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Crystal bases of modified quantized enveloping algebras and a double RSK correspondence ☆

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A B S T R A C T

We give a new combinatorial realization of the crystal base of the modified quantized enveloping algebras of type $A_{+\infty}$ or A_∞. It is obtained by describing the decomposition of the tensor product of a highest weight crystal and a lowest weight crystal into extremal weight crystals, and taking its limit using a tableaux model of extremal weight crystals. This realization induces in a purely combinatorial way a bicrystal structure of the crystal base of the modified quantized enveloping algebras and hence its Peter–Weyl type decomposition generalizing the classical RSK correspondence.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}$. In [17], Lusztig introduced the modified quantized enveloping algebra $\tilde{U}_q(\mathfrak{g}) = \bigoplus_\Lambda U_q(\mathfrak{g})a_\Lambda$, where Λ runs over all integral weight for $\mathfrak{g}$, and proved the existence of its global crystal basis or canonical basis. In [10], Kashiwara studied the crystal structure of $\tilde{U}_q(\mathfrak{g})$ in detail, and showed that

$$\mathbf{B}\big(U_q(\mathfrak{g})a_\Lambda\big) \simeq \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty),$$

where $\mathbf{B}(U_q(\mathfrak{g})a_\Lambda)$ denotes the crystal base of $U_q(\mathfrak{g})a_\Lambda$, $\mathbf{B}(\pm\infty)$ is the crystal base of the negative (resp. positive) part of $U_q(\mathfrak{g})$ and $T_\Lambda = \{t_\Lambda\}$ is a crystal with $\mathrm{wt}(t_\Lambda) = \Lambda$ and $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$. It is also shown that the Lusztig's involution on $\tilde{U}_q(\mathfrak{g})$ provides the crystal $\mathbf{B}(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_\Lambda \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ with another crystal structure so-called $*$-crystal structure and therefore a regular $(\mathfrak{g},\mathfrak{g})$-bicrystal structure [10]. With respect to this bicrystal structure, a Peter–Weyl type decomposition for $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ was obtained when it is of finite type or affine type at non-zero levels by Kashiwara [10]

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and of affine type at level zero by Beck and Nakajima [1] (see also [21,22] for partial results). Note that the crystal base of the quantized coordinate ring for $\mathfrak{g}$ [9] is a subcrystal of $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$, and equal to $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ if and only if $\mathfrak{g}$ is of finite type [10].

One of the essential ingredients for understanding the structure of $\tilde{U}_q(\mathfrak{g})$ is the notion of extremal weight $U_q(\mathfrak{g})$-module introduced by Kashiwara [10]. An extremal weight module associated with an integral weight Λ for $\mathfrak{g}$ is an integrable $U_q(\mathfrak{g})$-module, which is a generalization of a highest weight and a lowest weight module, and it also has a (global) crystal base. When $\mathfrak{g}$ is an affine algebra of finite rank, it is shown by Kashiwara [19, Remark 2.15] that a level zero extremal weight module is isomorphic to a Weyl module introduced by Chari and Pressley [3].

The main purpose of this work is to study the structure of $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ when $\mathfrak{g}$ is a general linear Lie algebra of type $A_{+\infty}$ or A_∞ (affine type of infinite rank following [7]) using the combinatorics of Young tableaux, and understand its connection with the classical RSK correspondence. From now on, we denote $\mathfrak{g}$ by $\mathfrak{gl}_{>0}$ and $\mathfrak{gl}_\infty$ when it is of type $A_{+\infty}$ and A_∞, respectively.

The main result in this paper gives a new combinatorial realization of $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ for all integral $\mathfrak{gl}_{>0}$-weights and all level zero integral $\mathfrak{gl}_\infty$-weights Λ as a set of certain bimatrices. This also implies directly Peter–Weyl type decompositions of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$ and $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))_0$, the level zero part of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))$, without using the $*$-crystal structure. Our approach is based on the combinatorial models of extremal weight crystals of type $A_{+\infty}$ and A_∞ developed in [14,15].

Let us state our results more precisely. Let $\mathcal{M}$ be the set of $\mathbb{N} \times \mathbb{N}$ matrices with non-negative integral entries and finitely many positive entries. Recall that $\mathcal{M}$ has a $\mathfrak{gl}_{>0}$-crystal structure where each row of a matrix in $\mathcal{M}$ is identified with a single row Young tableau or a crystal element associated with the symmetric power of the natural representation. Let $\mathcal{M}^\vee = \{M^\vee \mid M \in \mathcal{M}\}$ be the dual crystal of $\mathcal{M}$. For each integral weight Λ, let

$$\tilde{\mathcal{M}}_\Lambda = \{M^\vee \otimes N \mid \mathrm{wt}(N^t) - \mathrm{wt}(M^t) = \Lambda\} \subset \mathcal{M}^\vee \otimes \mathcal{M}.$$

Here wt denotes the weight with respect to $\mathfrak{gl}_{>0}$-crystal structure and A^t denotes the transpose of $A \in \mathcal{M}$. Then we show that

$$\tilde{\mathcal{M}}_\Lambda \simeq \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$$

(Theorem 5.5). The crucial step in the proof is the description of the tensor product $\mathbf{B}(\Lambda') \otimes \mathbf{B}(-\Lambda'')$ for dominant integral weights Λ', Λ'' with $\Lambda = \Lambda' - \Lambda''$ in terms of skew Young bitableaux (Proposition 5.1), and its embedding into $\mathbf{B}(\Lambda' + \xi) \otimes \mathbf{B}(-\xi - \Lambda'')$ for arbitrary dominant integral weight ξ (Proposition 5.4). In fact, $\mathbf{B}(\Lambda' + \xi) \otimes \mathbf{B}(-\xi - \Lambda'')$ is realized as a set of skew Young bitableaux whose shapes are almost horizontal strips as ξ goes to infinity. This establishes the above isomorphism and as a consequence

$$\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0})) \simeq \mathcal{M}^\vee \otimes \mathcal{M},$$

since $\bigsqcup_\Lambda \tilde{\mathcal{M}}_\Lambda = \mathcal{M}^\vee \otimes \mathcal{M}$.

Now, for partitions μ, ν, let $\mathcal{B}_{\mu,\nu}$ be the extremal weight crystal with the Weyl group orbit of its extremal weight corresponding to the pair (μ, ν). Note that $\mathcal{B}_{\mu,\emptyset}$ (resp. $\mathcal{B}_{\emptyset,\nu}$) is a highest (resp. lowest) weight crystal and $\mathcal{B}_{\mu,\nu} \simeq \mathcal{B}_{\emptyset,\nu} \otimes \mathcal{B}_{\mu,\emptyset}$ [14]. Then a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal structure of $\mathcal{M}$ and $\mathcal{M}^\vee$ arising from the RSK correspondence [4] naturally induces a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal structure of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$ and the following Peter–Weyl type decomposition (Corollary 5.7)

$$\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0})) \simeq \bigsqcup_{\mu,\nu} \mathcal{B}_{\mu,\nu} \times \mathcal{B}_{\mu,\nu}.$$

Hence the decomposition of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$ into extremal weight crystals can be understood as the tensor product of two RSK correspondences, which are dual to each other as a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal.

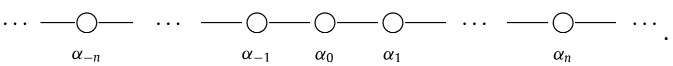
Next, we prove analogues for $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))_0$. This is done by taking the limit of the results in $\mathfrak{gl}_{>0}$. In this case, $\mathcal{M}$ is replaced by $\mathbb{Z} \times \mathbb{Z}$-matrices and $\mathcal{B}_{\mu,\nu}$ is replaced by the level zero extremal weight crystal with the same parameter (μ, ν). Finally, we conjecture that the second crystal structures arising from the RSK correspondence is compatible with the dual of $*$-crystal structure.

There are several nice combinatorial descriptions of $\mathbf{B}(\infty)$ for $\mathfrak{gl}_{>0}$ and $\mathfrak{gl}_\infty$ (see e.g. [16,23,24]), by which one can understand the structure of $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$. But our description of $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ enables us to explain more explicitly the connected component of a given element by applying usual Young tableaux insertion to the row word of its matrix form, an embedding of a tensor product of a highest weight crystal and a lowest weight crystal into $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ in terms of skew Young tableaux and hence a bicrystal structure on $\mathbf{B}(U_q(\mathfrak{gl}_{>0}))$ and $\mathbf{B}(U_q(\mathfrak{gl}_\infty))_0$ in connection with RSK algorithm.

