm{Lip}_0(X)$ is isometrically isomorphic to some $\mathrm{Lip}(Y)$ precisely if it has a multiplicative unit which is the greatest element of the real part of its unit ball ([77], Proposition 1). Thus, Lip spaces are Lip_0 spaces with well-behaved units. Because of this it is most convenient to develop the general theory for Lip_0 spaces, and only specialize to Lip spaces when necessary.

Throughout this section X and Y are assumed to be complete finite-diameter metric spaces, unless we state otherwise. Completing a metric space does not change the class of Lipschitz functions in any way, so there is no harm in consistently assuming completeness. Less natural is the restriction to finite diameters, which is needed to make the multiplication in $\mathrm{Lip}_0(X)$ continuous; because of this restriction the class of Lip_0 spaces is, for instance, not closed under the formation of infinite direct sums. It seems likely that the theory of Lip_0 algebras can be extended in some way to include infinite diameter metric spaces, but this has not been attempted.

The Arens–Eells Space

For every X the space $\mathrm{Lip}_0(X)$ has a predual, the *Arens-Eells space* $\mathrm{AE}(X)$ ([1], Proposition 1). This Banach space is concretely defined in [1] and it may also be abstractly characterized by the following universal property ([79], Proposition III.1.4): $\mathrm{AE}(X)$ isometrically contains X (identifying e with 0), and for every Banach space E and Lipschitz map $f: X \to E$ such that $f(e) = 0$ there is a unique bounded linear extension $\tilde{f}: \mathrm{AE}(X) \to E$, which satisfies $\|\tilde{f}\| = L(f)$.

On norm bounded subsets of $\mathrm{Lip}_0(X)$, the weak* topology is just the topology of pointwise convergence ([39], Theorem 4.3). By the Krein–Smulian theorem this is in practice generally a sufficient description of the weak* topology.

Duality between X and $\mathrm{Lip}_0(X)$

The metric space X can be recovered from $\mathrm{Lip}_0(X)$ as the set of its weak* continuous complex homomorphisms, with metric inherited from the dual space $\mathrm{Lip}_0(X)^*$ ([77], Theorem A). Furthermore, the set of Lipschitz functions $f: X \to Y$ which preserve base point is in 1–1 correspondence with the set of weak* continuous homomorphisms $T: \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$, via the formula $T(g) = g \circ f$ ([77], Corollary A).

The weak* closed ideals of $\mathrm{Lip}_0(X)$ are in 1–1 correspondence with the closed subsets of X containing e, by pairing each ideal with its hull ([77], Theorem C). Finally, for every weak* closed self-adjoint subalgebra $\mathscr{A}$ of $\mathrm{Lip}_0(X)$, there is a metric space Y and a surjective map $f: X \to Y$ such that composition with f defines an isometric isomorphism from $\mathrm{Lip}_0(Y)$ onto $\mathscr{A}$ ([77], Corollary B). This last result is proved with the aid of a Stone–Weierstrass type theorem ([77], Theorem B) which states that any self-adjoint subalgebra of $\mathrm{Lip}_0(X)$ which separates points uniformly is weak* dense in $\mathrm{Lip}_0(X)$.

These results imply that every weak* closed self-adjoint subalgebra of $\mathrm{Lip}_0(X)$ is isometrically isomorphic to some $\mathrm{Lip}_0(Y)$, and the quotient of $\mathrm{Lip}_0(X)$ by any weak* closed ideal is isomorphic to some $\mathrm{Lip}_0(Y)$. The latter isomorphism is isometric on self-adjoint elements ([77], Corollary C).

Little Lipschitz Spaces

If X is compact it is also interesting to consider the *little Lipschitz space* $\mathrm{lip}_0(X)$, which is the subspace of $\mathrm{Lip}_0(X)$ consisting of those functions with the property that

$$|f(x) - f(y)|/\rho(x, y) \to 0 \qquad \text{as} \quad \rho(x, y) \to 0.$$

This may be thought of as a “local flatness” condition.

$\mathrm{lip}_0(X)$ is mainly of interest when it separates points uniformly ([33], [74]); by ([74], Theorem 3.4) every $\mathrm{lip}_0(X)$ is isometrically isomorphic to some $\mathrm{lip}_0(Y)$ which has this property. Under this hypothesis the double dual of $\mathrm{lip}_0(X)$ is isometrically isomorphic to $\mathrm{Lip}_0(X)$ ([39], Theorem 4.7; [2], Theorem 3.5).

Still assuming that $\mathrm{lip}_0(X)$ separates points uniformly, we have analogues of all but the last of the facts about $\mathrm{Lip}_0(X)$ stated in the preceding subsection, now replacing hypotheses of weak* continuity and closure with norm continuity and closure ([70], Proposition 2.1; [79], Corollary 5.4; [74], Theorem 3.4; [76], Theorem 1.4; [70], Corollary 4.3). The question whether the quotient of $\mathrm{lip}_0(X)$ by any norm closed ideal is isomorphic to some $\mathrm{lip}_0(Y)$, isometrically on the self-adjoint part, is apparently open.

Measurable Metric Spaces

It is also useful to have the following more general version of a metric. Let (X, μ) be a σ-finite measure space and let Ω be the collection of positive measure subsets of X. Then a *measurable pseudometric* on X is a map $\rho\colon \Omega^2 \to \mathbf{R}^+$ such that $\rho(A, B) = \rho(A', B)$ if A and A' differ by a null set and such that

$$\rho(A, A) = 0$$

$$\rho(A, B) = \rho(B, A)$$

$$\rho\left(\bigcup_{n=1}^{\infty} A_n, B\right) = \inf_n (\rho(A_n, B))$$

$$\rho(A, C) \leqslant \sup_{B' \subset B} (\rho(A, B') + \rho(B', C))$$

for all $A, B, C, A_n \in \Omega$ [78].

(For non-σ-finite measure spaces the third displayed equation must be strengthened. If $L^\infty(X, \mu)$ is a von Neumann algebra, this may be done by replacing the countable union with an uncountable join, but to accomodate some pathological measure spaces a more complicated axiom is needed.)

The space X (more precisely, the triple (X, μ, ρ)) is called a *measurable pseudometric space*. Notice that if μ is atomic then every measurable pseudometric is just the ordinary distance between sets for some pseudometric on X. But this is not the case for nonatomic measures.

For $f \in L^\infty(X, \mu)$ and $A, B \in \Omega$ let $\rho_f(A, B)$ be the distance (in $\mathbf{C}$) between the essential ranges of $f|_A$ and $f|_B$. (The *essential range* of a complex-valued measurable function f is the set of $a \in \mathbf{C}$ such that $f^{-1}(U)$ has positive measure, for all neighborhoods U of a.) Then the Lipschitz number of f is defined to be

$$L(f) = \sup\{\rho_f(A, B)/\rho(A, B) \colon A, B \in \Omega, \rho(A, B) > 0\}$$

and $\mathrm{Lip}(X)$ is the set of all $f \in L^\infty(X, \mu)$ for which $L(f)$ is finite, with norm $\|f\|_L = \max(L(f), \|f\|_\infty)$. No version of $\mathrm{Lip}_0(X)$ has been defined for measurable pseudometric spaces.

If $\mathrm{Lip}(X)$ is weak* dense in $L^\infty(X, \mu)$, we call ρ a *measurable metric* and X a *measurable metric space*. (This is a stronger definition than the one given in [78].) If μ is atomic then every measurable metric on X is just the ordinary distance between sets for some metric on X.

Abstract Characterizations

The spaces $\mathrm{Lip}(X)$ for ordinary metric spaces X are characterized up to isomorphism by the property

(1) every subset S of the real part of the unit ball has a supremum $\bigvee S$, which is also in the unit ball

plus either of the properties

(2a) $\bigvee$ and $\bigwedge$ are completely distributive, or

(2b) there is a separating family of pure normal states

([73], Theorem 3; [78], Theorem 4). The spaces Lip(X) for measurable metric spaces X are characterized by (1) plus

(2c) there is a separating family of normal states

([78], Theorems 6 and 10). There are no known abstract characterizations of $\mathrm{Lip}_0(X)$.

This completes our short survey of Lipschitz algebras.

II. W^* Domain Algebras

In this section we define the kind of derivations to be studied and prove some general results about their domains.

W^*-Derivations

Let $\mathcal{M} \subset \mathcal{N}$ be von Neumann algebras with the same unit. An *unbounded $*$-derivation* $\delta\colon \mathcal{M} \to \mathcal{N}$ is then an unbounded linear map whose domain is a unital $*$-subalgebra of $\mathcal{M}$ and which satisfies

$$\delta(x^*) = \delta(x)^* \qquad \text{and} \qquad \delta(xy) = x\delta(y) + \delta(x)\,y$$

for all $x, y \in \mathrm{dom}(\delta)$. It is natural to ask dom(δ) to be ultraweakly dense in $\mathcal{M}$ and the graph of δ to be ultraweakly closed in $\mathcal{M} \oplus \mathcal{N}$.

The most basic example of a von Neumann algebra derivation is the derivative map $\delta\colon L^\infty[0,1] \to L^\infty[0,1]$ taking f to f', with domain Lip[0, 1]. This definition makes sense since the Lipschitz functions on [0,1] are precisely those absolutely continuous functions f such that f' is essentially bounded ([53], Proposition 1.1).

Unfortunately, derivations of the above type are not general enough to encompass all of the examples to be discussed in the next section. Recalling that the most general derivation of an associative algebra $\mathcal{M}$ is not into a larger algebra, but rather into an $\mathcal{M}$-bimodule, it is natural to replace $\mathcal{N}$ with an operator $\mathcal{M}$-bimodule. More precisely, let $\mathcal{M}$ be a von Neumann algebra and let E be a normal dual operator $\mathcal{M}$-bimodule which is self-adjoint. (See [18] and [29] for background on operator bimodules.) We call E a *W^* $\mathcal{M}$-bimodule.*

Concretely, E may be thought of as a self-adjoint, ultraweakly closed subspace of $B(H)$, where we have two normal representations $\mathscr{M}_l, \mathscr{M}_r \subset B(H)$ such that $\mathscr{M}_l E \mathscr{M}_r \subset E$. Whenever it is convenient we will assume that we have such a concrete realization. In this case we write x_l, x_r for the images of $x \in \mathscr{M}$ in $B(H)$ given by the two representations of $\mathscr{M}$.

Of special interest in this paper is the case where $\mathscr{M}$ and E are abelian. This means that $\mathscr{M}_l$, $\mathscr{M}_r$, and E generate an abelian subalgebra of $B(H)$. There is also an abstract characterization of abelian operator modules [81].

One natural example of a W^* $\mathscr{M}$-bimodule, which we will consider again in Section III, is the tangent bimodule. Let X be a Riemannian manifold and let $\mathscr{M} = L^\infty(X)$; then the tangent bimodule E is the space of measurable complex vector fields on X. In this case the left and right actions of $\mathscr{M}$ are the same. There is an $\mathscr{M}$-valued inner product on E, defined pointwise on pairs of vectors, which reflects the Riemannian structure on X. This shows that E is an operator bimodule; in fact any Hilbert module can be realized as an operator module via the linking algebra construction [16]. One important point brought out by this example is the significance of the self-adjointness of E. If X is a topological space and we form a module over $C(X)$ from the continuous sections of a vector bundle over X, then self-adjointness of this module reflects the existence of real (not complex) structure on the fibers of the bundle. See [81] for more on this point.

Now given a von Neumann algebra $\mathscr{M}$ and a W^* $\mathscr{M}$-bimodule E, we define a *W^*-derivation* $\delta\colon \mathscr{M} \to E$ to be an unbounded linear map whose domain is an ultraweakly dense, unital $*$-subalgebra of $\mathscr{M}$, whose graph is an ultraweakly closed subspace of $\mathscr{M} \oplus E \subset B(n \oplus H)$, and which satisfies

$$\delta(x^*) = \delta(x)^* \qquad \text{and} \qquad \delta(xy) = x_l \delta(y) + \delta(x)\, y_r$$

for all $x, y \in \operatorname{dom}(\delta)$.

