Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **xxx**, xxx–xxx

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

Quintic spline collocation method for fractional boundary value problems

Ghazala Akram *, **Hira Tariq**

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan

Received 5 November 2015; revised 5 February 2016; accepted 13 March 2016

KEYWORDS

Fractional boundary value problems;
Caputo derivative;
Quintic spline;
Collocation method

Abstract The spline collocation method is a competent and highly effective mathematical tool for constructing the approximate solutions of boundary value problems arising in science, engineering and mathematical physics. In this paper, a quintic polynomial spline collocation method is employed for a class of fractional boundary value problems (FBVPs). The FBVPs are expressed in terms of Caputo's fractional derivative in this approach. The consistency relations are derived in order to compute the approximate solutions of FBVPs. Finally, numerical results are given, which demonstrate the effectiveness of the numerical scheme.

1. Introduction

The idea behind fractional derivatives (and integrals) is no more new now. Fractional derivative gives a perfect aid to characterize the memory and hereditary properties of various processes and materials, therefore differential equations of fractional order are being used in modeling of electrical and mechanical properties of various real materials, rock's rheological properties, and in many other areas. The most of the differential equations of fractional order do not have analytical solutions, this is the main reason for finding new numerical methods for the solutions of fractional differential equations becomes a hot topic for the research community. An extensive research has been carried out to obtain the numerical schemes which are numerically stable for both linear and nonlinear differential equations of fractional order.

Many authors used the spline technique to establish the accurate and efficient numerical schemes for the solution of boundary value problems. For example, Siddiqi and Akram constructed many numerical schemes with the help of different spline functions such as polynomial splines and non-polynomial splines for the solution of various BVPs (Siddiqi et al., 2007; Siddiqi and Akram, 2008, 2007). Also, Akram and Aslam (2014) established the Adomian decomposition method (ADM) and the reproducing kernel method (RKM) for the solution of fourth order three-point boundary value problem. The theory of FBVPs has received considerable interest in recent years. The interest towards the theory of existence and uniqueness of solutions to FBVPs is apparent from the recent publications (Ahmad and Nieto, 2009; Bai, 2010; Zhang, 2006). FBVPs occur in the explanation of many physical stochastic-transport processes and in the inspection of liquid filtration which arises in a strongly porous's medium (Taukenova and Shkhanukov-Lafishev, 2006). Also, boundary value problems with integral boundary conditions establish a very fascinating and predominant class of problems. Two, three, four, multi point and nonlocal boundary value problems

* Corresponding author.
E-mail addresses: toghazala2003@yahoo.com (G. Akram), hiratariq47@gmail.com (H. Tariq).
Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2016.03.003
1815-3852

Please cite this article in press as: Akram, G., Tariq, H. Quintic spline collocation method for fractional boundary value problems. Journal of the Association of Arab Universities for Basic and Applied Sciences (2016), http://dx.doi.org/10.1016/j.jaubas.2016.03.003

are the special cases of such problems. Cellular systems and population dynamics are some phenomena in which boundary conditions of integral type occur (Chen et al., 2010). Analysis and representation of many physical systems demand solutions of fractional boundary value problems. Recently, a great amount of effort has been employed in attempting to find stable and robust numerical and analytical methods in order to solve FBVPs (Al-Rabtah et al., 2012; Akram and Tariq, 2015; Almeida and Bastos, 2015; Doha et al., 2011; Edwards et al., 2002; Ford et al., 2014; Jin et al., 2015, 2016; Kumar et al., 2015a,b; Nouri and Siavashani, 2014; Rehmana and Khan, 2012; Secer et al., 2013; Seifollahi and Shamloo, 2013).

The fourth order boundary value problem explains an elastic-bending-beams's static deflection through a nonlinear loading (Gupta, 1988). It also depicts a prototype-equation's steady state for phase transitions in systems of condensed matter (Grinstein and Luther, 1976) and is helpful in understanding traveling waves in a suspension bridge (Lazer and McKenna, 1990). In this paper, consider the following fourth order boundary value problems for linear fractional differential equations:

$$y^{(4)}(x) + D^{\alpha}p(x)y(x) = g(x), \quad x \in [a,b], \tag{1}$$

subject to

$$y(a) - A_1 = y(b) - A_2 = y''(a) - B_1 = y''(b) - B_2 = 0, \tag{2}$$

where $A_i,\ B_i,\ i = 1,2$ are real constants. The functions $p(x)$ and $g(x)$ are continuous on the interval $[a,b]$ and D^{α} denotes fractional derivative in Caputo's sense. In general, the analytical solution of Eqs. (1) and (2) cannot be obtained for arbitrary choices of $p(x)$ and $g(x)$. The problem Eq. (1) arise in the plate deflection theory. When $\alpha = 0$, Eq. (1) is reduced to the classical fourth order boundary value problem.

In this study, Caputo's fractional derivative is used. This operator is widely applied in modeling of the material's mechanical properties, modeling of the viscoelastic behavior, signal processing, diffusion problems, bioengineering and mathematical finance models etc. Also in this study, we focus on providing a numerical scheme, based on quintic polynomial spline collocation method, to solve fourth order boundary value problems for linear fractional differential equations.

Quintic polynomial spline scheme is commonly used in order to solve differential equations. If the solutions of FBVPs are needed at various locations in the given region then the spline solutions guarantee to give the information of spline interpolation between mesh points. Also the strong advantage of this scheme is to provide smooth continuous approximations to exact solutions at every point of the range of integration.

The paper is organized as follows: some preliminaries of fractional calculus are given in Section 2. In Section 3, quintic polynomial spline method is developed for the solution of fractional differential equation. The matrix form of the proposed scheme is discussed in Section 4. In Section 5, numerical results are given to compare and illustrate the efficiency of the method.

2. Preliminaries

There are several definitions to the generalization of the notion of fractional differentiation. Riemann–Liouville and Caputo's are most common definitions. But Caputo's approach is suitable for real world physical problems because it defines integer order initial conditions for fractional differential equations. The Riemann- Liouville left and right fractional integral of order $\alpha > 0$ is defined as

$$I^{\alpha}_{a+}y(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}y(s)ds, \quad x > 0$$

and

$$I^{\alpha}_{b-}y(x) = -\frac{1}{\Gamma(\alpha)}\int_b^x (x-s)^{\alpha-1}y(s)ds,$$

respectively. Also, Caputo's fractional derivative of order α is defined as

$$D^{\alpha}y(x) = I^{m-\alpha}D^m y(x),$$

where D^m is ordinary differential operator.