The paper is organized as follows. In Section 2, we give necessary background on crystals. In Section 3, we recall some combinatorics of Littlewood–Richardson tableaux from a view point of crystals, which is necessary for our later arguments. In Section 4, we review a combinatorial model of extremal weight $\mathfrak{gl}_{>0}$-crystals [14] and their non-commutative Littlewood–Richardson rule. Then in Section 5 we prove the main theorem. In Section 6, we recall a combinatorial model of extremal weight $\mathfrak{gl}_\infty$-crystals [15] and describe the Littlewood–Richardson rule of the tensor product of a highest weight crystal and a lowest weight crystal. In Section 7, we prove analogues of the results in Section 5 for $\mathfrak{gl}_\infty$. We remark that the Littlewood–Richardson rule in Section 6 is not necessary for Section 7, but is of independent interest, which completes the discussion on tensor product of extremal weight $\mathfrak{gl}_\infty$-crystals in [15].

2. Crystals

2.1. Let $\mathfrak{gl}_\infty$ be the Lie algebra of complex matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many non-zero entries, which is spanned by E_{ij} $(i, j \in \mathbb{Z})$, the elementary matrix with 1 at the i-th row and the j-th column and zero elsewhere. Let $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}E_{ii}$ be the Cartan subalgebra of $\mathfrak{gl}_\infty$ and let $\langle\cdot,\cdot\rangle$ denote the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. We denote by $\{h_i = E_{ii} - E_{i+1\,i+1} \mid i \in \mathbb{Z}\}$ the set of simple coroots, and denote by $\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}\}$ the set of simple roots, where $\epsilon_i \in \mathfrak{h}^*$ is given by $\langle \epsilon_i, E_{jj}\rangle = \delta_{ij}$. The Dynkin diagram associated with the Cartan matrix $(\langle \alpha_j, h_i\rangle)_{i,j\in\mathbb{Z}}$ is



Let $P = \mathbb{Z}\Lambda_0 \oplus \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\epsilon_i = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\Lambda_i$ be the weight lattice of $\mathfrak{gl}_\infty$, where Λ_0 is given by $\langle \Lambda_0, E_{-j+1\,-j+1}\rangle = -\langle \Lambda_0, E_{jj}\rangle = \frac{1}{2}$ $(j \geqslant 1)$, and $\Lambda_i = \Lambda_0 + \sum_{k=1}^{i} \epsilon_k$, $\Lambda_{-i} = \Lambda_0 - \sum_{k=-i+1}^{0} \epsilon_k$ for $i \geqslant 1$. We call Λ_i the i-th fundamental weight.

For $k \in \mathbb{Z}$, let $P_k = k\Lambda_0 + \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\epsilon_i$ be the set of integral weights of level k. Let $P^+ = \{\Lambda \in P \mid \langle \Lambda, h_i\rangle \geqslant 0,\ i \in \mathbb{Z}\} = \sum_{i\in\mathbb{Z}} \mathbb{Z}_{\geqslant 0}\Lambda_i$ be the set of dominant integral weights. We put $P_k^+ = P^+ \cap P_k$ for $k \geqslant 0$. For $\Lambda = \sum_{i\in\mathbb{Z}} c_i\Lambda_i \in P$, the level of Λ is $\sum_{i\in\mathbb{Z}} c_i$. If we put $\Lambda_\pm = \sum_{i;\,c_i \gtrless 0} |c_i|\Lambda_i$, then $\Lambda = \Lambda_+ - \Lambda_-$ with $\Lambda_\pm \in P^+$.

For $i \in \mathbb{Z}$, let r_i be the simple reflection given by $r_i(\lambda) = \lambda - \langle \lambda, h_i\rangle\alpha_i$ for $\lambda \in \mathfrak{h}^*$. Let W be the Weyl group of $\mathfrak{gl}_\infty$, that is, the subgroup of $GL(\mathfrak{h}^*)$ generated by r_i for $i \in \mathbb{Z}$.

For $p, q \in \mathbb{Z}$, let $[p, q] = \{p, p+1, \ldots, q\}$ $(p < q)$, $[p, \infty) = \{p, p+1, \ldots\}$ and $(-\infty, q] = \{\ldots, q-1, q\}$. For simplicity, we denote $[1, n]$ by $[n]$ $(n \geqslant 1)$. For an interval S in $\mathbb{Z}$, let $\mathfrak{gl}_S$ be the subalgebra of $\mathfrak{gl}_\infty$ spanned by E_{ij} for $i, j \in S$. (We have $\mathfrak{gl}_\mathbb{Z} = \mathfrak{gl}_\infty$.) We denote by S° the index set of simple roots for $\mathfrak{gl}_S$. For example, $[p, q]^\circ = \{p, \ldots, q-1\}$. We also put $\mathfrak{gl}_{>r} = \mathfrak{gl}_{[r+1,\infty)}$ and $\mathfrak{gl}_{<r} = \mathfrak{gl}_{(-\infty,r-1]}$ for $r \in \mathbb{Z}$.

2.2. Let S be an interval in $\mathbb{Z}$. Let $U_q(\mathfrak{gl}_S)$ be the quantized enveloping algebra associated with $\mathfrak{gl}_S$. Then we can consider the crystal base of a $U_q(\mathfrak{gl}_S)$-module following Kashiwara [8]. Roughly speaking, the crystal base of a $U_q(\mathfrak{gl}_S)$-module V is an S°-colored oriented graph, which can be viewed as a limit of V at $q = 0$, but still has important combinatorial information of V. The existence of the crystal bases of $U_q(\mathfrak{gl}_S)$-modules which are related with the work in this paper can be found in [8–10,13].

Based on the properties of crystal bases, one can define the notion of crystal as follows (see [11] for a general review and references therein).

A $\mathfrak{gl}_S$-*crystal* is a set B together with the maps $\mathrm{wt}: B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{\mathbf{0}\}$ ($i \in S^\circ$) such that for $b \in B$

(1) $\varphi_i(b) = \langle \mathrm{wt}(b), h_i \rangle + \varepsilon_i(b)$,
(2) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq \mathbf{0}$,
(3) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq \mathbf{0}$,
(4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,
(5) $\tilde{e}_i b = \tilde{f}_i b = \mathbf{0}$ if $\varphi_i(b) = -\infty$,

where $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. For example, the crystal base of an integrable $U_q(\mathfrak{gl}_S)$-module is a $\mathfrak{gl}_S$-crystal.

Note that B is equipped with an S°-colored oriented graph structure, where $b \overset{i}{\to} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \in S^\circ$. For $b \in B$, we denote by $C(b)$ the connected component in B including b as an S°-colored graph. We say that B is *connected* if $C(b) = B$ for some $b \in B$.

The *dual crystal* $B^\vee$ *of* B is defined to be the set $\{b^\vee \mid b \in B\}$ with $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$, $\varepsilon_i(b^\vee) = \varphi_i(b)$, $\varphi_i(b^\vee) = \varepsilon_i(b)$, $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ and $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$ for $b \in B$ and $i \in S^\circ$. We assume that $\mathbf{0}^\vee = \mathbf{0}$.

Let B_1 and B_2 be crystals. A *morphism* $\psi : B_1 \to B_2$ is a map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$ such that for $b \in B_1$ and $i \in S^\circ$

(1) $\psi(\mathbf{0}) = \mathbf{0}$,
(2) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq \mathbf{0}$,
(3) $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{e}_i b) \neq \mathbf{0}$,
(4) $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{f}_i b) \neq \mathbf{0}$.

We call ψ an *embedding* and B_1 a *subcrystal of* B_2 when ψ is injective, and call ψ *strict* if $\psi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in S^\circ$, where we assume that $\tilde{e}_i \mathbf{0} = \tilde{f}_i \mathbf{0} = \mathbf{0}$. If ψ is a strict embedding, then B_2 is isomorphic to $B_1 \sqcup (B_2 \setminus B_1)$.

For $b_i \in B_i$ ($i = 1, 2$), we say that b_1 *is* ($\mathfrak{gl}_S$-)*equivalent to* b_2, and write $b_1 \equiv b_2$ if there exists an isomorphism of crystals $C(b_1) \to C(b_2)$ sending b_1 to b_2.