W^* Domain Algebras

Let $\mathscr{L}$ be the domain of a W*-derivation δ. Then $\mathscr{L}$ is a $*$-algebra, and by analogy with Lipschitz algebras we give it the norm

$$\|x\|_D = \max(\|\delta(x)\|, \|x\|).$$

With this norm we call $\mathscr{L}$ a *W^* domain algebra.*

In past work on derivations of C*-algebras it has often been fruitful to define a new norm on the domain and study it as an algebra in its own right (e.g. see [5], [31], [45], [58]). At least three different norms have been used in this connection; of course, all reasonable norms will in general be equivalent, hence for many purposes interchangeable. However, the isometric character of our main results (Theorems 9 and 16) requires the specific norm given above. (The specific norm used for Lipschitz algebras is also needed for some of the results mentioned in Section I.)

As we indicated in Section I, there is some question as to the completeness of the theory of Lip_0 spaces, and in particular we do not have a definition of $\mathrm{Lip}_0(X)$ for measurable metric spaces. For this reason we will only consider W* domain algebras with the norm $\|\cdot\|_D$, which are the analogues of Lip spaces.

Lipschitz Functional Calculus

It is well-known that there is a C^1 functional calculus in the domain of any norm closed derivation of an abelian C*-algebra into itself [61]. (See [12] and [51] for discussion of the nonabelian case.) We will now show that in W* domain algebras it can be extended to a Lipschitz functional calculus; this is analogous to the extension of the continuous functional calculus in C*-algebras to the Borel functional calculus in von Neumann algebras.

The result does not require that the entire domain or range be abelian, but only that $\delta(x)$ commutes with either x_l or x_r. A version of this condition has also been used in [23] and [64].

THEOREM 1. *Let $\delta: \mathcal{M} \to E$ be a W^*-derivation, let $x \in \mathrm{dom}(\delta)$ be self-adjoint, and let $f \in \mathrm{Lip}(\mathrm{spec}(x))$. Suppose also that $\delta(x)$ commutes with either x_l or x_r. Then $f(x) \in \mathrm{dom}(\delta)$ and*

$$\|\delta(f(x))\| \leqslant L(f) \cdot \|\delta(x)\|.$$

Proof. Without loss of generality suppose that $\delta(x)$ commutes with x_r.

First suppose f is a polynomial, $f(t) = \sum a_n t^n$. Then obviously $f(x) \in \mathrm{dom}(\delta)$, and

$$\begin{aligned}
\delta(f(x)) &= \sum a_n \delta(x^n) \\
&= \sum a_n (x_l^{n-1}\delta(x) + x_l^{n-2}\delta(x)x_r + \cdots + \delta(x)x_r^{n-1}) \\
&= \sum a_n (x_l^{n-1} + x_l^{n-2}x_r + \cdots + x_r^{n-1})\, \delta(x) \\
&= h(x_l, x_r)\, \delta(x),
\end{aligned}$$

where h is the function

$$h(s,t)=\begin{cases}(f(s)-f(t))/(s-t) & \text{if} \quad s\neq t\\ f'(s) & \text{if} \quad s=t.\end{cases}$$

Let $K=[-\|x\|, \|x\|]\subset \mathbf{R}$; then clearly $|h(s,t)|\leqslant \|f'|_K\|_\infty$ for all s and t in the spectrum of x. Thus

$$\|\delta(f(x))\|\leqslant \|f'|_K\|_\infty\cdot\|\delta(x)\|$$

for all polynomials f.

Now let $f\in \mathrm{Lip}(\mathrm{spec}(x))$. By ([30], Theorem 2.10.43) we can extend f to K without increasing its Lipschitz number; then by ([53], Proposition 1.1) the derivative of f exists a.e. and belongs to $L^\infty(K)$. Let (g_n) be a sequence of polynomials such that $g_n|_K\to f'$ in L^1-norm and $\|g_n|_K\|_\infty\leqslant\|f'\|_\infty = L(f)$ for all n. Then letting

$$f_n(t)=f(0)+\int_0^t g_n$$

we get a sequence of polynomials (f_n) such that $\|(f'_n)|_K\|_\infty\leqslant L(f)$ for all n and $\|f_n|_K-f\|_\infty\to 0$.

By the first part of the proof, $f_n(x)\in\mathrm{dom}(\delta)$ and

$$\|\delta(f_n(x))\|\leqslant L(f)\cdot\|\delta(x)\|$$

for all n. We may therefore take a subnet (f_α) such that $\delta(f_\alpha(x))$ ultraweakly converges to some $y\in E$. As $f_n(x)\to f(x)$ uniformly, it then follows by ultraweak closure of the graph of δ that $f(x)\in\mathrm{dom}(\delta)$ and $\delta(f(x))=y$. Finally

$$\begin{aligned}\|y\|&\leqslant \liminf\|\delta(f_\alpha(x))\|\\&\leqslant L(f)\cdot\|\delta(x)\|,\end{aligned}$$

as desired. ∎

The Weak Topology*

The following result is trivial but important. Together with material in Section III it furnishes an easy proof that every $\mathrm{Lip}(X)$ is a dual space (for X a metric space or measurable metric space).

PROPOSITION 2. *Every W^* domain algebra is a dual space.*

Proof. Let $\mathscr{L}$ be the W* domain algebra associated with a W*-derivation $\delta\colon\mathscr{M}\to E$. Then the graph of δ is an ultraweakly closed subspace of

$\mathcal{M} \oplus E$, and therefore it is a dual space. But as a Banach space, $\mathcal{L}$ is isometrically isomorphic to the graph of δ, by the map $x \mapsto (x, \delta(x))$. So $\mathcal{L}$ is a dual space too. ∎

There is a possibility for confusion here because $\mathcal{L}$ and $\mathcal{M}$ are both dual spaces and their weak* topologies do not agree. (Specifically, $x_\alpha \to x$ in the weak* topology on $\mathcal{L}$ if and only if $x_\alpha \to x$ in the weak* topology on $\mathcal{M}$ and $\delta(x_\alpha) \to \delta(x)$ in the weak* topology on E.) To avoid ambiguity, henceforth "weak*" will always refer to the weak* topology on $\mathcal{L}$ given by Proposition 2.

We now give a result about weak* closed ideals of W* domain algebras. Only its first corollary is needed in later sections (and that only in the commutative case), but the other corollaries, and the theorem itself, may be of independent interest.

THEOREM 3. *Let $\mathcal{L}$ be the W* domain algebra of a W*-derivation $\delta \colon \mathcal{M} \to E$ and let $\mathcal{I}$ be a self-adjoint weak* closed ideal of $\mathcal{L}$. Then $\mathcal{I}$ equals the weak* closure of $\mathcal{I}^2$.*

Proof. It is clear that $\mathcal{I}$ contains the weak* closure of $\mathcal{I}^2$. For the converse let x be any self-adjoint element of $\mathcal{I}$; we must approximate x by elements of $\mathcal{I}^2$.

Define $f_n(t) = \exp(-nt^2)$ and $g_n(t) = t f_n(t)$ for $t \in \mathbf{R}$. We claim that $1 - f_n(x) \in \mathcal{I}$, $g_n(x) \in \mathcal{L}$, and $\|\delta(g_n(x))\| \leqslant C\,\|\delta(x)\|$ for some $C > 0$ which is independent of n. The first claim follows from the ordinary holomorphic functional calculus, since $1 - f_n$ is an entire analytic function which vanishes at 0 and $\mathcal{I}$ is weak* closed hence complete.

The rest will follow from ([15], Theorem 3.2.32). (This result was stated for derivations of $\mathcal{M}$ into itself, but its proof works just as well for derivations into a W* $\mathcal{M}$-bimodule.) The Fourier transform of $g_n(t)$ is

$$\hat{g}_n(p) = \frac{i}{(2n)^{3/2}}\, p e^{-p^2/4n},$$

so

$$\begin{aligned} |\hat{g}_n| &\equiv \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} |\hat{g}_n(p)|\,|p|\,dp \\ &= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^\infty \frac{1}{(2n)^{3/2}}\, p^2 e^{-p^2/4n}\,dp \\ &= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^\infty \frac{1}{2^{3/2}}\, q^2 e^{-q^2/4}\,dq, \end{aligned}$$

where $q = p/\sqrt{n}$. The last integral converges and is independent of n, so the numbers $|\hat{g}_n|$ are all equal to some positive, finite number C. Then ([15], Theorem 3.2.32) implies that $g_n(x) \in \mathscr{L}$ and $\|\delta(g_n(x))\| \leqslant C\,\|\delta(x)\|$.

Finally, let $x_n = 1 - f_n(x)$. Then $xx_n = x - g_n(x) \to x$ in norm, and the sequence $(\delta(xx_n))$ is bounded, so there is a subnet such that $\delta(xx_\alpha) \to \delta(x)$ ultraweakly. This shows that x is in the weak* closure of $\mathscr{I}^2$, as desired. ∎

Note that the corresponding statement for derivations of C*-algebras and $\|\cdot\|_D$-closed ideals is false. For instance, let $\delta\colon C[0,1] \to C[0,1]$ be differentiation, $\delta(f) = f'$, with domain $C^1[0,1]$, and consider the ideal $\mathscr{I}$ of functions in $\mathrm{dom}(\delta_1)$ which vanish at 0. The $\|\cdot\|_D$-closure of $\mathscr{I}^2$ is then the ideal of functions which vanish at 0 and whose derivative vanishes at 0.

COROLLARY 4. *Let $\mathscr{I}$ be a self-adjoint ideal of the domain of a W^*-derivation $\delta\colon \mathscr{M} \to E$. Then $\delta(\mathscr{I})$ is contained in the ultraweak closure of $\mathscr{I}_l E + E\mathscr{I}_r$.*

If $E = \mathscr{M}$ and $\delta\colon \mathscr{M} \to \mathscr{M}$ is a W^*-derivation then $\delta(\mathscr{I})$ is contained in the ultraweak closure of $\mathscr{I}$.

Proof. Since $\mathscr{I}$ is contained in the weak* closure of $\mathscr{I}^2$, it follows that $\delta(\mathscr{I})$ is contained in the ultraweak closure of $\delta(\mathscr{I}^2)$, and the derivation law shows that $\delta(\mathscr{I}^2) \subset \mathscr{I}_l E + E\mathscr{I}_r$. If δ is a W* derivation from $\mathscr{M}$ into itself then $\mathscr{I}_l E + E\mathscr{I}_r = \mathscr{I}\mathscr{M} + \mathscr{M}\mathscr{I} = \mathscr{I}$. ∎

In the abelian case, the second part of Corollary 4 implies a kind of locality condition. Thus, if δ is a W*-derivation of $L^\infty(X, \mu)$ into itself and $f \in \mathrm{dom}(\delta)$ vanishes on some positive measure subset $U \subset X$, then so does $\delta(f)$. A similar result is true for derivations of $C(X)$, for open subsets $U \subset X$ ([4], Lemma 2.4).

COROLLARY 5. *Let $\mathscr{I}$ and $\mathscr{J}$ be self-adjoint ideals of a W^* domain algebra. Then $\mathscr{I} \cap \mathscr{J}$ is contained in the weak* closure of $\mathscr{I}\mathscr{J}$. If $\mathscr{I}$ and $\mathscr{J}$ are weak* closed then $\mathscr{I} \cap \mathscr{J}$ equals the weak* closure of $\mathscr{I}\mathscr{J}$.*

Proof. $\mathscr{I} \cap \mathscr{J}$ is contained in the weak* closure of $(\mathscr{I} \cap \mathscr{J})^2 \subset \mathscr{I}\mathscr{J}$ by Theorem 3. Conversely, if $\mathscr{I}$ and $\mathscr{J}$ are weak* closed then $\mathscr{I} \cap \mathscr{J}$ clearly contains $\mathscr{I}\mathscr{J}$ and hence also its weak* closure. ∎

COROLLARY 6. *Let $\delta\colon \mathscr{M} \to E$ be a W^*-derivation and $\mathscr{I}$ and $\mathscr{J}$ the kernels of the maps $x \mapsto x_l$ and $x \mapsto x_r$. Then δ vanishes on $\mathscr{I} \cap \mathscr{J}$.*

Proof. Let $\mathscr{I}' = \mathscr{I} \cap \mathrm{dom}(\delta)$ and $\mathscr{J}' = \mathscr{J} \cap \mathrm{dom}(\delta)$. Then it is clear that $\delta(\mathscr{I}'\mathscr{J}') = 0$, hence δ vanishes on the weak* closure of $\mathscr{I}'\mathscr{J}'$, hence $\delta(\mathscr{I}' \cap \mathscr{J}') = 0$ by Corollary 5. ∎

The point of Corollary 6 is that one can essentially ignore $\mathcal{I} \cap \mathcal{J}$. This is of particular interest when E is *one-sided* in the sense that $x_l = x_r$ for all $x \in \mathcal{M}$, so that $\mathcal{I} = \mathcal{J}$. Since $\mathcal{I}$ is an ultraweakly closed ideal, we can write $\mathcal{M} = \mathcal{I} \oplus \mathcal{I}'$, where $\mathcal{I}'$ is the complementary ultraweakly closed ideal; furthermore, $\mathcal{I}'$ naturally embeds in E by the map $x \mapsto x_l$, and Corollary 6 tells us that δ vanishes on $\mathcal{I}$. Thus, we conclude that δ is the sum of a W*-derivation (from $\mathcal{I}'$ into E) and an identically zero derivation (on $\mathcal{I}$).