If $\alpha > 0,\ m-1 \leqslant \alpha < m,\ \delta > -1,\ m \in \mathbb{N},\ \lambda,\ \mu \in \mathbb{R}$ and $y(x)$ is continuous function, then the following results hold:

$$D^{\alpha}C = 0, \quad C \text{ is constant}$$

$$D^{\alpha}(\lambda y(x) + \mu q(x)) = \lambda D^{\alpha}y(x) + \mu D^{\alpha}q(x)$$

$$I^{\alpha}x^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1+\alpha)}x^{\delta+\alpha}$$

Theorem 1. *Let $0 < \alpha < 1$ and assume that f and g are analytic on $(a-h, a+h)$. Then*

$$D^{\alpha}_a[fg](x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}g(a)(f(x)-f(a)) + (D^{\alpha}_a g(x))f(x) + \sum_{k=1}^{\infty}\binom{\alpha}{k}(I^{k-\alpha}_a g(x))D^k_a f(x).$$

For more properties of fractional derivatives, we refer to (Diethelm, 2010; Kilbas et al., 2006; Lakshmikantham and Vatsala, 2008; Ortigueira and Machado, 2015; Ortigueira and Trujillo, 2012; Podlubny, 1999).

3. Quintic spline functions

Let $x_i = ih\ \left(i = 0,1,\ldots,n,\ h = \frac{b-a}{n},\ n > 0\right)$ be grid points of the uniform partition of $[a,b]$ into the subintervals $[x_{i-1}, x_i]$. Let $y(x)$ be the exact solution of Eq. (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $T_i(x)$ passing through the points $(x_i,\ S_i)$ and $(x_{i+1},\ S_{i+1})$. Consider that each quintic polynomial spline segment $T_i(x)$ has the following form:

$$T_i(x) = a_i(x-x_{i-1})^5 + b_i(x-x_{i-1})^4 + c_i(x-x_{i-1})^3 + d_i(x-x_{i-1})^2 + e_i(x-x_{i-1}) + f_i, \tag{3}$$

$i = 1,2,\ldots,n$, along with the requirement that $T_i(x) \in C^4[a,b]$ and

$$S(x) = T_i(x)\ \forall x\ [x_{i-1}, x_i], \quad i = 1,2,\ldots,n. \tag{4}$$

In order to develop the consistency relations between the values of spline and its derivatives at knots, let

$$T_i(x_{i-1}) = S_{i-1}, \quad T_i(x_i) = S_i,$$
$$T_i^{(2)}(x_{i-1}) = M_{i-1}, \quad T_i^{(2)}(x_i) = M_i,$$
$$T_i^{(4)}(x_{i-1}) = F_{i-1}, \quad T_i^{(4)}(x_i) = F_i.$$

It is to be noticed that the spline can be written in terms of S_is and any three derivatives at the boundaries of each subinterval. To define spline in terms of S_is, M_is, and F_is, the coefficients introduced in Eq. (3) are calculated, as

$$a_i = \frac{1}{120h}(F_i - F_{i-1}), \quad b_i = \frac{1}{24}F_{i-1},$$
$$c_i = \frac{1}{6h}(M_i - M_{i-1}) - \frac{h}{36}(F_i + 2F_{i-1}),$$
$$d_i = \frac{1}{2}M_{i-1}, e_i = \frac{1}{h}(S_i - S_{i-1}) - \frac{h}{6}(M_i + 2M_{i-1}) + \frac{h^3}{360}(7F_i + 8F_{i-1}), \quad f_i = S_{i-1}.$$

Applying the first and third derivative continuities at the knots, i.e. $T_i^{\rho}(x_i) = T_{i+1}^{\rho}(x_i)$, $\rho = 1$ and 3, the following useful relations are obtained, as

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}(S_{i+1} - 2S_i + S_{i-1}) + \frac{h^2}{60}(7F_{i+1} + 16F_i + 7F_{i-1}), \tag{5}$$

$$M_{i+1} - 2M_i + M_{i-1} = \frac{h^2}{6}(F_{i+1} + 4F_i + F_{i-1}). \tag{6}$$

Using Eqs. (5) and (6), the following consistency relation in terms of the fourth derivative of spline F_i and S_i, $i = 1, 2, \ldots, n$, is derived, as

$$S_{i+2} - 4S_{i+1} + 6S_i - 4S_{i-1} + S_{i-2} = \frac{h^4}{120}(F_{i+2} + 26F_{i+1} + 66F_i + 26F_{i-1} + F_{i-2}) \quad i = 2, 3, \ldots, n-2, \tag{7}$$

where

$$F_i = [-D^{\alpha}_{x_{i-1}} p(x)T(x) + g(x)]|_{x=x_i}. \tag{8}$$

Since the system (7) gives $(n-3)$ linear algebraic equations in the $(n-1)$ unknowns (S_i, $i = 1, 2, \ldots, n-1$), therefore two more equations (end conditions) are required. The two end conditions can be obtained using Taylor series and the method of undetermined coefficients. Two end equations are

$$-2S_0 + 5S_1 - 4S_2 + S_3 = -h^2 M_0 + \frac{h^4}{120}(18F_0 + 65F_1 + 26F_2 + F_3) \tag{9}$$

and

$$S_{n-3} - 4S_{n-2} + 5S_{n-1} - 2S_n = -h^2 M_n + \frac{h^4}{120}(F_{n-3} + 26F_{n-2} + 65F_{n-1} + 18F_n). \tag{10}$$

Suppose that F_0 is linearly approximated between F_1 and F_2 as, $F_0 = 2F_1 - F_2$ and also F_n is linearly approximated between F_{n-1} and F_{n-2} as, $F_n = 2F_{n-1} - F_{n-2}$.