For a crystal B and $m \in \mathbb{Z}_{\geqslant 0}$, we denote by $B^{\oplus m}$ the disjoint union $B_1 \sqcup \cdots \sqcup B_m$ with $B_i \simeq B$, where $B^{\oplus 0}$ means the empty set.

We say that a crystal B is *regular* if B is as a $\mathfrak{gl}_{S'}$-crystal, isomorphic to the crystal base of an integrable $U_q(\mathfrak{gl}_{S'})$-module for any finite subinterval $S' \subset S$. In particular, if B is regular, then $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq \mathbf{0}\}$ for $b \in B$ and $i \in S^\circ$. Note that an embedding between regular crystals is always strict.

A *tensor product* $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\bigl(\varepsilon_i(b_1), \varepsilon_i(b_2) - \bigl\langle \mathrm{wt}(b_1), h_i \bigr\rangle\bigr),$$

$$\varphi_i(b_1 \otimes b_2) = \max\bigl(\varphi_i(b_1) + \bigl\langle \mathrm{wt}(b_2), h_i \bigr\rangle, \varphi_i(b_2)\bigr),$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \end{cases}$$

for $i \in S^\circ$ and $b_1 \otimes b_2 \in B_1 \otimes B_2$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is a crystal. Note that $B_1 \otimes B_2$ is regular if B_1 and B_2 are regular, and $(B_1 \otimes B_2)^\vee \simeq B_2^\vee \otimes B_1^\vee$.

2.3. Let us briefly review the crystal bases of an extremal weight module and a modified quantized enveloping algebra. We refer the reader to Kashiwara's papers [8,10,12] for more details.

Let S be an interval in $\mathbb{Z}$. Let $\Lambda \in P$ be given. We may regard Λ as an integral weight for $\mathfrak{gl}_S$ by restricting it to the weight lattice of $\mathfrak{gl}_S$ (i.e. $\bigoplus_{i\in S}\mathbb{Z}\epsilon_i$ when $S \neq \mathbb{Z}$). Let $\mathbf{B}(\Lambda)$ be the crystal base of the extremal weight $U_q(\mathfrak{gl}_S)$-module with extremal weight vector u_Λ of weight Λ, which is a regular $\mathfrak{gl}_S$-crystal. When $\pm\Lambda$ is a dominant integral weight for $\mathfrak{gl}_S$, $\mathbf{B}(\Lambda)$ is the crystal base of the integrable highest (resp. lowest) weight $U_q(\mathfrak{gl}_S)$-module with highest (resp. lowest) weight Λ. Also we have $\mathbf{B}(\Lambda) \simeq \mathbf{B}(w\Lambda)$ for $w \in W$. When S is finite, Λ is Weyl group conjugate to a $\mathfrak{gl}_S$-dominant integral weight and hence $\mathbf{B}(\Lambda)$ is isomorphic to the crystal base of a highest weight module and in particular it is connected. When S is infinite, $\mathbf{B}(\Lambda)$ does not necessarily contain a highest weight or lowest weight element, but it is shown in [14, Proposition 3.1] and [15, Proposition 4.1] that $\mathbf{B}(\Lambda)$ is also connected.

Let $\mathbf{B}(\pm\infty)$ be the crystal base of the negative (resp. positive) part of $U_q(\mathfrak{gl}_S)$ with the highest (resp. lowest) weight element $u_{\pm\infty}$, which is a $\mathfrak{gl}_S$-crystal, and let $T_\Lambda = \{t_\Lambda\}$ $(\Lambda \in P)$ be the crystal with $\mathrm{wt}(t_\Lambda) = \Lambda$, $\tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = \mathbf{0}$ and $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$ for $i \in S^\circ$. Let $\tilde{U}_q(\mathfrak{gl}_S) = \bigoplus_\Lambda U_q(\mathfrak{gl}_S)a_\Lambda$ be the modified quantized enveloping algebra associated with $\mathfrak{gl}_S$, where Λ runs over all integral weights for $\mathfrak{gl}_S$, and let $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_S)) = \bigsqcup_\Lambda \mathbf{B}(U_q(\mathfrak{gl}_S)a_\Lambda)$ denote the crystal base of $\tilde{U}_q(\mathfrak{gl}_S)$. Then it was shown by Kashiwara that

$$\mathbf{B}\big(U_q(\mathfrak{gl}_S)a_\Lambda\big) \simeq \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty).$$

Note that $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ is regular, and there is a strict embedding of $\mathbf{B}(\Lambda)$ into $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ sending u_Λ to $u_\infty \otimes t_\Lambda \otimes u_{-\infty}$. Hence $\mathbf{B}(\Lambda)$ is isomorphic to $C(u_\infty \otimes t_\Lambda \otimes u_{-\infty})$ since $\mathbf{B}(\Lambda)$ is connected.

The crystal $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ can be understood as a limit of $\mathbf{B}(\Lambda') \otimes \mathbf{B}(-\Lambda'')$ for $\mathfrak{gl}_S$-dominant weights Λ', Λ'' with $\Lambda' - \Lambda'' = \Lambda$. First recall that there is an embedding $\mathbf{B}(\Lambda_+) \to \mathbf{B}(\infty) \otimes T_{\Lambda_+}$ (resp. $\mathbf{B}(-\Lambda_-) \to T_{\Lambda_-} \otimes \mathbf{B}(-\infty)$) sending u_{Λ_+} to $u_\infty \otimes t_{\Lambda_+}$ (resp. $u_{-\Lambda_-}$ to $t_{-\Lambda_-} \otimes u_{-\infty}$). This gives a strict embedding

$$\iota_{\Lambda_+,\Lambda_-} : \mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-) \to \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty) \tag{2.1}$$

sending $u_{\Lambda_+} \otimes u_{-\Lambda_-}$ to $u_\infty \otimes t_\Lambda \otimes u_{-\infty}$ since $t_\Lambda \equiv t_{\Lambda_+} \otimes t_{-\Lambda_-}$. For a $\mathfrak{gl}_S$-dominant weight $\xi \in P$, let

$$\iota^\xi_{\Lambda_+,\Lambda_-} : \mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-) \to \mathbf{B}(\Lambda_+ + \xi) \otimes \mathbf{B}(-\xi - \Lambda_-) \tag{2.2}$$

be a strict embedding given by the composition of the following two morphisms

$$\begin{aligned}\mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-) &\to \mathbf{B}(\Lambda_+) \otimes \mathbf{B}(\xi) \otimes \mathbf{B}(-\xi) \otimes \mathbf{B}(-\Lambda_-)\\ &\to \mathbf{B}(\Lambda_+ + \xi) \otimes \mathbf{B}(-\xi - \Lambda_-),\end{aligned}$$

where

$$\begin{aligned}&\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+} \otimes \tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\Lambda_-}\\ &\quad\mapsto (\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+}) \otimes u_\xi \otimes u_{-\xi} \otimes (\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\Lambda_-})\\ &\quad\mapsto \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+ + \xi} \otimes \tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\xi-\Lambda_-}\end{aligned}$$

for $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+} \neq \mathbf{0}$ and $\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\Lambda_-} \neq \mathbf{0}$. Note that

$$\begin{aligned}&\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+ + \xi} \equiv (\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+}) \otimes u_\xi, \quad \text{if } \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_+} \neq \mathbf{0},\\ &\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\xi-\Lambda_-} \equiv u_{-\xi} \otimes (\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\Lambda_-}), \quad \text{if } \tilde{e}_{j_1} \cdots \tilde{e}_{j_s} u_{-\Lambda_-} \neq \mathbf{0}.\end{aligned}$$

Since

$$\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty) = \bigcup_{\substack{\Lambda',\Lambda'':\mathfrak{gl}_S\text{-dominant}\\ \Lambda'-\Lambda''=\Lambda}} \mathrm{Im}(\iota_{\Lambda',\Lambda''}),$$

$$\iota_{\Lambda',\Lambda''} = \iota_{\Lambda'+\xi,\Lambda''+\xi} \circ \iota^\xi_{\Lambda',\Lambda''}, \tag{2.3}$$

$\{\mathbf{B}(\Lambda') \otimes \mathbf{B}(-\Lambda'') \mid \Lambda', \Lambda'' : \mathfrak{gl}_S\text{-dominant with } \Lambda = \Lambda' - \Lambda''\}$ together with $\iota^{\xi}_{\Lambda',\Lambda''}$'s forms a direct system, whose limit is isomorphic to $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$. Note that $\mathbf{B}(\Lambda)$ is also isomorphic to $C(u_{\Lambda_+ + \xi} \otimes u_{-\xi-\Lambda_-})$ in $\mathbf{B}(\Lambda_+ + \xi) \otimes \mathbf{B}(-\xi - \Lambda_-)$ for any $\mathfrak{gl}_S$-dominant weight ξ.