COROLLARY 7. *Every W*-derivation of $l^\infty(X)$ into itself is identically zero.*

Proof. Let $\delta: l^\infty(X) \to l^\infty(X)$ be a W*-derivation. (We consider $l^\infty(X)$ as acting by multiplication on $l^2(X)$.) For each $p \in X$ let $\mathcal{I}_p = \{f \in \mathrm{dom}(\delta): f(p) = 0\}$; then the second part of Corollary 4 implies that $\delta(f)(p) = 0$ for all $f \in \mathcal{I}_p$. But for any $f \in \mathrm{dom}(\delta)$ we have $f - f(p) \in \mathcal{I}_p$; since δ vanishes on constant functions this implies $\delta(f)(p) = 0$. As this holds for all $f \in \mathrm{dom}(\delta)$ and $p \in X$, δ must be identically zero. ∎

By ([3], Theorem 1), every unbounded derivation of an abelian C*-algebra $C(X)$ is identically zero if X is totally disconnected. However, the proof of this relies on norm density of the domain and it seems that different techniques are needed to prove Corollary 7. A complete description of W*-derivations of atomic abelian von Neumann algebras into abelian W* bimodules is given in Theorem 17.

III. LIPSCHITZ ALGEBRAS AS DOMAINS

We next give a construction which exhibits $\mathrm{Lip}(X)$ as the domain algebra of a W*-derivation of abelian von Neumann algebras. Although this can be done for X any measurable metric space, ordinary metric spaces are a bit more tranparent, so we discuss this case first. After this we give a W*-derivation into a one-sided bimodule, whose domain algebra is $\mathrm{Lip}(X)$, for X a connected Riemannian manifold.

Metric Spaces

Let X be a metric space and let $Y = \{(x, y) \in X^2 : x \neq y\}$. Then we have two natural maps from $l^\infty(X)$ into $l^\infty(Y)$; thus, for $f \in l^\infty(X)$ we define $f_l, f_r \in l^\infty(Y)$ by $f_l(x, y) = f(x)$ and $f_r(x, y) = f(y)$. This makes $l^\infty(Y)$ into a W* $l^\infty(X)$-bimodule.

Now consider the map $\delta: l^\infty(X) \to l^\infty(Y)$ given by

$$\delta(f)(x, y) = \frac{f(x) - f(y)}{\rho(x, y)},$$

with domain Lip(X). This map was introduced in [26] and has played a central role in the theory of Lipschitz spaces.

Note that we consider $l^\infty(X)$ and $l^\infty(Y)$ as acting by multiplication on $l^2(X)$ and $l^2(Y)$. (This is not really necessary, as the relevant concepts are all actually representation-independent.)

Proposition 8. *Let X be a metric space and let Y and δ be as above. Then δ is a W^*-derivation from $l^\infty(X)$ into $l^\infty(Y)$, and its domain algebra is* Lip(X).

Proof. The derivation law is verified as follows:

$$\begin{aligned}\delta(fg)(x, y) &= \frac{f(x)\, g(x) - f(y)\, g(y)}{\rho(x, y)} \\ &= f(x)\, \frac{g(x) - g(y)}{\rho(x, y)} + \frac{f(x) - f(y)}{\rho(x, y)}\, g(y) \\ &= (f_l \delta(g) + \delta(f)\, g_r)(x, y).\end{aligned}$$

It is also obvious that δ is self-adjoint.

For any distinct points $x, y \in X$, it is easy to find a function $f \in \mathrm{Lip}(X)$ which separates them; for instance take $f(z) = \min(\rho(x, z), 1)$. Since $\mathrm{dom}(\delta) = \mathrm{Lip}(X)$ is a $*$-subalgebra of $l^\infty(X)$ this implies that its ultraweak closure equals $l^\infty(X)$. Thus the domain of δ is ultraweakly dense in $l^\infty(X)$.

To show ultraweak closure of the graph, let $(f_\alpha) \subset l^\infty(X)$ and suppose $f_\alpha \to f$ and $\delta(f_\alpha) \to g$, both in the ultraweak sense (hence pointwise). Then for any distinct $x, y \in X$ we have

$$\frac{f(x) - f(y)}{\rho(x, y)} = \lim_\alpha \frac{f_\alpha(x) - f_\alpha(y)}{\rho(x, y)} = g(x, y).$$

This shows that f is Lipschitz (in fact $L(f) = \|g\|_\infty$) and $\delta(f) = g$, as desired. We conclude that δ is a W*-derivation.

By definition, $\mathrm{dom}(\delta) = \mathrm{Lip}(X)$ as sets. To see that the norms agree, simply observe that $\|\delta(f)\|_\infty = L(f)$ for all $f \in \mathrm{Lip}(X)$. ∎

A variation of the above map was given in [64]. For $f \in \mathrm{Lip}(X)$ define the operator $\delta_1(f) \in B(l^2(Y))$ to be the coordinate exchange followed by multiplication by $\delta(f)$; thus,

$$(\delta_1(f)(v))(x, y) = \frac{f(x) - f(y)}{\rho(x, y)}\, v(y, x)$$

for $v \in l^2(Y)$. Equivalently, $\delta_1(f) = i[D, M_{f_l}]$ where D is the unbounded self-adjoint operator on $l^2(Y)$ defined by

$$(Dv)(x, y) = iv(y, x)/\rho(x, y)$$

and M_{f_l} is multiplication by f_l. The advantage to this definition is that $B(l^2(Y))$ is thought of as a one-sided $l^\infty(X)$ bimodule, via the left inclusion of $l^\infty(X)$ in $l^\infty(Y)$. In contrast to δ, however, the range of δ_1 is in general noncommutative. (Commutativity of both domain and range seems to be an important feature of δ; see Section IV.)

Measurable Metric Spaces

The construction given above can be generalized to handle measurable metric spaces.

Let X be a measurable metric space. For any subsets $A, B \subset X$ such that $\rho(A, B) > 0$ let $Y_{AB} = A \times B$, and define a bounded W*-derivation δ_{AB}: $L^\infty(X) \to L^\infty(Y_{AB})$ by

$$\delta_{AB}(f)(x, y) = \frac{f(x) - f(y)}{\rho(A, B)},$$

with domain all of $L^\infty(X)$. Here the bimodule structure is defined just as in the atomic case, i.e. for $f \in L^\infty(X)$ we have $f_l(x, y) = f(x)$ and $f_r(x, y) = f(y)$.

Finally, let Y be the disjoint union of the Y_{AB}'s and let δ: $L^\infty(X) \to L^\infty(Y)$ be the direct sum of the δ_{AB}'s, with domain $\mathrm{Lip}(X)$.

THEOREM 9. *Let X be a measurable metric space and let Y and δ be as above. Then δ is a W*-derivation from $L^\infty(X)$ into $L^\infty(Y)$, and its domain algebra is* Lip(X).

Proof. The derivation law and self-adjointness are separately verified on each Y_{AB} as in Proposition 8, and ultraweak density of the domain is assumed, since X is a measurable metric space. To show ultraweak closure of the graph, let $(f_\alpha) \subset \mathrm{Lip}(X)$ and suppose $f_\alpha \to f$ and $\delta(f_\alpha) \to g$, both in the ultraweak sense. Then for all $A, B \subset X$ we have $\delta_{AB}(f_\alpha) \to \delta_{AB}(f)$ and $\delta_{AB}(f_\alpha) \to g|_{Y_{AB}}$, hence $\delta_{AB}(f) = g|_{Y_{AB}}$. This shows that $f \in \mathrm{Lip}(X)$ and $\delta(f) = g$. Thus δ is a W*-derivation as claimed.

To show that the domain algebra norm is $\|\cdot\|_L$, it suffices to observe that $\|\delta(f)\|_\infty = L(f)$ for all $f \in \mathrm{Lip}(X)$. ▮

The Unit Circle

When X is a connected Riemannian manifold, it is possible to exhibit $\mathrm{Lip}(X)$ as the domain of a W*-derivation into a one sided bimodule. To illustrate the method we begin with the simplest case, where $X = \mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the unit circle.

By ([53], Proposition 1.1), the Lipschitz functions on $[0, 1]$ are precisely those absolutely continuous functions f such that $f' \in L^\infty[0, 1]$, and furthermore $L(f) = \|f'\|_\infty$ for any such function. The same is obviously also true with $\mathbf{T}$ in place of $[0,1]$. Thus define $\delta\colon L^\infty(\mathbf{T}) \to L^\infty(\mathbf{T})$ by $\delta(f) = f'$, with domain $\mathrm{Lip}(\mathbf{T})$.

Also let $D = id/dx$ be the differentiation operator on $L^2(\mathbf{T})$ and recall that its domain is the set of absolutely continuous functions in $L^2(\mathbf{T})$ whose derivative is also in $L^2(\mathbf{T})$.

It is shown in ([15], Proposition 3.2.55) that for any unbounded self-adjoint operator D on a Hilbert space H, and any $x \in B(H)$, the following conditions are equivalent:

(1) the sesquilinear form $(\xi, \eta) \mapsto \langle x\xi, D\eta\rangle - \langle xD\xi, \eta\rangle$, defined for $\xi, \eta \in \mathrm{dom}(D)$, is bounded;

(2) x preserves the domain of D and the commutator $Dx - xD$ is bounded on $\mathrm{dom}(D)$;

and if they hold they define the same bounded operator. In this case we denote this operator by $[D, x]$ and say that $[D, x]$ is bounded.

Proposition 10. *Let D and δ be as defined above. Then*

$$\{f \in L^\infty(\mathbf{T})\colon [D, M_f] \text{ is bounded}\} = \mathrm{Lip}(\mathbf{T})$$

and

$$[D, M_f] = iM_{f'}$$

for all such f, where M denotes a multiplication operator on $L^2(\mathbf{T})$. The map δ is a W^*-derivation and its domain algebra is $\mathrm{Lip}(\mathbf{T})$.

Proof. Observe that if $f \in \mathrm{Lip}(\mathbf{T})$ and $g \in \mathrm{dom}(D)$ then f is absolutely continuous, hence fg is absolutely continuous, and $(fg)' = fg' + f'g \in L^2(\mathbf{T})$, hence $fg \in \mathrm{dom}(D)$. Thus M_f preserves the domain of D, and for any $g \in \mathrm{dom}(D)$ we have

$$DM_f g - M_f Dg = i(fg' + f'g) - ifg' = iM_{f'} g.$$

So for any $f \in \mathrm{Lip}(\mathbf{T})$ the commutator $[D, M_f]$ is bounded and we have $[D, M_f] = iM_{f'}$. This shows one direction of the first assertion.

For the other direction, let $f \in L^\infty(\mathbf{T})$ and suppose $[D, M_f]$ is bounded. This immediately implies that f is absolutely continuous, since M_f preserves the domain of D, which contains 1. The preceding calculation then shows that $[D, M_f] = iM_{f'}$, so that f' must be in $L^\infty(\mathbf{T})$. This implies that $f \in \mathrm{Lip}(\mathbf{T})$ and completes the proof of the first statement.