For $i = 1$, the consistency relation can be taken, as

$$-2S_0 + 5S_1 - 4S_2 + S_3 = -h^2 M_0 + \frac{h^4}{120}(101F_1 + 8F_2 + F_3). \tag{11}$$

For $i = 2$, the consistency relation can be written, as

$$S_0 - 4S_1 + 6S_2 - 4S_3 + S_4 = \frac{h^4}{120}(28F_1 + 65F_2 + 26F_3 + F_4). \tag{12}$$

For $i = n-2$, the consistency relation can be taken, as

$$S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = \frac{h^4}{120}(F_{n-4} + 26F_{n-3} + 65F_{n-2} + 28F_{n-1}). \tag{13}$$

For $i = n-1$, the consistency relation can be written, as

$$S_{n-3} - 4S_{n-2} + 5S_{n-1} - 2S_n = -h^2 M_n + \frac{h^4}{120}(F_{n-3} + 8F_{n-2} + 101F_{n-1}). \tag{14}$$

Lemma 1. *Let $y \in C^6[a,b]$ then the local truncation errors t_i, $i = 1, 2, \ldots, n-1$ associated with the* Eqs. (11), (12), (7), (13) and (14) *are*

$$t_i = \begin{cases} \frac{13}{180}h^6 y^{(6)}(x_1) + O(h^7), & i = 1, \\ \frac{113}{360}h^6 y^{(6)}(x_1) + O(h^7), & i = 2, \\ \frac{-1}{12}h^6 y^{(6)}(x_i) + O(h^7), & i = 3, 4, \ldots, n-3, \\ \frac{113}{360}h^6 y^{(6)}(x_{n-1}) + O(h^7), & i = n-2, \\ \frac{13}{180}h^6 y^{(6)}(x_{n-1}) + O(h^7), & i = n-1. \end{cases}$$

Proof. In order to obtain the local truncation errors t_i, $i = 1, 2, \ldots, n-1$, firstly rewrite Eqs. (11), (12), (7), (13) and (14) in the following form, as

$$t_1 = -2y_0 + 5y_1 - 4y_2 + y_3 + h^2 M_0 - \frac{h^4}{120}[101y_1^{(iv)} + 8y_2^{(iv)} + y_3^{(iv)}],$$

$$t_2 = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^4}{120}[28y_1^{(iv)} + 65y_2^{(iv)} + 26y_3^{(iv)} + y_4^{(iv)}],$$

$$t_i = y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} - \frac{h^4}{120}[y_{i+2}^{(iv)} + 26y_{i+1}^{(iv)} + 66y_i^{(iv)} + 26y_{i-1}^{(iv)} + y_{i-2}^{(iv)}],$$

$$t_{n-2} = y_n - 4y_{n-1} + 6y_{n-2} - 4y_{n-3} + y_{n-4} - \frac{h^4}{120}[y_{n-4}^{(iv)} + 26y_{n-3}^{(iv)} + 65y_{n-2}^{(iv)} + 28y_{n-1}^{(iv)}]$$

and

$$t_{n-1} = y_{n-3} - 4y_{n-2} + 5y_{n-1} - 2y_n + h^2 M_n - \frac{h^4}{120}[y_{n-3}^{(iv)} + 8y_{n-2}^{(iv)} + 101y_{n-1}^{(iv)}].$$

The terms y_0, y_1, $y_1^{(iv)}$, y_2, $y_2^{(iv)}$, etc are expanded about the point x_1 using Taylor's series and the expressions for

$t_i,\ i=1$, can be obtained. The expressions for $t_i,\ i=2$, can be derived by expanding the terms $y_0,\ y_1,\ y_1^{(iv)},\ y_2,\ y_2^{(iv)}$, etc about the point x_2 using Taylor's series. Also, the expressions for $t_i,\ i=3,4,\ldots,n-3$ can be obtained by expanding $y_{i+2},\ y_{i+2},\ y_i,\ y_i^{(iv)}$ etc around the point x_i using Taylor series and the expression for $t_i,\ i=3,4,\ldots,n-3$, can be obtained. The terms $y_n,\ y_{n-1},\ y_{n-1}^{(iv)},\ y_{n-2},\ y_{n-2}^{(iv)}$, etc are expanded about the point x_{n-2} using Taylor's series and the expressions for $t_i,\ i=n-2$, can be obtained. Finally the expressions for $t_i,\ i=n-1$, can be derived as expanding about the point x_{n-1}. □

In order to obtain the values of F_i, the following equation is obtained using Eqs. (4), (8) and Theorem 1.

$$\begin{aligned} D^{\alpha}_{x_{i-1}}p(x)y(x) &= D^{\alpha}_{x_{i-1}}p(x)T_i(x) \\ &= \frac{(x-x_{i-1})^{-\alpha}}{\Gamma(1-\alpha)}T_i(x_{i-1})(p(x)-p(x_{i-1})) \\ &\quad + (D^{\alpha}_{x_{i-1}}T_i(x))p(x) + \sum_{k=1}^{\infty}\binom{\alpha}{k}(J^{k-\alpha}_{x_{i-1}}T_i(x))D^{k}_{x_{i-1}}p(x). \end{aligned} \tag{15}$$

Substitute the spline function for the values of $a_i,\ b_i, c_i,\ d_i$ and e_i and $x=x_i$, the Eq. (15) can be written as

$$\begin{aligned} &(1+\mu_3)F_i + \mu_1F_{i-2} + \mu_2F_{i-1} + \mu_4F_{i+1} + \eta_1S_{i-2} + \eta_2S_{i-1} \\ &\quad + \eta_3S_i + \eta_4S_{i+1} = g_i, \end{aligned} \tag{16}$$

where the values of $\mu_i,\ \eta_i,\ i=1,2,3,4$, are given in Appendix A. Also,

$$g_i = g(x_i).$$

One more equation is needed to complete the system. This end condition is obtained by computing the values of the constants in Eq. (3) at end with the help of Taylor series,

$$F_1 + \mu_{11}F_1 + \mu_{12}F_2 + \eta_{10}S_0 + \eta_{11}S_1 + \eta_{12}S_2 + \theta M_0 = g_1, \tag{17}$$

where the values of $\mu_{11},\ \mu_{12}, \eta_{10},\ \eta_{11},\ \eta_{12}$ and θ are given in Appendix A.