3. Young and Littlewood–Richardson tableaux

3.1. Let $\mathscr{P}$ denote the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i \geqslant 1}$ with a *Young diagram* or a subset $\{(i, j) \mid 1 \leqslant j \leqslant \lambda_i\}$ of $\mathbb{N} \times \mathbb{N}$ following [18]. Let $\ell(\lambda) = |\{i \mid \lambda_i \neq 0\}|$. We denote by $\lambda' = (\lambda'_i)_{i \geqslant 1}$ the conjugate partition of λ whose Young diagram is $\{(i, j) \mid (j, i) \in \lambda\}$. For $\mu, \nu \in \mathscr{P}$, $\mu \cup \nu$ is the partition obtained by rearranging $\{\mu_i, \nu_i \mid i \geqslant 1\}$, and $\mu + \nu = (\mu_i + \nu_i)_{i \geqslant 1}$.

Let $\mathcal{A}$ be a linearly ordered set and λ/μ a skew Young diagram. A tableau T obtained by filling λ/μ with entries in $\mathcal{A}$ is called a *semistandard tableau or Young tableau of shape* λ/μ if the entries in each row (resp. column) are weakly (resp. strictly) increasing from left to right (resp. from top to bottom). We denote by $T(i, j)$ the entry of T at $(i, j) \in \lambda/\mu$. Let $SST_{\mathcal{A}}(\lambda/\mu)$ denote the set of all semistandard tableaux of shape λ/μ with entries in $\mathcal{A}$.

Suppose that $\mathcal{A}$ is an interval in $\mathbb{Z}$ with a usual linear ordering. Then $\mathcal{A}$ is a regular $\mathfrak{gl}_{\mathcal{A}}$-crystal, where $\mathrm{wt}(i) = \epsilon_i$ ($i \in \mathcal{A}$) and $i \xrightarrow{i} i+1$ ($i \in \mathcal{A}^\circ$). The image of $SST_{\mathcal{A}}(\lambda/\mu)$ in $\mathcal{A}^{\otimes r}$ ($r = |\lambda/\mu|$) under the map $T \mapsto w(T) = w_1 \cdots w_r$ or $w_1 \otimes \cdots \otimes w_r$ together with $\{\mathbf{0}\}$ is invariant under $\tilde{e}_i, \tilde{f}_i$ ($i \in \mathcal{A}^\circ$), where $w(T)$ is the word obtained by reading the entries of T column by column from right to left, and in each column from top to bottom. Hence $SST_{\mathcal{A}}(\lambda/\mu)$ is a subcrystal of $\mathcal{A}^{\otimes r}$ [13]. We may identify the dual crystal element $T^\vee \in SST_{\mathcal{A}}(\lambda/\mu)^\vee$ with the tableau obtained from T by $180°$-rotation and replacing each entry a with $a^\vee$. So we have $SST_{\mathcal{A}}(\lambda/\mu)^\vee \simeq SST_{\mathcal{A}^\vee}((\lambda/\mu)^\vee)$, where $a^\vee < b^\vee$ if and only if $b < a$ for $a, b \in \mathcal{A}$ and $(\lambda/\mu)^\vee$ is the skew Young diagram obtained from λ/μ by $180°$-rotation. We use the convention $(a^\vee)^\vee = a$ and hence $(T^\vee)^\vee = T$.

3.2. For $\lambda, \mu, \nu \in \mathscr{P}$ with $|\lambda| = |\mu| + |\nu|$, let $\mathbf{LR}^\lambda_{\mu\nu}$ be the set of tableaux U in $SST_{\mathbb{N}}(\lambda/\mu)$ such that

(1) the number of occurrences of each $i \geqslant 1$ in U is ν_i,
(2) for $1 \leqslant k \leqslant |\nu|$, the number of occurrences of each $i \geqslant 1$ in $w_1 \cdots w_k$ is no less than that of $i+1$ in $w_1 \cdots w_k$, where $w(U) = w_1 \cdots w_{|\nu|}$.

We call $\mathbf{LR}^\lambda_{\mu\nu}$ the set of *Littlewood–Richardson tableaux of shape* λ/μ *with content* ν and put $c^\lambda_{\mu\nu} = |\mathbf{LR}^\lambda_{\mu\nu}|$ [18]. Let us introduce a variation of $\mathbf{LR}^\lambda_{\mu\nu}$, which is necessary for our later arguments. Let $\overline{\mathbf{LR}}^\lambda_{\mu\nu}$ be the set of tableaux U in $SST_{-\mathbb{N}}(\lambda/\mu)$ such that

(1) the number of occurrences of each $-i \leqslant -1$ in U is ν_i,
(2) for $1 \leqslant k \leqslant |\nu|$, the number of occurrences of each $-i \leqslant -1$ in $w_k \cdots w_{|\nu|}$ is no less than that of $-(i+1)$ in $w_k \cdots w_{|\nu|}$, where $w(U) = w_1 \cdots w_{|\nu|}$.

There are characterizations of $\mathbf{LR}^\lambda_{\mu\nu}$ and $\overline{\mathbf{LR}}^\lambda_{\mu\nu}$ using crystals. For $U \in SST_{\mathbb{N}}(\lambda/\mu)$, we can check that $U \in \mathbf{LR}^\lambda_{\mu\nu}$ if and only if U is $\mathfrak{gl}_{>0}$-equivalent (or Knuth equivalent) to the highest weight element H_ν in $SST_{\mathbb{N}}(\nu)$, that is, $H_\nu(i, j) = i$ for $(i, j) \in \nu$. Similarly, for $U \in SST_{-\mathbb{N}}(\lambda/\mu)$, we have $U \in \overline{\mathbf{LR}}^\lambda_{\mu\nu}$ if and only if U is $\mathfrak{gl}_{<0}$-equivalent (or Knuth equivalent) to the lowest weight element L_ν in $SST_{-\mathbb{N}}(\nu)$, that is, $L_\nu(i, j) = -\nu'_j + i - 1$ for $(i, j) \in \nu$.

There is a one-to-one correspondence between the set of $V \in SST_{\mathbb{N}}(\nu)$ such that $H_\mu \otimes V \equiv H_\lambda$ and $\mathbf{LR}^\lambda_{\mu\nu}$. Indeed, V corresponds to $\imath(V) = U \in \mathbf{LR}^\lambda_{\mu\nu}$, where the number of k's in the i-th row of V is equal to the number of i's in the k-th row of U for $i, k \geqslant 1$ [20].

Example 3.1. Consider

$$V = \begin{matrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & \end{matrix} \in SST_{\mathbb{N}}((3,3,2)).$$

Then $H_{(3,1)} \otimes V \equiv H_{(5,4,2,1)}$ and

$$\imath(V) = \begin{matrix} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 1 & 2 & 2 & \\ 2 & 3 & & & \\ 3 & & & & \end{matrix} \quad \in \mathbf{LR}^{(5,4,2,1)}_{(3,1)\ (3,3,2)}.$$

3.3. Next, let us briefly recall the *switching algorithm* [2]. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two linearly ordered sets. Let λ/μ be a skew Young diagram. Let U be a tableau of shape λ/μ with entries in $\mathcal{A} \sqcup \mathcal{B}$, satisfying the following conditions:

(S1) $U(i,j) \leqslant U(i',j')$ whenever $U(i,j), U(i',j') \in \mathcal{X}$ for $(i,j),(i',j') \in \lambda/\mu$ with $i \leqslant i'$ and $j \leqslant j'$,
(S2) in each column of U, entries in $\mathcal{X}$ increase strictly from top to bottom,

where $\mathcal{X} = \mathcal{A}$ or $\mathcal{B}$. Suppose that $b \in \mathcal{B}$ and $a \in \mathcal{A}$ are two adjacent entries in U such that b is placed above or to the left of a. Interchanging a and b is called a *switching* if the resulting tableau still satisfies the conditions (S1) and (S2).