It is easy to check that δ is an unbounded $*$-derivation, and its domain contains the trigonometric polynomials hence is ultraweakly dense in $L^\infty(\mathbf{T})$. It is also clear that the domain of δ is isometrically identified with $\mathrm{Lip}(\mathbf{T})$, since $L(f) = \|f'\|_\infty$ for any $f \in \mathrm{Lip}(\mathbf{T})$. To check ultraweak closure of the graph of δ, suppose $(f_\alpha) \subset \mathrm{Lip}(\mathbf{T})$ such that $f_\alpha \to f$ and $f'_\alpha \to g$ (both in the ultraweak sense, in $L^\infty(\mathbf{T})$). By the Kaplansky density theorem applied to the graph of δ, we may assume (f_α) is bounded in $\|\cdot\|_D$-norm.

It follows from the sesquilinear form definition of the commutator that $[D, M_{f_\alpha}] \to [D, M_f]$ weak operator. In particular, this implies that the latter is bounded and thus $f \in \mathrm{Lip}(\mathbf{T})$ and $[D, M_f] = iM_{f'}$. Also, since $[D, M_{f_\alpha}] = iM_{f'_\alpha} \to iM_g$, it follows that $f' = g$. So the graph of δ is indeed ultraweakly closed. ∎

Riemannian Manifolds

The unit circle derivation can be generalized to connected Riemannian manifolds. A different generalization, to compact Riemannian spin manifolds, was given in [20] and our construction may be viewed as a simpler version of this one. For basic material on differential geometry see [7], [36], or [46].

Let X be a connected Riemannian manifold, let $\mathcal{M} = L^\infty(X)$, and let E be the tangent bimodule discussed in Section II. (Thus, E is the $\mathcal{M}$-bimodule of bounded measurable complex vector fields on X.) Recall that E is a Hilbert module over $\mathcal{M}$. Let $\delta\colon \mathcal{M} \to E$ be the exterior derivative, i.e. $\delta(f)$ is the gradient vector field of f. Equivalently, $\delta(f)$ is the vector field which satisfies

$$\langle \delta(f)(p), v \rangle = d_p f(v)$$

for almost all $p \in X$ and all tangent vectors v at p. Here $d_p f(v)$ is the derivative of f at p in the direction v. The domain algebra of δ is precisely $\mathrm{Lip}(X)$.

There is a slightly different version of this construction which exhibits it as an inner derivation. To achieve this let Y be the "unit sphere bundle" over X,

$$Y = \{(p, v) : p \in X, v \in T_p X, \|v\| = 1\}$$

where $T_p X$ is the tangent space at p. Then we have a natural map $f \mapsto f_l$ from $L^\infty(X)$ into $L^\infty(Y)$ given by $f_l(p, v) = f(p)$ and we define a derivation $\delta' : L^\infty(X) \to L^\infty(Y)$ by

$$\delta'(f)(p, v) = d_p f(v),$$

with domain Lip(X).

The first thing to check is that if $f \in \mathrm{Lip}(X)$ then $d_p f(v)$ exists for almost every p and v. To see this, recall that every $(p, v) \in Y$ is tangent to exactly one (oriented) maximal geodesic in X. Therefore Y has a natural fibering, with each fiber being the set of all vectors tangent to a given maximal geodesic γ. Now each such γ is locally isometric to a line segment, and the restriction of f to γ is Lipschitz hence differentiable almost everywhere. Thus $d_p f(v)$ exists for almost every (p, v) in each fiber, hence for almost every $(p, v) \in Y$ by Fubini's theorem.

Observe that there is a natural flow on Y along the fibering just described. For $(p, v) \in Y$ let γ be the maximal geodesic to which (p, v) is tangent, parametrized by arc length so that $\gamma(0) = p$ and $d\gamma/dt(0) = v$, and define

$$\alpha_{t0}(p, v) = (\gamma(t_0), d\gamma/dt(t_0)).$$

Then α_t is an isometric flow on Y, and it induces a strongly continuous one-parameter unitary group U_t on $L^2(Y)$, by $U_t(f) = f \circ \alpha_t$. The generator D of this group is differentiation in the direction of the flow, and just as for the unit circle $[D, M_{f_l}]$ is bounded if and only if $f \in \mathrm{Lip}(X)$, in which case $[D, M_{f_l}] = iM_{\delta'(f)}$. Thus, by the same argument as for the unit circle, δ' is a W*-derivation.

In addition, $L(f) = \|\delta'(f)\|_\infty$ for all $f \in \mathrm{Lip}(X)$, which implies that the domain algebra norm equals the Lipschitz norm. To see this, let $(p, v) \in Y$ be a point for which $d_p f(v)$ exists and let γ be a geodesic through p tangent to v. Then near p, γ is isometric to a line segment, and letting g be the restriction of f to this line segment we see

$$|d_p f(v)| \leqslant \|g'\|_\infty = L(g) \leqslant L(f).$$

This shows that $L(f) \geqslant \|\delta'(f)\|_\infty$. Conversely, let $p, q \in X$, let γ be a geodesic passing through p and q, and let g be the restriction of f to γ. Then if a is the length of the segment of γ between p and q, there must exist $p_0 \in \gamma$ with tangent vector v_0 along γ such that

$$|d_{p_0} f(v_0)| \geqslant |f(p) - f(q)|/a.$$

As $\rho(p, q)$ is the infimum of all such a, it follows that $|f(p) - f(q)|/\rho(p, q) \leqslant \|\delta'(f)\|_\infty$. We conclude that $L(f) = \|\delta'(f)\|_\infty$, as claimed.

The above conclusions are summarized in the following.

THEOREM 11. *Let X be a connected Riemannian manifold and let D and δ' be defined as above. Then*

$$\{f \in L^\infty(X) : [D, M_f] \textit{ is bounded}\} = \mathrm{Lip}(X)$$

and

$$[D, M_f] = iM_{f'}$$

for all such f, where M denotes a multiplication operator on $L^2(X)$. The map δ' is a W^*-derivation and its domain algebra is Lip(X). ∎

The derivation described here may be loosely thought of as an infinitesimal version of the derivation discussed in Proposition 8. The idea is that if X is a Riemannian manifold then X^2 is an "approximation" to the tangent manifold TX, with the pair $(p, q) \in X^2$ thought of as approximating a tangent vector at p of length $\rho(p, q)$, pointing towards q. (Indeed, for small ε the set of tangent vectors at p of length at most ε is a good approximation to the ε-ball about p in X. This intuition seems to be a motivation in [64].) According to this analogy, the tangent manifold version of the derivation in Proposition 8 will be the derivation $\delta: L^\infty(X) \to L^\infty(TX)$ defined by

$$\delta(f)(p, v) = \frac{d_p f(v)}{\|v\|}.$$

Restricting to the sphere bundle then yields the derivation of Theorem 11.

It would be interesting to know for exactly which metric spaces X we can realize Lip(X) as the domain algebra of a W*-derivation into a one-sided $L^\infty(X)$-bimodule. This would seem to require some sort of differential or "local" structure in X.

IV. Domains as Lipschitz Algebras

We will now prove that the domain algebra of any W*-derivation of an abelian von Neumann algebra $\mathcal{M}$ into an abelian W* $\mathcal{M}$-bimodule is isometrically isomorphic to a Lipschitz algebra. More precisely, if $\mathcal{M} = L^\infty(X, \mu)$ then there is a measurable metric on X such that the domain algebra is isometrically identified with $\mathrm{Lip}(X)$.

As in the last section, the material simplifies in the atomic case. First of all, the measurable metric then obviously reduces to an ordinary metric. More interesting is that we can actually give a complete description of W*-derivations of atomic abelian von Neumann algebras; this is done in Theorem 17.

Since we are assuming $\mathcal{M}$ and E are abelian, there is no loss in generality in replacing E with the von Neumann algebra generated by $\mathcal{M}_l$, $\mathcal{M}_r$, and E and then taking $\mathcal{M} = L^\infty(X)$ and $E = L^\infty(Y)$. For the next four lemmas, fix a W*-derivation $\delta\colon L^\infty(X) \to L^\infty(Y)$ and let $\mathcal{L}$ be its domain algebra. We write $\mathrm{Re}(\mathcal{L})$ for the self-adjoint part of $\mathcal{L}$ (i.e. the real-valued functions in $\mathcal{L}$).

Lemma 12. *Let $f, g \in \mathrm{Re}(\mathcal{L})$. Then $f \vee g \in \mathcal{L}$ and*

$$\|\delta(f \vee g)\|_\infty \leqslant \max(\|\delta(f)\|_\infty, \|\delta(g)\|_\infty).$$

Proof. Since the function $h(t) = |t|$ is Lipschitz on $\mathbf{R}$, Theorem 1 implies that $|f - g| \in \mathcal{L}$; $f \vee g \in \mathcal{L}$ then follows from the formula $f \vee g = (f + g + |f - g|)/2$.

To show the norm inequality, let

$$A = \{x \in X\colon f(x) \geqslant g(x)\}, \qquad B = \{x \in X\colon f(x) \leqslant g(x)\}$$

and

$$A' = \{x \in X\colon f(x) > g(x)\} \qquad B' = \{x \in X\colon f(x) < g(x)\}.$$

Let Y_1 be the support of $(\chi_A)_l(\chi_A)_r$, Y_2 the support of $(\chi_B)_l(\chi_B)_r$, Y_3 the support of $(\chi_{A'})_l(\chi_{B'})_r$ and Y_4 the support of $(\chi_{B'})_l(\chi_{A'})_r$. (Here χ denotes characteristic function.) Then $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$.

Now $f \vee g - f$ is zero an A, hence by Corollary 4, $\delta(f \vee g - f)$ is zero on Y_1; that is, $\delta(f \vee g) = \delta(f)$ on Y_1. Similarly $\delta(f \vee g) = \delta(g)$ on Y_2. Thus the norm inequality only needs to be verified on Y_3 and Y_4; by symmetry it will be enough to show it on Y_3.

For $\varepsilon > 0$ define

$$A_\varepsilon = \{x \in X : f(x) \geqslant g(x) + \varepsilon\}, \qquad B_\varepsilon = \{x \in X : f(x) \leqslant g(x) - \varepsilon\}$$

and let $Y_3^\varepsilon = (\chi_{A_\varepsilon})_l (\chi_{B_\varepsilon})_r$. Then $Y_3 = \bigcup_{\varepsilon > 0} Y_3^\varepsilon$, so it will be enough to show the inequality on Y_3^ε. This will be done by finding an explicit formula for the restriction of δ to Y_3^ε.

Let

$$h^A = ((f - g)/\varepsilon \vee 0) \wedge 1$$

and

$$h^B = ((g - f)/\varepsilon \vee 0) \wedge 1;$$

then for all $k \in \mathscr{L}$ we have $k = (h^A + h^B)k$ on $A'_\varepsilon \cup B'_\varepsilon$. Hence $\delta(k) = \delta(h^A k + h^B k)$ on Y_3^ε, again by Corollary 4. Since h^A vanishes on B'_ε and h^B vanishes on A'_ε, it follows that h_r^A and h_l^B vanish on Y_3^ε, so that

$$\delta(k) = k_l \delta(h^A) + k_r \delta(h^B)$$

on Y_3^ε. With $k = 1$ this implies that $\delta(h^A) + \delta(h^B) = 0$ on Y_3^ε. Thus, letting h be the restriction of $\delta(h^A)$ to Y_3^ε, we get $\delta(k) = h(k_l - k_r)$ on Y_3^ε, for all $k \in \mathscr{L}$.