4. Quintic spline solution

The spline solution of boundary value problem (1) is determined, using Eqs. (11), (12), (7), (13), (14), (16) and (17). Considering $S=[S_1,S_2,\ldots,S_{n-1}]^T$ and $F=[F_1,F_2,\ldots,F_{n-1}]^T, S_i$ satisfies the following matrix equation

$$CS = h^4DF, \tag{18}$$

where $C,\ D$ are $(n-1)\times(n-1)$ matrices and

$$C = \begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & & & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ & & & & & 1 & -4 & 5 \end{pmatrix},$$

$$D = \frac{1}{120}\begin{pmatrix} 101 & 8 & 1 & & & & & \\ 28 & 65 & 26 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & & & & \ddots & & & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 26 & 65 & 28 \\ & & & & & 1 & 8 & 101 \end{pmatrix}.$$

The Eqs. (16) and (17) in matrix form can be written, as

$$NS + MF = G, \tag{19}$$

where $N,\ M$ are matrices of order $(n-1)\times(n-1)$ and

$$N = \begin{pmatrix} \eta_{11} & \eta_{12} & & & & & \\ \eta_2 & \eta_3 & \eta_4 & & & & \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & & & \\ & & & \ddots & & & \\ & & & \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ & & & & \eta_1 & \eta_2 & \eta_3 \end{pmatrix},$$

$$M = \begin{pmatrix} \mu_{11}+1 & \mu_{12} & & & & & \\ 2\mu_1+\mu_2 & \mu_3-\mu_1+1 & \mu_4 & & & & \\ \mu_1 & \mu_2 & \mu_3+1 & \mu_4 & & & \\ & & & \ddots & & & \\ & & & & \mu_1 & \mu_2 & \mu_3+1 & \mu_4 \\ & & & & & \mu_1 & \mu_2-\mu_4 & 2\mu_4+\mu_3+1 \end{pmatrix}.$$

Moreover $G=(g_i)$ is $(n-1)$ dimensional column vector such that

$$G = \begin{cases} g_1 - \theta M_0, & i=1, \\ g_i, & i=2,3,\ldots n-1. \end{cases}$$

The Eq. (19) can be written, as

$$F = M^{-1}G - M^{-1}NS. \tag{20}$$

From Eqs. (18) and (20), it can be written, as

$$(C + h^4DM^{-1}N)S = h^4DM^{-1}G. \tag{21}$$

5. Convergence of the method

Let $Y=(y_i)$ and $E=(e_i)=Y–S$ be an (n-1)-dimensional column vectors.

In order to get a bound on $\|E\|_\infty$, consider

$$(C + h^4DM^{-1}N)Y = h^4DM^{-1}G + T. \tag{22}$$

where the vector T is defined, as

$$T = h^6\left(\frac{13}{180}y_1^{(6)}, \frac{113}{360}y_2^{(6)}, \frac{-1}{12}y_3^{(6)}, \ldots, \frac{-1}{12}y_{n-3}^{(6)}, \frac{113}{360}y_{n-2}^{(6)}, \frac{13}{180}y_{n-1}^{(6)}\right).$$

Moreover,

$$||T||_\infty = c_3h^6Z_6, \quad Z_6 = max_{x\in[0,1]}|y^{(6)}(x)|, \tag{23}$$

where c_3 is a constant and also independent of h.

From Eqs. (21) and (22),

$$(C + h^4DM^{-1}N)E = T. \tag{24}$$

From Eq. (24), E can be expressed, as

$$E = (I + h^4 C^{-1} D M^{-1} N)^{-1} C^{-1} T. \tag{25}$$

For the sake of simplicity, consider the case where $p(x)$ is a constant function. Then Eq. (8) becomes

$$F_i = [-pD^{\alpha}_{x_{i-1}} T(x) + g(x)]|_{x=x_i}.$$

Lemma 2 (Siddiqi and Akram, 2008). *If Z is a matrix of order n and $\|Z\| < 1$, then $(I+Z)^{-1}$ exists and*

$$\|(I+Z)^{-1}\| < \frac{1}{1-\|Z\|}.$$

Lemma 3. *The infinite norm of M^{-1} satisfies the inequality*

$$\|M^{-1}\|_\infty \leqslant \frac{720\Gamma(6-\alpha)}{720\Gamma(6-\alpha) - p\alpha h^{4-\alpha}(3180 - 2616\alpha + 636\alpha^2 - 48\alpha^3)}, \tag{26}$$

provided that $\frac{p\alpha h^{4-\alpha}(3180-2616\alpha+636\alpha^2-48\alpha^3)}{720\Gamma(6-\alpha)} \leqslant 1$.

Proof. The matrix M can be written, as

$$M = I + \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M},$$

where matrix $\widetilde{M}$ is

$$\begin{pmatrix} \mu_{11}^* & \mu_{12}^* & & & & & & \\ 2\mu_1^* + \mu_2 & \mu_3 - \mu_1^* & \mu_4 & & & & & \\ \mu_1^* & \mu_2 & \mu_3 & \mu_4 & & & & \\ & & & \ddots & & & & \\ & & & & \mu_1^* & \mu_2 & \mu_3 & \mu_4 \\ & & & & & \mu_1^* & \mu_2 - \mu_4 & 2\mu_4 + \mu_3 \end{pmatrix},$$

where

$$\mu_{11}^* = 4(-931 + \alpha(764 + 13(-14+\alpha)\alpha)),$$

$$\mu_{12}^* = \frac{14\Gamma(6-\alpha)}{\alpha},$$

$$\mu_1^* = \frac{2(-2+\alpha)\Gamma(6-\alpha)}{\Gamma(4-\alpha)}.$$

The matrix M^{-1} can be expressed, as

$$M^{-1} = \left(I + \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M}\right)^{-1},$$

Using the Lemma 2, if

$$\left\| \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M} \right\|_\infty < 1, \tag{27}$$

then

$$\|M^{-1}\|_\infty \leqslant \frac{1}{1 - \left\| \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M} \right\|_\infty},$$

where

$$\left\| \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M} \right\|_\infty = \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} (3180 - 2616\alpha + 636\alpha^2 - 48\alpha^3). \tag{28}$$

In this case, at $\alpha = 1$, maximum value of Eq. (28) is

$$\left\| \frac{p\alpha h^{4-\alpha}}{720\Gamma(6-\alpha)} \widetilde{M} \right\|_\infty \leqslant p\frac{h^3}{15}.$$