Let λ/μ and μ/η be two skew Young diagrams. For $S \in SST_{\mathcal{B}}(\mu/\eta)$ and $T \in SST_{\mathcal{A}}(\lambda/\mu)$, we denote by $S * T$ the tableau of shape λ/η with entries $\mathcal{A} \sqcup \mathcal{B}$ obtained by gluing S and T, that is, $(S*T)(i,j) = S(i,j)$ if $(i,j) \in \mu/\eta$, and $T(i,j)$ if $(i,j) \in \lambda/\mu$. Let U be a tableau obtained from $S * T$ by applying switching procedures as far as possible. Then it is shown in [2, Theorems 2.2 and 3.1] that

(1) $U = T' * S'$, where $T' \in SST_{\mathcal{A}}(\nu/\eta)$ and $S' \in SST_{\mathcal{B}}(\lambda/\nu)$ for some ν,
(2) U is uniquely determined by S and T,
(3) $w(S)$ (resp. $w(T)$) is Knuth equivalent to $w(S')$ (resp. $w(T')$).

Suppose that $\eta = \emptyset$ and $S = H_\mu \in SST_{\mathbb{N}}(\mu)$. We put

$$\jmath(T) = T', \qquad \jmath(T)_R = S'. \tag{3.1}$$

Then we have the following.

Proposition 3.2. *Suppose that $\mathcal{A}$ is an interval in $\mathbb{Z}$. The map sending T to $(\jmath(T), \jmath(T)_R)$ is an isomorphism of $\mathfrak{gl}_{\mathcal{A}}$-crystals*

$$SST_{\mathcal{A}}(\lambda/\mu) \to \bigsqcup_{\nu \in \mathscr{P}} SST_{\mathcal{A}}(\nu) \times \mathbf{LR}^{\lambda}_{\nu\,\mu},$$

where $\tilde{x}_i(T',S') = (\tilde{x}_i T', S')$ for $i \in \mathcal{A}^\circ$ and $x = e, f$ on the right-hand side. In particular, the map $Q \mapsto \jmath(Q)_R$ restricts to a bijection from $\mathbf{LR}^{\lambda}_{\mu\,\nu}$ to $\mathbf{LR}^{\lambda}_{\nu\,\mu}$, and from $\overline{\mathbf{LR}}^{\lambda}_{\mu\,\nu}$ to $\mathbf{LR}^{\lambda}_{\nu\,\mu}$ when $\mathcal{A} = \pm\mathbb{N}$, respectively.

Proof. The map is clearly a bijection by [2, Theorem 3.1]. Moreover, $\jmath(T)$ is $\mathfrak{gl}_{\mathcal{A}}$-equivalent to T and $\jmath(T)_R$ is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in \mathcal{A}^\circ$ (cf. [6, Theorem 5.9]). Hence the bijection is an isomorphism of $\mathfrak{gl}_{\mathcal{A}}$-crystals. $\square$

Remark 3.3. The inverse of the isomorphism in Proposition 3.2 is given directly by applying the switching process in a reverse way.

4. Extremal weight crystals of type $A_{+\infty}$

Note that for $r \in \mathbb{Z}$ the $\mathfrak{gl}_{>r}$-crystals $[r+1,\infty)$ and $[r+1,\infty)^\vee$ are given by

$$r+1 \xrightarrow{r+1} r+2 \xrightarrow{r+2} r+3 \xrightarrow{r+3} \cdots,$$
$$\cdots \xrightarrow{r+3} (r+3)^\vee \xrightarrow{r+2} (r+2)^\vee \xrightarrow{r+1} (r+1)^\vee.$$

For $\mu \in \mathscr{P}$, let

$$\mathbf{B}_{\mu}^{>r} = SST_{[r+1,\infty)}(\mu). \tag{4.1}$$

Then $\mathbf{B}_{\mu}^{>r}$ is a highest weight $\mathfrak{gl}_{>r}$-crystal with highest weight element $H_{\mu}^{>r}$ of weight $\sum_{i\geqslant 1}\lambda_i\epsilon_{r+i}$, where $H_{\mu}^{>r}(i,j) = r+i$ for $(i,j)\in\mu$. We identify $(\mathbf{B}_{\mu}^{>r})^{\vee}$ with $SST_{[r+1,\infty)^{\vee}}(\mu^{\vee})$.

For $\nu \in \mathscr{P}$ and $s \geqslant \ell(\nu)$, let $E_{\nu}^{>r}(s) \in (\mathbf{B}_{\nu}^{>r})^{\vee}$ be given by

$$\big(E_{\nu}^{>r}(s)\big)^{\vee}(i,j) = r+s-\nu'_j+i \tag{4.2}$$

for $(i,j)\in\nu$. For $s \geqslant \ell(\mu)+\ell(\nu)$, let

$$\mathbf{B}_{\mu,\nu}^{>r} = C\big(H_{\mu}^{>r}\otimes E_{\nu}^{>r}(s)\big) \subset \mathbf{B}_{\mu}^{>r}\otimes\big(\mathbf{B}_{\nu}^{>r}\big)^{\vee} \tag{4.3}$$

be the connected component including $H_{\mu}^{>r}\otimes E_{\nu}^{>r}(s)$ as a $\mathfrak{gl}_{>r}$-crystal. Then we have the following by [14, Proposition 3.4] and [14, Theorem 3.5].

Theorem 4.1. *For $\mu,\nu\in\mathscr{P}$,*

(1) *$\mathbf{B}_{\mu,\nu}^{>r}$ is the set of $S\otimes T\in\mathbf{B}_{\mu}^{>r}\otimes(\mathbf{B}_{\nu}^{>r})^{\vee}$ such that for each $k\geqslant 1$,*

$$\big|\big\{i \bigm| S(i,1)\leqslant r+k\big\}\big| + \big|\big\{i \bigm| T^{\vee}(i,1)\leqslant r+k\big\}\big| \leqslant k,$$

(2) *$\mathbf{B}_{\mu,\nu}^{>r}$ is isomorphic to an extremal weight $\mathfrak{gl}_{>r}$-crystal with extremal weight*

$$\sum_{i=1}^{\ell(\mu)}\mu_i\epsilon_{r+i} - \sum_{j=1}^{\ell(\nu)}\nu_j\epsilon_{r+\ell(\mu)+\ell(\nu)-j+1}.$$

Note that $\mathbf{B}_{\mu,\nu}^{>r}$ does not depend on the choice of s. Moreover, $\{\mathbf{B}_{\mu,\nu}^{>r} \mid \mu,\nu\in\mathscr{P}\}$ is a complete list of pairwise non-isomorphic extremal weight $\mathfrak{gl}_{>r}$-crystals [14, Theorem 3.5 and Lemma 5.1] and the tensor product of extremal weight $\mathfrak{gl}_{>r}$-crystals is isomorphic to a finite disjoint union of extremal weight crystals [14, Theorem 4.10].

To describe the tensor product of extremal weight $\mathfrak{gl}_{>r}$-crystals, let us review an insertion algorithm for extremal weight crystal elements [14, Section 4], which is an infinite analogue of [25,26]. Recall that for $a\in\mathcal{A}$ and $T\in SST_{\mathcal{A}}(\lambda)$ ($\lambda\in\mathscr{P}$), $a\to T$ (resp. $T\leftarrow a$) denotes the tableau obtained by the Schensted column (resp. row) insertion, where $\mathcal{A}$ is a linearly ordered set (see for example [5, Appendix A.2]).

From now on, we denote $S\otimes T\in\mathbf{B}_{\mu,\nu}^{>r}$ by (S,T) following [14]. For $a\in[r+1,\infty)$, we define $a\to(S,T)$ in the following way.

Suppose first that S is the empty tableau $\emptyset$ and T is a single column tableau. Let (T',a') be the pair obtained by the following process:

(1) If T contains $a^{\vee},(a+1)^{\vee},\dots,(b-1)^{\vee}$ as its entries but not $b^{\vee}$, then T' is the tableau obtained from T by replacing $a^{\vee},(a+1)^{\vee},\dots,(b-1)^{\vee}$ with $(a+1)^{\vee},(a+2)^{\vee},\dots,b^{\vee}$, and put $a'=b$.
(2) If T does not contain $a^{\vee}$, then leave T unchanged and put $a'=a$.