In particular,

$$\delta(f \vee g) = h(f_l - g_r), \qquad \delta(f) = h(f_l - f_r), \qquad \delta(g) = h(g_l - g_r)$$

on Y_3^ε. But since $f_l > g_l$ and $f_r < g_r$ on Y_3^ε, it follows that

$$|(f_l - g_r)(x, y)| \leqslant \max(|(f_l - f_r)(x, y)|, |(g_l - g_r)(x, y)|)$$

for all $(x, y) \in Y_3^\varepsilon$. Therefore

$$|\delta(f \vee g)(x, y)| \leqslant \max(|\delta(f)(x, y)|, |\delta(g)(x, y)|)$$

for all $(x, y) \in Y_3^\varepsilon$, as desired. ∎

Lemma 13. *Let (f_α) be a $\|\cdot\|_D$-bounded subset of* Re$(\mathscr{L})$. *Then $f = \bigvee f_\alpha$ also belongs to $\mathscr{L}$ and satisfies $\|\delta(f)\|_\infty \leqslant \sup \|\delta(f_\alpha)\|_\infty$.*

Proof. First let (g_β) be the net of joins of finite subsets of (f_α). By Lemma 12, (g_β) satisfies $\sup \|\delta(g_\beta)\|_\infty = \sup \|\delta(f_\alpha)\|_\infty$. Since (f_α) is bounded in the domain algebra norm, so is (g_β), and so a subnet of (g_β) can be chosen which converges weak*. But since (g_β) already converges ultraweakly to f, it follows that f also belongs to $\mathscr{L}$ and satisfies the desired inequality. ∎

At this point we could invoke ([78], Theorem 10) to conclude that $\mathcal{L}$ is isomorphic to some Lipschitz algebra. However, in the present situation this conclusion can be strengthened, so we proceed with the argument.

LEMMA 14. *Let $f \in \mathcal{L}$. Then $|f| \in \mathcal{L}$ and $\|\delta(|f|)\|_\infty \leqslant \|\delta(f)\|_\infty$.*

Proof. For every complex number a of modulus one, let $g_a = \mathrm{Re}(af)$. Then $g_a \in \mathrm{Re}(\mathcal{L})$, $\|\delta(g_a)\|_\infty \leqslant \|\delta(f)\|_\infty$, and $\|g_a\|_\infty \leqslant \|f\|_\infty$. Since $|f| = \bigvee_a g_a$, the conclusion follows from Lemma 13. ∎

We now define a measurable pseudometric ρ on X by setting

$$\rho(A, B) = \sup\{\rho_f(A, B) : f \in \mathcal{L}, \|f\|_D \leqslant 1\},$$

where $\rho_f(A, B)$ is the distance (in $\mathbf{C}$) between the essential ranges of $f|_A$ and $f|_B$. It is then easy to see that $\mathcal{L} \subset \mathrm{Lip}(X)$. Since $\mathcal{L}$ is ultraweakly dense in $L^\infty(X)$ by hypothesis, it follows that ρ is a measurable metric (not just a pseudometric).

LEMMA 15. *Let $A \subset X$ be a positive measure subset. Then there exists a positive function $f_A \in \mathcal{L}$ such that $\|\delta(f_A)\|_\infty \leqslant 1$ and for any $B \subset X$ we have $\rho(A, B) = \operatorname{ess\,inf}(f_A|_B)$.*

Proof. Define

$$f_A = \bigvee \{f \in \mathrm{Re}(\mathcal{L}) : \|\delta(f)\|_\infty \leqslant 1, \|f\|_\infty \leqslant 2, f|_A = 0 \text{ a.e.}\}.$$

Then $f_A \in \mathcal{L}$ and $\|\delta(f)\|_\infty \leqslant 1$ by Lemma 13. It is clear that f_A is positive.

Let $B \subset X$ be a positive measure subset and let $\varepsilon > 0$. By the definition of ρ there exists $f \in \mathcal{L}$ such that $\|f\|_D \leqslant 1$ and $\rho_f(A, B) \geqslant \rho(A, B) - \varepsilon$. Let S be the essential range of $f|_A$ and define

$$g = \bigwedge_{a \in S} |f - a| \qquad \left(= -\bigvee_{a \in S} -|f - a|\right).$$

Note that $\delta(f - a) = \delta(f)$ and $|a| \leqslant 1$ for all $a \in S$. Thus $\|g\|_\infty \leqslant 2$ and $\|\delta(g)\|_\infty \leqslant 1$ by Lemmas 13 and 14. Also, g is positive and satisfies $g|_A = 0$ a.e. Thus g belongs to the join which defines f_A, and so

$$f_A|_B \geqslant g|_B \geqslant \rho(A, B) - \varepsilon$$

almost everywhere. Taking $\varepsilon \to 0$ shows that $\operatorname{ess\,inf}(f_A|_B) \geqslant \rho(A, B)$.

Conversely, $\|f_A - 1\|_\infty \leqslant 1$, so $\|f_A - 1\|_D \leqslant 1$ and therefore

$$\rho(A, B) \geqslant \rho_{f_A - 1}(A, B) = \rho_{f_A}(A, B) = \text{ess inf}\,(f_A|_B).$$

Thus $\rho(A, B) = \text{ess inf}\,(f_A|_B)$. ∎

THEOREM 16. *Let $\delta: L^\infty(X) \to L^\infty(Y)$ be a W^*-derivation of an abelian von Neumann algebra into an abelian W^* bimodule. Then there exists a measurable metric ρ on X such that the domain algebra isometrically equals* Lip(X).

Proof. Let $\mathscr{L}$ be the domain algebra and define the measurable metric ρ and the functions f_A as above. By the definition of ρ, it is easy to see that the identity map carries $\mathscr{L}$ nonexpansively into Lip(X), i.e. $\|f\|_L \leqslant \|f\|_D$ for all $f \in \mathscr{L}$.

Let $f \in \text{Re}(\text{Lip}(X))$ and suppose $L(f) \leqslant 1$. For every positive measure subset $A \subset X$ let a_A be the essential infimum of $f|_A$. Then we claim that $a_A - f_A \leqslant f$ a.e. For if $B \subset X$ is a positive measure subset and $f \leqslant a_A - f_A - \varepsilon$ a.e. on B, then since $f \geqslant a_A$ a.e. on A we have (using Lemma 15)

$$\rho_f(A, B) \geqslant \text{ess inf}\,(f_A|_B) + \varepsilon = \rho(A, B) + \varepsilon,$$

contradicting the assumption that $L(f) \leqslant 1$. This proves the claim.

Since $f_A|_A = 0$ a.e., it follows that

$$f = \bigvee_{A \subset X} (a_A - f_A).$$

This equation shows that $f \in \mathscr{L}$ and $\|\delta(f)\|_\infty \leqslant 1$, by Lemma 13. We conclude that $\text{Re}(\text{Lip}(X)) = \text{Re}(\mathscr{L})$ (hence $\text{Lip}(X) = \mathscr{L}$) as sets, and $\|\delta(f)\|_\infty \leqslant L(f)$ for all $f \in \text{Re}(\mathscr{L})$.

Now for any $f \in \mathscr{L}$ we can find a complex number a of modulus one such that

$$\|\text{Re}(a\delta(f))\|_\infty = \|\delta(f)\|_\infty;$$

then

$$L(f) \geqslant L(\text{Re}(af)) = \|\delta(\text{Re}(af))\|_\infty = \|\delta(f)\|_\infty.$$

It follows that $\|f\|_L = \|f\|_D$ for all $f \in \mathscr{L}$, so that $\mathscr{L}$ is isometrically identified with Lip(X). ∎

We mentioned in section II that other norms besides $\|\cdot\|_D$ have been used on domain algebras, and it is also true that other norms besides $\|\cdot\|_L$ have been used on Lipschitz algebras. However, there is no obvious way of changing either or both of $\|\cdot\|_D$ and $\|\cdot\|_L$ in such a way as to retain the isometric character of Theorem 16. Thus for present purposes we are essentially forced to work with the norms we have chosen. (In the general theory of Lipschitz algebras there are also many reasons for preferring $\|\cdot\|_L$ over alternate norms; see in particular §§ II.2 and VII.1 of [79].)

A Counterexample

Theorem 16 requires that both the domain and range of δ must be abelian. That commutativity of the domain is not enough is shown by the following example. Let n be a positive integer and let D be the self-adjoint operator on an n-dimensional Hilbert space H given by the $n \times n$ matrix

$$D = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\mathcal{M}$ be the (abelian) algebra of diagonal matrices. Then $\delta(x) = i[D, x]$ defines a bounded derivation from $\mathcal{M}$ into $B(H)$.

For $2 \leqslant i \leqslant n$ let $x_i \in \mathcal{M}$ be the matrix all of whose entries are zero, except for the (i, i) entry, which is one. Then a direct calculation shows that $\|\delta(x_i)\| = 1$ for all i but $\|\delta(y)\| = \sqrt{n-1}$, where $y = x_2 \vee \cdots \vee x_n$.

Thus, we have an example of a W*-derivation with abelian domain, and a finite subset of the unit ball of the domain algebra whose join has arbitrarily large $\|\cdot\|_D$-norm. This shows that the domain algebra cannot be isometrically isomorphic to a Lipschitz algebra. Furthermore, by taking a direct sum as $n \to \infty$ we can get an example in which there is a subset of the unit ball of the domain algebra whose join does not even belong to the domain. Thus, the domain algebra now cannot even be isomorphic to a Lipschitz algebra.

Some Conjectures

Theorem 16 raises the possibility of generalizing known facts about Lipschitz algebras to domains of W*-derivations. There are three interesting problems relating to weak* closed ideals which arise in this way. First, is every weak* closed ideal of a W* domain algebra automatically self-adjoint? By ([77], comment preceding Lemma 3.1) this is the case for Lipschitz algebras, hence, by Theorem 16, for W* domain algebras in the commutative case.

Second, is every weak* closed ideal of dom(δ) equal to the intersection of its ultraweak closure in $\mathcal{M}$ with dom(δ)? By ([77], Theorem C) this again holds for Lipschitz algebras. (The corresponding statement for weak* closed subalgebras is not true; see the example in section 2 of [77].)

Finally, is the quotient of a W* domain algebra by any weak* closed ideal isomorphic to another W* domain algebra? Again, this is true in the commutative case, and in such a way that the isomorphism is isometric on self-adjoint elements. (For a Lipschitz algebra example in which the isomorphism is not simply isometric, see the end of § II.1 of [79].)

Atomic Abelian von Neumann Algebras

By combining the techniques of Corollary 7 and Lemma 12, we can obtain a complete description of all W*-derivations of atomic abelian von Neumann algebras. It turns out that they are all more or less of the type described at the start of section III.

THEOREM 17. *Let $\delta : l^\infty(X) \to l^\infty(Y)$ be a W*-derivation of atomic abelian von Neumann algebras. Then there is a pair of maps $\alpha, \beta \colon Y \to X$ and a map $\gamma \colon Y \to \mathbf{R}$ such that*

$$\delta(f) = \gamma \cdot (f_l - f_r) = \gamma \cdot (f \circ \alpha - f \circ \beta)$$

for all $f \in \mathrm{dom}(\delta)$.

Proof. As the maps $f \mapsto f_l, f_r$ are unital normal $*$-homomorphisms of atomic abelian von Neumann algebras, they are adjoint to maps $\alpha, \beta \colon Y \to X$; that is, $f_l(p) = f(\alpha(p))$ and $f_r(p) = f(\beta(p))$ for all $p \in Y$. (For instance, this follows from Corollary A of [77].)

Let $p \in Y$ and suppose first that $\alpha(p) = \beta(p)$, so that $f_l(p) = f_r(p)$ for all $f \in l^\infty(X)$. Let $\mathscr{I}_p = \{f \in \mathrm{dom}(\delta) : f_l(p) = 0\}$ and observe that $\delta(f)(p) = 0$ for all $f \in \mathscr{I}_p$ by the first part of Corollary 4. For any $f \in \mathrm{dom}(\delta)$ we have $f - f(p) \in \mathscr{I}_p$, and since δ vanishes on constant functions this implies that $\delta(f)(p) = 0$ for all $f \in \mathrm{dom}(\delta)$. Thus the desired equality is satisfied if we define $\gamma(p) = 0$.