In order to satisfy the Lemma 3, the parameter p must satisfy the following condition:

$$p_{max} < \frac{15}{h^3}$$

and

$$\|M^{-1}\|_\infty \leqslant \frac{720\Gamma(6-\alpha)}{720\Gamma(6-\alpha) - p\alpha h^{4-\alpha}(3180 - 2616\alpha + 636\alpha^2 - 48\alpha^3)}. \quad \square$$

Lemma 4. *The matrix $(C + h^4 D M^{-1} N)$ in Eq. (24) is nonsingular, provided that:*

$$\frac{720\Gamma(6-\alpha)\xi((b-a)^3 + 8h^3)\lambda_2 h^{-\alpha}}{(720\Gamma(6-\alpha) - h^{4-\alpha}\lambda_1)} < 1,$$

where $\lambda_1 = p\alpha(3180 - 2616\alpha + 636\alpha^2 - 48\alpha^3)$, $\lambda_2 = \frac{1}{\Gamma(6-\alpha)}(240 - 288\alpha + 113\alpha^2 - 18\alpha^3 + \alpha^4)$ *and* $\xi = \frac{(b-a)^4}{384(b-a)^3}$.

Table 1 Maximum absolute errors.

n	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$
10	$8.3236E-004$	$9.0723E-004$	$9.8179E-004$
20	$4.2843E-004$	$5.0369E-004$	$5.8857E-004$
40	$2.1281E-004$	$2.7132E-004$	$3.4350E-004$

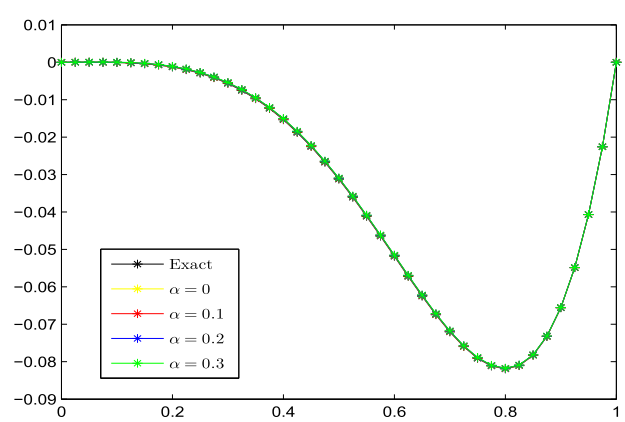

Figure 1 Exact and approximate solutions of Example 1 with different values of α.

Table 2 Maximum absolute errors and order of convergence (O. C.) for Example 2.

n	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
	Error	O.C.	Error	O.C.	Error	O.C.
10	$8.44E-002$		$7.61E-002$		$6.68E-002$	
20	$2.37E-002$	1.8324	$2.19E-002$	1.7970	$2.01E-002$	1.7318
40	$6.90E-003$	1.7802	$6.90E-003$	1.6663	$6.80E-003$	1.5636

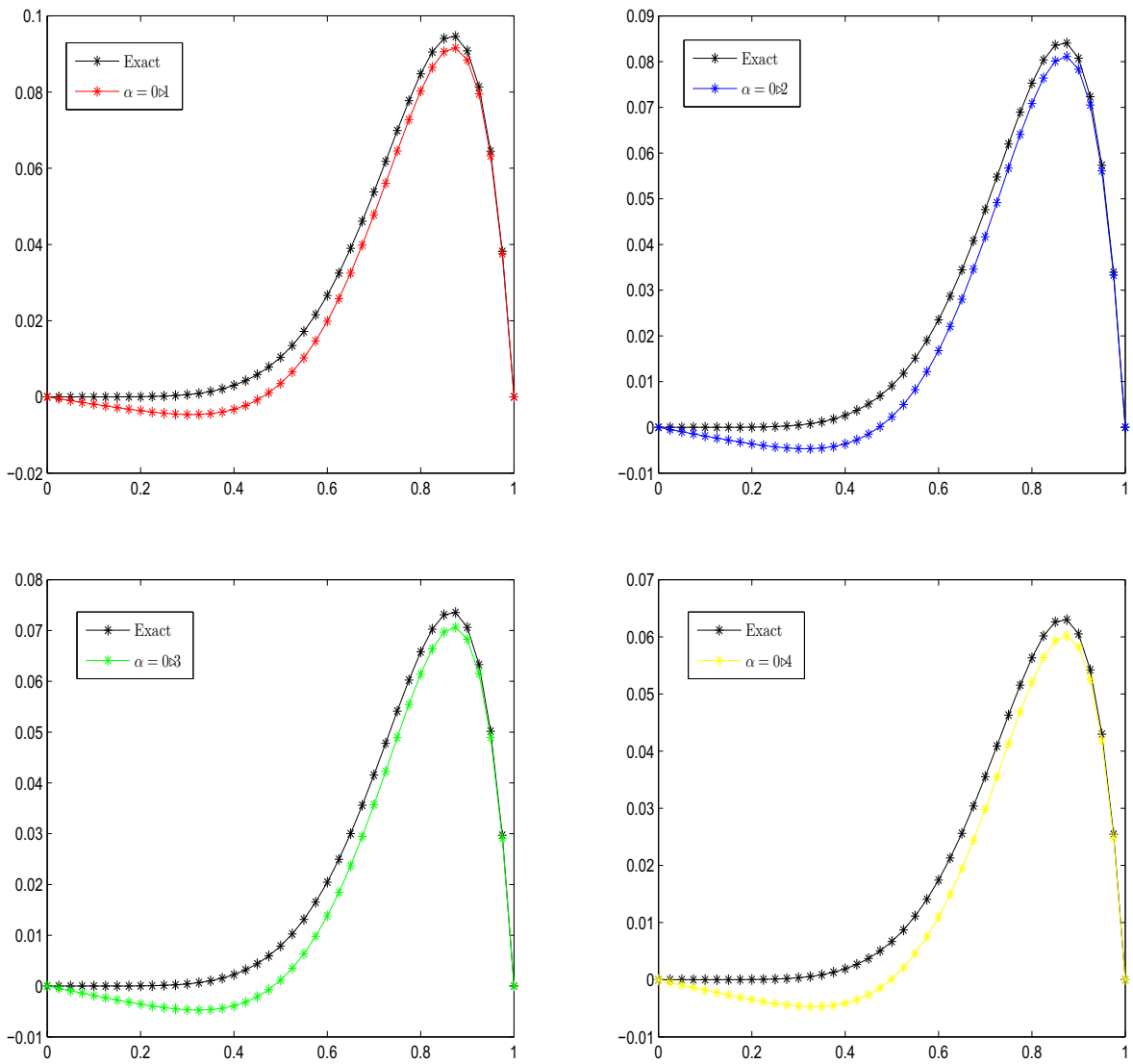

Figure 2 Exact and approximate solutions of Example 2 with different values of α.