Now, we suppose that S and T are arbitrary.

(1) Apply the above process to the left-most column of T with a.
(2) Repeat (1) with a' and the next column to the right.
(3) Continue this process to the right-most column of T to get a tableau T' and a''.
(4) Define $a\to(S,T)$ to be $((a''\to S),T')$.

Then $(a \to (S,T)) \in \mathbf{B}^{>r}_{\sigma,\nu}$ for some $\sigma \in \mathscr{P}$ with $|\sigma/\mu| = 1$ ($\mu \subset \sigma$). For a finite word $w = w_1 \cdots w_n$ with letters in $[r+1,\infty)$, we let $(w \to (S,T)) = (w_n \to (\cdots (w_1 \to (S,T)) \cdots))$.

For $a \in [r+1,\infty)$ and $(S,T) \in \mathbf{B}^{>r}_{\mu,\nu}$, we define $(S,T) \leftarrow a^\vee$ to be the pair (S',T') obtained in the following way:

(1) If the pair $(S,(T^\vee \leftarrow a)^\vee)$ satisfies the condition in Theorem 4.1(1), then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.
(2) Otherwise, choose the smallest k such that a_k is bumped out of the k-th row in the row insertion of a into $T^\vee$ and the insertion of a_k into the $(k+1)$-st row violates the condition in Theorem 4.1(1).
(2-a) Stop the row insertion of a into $T^\vee$ when a_k is bumped out and let T' be the resulting tableau after taking $\vee$.
(2-b) Remove a_k in the left-most column of S, which necessarily exists, and then apply the *jeu de taquin* (see for example [5, Section 1.2]) to obtain a tableau S'.

In this case, $((S,T) \leftarrow a^\vee) \in \mathbf{B}^{>r}_{\sigma,\tau}$, where either (1) $|\mu/\sigma| = 1$ ($\sigma \subset \mu$) and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau/\nu| = 1$ ($\nu \subset \tau$). For a finite word $w = w_1 \cdots w_n$ with letters in $[r+1,\infty)^\vee$, we let $((S,T) \leftarrow w) = ((\cdots ((S,T) \leftarrow w_1) \cdots) \leftarrow w_n)$.

Let $\mu,\nu,\sigma,\tau \in \mathscr{P}$ be given. For $(S,T) \in \mathbf{B}^{>r}_{\mu,\nu}$ and $(S',T') \in \mathbf{B}^{>r}_{\sigma,\tau}$, we define

$$\big((S',T') \to (S,T)\big) = \big(\big(w(S') \to (S,T)\big) \leftarrow w(T')\big).$$

Then $((S',T') \to (S,T)) \in \mathbf{B}^{>r}_{\zeta,\eta}$ for some $\zeta,\eta \in \mathscr{P}$. Assume that $w(S') = w_1 \cdots w_s$ and $w(T') = w_{s+1} \cdots w_{s+t}$. For $1 \leqslant i \leqslant s+t$, let

$$\big(S^i,T^i\big) = \begin{cases} w_1 \cdots w_i \to (S,T), & \text{if } 1 \leqslant i \leqslant s, \\ (S^s,T^s) \leftarrow w_{s+1} \cdots w_i, & \text{if } s+1 \leqslant i \leqslant s+t, \end{cases}$$

and $(S^0,T^0) = (S,T)$. We define

$$\big((S',T') \to (S,T)\big)_R = (U,V),$$

where (U,V) is the pair of tableaux with entries in $\mathbb{Z} \setminus \{0\}$ determined by the following process:

(1) U is of shape σ and V is of shape τ.
(2) Let $1 \leqslant i \leqslant s$. If w_i is inserted into (S^{i-1},T^{i-1}) to create a dot (or box) in the k-th row of the shape of S^{i-1}, then we fill the dot in σ corresponding to w_i with k.
(3) Let $s+1 \leqslant i \leqslant s+t$. If w_i is inserted into (S^{i-1},T^{i-1}) to create a dot in the k-th row (from the bottom) of the shape of T^{i-1}, then we fill the dot in τ corresponding to w_i with $-k$. If w_i is inserted into (S^{i-1},T^{i-1}) to remove a dot in the k-th row of the shape of S^{i-1}, then we fill the corresponding dot in τ with k.

We call $((S',T') \to (S,T))_R$ *the recording tableau of* $((S',T') \to (S,T))$. By [14, Theorem 4.10], we have the following.

Proposition 4.2. *Under the above hypothesis, we have*

(1) $((S',T') \to (S,T)) \equiv (S,T) \otimes (S',T')$,
(2) $((S',T') \to (S,T))_R \in SST_{\mathbb{N}}(\sigma) \times SST_{\mathcal{Z}}(\tau)$, *where* $\mathcal{Z}$ *is the set of non-zero integers with a linear ordering* $1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1$,
(3) *the recording tableaux are constant on the connected component of* $\mathbf{B}^{>r}_{\mu,\nu} \otimes \mathbf{B}^{>r}_{\sigma,\tau}$ *including* $(S,T) \otimes (S',T')$.

Suppose that $\mu,\nu \in \mathscr{P}$ and $W \in SST_{\mathcal{Z}}(\nu)$ are given with $w(W) = w_{|\nu|} \cdots w_1$. Let (α^0,β^0), $(\alpha^1,\beta^1),\ldots,(\alpha^{|\nu|},\beta^{|\nu|})$ be the sequence, where $\alpha^i = (\alpha^i_j)_{j\geqslant 1}$ and $\beta^i = (\beta^i_j)_{j\geqslant 1}$ ($1 \leqslant i \leqslant |\nu|$) are sequences of integers defined inductively as follows:

(1) $\alpha^0 = \mu$ and $\beta^0 = (0, 0, \ldots)$.
(2) If w_i is positive, then α^i is obtained by subtracting 1 in the w_i-th part of α^{i-1}, and $\beta^i = \beta^{i-1}$. If w_i is negative, then $\alpha^i = \alpha^{i-1}$ and β^i is obtained by adding 1 in the $(-w_i)$-th part of β^{i-1}.

Then for $\sigma, \tau \in \mathscr{P}$ we define $\mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)}$ to be the set of $W \in SST_{\mathbb{Z}}(\nu)$ such that $\alpha^i, \beta^i \in \mathscr{P}$ for $1 \leqslant i \leqslant |\nu|$, and $(\alpha^{|\nu|}, \beta^{|\nu|}) = (\sigma, \tau)$.

For $S \in \mathbf{B}^{>r}_{\mu}$ and $T \in (\mathbf{B}^{>r}_{\nu})^{\vee}$, we have $((\emptyset, T) \to (S, \emptyset))_R = (\emptyset, W)$ for some $W \in \mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)}$ by Proposition 4.2(2). For convenience, we identify W with $((\emptyset, T) \to (S, \emptyset))_R$. Then, we have the following decomposition as a special case of [14, Theorem 4.10].

Proposition 4.3. *For $\mu, \nu \in \mathscr{P}$, we have an isomorphism of $\mathfrak{gl}_{>r}$-crystals*

$$\mathbf{B}^{>r}_{\mu} \otimes \left(\mathbf{B}^{>r}_{\nu}\right)^{\vee} \to \bigsqcup_{\sigma,\tau \in \mathscr{P}} \mathbf{B}^{>r}_{\sigma,\tau} \times \mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)},$$

where $S \otimes T$ is sent to $(((\emptyset, T) \to (S, \emptyset)), ((\emptyset, T) \to (S, \emptyset))_R)$.

Further, we can characterize $\mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)}$ as follows.

Proposition 4.4. *For $\mu, \nu, \sigma, \tau \in \mathscr{P}$, there exists a bijection*

$$\mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)} \to \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{LR}^{\mu}_{\sigma\lambda} \times \mathbf{LR}^{\nu}_{\tau\lambda}.$$

Proof. Suppose that $W \in \mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)}$ is given. Let W_+ (resp. W_-) be the subtableau in W consisting of positive (resp. negative) entries.