Now suppose $\alpha(p) \neq \beta(p)$. Then since $\mathrm{dom}(\delta)$ is ultraweakly dense in $l^\infty(X)$, there exists $g \in \mathrm{dom}(\delta)$ such that $g(\alpha(p)) \neq g(\beta(p))$, i.e. $g_l(p) \neq g_r(p)$. Define

$$h^\alpha = \mathrm{Re}\left(\frac{g - g_r(p)}{g_l(p) - g_r(p)}\right)$$

and

$$h^\beta = \mathrm{Re}\left(\frac{g - g_l(p)}{g_r(p) - g_l(p)}\right);$$

then $h^\alpha_l(p) = h^\beta_r(p) = 1$ and $h^\alpha_r(p) = h^\beta_l(p) = 0$. Thus for any $f \in \mathrm{dom}(\delta)$ we have $f - f(h^\alpha + h^\beta) = 0$ on $\alpha(p)$ and $\beta(p)$, so by Corollary 4

$$\delta(f - f(h^\alpha + h^\beta))(p) = 0$$

hence

$$\delta(f)(p) = \delta(f(h^\alpha + h^\beta))(p)$$
$$= f_l\,\delta(h^\alpha)(p) + f_r\,\delta(h^\beta)(p).$$

With $f = 1$ this implies that $\delta(h^\alpha)(p) = -\delta(h^\beta)(p)$, and we define $\gamma(p)$ to be this number. Since h^α is real-valued, so is $\delta(h^\alpha)$, hence $\gamma(p) \in \mathbf{R}$. Thus,

$$\delta(f)(p) = \gamma(p) \cdot (f_l(p) - f_r(p)) = \gamma(p) \cdot (f \circ \alpha(p) - f \circ \beta(p))$$

for all $f \in \mathrm{dom}(\delta)$. ∎

V. Noncommutative Metric Spaces

In this section we make some general comments about noncommutative metric spaces.

Noncommutative Metric Spaces

By the duality between X and $\mathrm{Lip}_0(X)$ discussed in Section I, Lipschitz algebras play more or less the same role with respect to metric spaces as do abelian C*-algebras with respect to topological spaces. It is therefore reasonable to expect that Lipschitz algebras will be important in the theory of noncommutative metric spaces.

A standard sort of idea is that we might describe a "noncommutative metric space" by a norm-dense $*$-subalgebra $\mathscr{L}$ of a C*-algebra $\mathscr{A}$. This would be analogous to the subalgebra $\mathrm{Lip}(X) \subset C(X)$ for X a compact metric space, so that we would loosely think of $\mathscr{A}$ as the set of "noncommutative continuous functions" and $\mathscr{L}$ as the set of "noncommutative Lipschitz functions." This set-up could be thought of as describing a noncommutative metric on the grounds that in the commutative case one way of defining a metric on X is by specifying the algebra of Lipschitz functions.

The use of a C*-algebra here in a sense runs counter to the commutative theory, for one does not typically bestow a metric on a topological space. Rather, one defines a metric on a set, or more generally on a measure space (see Section I), and this seems to be a more basic operation than metrizing a topological space. Thus, we might instead ask for an ultraweakly dense $*$-subalgebra $\mathscr{L}$ of a von Neumann algebra $\mathscr{M}$. Of course, once $\mathscr{L}$ is given we can take $\mathscr{A}$ to be the operator norm closure of $\mathscr{L}$ in $\mathscr{M}$ and in this way describe a "noncommutative topology" associated to the "noncommutative metric" given by $\mathscr{L}$.

The main question is exactly which subalgebras $\mathscr{L}$ should be thought of as describing noncommutative metrics. Obviously we would like a definition which, in the commutative case, specifies exactly the Lipschitz algebras. One natural approach would be to try to define the relevant subalgebras $\mathscr{L}$ axiomatically, based on some abstract characterization of Lipschitz algebras; however, all known characterizations involve lattice structure and so there is no obvious way of removing the commutativity hypothesis to get a noncommutative definition.

The seminal idea of using derivations in this connection is implicit in the work of Connes (see [22], Chapter VI). It seems well corroborated here by our characterization of Lipschitz algebras as being precisely the domains of W*-derivations of von Neumann algebras. This leads to the following prescription:

> *A noncommutative metric space is described by a von Neumann algebra $\mathscr{M}$ and a subalgebra $\mathscr{L}$ which is the domain algebra of some W*-derivation of $\mathscr{M}$.*

The analogy to noncommutative topology is summarized in the following diagram.

$$\begin{tikzcd} \text{metric spaces} \arrow[r, leftrightarrow] & \text{Lipschitz algebras} \arrow[d] & \text{topological spaces} \arrow[r, leftrightarrow] & \text{abelian C*-algebras} \arrow[d] \\ & \text{W* domain algebras} & & \text{C*-algebras} \end{tikzcd}$$

Relation to Previous Work

The theory of noncommutative metric spaces was initiated by Connes and has been developed in [20], [21], [22], [23], [59], [64]. His approach is motivated by a construction involving a compact connected Riemannian spin manifold X. Namely, there is a derivation of $C(X)$ into $B(L^2(X, S))$ (the bounded operators on the Hilbert space of L^2 spinors on X) given by commutation with the Dirac operator D. The domain of this derivation is $\mathrm{Lip}(X)$. (A simpler version of this example was given in Section III.)

The idea at this point is that the metric on X can be recovered from $\mathrm{Lip}(X)$, hence from the Dirac operator. Thus, the metric is encoded in the triple $(C(X), L^2(X, S), D)$. Based on this fact Connes argues that noncommutative metric spaces should be modelled by "unbounded Fredholm modules," i.e. triples $(\mathscr{A}, H, D)$ consisting of a C*-algebra $\mathscr{A}$ represented on a Hilbert space H together with an unbounded self-adjoint operator D on H.

Our approach in this paper is based on Connes', but differs from it in some important ways. First, we take as fundamental not the operator D, but rather the derivation $x \mapsto i[D, x]$. This then leads us to the more general case of abstract derivations, which can be used to model $\mathrm{Lip}(X)$ for any metric space X (not just Riemannian manifolds). The abstract point of view also has the advantage of allowing us to distinguish the case where E is abelian. The passage from C*-algebras to von Neumann algebras is needed for technical reasons in the proof of Theorem 16 and also is supported by the argument given in the last subsection. (See Section VI for more on the C*-algebra versus von Neumann algebra question.)

"*Noncommutative*" *Spaces*

There are already several kinds of operator-theoretic constructs which are viewed as "noncommutative" or "quantized" versions of various types of classical spaces. As Effros has emphasized [28], this can generally be viewed as a consequence of the idea that an operator is a "quantized" function. It is possible to trace this whole circle of ideas back to the basic idea of Birkhoff and von Neumann [6] that Hilbert spaces are analogous to sets, with subsets corresponding to closed linear subspaces, union of subsets to closed linear span, etc. (This idea is motivated by the fact that in classical physics the phase space of a system is modelled by a set, whereas in quantum mechanics the phase space is modelled by a Hilbert space.) Next we have Mackey's analysis of the Hilbert space version of a real-valued function on a set (or a measurable real-valued function on a measure space) ([49], pp. 162–163). This argument shows how the Birkhoff–von Neumann analogy leads to an analogy (again with physical motivation) between real-valued functions and spectral measures, i.e. self-adjoint operators. Thus, the Hilbert space version of $l^\infty(X)$ is $B(H)$.

From here we can move on to the idea that a concrete C*-algebra is the Hilbert space version of a topological space. This is done by observing that one way to specify a completely regular topology on a set is by indicating which bounded complex-valued functions are continuous. That is, one specifies a C*-subalgebra of $l^\infty(X)$. By the last paragraph, the Hilbert space version of a topology is therefore given by a C*-subalgebra of $B(H)$.

VI. From C*-Algebras to von Neumann Algebras

As we mentioned in the introduction, it seems to be felt that the unbounded derivations of greatest interest are C*-algebraic in character. To a large extent abelian examples come from differential geometry and nonabelian examples come from physics.

On the other hand, the general theory may be more tractable in the von Neumann algebra case. For bounded derivations this is seen in the celebrated derivation theorem of Kadison and Sakai, which states that every bounded derivation of von Neumann algebras is inner ([68], Theorem 2.5.3). (A survey of related results for C*-algebras is given in

[82].) Our characterization of domains is further evidence in this direction, for there seems little hope of characterizing the domains of derivations of C*-algebras. (In the first place, it is not clear what class of spaces would replace the Lipschitz algebras; secondly, even in the rather thorough literature on derivations of $C[0, 1]$ (see [4], [25], [31], [47], [48]) there is no general characterization of the domain, nor even a conjecture along this line.)

Let $\mathscr{A}$ be a unital C*-algebra and F a self-adjoint operator $\mathscr{A}$-bimodule. Then we define a *C*-derivation* $\delta: \mathscr{A} \to F$ to be an unbounded linear map whose domain is a norm dense, unital $*$-subalgebra of $\mathscr{A}$, whose graph is a norm closed subspace of $\mathscr{A} \oplus F$, and which satisfies

$$\delta(x^*) = \delta(x)^* \qquad \text{and} \qquad \delta(xy) = x\,\delta(y) + \delta(x)\,y$$

for all $x, y \in \mathrm{dom}(\delta)$.

Because of the above reasons we are interested in whether examples of C*-derivations can be extended to a von Neumann algebra setting. An ideal model for such an extension is provided by the derivative map $\delta_1: C[0, 1] \to C[0, 1]$, defined by $\delta_1(f) = f'$, with domain $C^1[0, 1]$. This C*-derivation can be extended to a W*-derivation $\delta_2: L^\infty[0, 1] \to L^\infty[0, 1]$ by considering $C[0, 1]$ as contained in $L^\infty[0, 1]$ and taking the ultraweak closure of the graph. The result is an enlargement of the domain to $\mathrm{Lip}[0, 1]$. (Ultraweak density of the graph of δ_1 in the graph of δ_2 follows from Theorem B of [77].)

This suggests that differential geometric examples, at least, can be formulated in a von Neumann algebra setting, in general by expanding the domain from $C^1(X)$ to $\mathrm{Lip}(X)$. It is interesting that one does not "lose the topology" by doing this. For instance, in the above example, although δ_2 is obtained by taking the ultraweak closure of the graph of δ_1, its domain is still contained in $C[0, 1]$. (Every Lipschitz function is continuous.)

We consider this an important phenomenon and in general pose the following question: given a C*-derivation $\delta: \mathscr{A} \to F$, are there a natural ambient von Neumann algebra $\mathscr{M} \supset \mathscr{A}$ and W*$\mathscr{M}$-bimodule $E \supset F$ such that the graph of δ is ultraweakly closable in $\mathscr{M} \oplus E$, and the domain of the ultraweak closure of δ is still contained in $\mathscr{A}$?

W-Dynamical Systems*

The following results give a means of constructing examples of this sort. They are based on material in ([15], Chapter 3).

Let $(\mathscr{M}, \mathbf{R}, \alpha)$ be a W*-dynamical system. Then define

$$\mathscr{A} = \{x \in \mathscr{M}: \|\alpha_t(x) - x\| \to 0 \text{ as } t \to 0\}$$

and define unbounded derivations $\delta_1, \delta_2 \colon \mathcal{M} \to \mathcal{M}$ by

$$\delta_k(x) = \lim_{t \to 0} \frac{\alpha_t(x) - x}{t},$$

where the limit is taken in the norm topology for $k = 1$ and the ultraweak topology for $k = 2$, with domains consisting of all x for which these limits exist.

PROPOSITION 18. *Let $(M, \mathbf{R}, \alpha)$ be a W^*-dynamical system and define $\mathcal{A}$, δ_1, and δ_2 as above. Then $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{A}$,* $\mathrm{dom}(\delta_1)$, *and* $\mathrm{dom}(\delta_2)$ *are invariant under α. We have*

$$\mathcal{A} = \{x \in \mathcal{M} : t \mapsto \alpha_t(x) \text{ is norm continuous}\},$$

$$\mathrm{dom}(\delta_1) = \{x \in \mathcal{M} : t \mapsto \alpha_t(x) \text{ is norm continuously differentiable}\}$$

$$\mathrm{dom}(\delta_2) = \{x \in \mathcal{M} : t \mapsto \alpha_t(x) \text{ is ultraweakly continuously differentiable}\},$$

in the sense that there exists a norm continuous map $\beta \colon \mathbf{R} \to \mathcal{A}$ such that $\delta_1(\alpha_t(x)) = \beta(t)$ for all t (respectively, ultraweakly continuous map $\beta \colon \mathbf{R} \to \mathcal{M}$ such that $\delta_2(\alpha_t(x)) = \beta(t)$ for all t).