Then

$$\|E\|_\infty \leqslant \frac{\|C^{-1}\|_\infty \|T\|_\infty}{1 - h^4 \|C^{-1}\|_\infty \|D\|_\infty \|M^{-1}\|_\infty \|N\|_\infty} \cong O(h^2). \tag{29}$$

Proof. From Lemma 2,

$$\|E\|_\infty = max_{1 \leqslant i \leqslant n-1} |e_i| \leqslant \frac{\|C^{-1}\|_\infty \|T\|_\infty}{1 - h^4 \|C^{-1}\|_\infty \|D\|_\infty \|M^{-1}\|_\infty \|N\|_\infty}, \tag{30}$$

provided that $h^4\|C^{-1}\|_\infty\|D\|_\infty\|M^{-1}\|_\infty\|N\|_\infty < 1$. From Siddiqi and Akram (2008),

$$\|C^{-1}\|_\infty = \frac{1}{384h^4}(b-a)^4\left(1+\frac{8h^3}{(b-a)^3}\right).$$

Also,

$$\|D\|_\infty = 1 \text{ and } \|N\|_\infty = \frac{h^{-\alpha}}{\Gamma(6-\alpha)}(240-288\alpha+113\alpha^2-18\alpha^3+\alpha^4).$$

By substituting the values of $\|C^{-1}\|_\infty$, $\|D\|_\infty$, $\|M^{-1}\|_\infty$ and $\|N\|_\infty$ in Eq. (30) and using Eq. (23), it can be written as

$$\|E\|_\infty \leqslant \frac{h^2c_3Z_6\xi(720\Gamma(6-\alpha)-h^{4-\alpha}\lambda_1)((b-a)^3+8h^3)}{(720\Gamma(6-\alpha)-h^{4-\alpha}\lambda_1)-720\Gamma(6-\alpha)\xi((b-a)^3+8h^3)\lambda_2h^{-\alpha}} \cong O(h^2), \tag{31}$$

where $Z_6 = max_{a\leqslant x\leqslant b}|y^6(x)|$. □

Theorem 2. *Let $y(x)$ be the exact solution of the FBVP* Eq. (1) *with boundary condition* Eq. (2) *and y_i, $i = 0, 1, 2, \ldots, n-1$, satisfy the discrete BVP* Eq. (22). *Moreover, if $e_i = y_i - S_i$, then*

$$\|E\|_\infty = O(h^2).$$

6. Numerical results

In this section, to check the accuracy, efficiency and validity of the method, some examples of suggested method are given.

Example 1. Consider the following FBVP:

$$y^{(4)}(x) + 0.05D^\alpha y(x) = g(x), \quad x \in [0,1],$$

with

$$y(0) = 0, \quad y(1) = 0,$$

$$y''(0) = 0, \quad y''(1) = 8.$$

The exact solution of this problem is $x^5 - x^4$. The present scheme is applied with different values of α and results are shown in Table 1 and Fig. 1.

Furthermore in the limit, as α goes to zero, the method provides a solution for the integer order system. From numerical results, it is observed that suggested scheme is of $O(h^2)$.

Example 2. Consider the following FBVP:

$$y^{(4)}(x) + D^\alpha xy(x) = g(x), \quad x \in [0,1],$$

with

$$y(0) = 0, \quad y(1) = 0,$$

$$y''(0) = 0, \quad y''(1) = 26(\alpha-1).$$

The exact solution of this problem is $x^{6+\alpha} - x^{8-\alpha}$. The present scheme is applied with different values of α and results are shown in Table 2 and Fig. 2.

7. Conclusion

Collocation method is established for the approximate solution of fractional differential equation along with boundary conditions, using quintic spline. The suggested method also utilizes the properties of fractional derivatives in order to solve this problem. This numerical scheme is computationally captivating. Descriptive examples show applications of this problem. It is proved that the method is of $O(h^2)$.

Appendix A

$$\begin{aligned}
D^\alpha_{x_{i-1}}p(x)T_i(x)|_{x=x_i} = {} & \frac{h^{-\alpha}}{\Gamma(1-\alpha)}S_{i-1}(p(x_i)-p(x_{i-1})) + p(x_i)\left(\frac{2h^{4-\alpha}(-2+\alpha)\alpha}{720\Gamma(4-\alpha)}F_{i-2} + \frac{h^{4-\alpha}(-4+\alpha)\alpha(361+\alpha(-226+25\alpha))}{720\Gamma(6-\alpha)}F_{i-1}\right. \\
& + \frac{2h^{4-\alpha}\alpha(-778+\alpha(605+\alpha(-137+10\alpha)))}{720\Gamma(6-\alpha)}F_i + \frac{(-5+\alpha)^2(-4+\alpha)h^{4-\alpha}\alpha}{720\Gamma(6-\alpha)}F_{i+1} \\
& + \frac{-240(-2+\alpha)(-5+\alpha)(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)}S_{i-2} + \frac{-360h^{-\alpha}(12+(-9+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)}S_{i-1} \\
& \left. + \frac{h^{-\alpha}720(6+(-6+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{\Gamma(6-\alpha)}S_i + \frac{-120(-5+\alpha)^2(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)}S_{i+1}\right) \\
& + \sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{h^{4+k-\alpha}}{\Gamma(5+k-\alpha)}F_{i-1} + \frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)}(F_i - F_{i-1}) + \frac{h^{4+k-\alpha}}{\Gamma(1+k-\alpha)}S_{i-1}\right. \\
& + \frac{(6(h^{4+k-\alpha}(F_{i-2}-37F_{i-1}-23F_i-F_{i+1}) - 120(S_{i-2}-3S_{i-1}+3S_i-S_{i+1})))}{720\Gamma(4+k-\alpha)} \\
& - \frac{(6(h^{4+k-\alpha}(F_{i-2}+4F_{i-1}+F_i) - 120(S_{i-2}-2S_{i-1}+S_i)))}{720\Gamma(3+k-\alpha)} - \frac{1}{720\Gamma(2+k-\alpha)}(h^{4+k-\alpha}(2F_{i-2}+25F_{i-1}+20F_i+F_{i+1}) \\
& \left. - 120(2S_{i-2}+3S_{i-1}-6S_i+S_{i+1}))\right)D^k_{x_{i-1}}p(x).
\end{aligned}$$