We have $W_+ \in SST_{\mathbb{N}}(\lambda)$ and $W_- \in SST_{-\mathbb{N}}(\nu/\lambda)$ for some $\lambda \subset \nu$. By definition of $W \in \mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)}$, we have $\imath(W_+) \in \mathbf{LR}^{\mu}_{\sigma\lambda}$ and $W_- \in \overline{\mathbf{LR}}^{\nu}_{\lambda\tau}$, hence $\jmath(W_-)_R \in \mathbf{LR}^{\nu}_{\tau\lambda}$ by Proposition 3.2.

We can check that the correspondence

$$W \mapsto (W_1, W_2) := \left(\imath(W_+), \jmath(W_-)_R\right) \tag{4.4}$$

is reversible and hence gives a bijection $\mathcal{C}^{(\mu,\nu)}_{(\sigma,\tau)} \to \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{LR}^{\mu}_{\sigma\lambda} \times \mathbf{LR}^{\nu}_{\tau\lambda}$. $\square$

Example 4.5. Consider

$$S = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array} \in \mathbf{B}^{>0}_{(3,2)}, \qquad T = \begin{array}{ccc} & & 4^{\vee} \\ & 3^{\vee} & 2^{\vee} \\ 2^{\vee} & 2^{\vee} & 1^{\vee} \end{array} \in \left(\mathbf{B}^{>0}_{(3,2,1)}\right)^{\vee}.$$

Then we have

$$\left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array}, \ \emptyset\right) \leftarrow 4^{\vee} = \left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array}, \ 4^{\vee}\right) \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ -1 & & \end{array}$$

$$\left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array}, \ 4^{\vee}\right) \leftarrow 2^{\vee} = \left(\begin{array}{ccc} 1 & 1 & 2 \\ 3 & & \end{array}, \ 4^{\vee}\right) \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ 2 & \bullet & \\ -1 & & \end{array}$$

$$\left(\begin{array}{ccc} 1 & 1 & 2 \\ 3 & & \end{array}, \ 4^{\vee}\right) \leftarrow 1^{\vee} = \left(\begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \ 4^{\vee}\right) \qquad \begin{array}{ccc} 1 & \bullet & \bullet \\ 2 & \bullet & \\ -1 & & \end{array}$$

$$\left(\begin{array}{cc} 1 & 2 \\ 3 & \end{array}\ ,\ \begin{array}{c} \\ 4^\vee \end{array}\right) \leftarrow 3^\vee = \left(\begin{array}{cc} 1 & 2 \\ 3 & \end{array}\ ,\ \begin{array}{c} 4^\vee \\ 3^\vee \end{array}\right) \qquad \begin{array}{ccc} 1 & \bullet & \bullet \\ 2 & -2 & \\ -1 & & \end{array}$$

$$\left(\begin{array}{cc} 1 & 2 \\ 3 & \end{array}\ ,\ \begin{array}{c} 4^\vee \\ 3^\vee \end{array}\right) \leftarrow 2^\vee = \left(\begin{array}{cc} 1 & 2 \\ & \end{array}\ ,\ \begin{array}{c} 4^\vee \\ 2^\vee \end{array}\right) \qquad \begin{array}{ccc} 1 & 2 & \bullet \\ 2 & -2 & \\ -1 & & \end{array}$$

$$\left(\begin{array}{cc} 1 & 2 \\ & \end{array}\ ,\ \begin{array}{c} 4^\vee \\ 2^\vee \end{array}\right) \leftarrow 2^\vee = \left(\begin{array}{cc} 1 & 2 \\ & \end{array}\ ,\ \begin{array}{cc} & 4^\vee \\ 2^\vee & 2^\vee \end{array}\right) \qquad \begin{array}{ccc} 1 & 2 & -1 \\ 2 & -2 & \\ -1 & & \end{array}.$$

Hence,

$$\big((\emptyset, T) \to (S, \emptyset)\big) = \left(\begin{array}{cc} 1 & 2 \\ & \end{array}\ ,\ \begin{array}{cc} & 4^\vee \\ 2^\vee & 2^\vee \end{array}\right) \in \mathbf{B}^{>0}_{(2),(2,1)},$$

$$\big((\emptyset, T) \to (S, \emptyset)\big)_R = \begin{array}{ccc} 1 & 2 & -1 \\ 2 & -2 & \\ -1 & & \end{array} \in \mathcal{C}^{(3,2),(3,2,1)}_{(2),(2,1)}.$$

If we put $W = ((\emptyset, T) \to (S, \emptyset))_R$, then

$$W_+ = \begin{array}{cc} 1 & 2 \\ 2 & \end{array}\ , \qquad W_- = \begin{array}{ccc} \bullet & \bullet & -1 \\ \bullet & -2 & \\ -1 & & \end{array}.$$

Since

$$\imath(W_+) = \begin{array}{ccc} \bullet & \bullet & 1 \\ 1 & 2 & \end{array}, \qquad \jmath(W_-) = \begin{array}{cc} -2 & -1 \\ -1 & \end{array}, \qquad \jmath(W_-)_R = \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \end{array}$$

(see Proposition 3.2), we have

$$(W_1, W_2) = \left(\begin{array}{ccc} \bullet & \bullet & 1 \\ 1 & 2 & \\ & & \end{array}\ ,\ \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \end{array}\right) \in \mathbf{LR}^{(3,2)}_{(2)(2,1)} \times \mathbf{LR}^{(3,2,1)}_{(2,1)(2,1)}.$$

Now, the multiplicity of each connected component can be written in terms of Littlewood–Richardson coefficient as follows. We remark that it was already given in [14, Corollary 7.3], while Proposition 4.4 gives a bijective proof of it.

Corollary 4.6. *For $\mu, \nu \in \mathscr{P}$, we have*

$$\mathbf{B}^{>r}_\mu \otimes \big(\mathbf{B}^{>r}_\nu\big)^\vee \simeq \bigsqcup_{\sigma, \tau \in \mathscr{P}} \big(\mathbf{B}^{>r}_{\sigma,\tau}\big)^{\oplus c^{(\mu,\nu)}_{(\sigma,\tau)}},$$

where

$$c^{(\mu,\nu)}_{(\sigma,\tau)} = \sum_{\lambda \in \mathscr{P}} c^\mu_{\sigma\lambda} c^\nu_{\tau\lambda}.$$

Proposition 4.7. *For $\mu, \nu \in \mathscr{P}$, we have an isomorphism of $\mathfrak{gl}_{>r}$-crystals*

$$\big(\mathbf{B}^{>r}_\nu\big)^\vee \otimes \mathbf{B}^{>r}_\mu \to \mathbf{B}^{>r}_{\mu,\nu},$$

where $T \otimes S$ is mapped to $((S, \emptyset) \to (\emptyset, T))$.

Proof. For $T \otimes S \in (\mathbf{B}_\nu^{>r})^\vee \otimes \mathbf{B}_\mu^{>r}$, it follows from Proposition 4.2(2) that

(1) $((S,\emptyset) \to (\emptyset,T))_R = (H_\mu,\emptyset)$,
(2) $((S,\emptyset) \to (\emptyset,T)) \in \mathbf{B}_{\mu,\nu}^{>r}$.

Therefore, by [14, Theorem 4.10] the map

$$\left(\mathbf{B}_\nu^{>r}\right)^\vee \otimes \mathbf{B}_\mu^{>r} \to \mathbf{B}_{\mu,\nu}^{>r} \times \left\{(H_\mu,\emptyset)\right\}$$

sending $T \otimes S$ to $(((S,\emptyset) \to (\emptyset,T)), ((S,\emptyset) \to (\emptyset,T))_R)$ is an isomorphism of $\mathfrak{gl}_{>r}$-crystals. □