Proof. It is easy to check that $\mathcal{A}$ is a unital C*-algebra, and invariance of $\mathcal{A}$, $\mathrm{dom}(\delta_1)$, and $\mathrm{dom}(\delta_2)$ under α follows from applying α_t to their original definitions.

The second statement follows from the invariance of $\mathcal{A}$, $\mathrm{dom}(\delta_1)$, and $\mathrm{dom}(\delta_2)$ under α, in the latter part by letting $\beta(t) = \alpha_t(\delta_k(x))$ $(k = 1, 2)$. ∎

The characterization of $\mathrm{dom}(\delta_2)$ given in this proposition has a physical interpretation. Suppose $\mathcal{M}$ is the algebra of observables of some physical system and α represents the time evolution of the system. Then δ_2 is the generator of α and Proposition 18 shows that $\mathrm{dom}(\delta_2)$ consists of precisely those observables whose expectation value in each normal state evolves in a continuously differentiable manner.

THEOREM 19. *Let $(\mathcal{M}, \mathbf{R}, \alpha)$, $\mathcal{A}$, δ_1, and δ_2 be as in the proposition. Then δ_1 is a C^*-derivation of $\mathcal{A}$ into itself, δ_2 is a W^*-derivation of $\mathcal{M}$ into itself,* $\mathrm{dom}(\delta_2) \subset \mathcal{A}$, *and the graph of δ_2 is the ultraweak closure of the graph of δ_1.*

Proof. If the norm limit $(\alpha_t(x) - x)/t$ exists then clearly $\alpha_t(x) \to x$ in norm, so $\mathrm{dom}(\delta_1) \subset \mathcal{A}$. Furthermore, $\delta_1(x)$ is then the norm limit of the elements $(\alpha_t(x) - x)/t$, which belong to $\mathcal{A}$, hence $\delta(x) \in \mathcal{A}$. So $\mathrm{ran}(\delta_1) \subset \mathcal{A}$ as well.

It follows from ([15], Proposition 3.1.6) that δ_2 is a W*-derivation of $\mathcal{M}$, and essentially the same argument shows that δ_1 is a C*-derivation of $\mathcal{A}$.

By ([15], Proposition 3.1.23) we have $\|\alpha_t(x) - x\| \leqslant t\, \|\delta_2(x)\|$ for all $x \in \mathrm{dom}(\delta_2)$, and this implies $\mathrm{dom}(\delta_2) \subset \mathcal{A}$. Since $\mathrm{dom}(\delta_2)$ is ultraweakly dense in $\mathcal{M}$ and $\mathrm{dom}(\delta_1)$ is norm dense in $\mathcal{A}$, it follows that $\mathrm{dom}(\delta_1)$ is ultraweakly dense in $\mathcal{M}$ and hence ([15], Corollary 3.1.7) the graph of δ_1 is ultraweakly dense in the graph of δ_2. ∎

The above results also apply to the slightly more general case of a W*-dynamical system of the form $(\mathcal{M}, \mathbf{R}^n, \alpha)$. For such a system choose a basis $\{f_1, ..., f_n\}$ of $\mathbf{R}^n$, and for $1 \leqslant j \leqslant n$ let α^j be the one-parameter action $\alpha^j_t = \alpha_{tf_j}$ obtained by restricting to the one-dimensional subgroup $\{tf_j\}$ of $\mathbf{R}^n$.

For each j we then define $\mathcal{A}_j$, δ^j_1, and δ^j_2 as in the one-parameter case. Let $F = \mathcal{A}^n$ and $E = \mathcal{M}^n$ be the canonical free Hilbert modules over $\mathcal{A}$ and $\mathcal{M}$. Finally let $\mathcal{A} = \bigcap A_j$ and define $\delta_1 \colon \mathcal{A} \to \mathcal{A}^n$ and $\delta_2 \colon \mathcal{M} \to \mathcal{M}^n$ by

$$\delta_k(x) = (\delta^1_k(x), ..., \delta^n_k(x)),$$

with $\mathrm{dom}(\delta_k) = \bigcap \mathrm{dom}(\delta^j_k)$. These definitions are motivated by the example $\mathcal{A} = C(\mathbf{T}^n)$, $\mathcal{M} = L^\infty(\mathbf{T}^n)$, and with α the natural translation of the n-torus by $\mathbf{R}^n$. In this case F and E are the continuous and measurable versions of the tangent bimodule, and δ_1 and δ_2 are continuous and measurable versions of the exterior derivative.

In general we have the following result.

THEOREM 20. *Let $(\mathcal{M}, \mathbf{R}^n, \alpha)$ be a W*-dynamical system and define $\mathcal{A}$, δ_1, and δ_2 as above. Then $\mathcal{A}$ is a unital C*-algebra and $\mathcal{A}$,* $\mathrm{dom}(\delta_1)$, *and* $\mathrm{dom}(\delta_2)$ *are invariant under α. We have*

$$\mathcal{A} = \{x \in \mathcal{M} : v \mapsto \alpha_v(x) \textit{ is norm continuous}\},$$

$$\mathrm{dom}(\delta_1) = \{x \in \mathcal{M} : v \mapsto \alpha_v(x) \textit{ is norm continuously differentiable}\},$$

$$\mathrm{dom}(\delta_2) = \{x \in \mathcal{M} : v \mapsto \alpha_v(x) \textit{ is ultraweakly continuously differentiable}\},$$

in the sense that there exists a norm continuous map $\beta \colon \mathbf{R}^n \to \mathcal{A}^n$ such that $\delta_1(\alpha_v(x)) = \beta(v)$ for all $v \in \mathbf{R}^n$ (respectively, an ultraweakly continuous map $\beta \colon \mathbf{R}^n \to \mathcal{M}^n$ such that $\delta_2(\alpha_v(x)) = \beta(v)$ for all $v \in \mathbf{R}^n$).

Furthermore, δ_1 is a C-derivation, δ_2 is a W*-derivation, and $\mathcal{A}$ is the norm-closure of* $\mathrm{dom}(\delta_2)$. ∎

Most of Theorem 20 is an immediate consequence of Proposition 18 and Theorem 19. The map β is defined by

$$\beta(v) = (\alpha_v(\delta_1^k(x)), ..., \alpha_v(\delta_n^k(x))).$$

(To show norm density of $\mathrm{dom}(\delta_1)$ in $\mathscr{A}$, the one-parameter argument is adapted as follows. For each $x \in \mathscr{A}$ and $m > 0$ define

$$x_m = \int_0^\infty \cdots \int_0^\infty e^{-m(t_1 + \cdots + t_n)} \alpha_{t_1 f_1 + \cdots + t_n f_n}(x)\, dt_1 \cdots dt_n.$$

Then since $\alpha_{v+w}(x) = \alpha_v(\alpha_w(x))$ for all $v, w \in \mathbf{R}^n$, for each j we can rearrange the integral so that

$$x_m = \int_0^\infty e^{-mt_j} \alpha_{t_j f_j}(y)\, dt_j$$

where $y \in \mathscr{A}$ is the rest of the integral. Then just as in the one-parameter case (see [15], Proposition 3.1.6) this implies that $x_m \in \mathrm{dom}(\delta_j^1)$. Also since $\alpha_v(x)$ is a norm continuous function of v it follows that $m^n x_m \to x$ in norm, and we conclude that $\mathrm{dom}(\delta_1)$ is norm dense in $\mathscr{A}$. An analogous argument shows that the domain of δ_2 is ultraweakly dense in $\mathscr{M}$. Finally, $\mathrm{dom}(\delta_2) \subset \mathrm{dom}(\delta_1) \subset \mathscr{A}$ and conversely, for any $x \in \mathscr{A}$ the sequence $m^n x_m$ is contained in $\mathrm{dom}(\delta_2)$, hence by the above $\mathrm{dom}(\delta_2)$ is norm-dense in $\mathscr{A}$.)

See [80] for examples of W*-dynamical systems of the above form which arise from quantum deformations a la Rieffel.

The Noncommutative Torus

The preceding ideas can be applied to the noncommutative torus. This will give us a specific example of a noncommutative metric space.

Our discussion of the noncommutative torus is based on [63]. Fix a real number $\theta \in (0, 1)$ and define unitary operators $U, V \in B(l^2(\mathbf{Z}^2))$ by setting

$$Uv_{mn} = v_{(m+1)n} \qquad \text{and} \qquad Vv_{mn} = e^{2\pi i\theta m} v_{m(n+1)},$$

where v_{mn} is the canonical basis of $l^2(\mathbf{Z}^2)$. Let $\mathscr{A}_\theta$ and $\mathscr{M}_\theta$ respectively be the C*-algebra and von Neumann algebra generated by U and V. In the $\theta = 0$ case the Fourier transform takes $\mathscr{A}_\theta$ and $\mathscr{M}_\theta$ to $C(\mathbf{T}^2)$ and $L^\infty(\mathbf{T}^2)$, respectively.

In general we define the *noncommutative Fourier series* of x to be the formal sum

$$\sum_{m,n} a_{mn} U^m V^n$$

where $a_{mn} = \langle xv_{00}, v_{mn} \rangle$. If x is a polynomial in U and V, the sum of this series evidently equals x. Also, for $N > 0$ define

$$s_N(x) = \sum_{|m|,\, |n| \leqslant N} a_{mn} U^m V^n$$

and

$$\sigma_N(x) = (s_0 + \cdots + s_N)/(N+1).$$

These are the partial sums and Cesaro means of the noncommutative Fourier series of x, respectively. (For basic material on harmonic analysis see [34], [44], or [83].)

The following proposition gives an alternative definition of $\mathscr{M}_\theta$. Its proof involves a noncommutative version of the fact that the Fourier series of any L^∞ function converges to that function weak*.

PROPOSITION 21. *Define $\mathscr{M}_\theta$ as above. Then $x \in \mathscr{B}(l^2(Z^2))$ belongs to $\mathscr{M}_\theta$ if and only if*

$$\langle xv_{mn}, v_{(m+j)(n+k)} \rangle = e^{2\pi i\theta mk} \langle xv_{00}, v_{jk} \rangle \tag{$\dagger$}$$

for all $j, k, m, n \in \mathbf{Z}$.

Proof. It is easy to see that every element of $\mathscr{M}_\theta$ satisfies ($\dagger$). To show the converse, suppose $x \in B(l^2(\mathbf{Z}^2))$ satisfies ($\dagger$); then we have $s_N(x) \to x$ (weak operator) since

$$\langle xv, w \rangle = \sum_{j,\, k,\, m,\, n} \langle v, v_{jk} \rangle \langle xv_{jk}, v_{mn} \rangle \langle v_{mn}, w \rangle$$

and

$$\langle s_N(x)v, w \rangle = \sum_{|j-m|,\, |k-n| \leqslant N} \langle v, v_{jk} \rangle \langle xv_{jk}, v_{mn} \rangle \langle v_{mn}, w \rangle$$

for any $v, w \in l^2(\mathbf{Z}^2)$. Since $s_N(x) \in \mathscr{M}_\theta$ it follows that $x \in \mathscr{M}_\theta$. ∎

Define unbounded self-adjoint operators D_1, D_2 on H by

$$D_1 v_{mn} = m v_{mn} \qquad \text{and} \qquad D_2 v_{mn} = n v_{mn}.$$

For $\theta = 0$ these correspond via the Fourier transform to $i\partial/\partial x$ and $i\partial/\partial y$. Then we have two flows of $\mathscr{M}_\theta$, given by

$$\alpha_t^k(x) = e^{-itD_k} x e^{itD_k}$$

($k = 1, 2$). For $\theta = 0$ these correspond to the translations of $L^\infty(\mathbf{T}^2)$ in the two variables.

Let δ_1 and δ_2 be the generators of these flows and define a W*-derivation $\delta\colon \mathcal{M}_\theta \to \mathcal{M}_\theta \oplus \mathcal{M}_\theta$ by $\delta(x) = (\delta_1(x), \delta_2(x))$, with domain $\mathrm{dom}(\delta) = \mathrm{dom}(\delta_1) \cap \mathrm{dom}(\delta_2)$. Equivalently, $\delta(x) = (i[D_1, x], i[D_2, x])$.