$$\mu_1=\frac{2h^{4-\alpha}(-2+\alpha)\alpha}{720\Gamma(4-\alpha)}(p(x))|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)}-\frac{h^{4+k-\alpha}}{120\Gamma(3+k-\alpha)}-\frac{h^{4+k-\alpha}}{360\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\mu_2=\frac{h^{4-\alpha}(-4+\alpha)\alpha(361+\alpha(-226+25\alpha))}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{h^{4+k-\alpha}}{\Gamma(5+k-\alpha)}-\frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)}-\frac{37h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)}-\frac{h^{4+k-\alpha}}{30\Gamma(3+k-\alpha)}\right.$$
$$\left.+\frac{5h^{4+k-\alpha}}{144\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\mu_3=\frac{2h^{4-\alpha}\alpha(-778+\alpha(605+\alpha(-137+10\alpha)))}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)}-\frac{23h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)}-\frac{h^{4+k-\alpha}}{120\Gamma(3+k-\alpha)}+\frac{h^{4+k-\alpha}}{36\Gamma(2+k-\alpha)}\right)$$
$$\times D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\mu_4=\frac{(-5+\alpha)^2(-4+\alpha)h^{4-\alpha}\alpha}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(-6\frac{h^{4+k-\alpha}}{\Gamma(4+k-\alpha)}+\frac{h^{4+k-\alpha}}{720\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\eta_1=\frac{-240(-2+\alpha)(-5+\alpha)(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{(-720h^{4+k-\alpha})}{\Gamma(4+k-\alpha)}+\frac{h^{4+k-\alpha}}{\Gamma(3+k-\alpha)}-\frac{h^{4+k-\alpha}}{3\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\eta_2=\frac{-360h^{-\alpha}(12+(-9+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\frac{h^{-\alpha}}{\Gamma(1-\alpha)}(p(x_i)-p(x_{i-1}))$$
$$+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{2160h^{4+k-\alpha}}{\Gamma(4+k-\alpha)}-\frac{2h^{4+k-\alpha}}{\Gamma(3+k-\alpha)}-\frac{h^{4+k-\alpha}}{2\Gamma(2+k-\alpha)}+\frac{h^{4+k-\alpha}}{\Gamma(1+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\eta_3=\frac{h^{-\alpha}720(6+(-6+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{-2160h^{4+k-\alpha}}{\Gamma(4+k-\alpha)}+\frac{h^{4+k-\alpha}}{\Gamma(3+k-\alpha)}+\frac{h^{4+k-\alpha}}{\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i},$$

$$\eta_4=\frac{-120(-5+\alpha)^2(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)}p(x)|_{x=x_i}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{720h^{4+k-\alpha}}{\Gamma(4+k-\alpha)}-\frac{h^{4+k-\alpha}}{6\Gamma(2+k-\alpha)}\right)D^k_{x_{i-1}}p(x)|_{x=x_i}$$

and

$$g_i=g(x_i).$$

$$\mu_{11}=h^{4-\alpha}\left(\frac{2}{\Gamma(5-\alpha)}-\frac{1}{\Gamma(6-\alpha)}+\frac{13}{180\Gamma(2-\alpha)}-\frac{53}{60\Gamma(4-\alpha)}\right)p(x)|_{x=x_1}$$
$$+\sum_{k=1}^{\infty}\binom{\alpha}{k}h^{4+k-\alpha}\left(\frac{2}{\Gamma(5+k-\alpha)}-\frac{1}{\Gamma(6+k-\alpha)}+\frac{13}{180\Gamma(2+k-\alpha)}-\frac{53}{60\Gamma(4+k-\alpha)}\right)D^k_{x_0}p(x)|_{x=x_1},$$

$$\mu_{12}=h^{4-\alpha}\left(\frac{-1}{\Gamma(5-\alpha)}+\frac{1}{\Gamma(6-\alpha)}-\frac{4}{180\Gamma(2-\alpha)}+\frac{20}{60\Gamma(4-\alpha)}\right)p(x)|_{x=x_1}$$
$$+\sum_{k=1}^{\infty}\binom{\alpha}{k}h^{4+k-\alpha}\left(\frac{-1}{\Gamma(5+k-\alpha)}+\frac{1}{\Gamma(6+k-\alpha)}-\frac{4}{180\Gamma(2+k-\alpha)}+\frac{20}{60\Gamma(4+k-\alpha)}\right)D^k_{x_0}p(x)|_{x=x_1},$$

$$\eta_{10}=\frac{h^{-\alpha}}{\Gamma(1-\alpha)}(p(x_1)-p(x_0))+h^{-\alpha}\left(\frac{-7}{6\Gamma(2-\alpha)}-\frac{1}{\Gamma(4-\alpha)}\right)p(x)|_{x=x_1}+\sum_{k=1}^{\infty}\binom{\alpha}{k}h^{k-\alpha}\left(\frac{-7}{6\Gamma(2+k-\alpha)}-\frac{1}{\Gamma(4+k-\alpha)}\right)D^k_{x_0}p(x)|_{x=x_1},$$

$$\eta_{11}=h^{-\alpha}\left(\frac{2}{45\Gamma(2-\alpha)}-\frac{1}{30\Gamma(4-\alpha)}\right)p(x)|_{x=x_1}+\sum_{k=1}^{\infty}\binom{\alpha}{k}h^{k-\alpha}\left(\frac{2}{45\Gamma(2+k-\alpha)}-\frac{1}{30\Gamma(4+k-\alpha)}\right)D^k_{x_0}p(x)|_{x=x_1},$$

$$\eta_{12}=h^{-\alpha}\left(\frac{1}{180\Gamma(2-\alpha)}-\frac{1}{60\Gamma(4-\alpha)}\right)p(x)|_{x=x_1}+\sum_{k=1}^{\infty}\binom{\alpha}{k}h^{k-\alpha}\left(\frac{1}{180\Gamma(2+k-\alpha)}-\frac{1}{60\Gamma(4+k-\alpha)}\right)D^k_{x_0}p(x)|_{x=x_1},$$

$$\theta=\left(\frac{-1}{\Gamma(4-\alpha)}+\frac{1}{\Gamma(3-\alpha)}-\frac{1}{3\Gamma(2-\alpha)}\right)h^{2-\alpha}p(x)|_{x=x_1}+\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{-1}{\Gamma(4+k-\alpha)}+\frac{1}{\Gamma(3+k-\alpha)}-\frac{1}{3\Gamma(2+k-\alpha)}\right)h^{2+k-\alpha}D^k_{x_0}p(x)|_{x=x_1}.$$

References

Al-Rabtah, A., Momani, S., Ramadan, M.A., 2012. Solving linear and nonlinear fractional differential equations using spline functions. Abstr. Appl. Anal. 2012, 9. http://dx.doi.org/10.1155/2012/426514. Article ID 426514.