Here $\mathcal{M}_\theta \oplus \mathcal{M}_\theta$ is naturally an $\mathcal{M}_\theta$-bimodule and it has corresponding left and right Hilbert module structure, given by the inner products

$$\langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle = x_1 x_2^* + y_1 y_2^*$$

and

$$\langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle = x_1^* x_2 + y_1^* y_2.$$

When embedded in the linking algebra in the natural self-adjoint manner, i.e. taking $x \oplus y \in \mathcal{M}_\theta \oplus \mathcal{M}_\theta$ to

$$\begin{pmatrix} 0 & x & y \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

in $M_3(\mathcal{M}_\theta)$, the norm becomes the max of the left and right Hilbert module norms.

Let $\mathcal{L}_\theta$ be the domain algebra of δ. We consider this to be the algebra of "noncommutative Lipschitz functions" on the noncommutative torus, on the grounds that for $\theta = 0$ it corresponds to $\mathrm{Lip}(\mathbf{T}^2)$.

We have the following characterization of $\mathcal{A}_\theta$, which will allow us to apply Theorem 20 to the present example. It is a version of the classical fact that every continuous function on the torus is the uniform limit of the Cesaro means of its Fourier series expansion.

(The first equality, showing that $\mathcal{A}_\theta$ is the norm-continuous part of the W*-dynamical system given by the torus action on $\mathcal{M}_\theta$, follows from Corollary 1 of [55]. This is actually all that we need, although the additional information about Fourier series may be of independent interest.)

THEOREM 22. *Let $\mathcal{A}_\theta$, $\mathcal{M}_\theta$, δ_k, and α_t^k be as above. Then*

$$\begin{aligned} \mathcal{A}_\theta &= \{x \in \mathcal{M}_\theta : t \mapsto \alpha_t^k(x) \textit{ is norm continuous for } k = 1, 2\} \\ &= \{x \in \mathcal{M}_\theta : \|x - \sigma_N(x)\| \to 0\}. \end{aligned}$$

Proof. Let

$$\mathcal{A}' = \{x \in \mathcal{M}_\theta : t \mapsto \alpha_t^k(x) \text{ is norm continuous for } k = 1, 2\}$$

and

$$\mathcal{A}'' = \{x \in \mathcal{M}_\theta : \|x - \sigma_N(x)\| \to 0\}.$$

Then it is easy to see that $\mathcal{A}'$ is a C*-algebra which contains the operators U and V, hence $\mathcal{A}_\theta \subset \mathcal{A}'$; and $\mathcal{A}'' \subset \mathcal{A}_\theta$ is also easy since each $\sigma_N(x)$ belongs to $\mathcal{A}_\theta$, hence $\sigma_N(x) \to x$ implies $x \in \mathcal{A}_\theta$.

It remains to show that $\mathcal{A}' \subset \mathcal{A}''$. Thus let $x \in \mathcal{M}_\theta$ and suppose $t \mapsto \alpha^1_t(x)$ and $t \mapsto \alpha^2_t(x)$ are continuous in the operator norm. We will show that $\|x - \sigma_N(x)\| \to 0$.

Let K_N be the Fejér kernel,

$$K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int} = \frac{1}{N+1} \left(\frac{\sin((N+1)\, t/2)}{\sin\,(t/2)}\right)^2.$$

Then

$$x = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x K_N(s)\, K_N(t)\, ds\, dt$$

since $\int_{-\pi}^{\pi} K_N(s)\, ds = 1$. (All operator integrals in this proof can be taken in the Riemann sense.) Also

$$\sigma_N(x) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \alpha^1_s(\alpha^2_t(x))\, K_N(s)\, K_N(t)\, ds\, dt;$$

this is verified by observing that, for $m, n \in \mathbf{Z}$, $|m|, |n| \leqslant N$,

$$\langle \sigma_N(x) v_{00}, v_{mn} \rangle = \left(1 - \frac{|m|}{N+1}\right)\left(1 - \frac{|n|}{N+1}\right) \langle x v_{00}, v_{mn} \rangle,$$

while

$$\langle \alpha^1_s(\alpha^2_t(x)) v_{00}, v_{mn} \rangle = e^{-i(ms+nt)} \langle x v_{00}, v_{mn} \rangle$$

hence

$$\iint \langle \alpha^1_s(\alpha^2_t(x))\, v_{00}, v_{mn} \rangle\, K_N(s)\, K_N(t)\, ds\, dt$$

$$= \langle x v_{00}, v_{mn} \rangle \iint e^{-i(ms+nt)} K_N(s)\, K_N(t)\, ds\, dt$$

$$= \left(1 - \frac{|m|}{N+1}\right)\left(1 - \frac{|n|}{N+1}\right) \langle x v_{00}, v_{mn} \rangle.$$

Therefore

$$\begin{aligned}
x-\sigma_N(x) &= \iint (x-\alpha_s^1(\alpha_t^2(x)))\, K_N(s)\, K_N(t)\, ds\, dt \\
&= \iint (x-\alpha_s^1(x))\, K_N(s)\, K_N(t)\, ds\, dt \\
&\quad + \iint \alpha_s^1(x-\alpha_t^2(x))\, K_N(s)\, K_N(t)\, ds\, dt \\
&= \int (x-\alpha_s^1(x))\, K_N(s)\, ds \\
&\quad + \int \alpha_s^1\left(\int (x-\alpha_t^2(x))\, K_N(t)\, dt\right) K_N(s)\, ds.
\end{aligned}$$

This reduces the problem to showing that both of these last two integrals go to zero as $N \to \infty$.

The first integral is small for large N since $s \mapsto (x-\alpha_s^1(x))$ is norm continuous and is zero for $s=0$, while $\int |K_N(s)|\, ds = 1$ and for any $\varepsilon > 0$ we have $\int_{|s| \geqslant \varepsilon} |K_N(s)|\, ds \to 0$ as $N \to \infty$. The same argument shows that the inner part of the second integral is small, from which it follows that the whole second integral is small because α_s^1 is an isometry and $\int |K_N(s)|\, ds = 1$. ▮

Since the two actions α^1 and α^2 commute, we actually have an action of $\mathbf{R}^2$ (in fact, $\mathbf{T}^2$) on $\mathcal{M}_\theta$, i.e. a W*-dynamical system $(\mathcal{M}_\theta, \mathbf{R}^2, \alpha)$. Using Proposition 22, we see that the above definitions of $\mathcal{A}_\theta$ and δ agree with the definitions of $\mathcal{A}$ and δ_2 given in the last subsection. Therefore all of the conclusions of Theorem 20 hold.

In particular, it follows that $\mathcal{L}_\theta \subset \mathcal{A}_\theta$, a welcome conclusion for the reasons discussed at the start of this section and in Section V. This shows that every "noncommutative Lipschitz function" is a "noncommutative continuous function."

It then follows automatically from Proposition 22 that every $x \in \mathcal{L}_\theta$ is the norm limit of the Cesaro means of its noncommutative Fourier series. But more is true; in fact, for $x \in \mathcal{L}_\theta$ we have $s_N(x) \to x$ in norm. In the $\theta = 0$ case this reduces to the nontrivial assertion that the partial sums of the Fourier series expansion of a Lipschitz function on the torus converge uniformly to that function. Our proof of the noncommutative result closely follows a proof of this commutative version which was kindly supplied by Thomas Wolff.

THEOREM 23. *Let* $x \in \mathcal{L}_\theta$. *Then* $\|x - s_N(x)\| \to 0$.

Proof. Let D_N be the Dirichlet kernel,

$$D_N(t) = \sum_{n=-N}^{N} e^{int} = \frac{\sin(n+1/2)t}{\sin(t/2)},$$

and V_N the de la Valée Poussin kernel,

$$V_N = 2K_{2N+1} - K_N.$$

(K_N is the Fejér kernel defined in the proof of Proposition 22.) Let $G_N(s,t) = D_N(s)D_N(t)$ and $W_N(s,t) = V_N(s)V_N(t)$ and define

$$s'_N(x) = \iint \alpha_s^1(\alpha_t^2(x))\, W_N(s,t)\, ds\, dt.$$

And observe that

$$s_N(x) = \iint \alpha_s^1(\alpha_t^2(x))\, G_N(s,t)\, ds\, dt.$$

Then $\|x - s'_N(x)\| \to 0$ by the same argument which showed that $\|x - \sigma_N(x)\| \to 0$ in the proof of Proposition 22. So we must show that $\|s_N(x) - s'_N(x)\| \to 0$.

Let $y_N = s_N(x) - x'_N(x)$ and let a_{mn} and b_{mn} be the noncommutative Fourier coefficients of x and y_N, respectively. Then the b_{mn} have the following properties:

(1) $|b_{mn}| \leqslant |a_{mn}|$ for all m, n;

(2) $b_{mn} = 0$ if $|m| \leqslant N$ and $|n| \leqslant N$; and

(3) $b_{mn} = 0$ if $|m| > 2N+1$ or $|n| > 2N+1$.

Therefore

$$\sum |b_{mn}| \leqslant \sum_{(m,n) \in A_N} |a_{mn}|$$

where $A_N = \{(m,n) : N < \max(|m|, |n|) \leqslant 2N+1\}$. For such (m,n), $|m|^2 + |n|^2 > N^2$ and therefore

$$\sum_{(m,n) \in A_N} (m^2 + n^2)^{-1} \leqslant C$$

where C is a universal constant. So

$$\begin{aligned}\|y_N\| &\leqslant \sum |b_{mn}| \\ &\leqslant \sum_{(m,\, n) \in A_N} |a_{mn}| \\ &\leqslant \left(C \sum_{(m,\, n) \in A_N} (m^2+n^2)\, |a_{mn}|^2 \right)^{1/2}.\end{aligned}$$

(The last line follows from the Cauchy–Schwarz inequality and the previous displayed equation.)

Finally, since $x \in \operatorname{dom}(\delta_1)$ the operator $\delta_1(x) = i[D_1, x]$ is bounded, and its noncommutative Fourier coefficients are ma_{mn}. Thus

$$\|\delta_1(x)(v_{00})\|^2 = \sum m^2\, |a_{mn}|^2$$

is finite. Likewise $\sum n^2\, |a_{mn}|^2$ is finite, and therefore $\sum_{(m,\, n) \in A_N} (m^2+n^2) \cdot |a_{mn}|^2$ goes to zero as N goes to infinity. Thus $\|y_N\| \to 0$, as desired. ∎

We close with a result about the unit ball of $\mathscr{L}_\theta$, which answers a question posed by Marc Rieffel. It is motivated by the fact that if X is a metric space, then the unit ball of $\mathrm{Lip}(X)$ is compact in sup norm if and only if X is compact.

THEOREM 24. *The unit ball of $\mathscr{L}_\theta$ is compact in operator norm.*

Proof. Let (x^k) be a sequence in $\mathscr{L}_\theta$ with $\|x^k\|_L \leqslant 1$ for all k. Let a^k_{mn} be the noncommutative Fourier series of x^k; then since $\|x^k\| \leqslant 1$ for all k it follows that $|a^k_{mn}| \leqslant 1$ for all k, m, n. Thus we can choose a subsequence x^{j_k} such that for all m and n the sequence $(a^{j_k}_{mn})$ converges.

Let x be a weak* limit of the subsequence in $\mathscr{L}_\theta$. Then x also belongs to the unit ball, and we have $a^{j_k}_{mn} \to a_{mn}$ for all m, n where a_{mn} is the noncommutative Fourier series of x. We must show that $x^{j_k} \to x$ in operator norm.

Now the estimates for $x - \sigma_N(x)$ in Theorem 22 only depended on the Lipschitz norm of x. Thus, for any positive ε we can choose N so that $n \geqslant N$ implies

$$\|x - \sigma_N(x)\|,\ \|x^{j_k} - \sigma^{j_k}_N\| \leqslant \varepsilon$$

for all k. Since the Fourier coefficients of x^{j_k} converge to those of x, we can also choose M so that $k \geqslant M$ implies

$$\|\sigma_N(x) - \sigma^{j_k}_N\| \leqslant \varepsilon.$$

This shows that $\|x - x^{j_k}\| \to 0$, as desired. ∎

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