Ahmad, B., Nieto, J.J., 2009. Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. Abstr. Appl. Anal. 2009, 9. http://dx.doi.org/10.1155/2009/494720. Article ID 494720.

Akram, G., Aslam, I.A., 2014. Solution of fourth order three-point boundary value problem using ADM and RKM. J. Assoc. Arab Univ. Basic Appl. Sci. http://dx.doi.org/10.1016/j.jaubas.2014.08.001.

Akram, G., Tariq, H., 2015. An exponential spline technique for solving fractional boundary value problem. Calcolo. http://dx.doi.org/10.1007/s10092-015-0161-0.

Almeida, R., Bastos, N.R.O., 2015. A numerical method to solve higher-order fractional differential equations. Mediterr. J. Math. http://dx.doi.org/10.1007/s00009-015-0550-2.

Bai, Z., 2010. On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. 72, 916–924.

Chen, W., Sun, H., Zhang, X., Korosak, D., 2010. Anomalous diffusion modeling by fractal and fractional derivatives. Comput. Math. Appl. 59, 1754–1758.

Diethelm, K., 2010. The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics.

Doha, E.H., Bhrawy, A.H., Ezz-Eldien, S.S., 2011. Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. Appl. Math. Modell. 35, 5662–5672.

Edwards, J.T., Ford, N.J., Simpson, A.C., 2002. The numerical solution of linear multi-term fractional differential equations: systems of equations. J. Comput. Appl. Math. 148, 401–418.

Ford, N.J., Morgado, M.L., Rebelo, M., 2014. High order numerical methods for fractional terminal value problems. Comput. Meth. Appl. Math. 14, 55–70.

Grinstein, G., Luther, A., 1976. Application of the renormalization group to phase transitions in disordered systems. Phys. Rev. B 13, 1329–1343.

Gupta, C.P., 1988. Existence and uniqueness theorem for the bending of an elastic beam equation. Appl. Anal. 26, 289–304.

Jin, B., Lazarov, R., Pasciak, J., Rundell, W., 2015. Variational formulation of problems involving fractional order differential operators. Math. Comput. 84, 2665–2700.

Jin, B., Lazarov, R., Lu, X., Zhou, Z., 2016. A simple finite element method for boundary value problems with a Riemann–Liouville derivative. J. Comput. Appl. Math. 293, 94–111.

Kilbasm, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory of Application of Fractional Differential Equations, first ed., Belarus.

Kumar, D., Singh, J., Kumar, S., 2015a. Numerical computation of fractional multidimensional diffusion equations by using a modified homotopy perturbation method. J. Assoc. Arab Univ. Basic Appl. Sci. 17, 20–26.

Kumar, D., Singh, J., Kumar, S., 2015b. A fractional model of Navier–Stokes equation arising in unsteady flow of a viscous fluid. J. Assoc. Arab Univ. Basic Appl. Sci. 17, 14–19.

Lakshmikantham, V., Vatsala, A.S., 2008. Basic theory of fractional differential equations. Nonlinear Anal. 69, 2677–2682.

Lazer, A.C., McKenna, P.J., 1990. Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. SIAM Rev. 32, 537–578.

Nouri, K., Siavashani, N.B., 2014. Application of Shannon wavelet for solving boundary value problems of fractional differential equations. Wavelets Linear Algebra 1, 25–32.

Ortigueira, M.D., Machado, J.A.T., 2015. What is a fractional derivative? J. Comput. Phys 293, 4–13.

Ortigueira, M.D., Trujillo, J.J., 2012. A unified approach to fractional derivatives. Commun. Nonlinear Sci. Numer. Simul. 17 (12), 5151–5157.

Podlubny, I., 1999. Fractional Differential Equation. Academic, San Diego.

Rehmana, M.U., Khan, R.A., 2012. A numerical method for solving boundary value problems for fractional differential equations. Appl. Math. Modell. 36, 894–907.

Secer, A., Alkan, S., Akinlar, M.A., Bayram, M., 2013. Sinc-Galerkin method for approximate solutions of fractional order boundary value problems. Boundary Value Probl. 2013, 281.

Seifollahi, M., Shamloo, A.S., 2013. Numerical solution of nonlinear multi-order fractional differential equations by operational matrix of Chebyshev polynomials. World Appl. Program 3 (3), 85–92.

Siddiqi, S.S., Akram, G., Nazeer, S., 2007. Quintic spline solution of linear sixth-order boundary value problems. Appl. Math. Comput. 189 (1), 887–892.

Siddiqi, S.S., Akram, G., 2008. Quintic spline solutions of fourth order boundary value problems. Int. J. Numer. Anal. Model. 5 (10), 101–111.

Siddiqi, S.S., Akram, G., 2007. Solution of tenth-order boundary value problems using eleventh degree spline. Appl. Math. Comput. 185 (2007), 115–127.

Taukenova, F.I., Shkhanukov-Lafishev, M.Kh., 2006. Difference methods for solving boundary value problems for fractional differential equations. Comput. Math. Math. Phys 46, 1785–1795.

Zhang, S., 2006. Existence of solution for boundary value problem of fractional order. Acta Math. Sci. 26, 220–228.

Please cite this article in press as: Akram, G., Tariq, H. Quintic spline collocation method for fractional boundary value problems. Journal of the Association of Arab Universities for Basic and Applied Sciences (2016), http://dx.doi.org/10.1016/j.jaubas.2016.03.003