Example 4.8. Let

$$(U,V) = \left(\begin{matrix} 1 & 2 \end{matrix}\ ,\ \begin{matrix} & 4^\vee \\ 2^\vee & 2^\vee \end{matrix}\right) \in \mathbf{B}_{(2),(2,1)}^{>0}$$

be as in Example 4.5. If we put

$$\tilde{V} \otimes \tilde{U} = \begin{matrix} & 4^\vee \\ 2^\vee & 1^\vee \end{matrix} \otimes \begin{matrix} 1 & 1 \end{matrix} \in \left(\mathbf{B}_{(2,1)}^{>0}\right)^\vee \otimes \mathbf{B}_{(2)}^{>0},$$

then

$$\left((\tilde{U},\emptyset) \to (\emptyset,\tilde{V})\right) = (U,V).$$

5. Combinatorial description of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$

5.1. For simplicity, we put for a skew Young diagram λ/μ

$$\mathcal{B}_{\lambda/\mu} = SST_{\mathbb{N}}(\lambda/\mu),$$

and for $\mu, \nu \in \mathscr{P}$

$$\mathcal{B}_{\mu,\nu} = \mathbf{B}_{\mu,\nu}^{>0}.$$

For $S \otimes T \in \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee$, suppose that

$$(U,V) = \left((\emptyset,T) \to (S,\emptyset)\right) \in \mathcal{B}_{\sigma,\tau},$$
$$W = \left((\emptyset,T) \to (S,\emptyset)\right)_R \in \mathcal{C}_{(\sigma,\tau)}^{(\mu,\nu)},$$

for some $\sigma, \tau \in \mathscr{P}$. (Recall that we identify W with $(\emptyset,W) = ((\emptyset,T) \to (S,\emptyset))_R$.) By Proposition 4.7, there exist unique $\tilde{U} \in \mathcal{B}_\sigma$ and $\tilde{V} \in \mathcal{B}_\tau^\vee$ such that $\tilde{V} \otimes \tilde{U} \equiv (U,V)$. The bijection (4.4) maps W to

$$(W_1,W_2) \in \mathbf{LR}_{\sigma\lambda}^{\mu} \times \mathbf{LR}_{\tau\lambda}^{\nu}$$

for some $\lambda \in \mathscr{P}$. By Proposition 3.2, there exist unique $X \in \mathcal{B}_{\mu/\lambda}$ and $Y \in \mathcal{B}_{\nu/\lambda}$ such that

$$\jmath(X) = \tilde{U}, \qquad \jmath(X)_R = W_1,$$
$$\jmath(Y)^\vee = \tilde{V}, \qquad \jmath(Y)_R = W_2.$$

Now, we define

$$\psi_{\mu,\nu}(S \otimes T) = Y^\vee \otimes X \in \mathcal{B}_{\nu/\lambda}^\vee \otimes \mathcal{B}_{\mu/\lambda}. \tag{5.1}$$

By construction, $\psi_{\mu,\nu}$ is bijective and commutes with $\tilde{x}_i$ for $x = e, f$ and $i \geqslant 1$. Hence we have the following.

Proposition 5.1. *For $\mu, \nu \in \mathscr{P}$, the map*

$$\psi_{\mu,\nu} : \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee \to \bigsqcup_{\lambda \subset \mu,\nu} \mathcal{B}_{\nu/\lambda}^\vee \otimes \mathcal{B}_{\mu/\lambda}$$

is an isomorphism of $\mathfrak{gl}_{>0}$-crystals.

Example 5.2. Let S and T be the tableaux in Example 4.5. Let

$$X = \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \end{array}, \qquad Y = \begin{array}{ccc} \bullet & \bullet & 2 \\ \bullet & 1 & \\ 4 & & \end{array}.$$

Following the above notations, we have

$$H_{(2,1)} * X = \begin{array}{ccc} \mathbf{1} & \mathbf{1} & 1 \\ \mathbf{2} & 1 & \end{array} \overset{\textit{switching}}{\leftrightsquigarrow} \begin{array}{ccc} 1 & 1 & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \end{array} = \jmath(X) * \jmath(X)_R = \tilde{U} * W_1,$$

$$H_{(2,1)} * Y = \begin{array}{ccc} \mathbf{1} & \mathbf{1} & 2 \\ \mathbf{2} & 1 & \\ 4 & & \end{array} \overset{\textit{switching}}{\leftrightsquigarrow} \begin{array}{ccc} 1 & 2 & \mathbf{1} \\ 4 & \mathbf{2} & \\ \mathbf{1} & & \end{array} = \jmath(Y) * \jmath(Y)_R = (\tilde{V})^\vee * W_2,$$

where $\tilde{U}$, $\tilde{V}$, W_i $(i = 1, 2)$ are as in Examples 4.5 and 4.8. Hence,

$$\begin{aligned} \psi_{\mu,\nu}(S \otimes T) &= Y^\vee \otimes X \\ &= \left(\begin{array}{ccc} \bullet & \bullet & 2 \\ \bullet & 1 & \\ 4 & & \end{array} \right)^\vee \otimes \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \end{array} \\ &= \begin{array}{ccc} & & 4^\vee \\ & 1^\vee & \bullet \\ 2^\vee & \bullet & \bullet \end{array} \otimes \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \end{array}. \end{aligned}$$

For a skew Young diagram λ/μ and $k \geqslant 1$, we define

$$\kappa_k : \mathcal{B}_{\lambda/\mu} \to \mathcal{B}_{(\lambda+(1^k))/(\mu+(1^k))} \tag{5.2}$$

by $\kappa_k(S) = S'$ with

$$S'(i,j) = \begin{cases} S(i,j), & \text{if } i > k, \\ S(i,j-1), & \text{if } i \leqslant k. \end{cases}$$

By definition, κ_k is a strict embedding of crystals.

Example 5.3.

$$\kappa_1 \left(\begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \end{array} \right) = \begin{array}{cccc} \bullet & \bullet & \bullet & 1 \\ \bullet & 2 & & \\ 1 & & & \end{array}, \qquad \kappa_2 \left(\begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \end{array} \right) = \begin{array}{cccc} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & 2 & \\ 1 & & & \end{array}.$$

For $k \geqslant 1$ and $\lambda \in \mathscr{P}$, we put

$$\begin{aligned} \omega_k &= \epsilon_1 + \cdots + \epsilon_k, \\ \omega_\lambda &= \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \cdots. \end{aligned}$$

Now, we have the following combinatorial interpretation of the embedding (2.2) in terms of *sliding* skew tableaux horizontally. It will play a crucial role in proving our main theorem.

Proposition 5.4. *For $\mu, \nu \in \mathscr{P}$ and $k \geqslant 1$, we have the following commutative diagram of $\mathfrak{gl}_{>0}$-crystal morphisms*

$$\begin{array}{ccc}
\mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee & \xrightarrow{\ \iota^{\omega_k}_{\omega_\mu,\omega_\nu}\ } & \mathcal{B}_{\mu+(1^k)} \otimes \mathcal{B}^\vee_{\nu+(1^k)} \\
{\scriptstyle \psi_{\mu,\nu}} \downarrow & & \downarrow {\scriptstyle \psi_{\mu+(1^k),\nu+(1^k)}} \\
\bigsqcup_\lambda \mathcal{B}^\vee_{\nu/\lambda} \otimes \mathcal{B}_{\mu/\lambda} & \xrightarrow{\ \kappa_k^\vee \otimes \kappa_k\ } & \bigsqcup_\eta \mathcal{B}^\vee_{(\nu+(1^k))/\eta} \otimes \mathcal{B}_{(\mu+(1^k))/\eta}
\end{array}$$

where $\iota^{\omega_k}_{\omega_\mu,\omega_\nu}$ is the strict embedding in (2.2) *and $\kappa_k^\vee = \vee \circ \kappa_k \circ \vee$.*

Proof. Let $S \otimes T \in \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee$ be given. We keep the previous notations. Note that

$$S \otimes u_{\omega_k} = S \otimes H_{(1^k)} \equiv S\{k\} := \big(k \to \big(\cdots (1 \to S) \cdots\big)\big) \in \mathcal{B}_{\mu+(1^k)},$$
$$u_{-\omega_k} \otimes T = H^\vee_{(1^k)} \otimes T \equiv T\{k\} := \big(k \to \big(\cdots \big(1 \to T^\vee\big) \cdots\big)\big)^\vee \in \mathcal{B}^\vee_{\nu+(1^k)}.$$

Hence by (2.2) we have $\iota^{\omega_k}_{\omega_\mu,\omega_\nu}(S \otimes T) = S\{k\} \otimes T\{k\}$. Since $S\{k\} \otimes T\{k\} \equiv S \otimes T$, we have

$$\big(U\{k\}, V\{k\}\big) := \big(\big(\emptyset, T\{k\}\big) \to \big(S\{k\}, \emptyset\big)\big) \equiv \big((\emptyset, T) \to (S, \emptyset)\big) = (U, V),$$

which implies that $(U\{k\}, V\{k\}) = (U, V)$ by [14, Lemma 5.1]. Put

$$W\{k\} = \big(\big(\emptyset, T\{k\}\big) \to \big(S\{k\}, \emptyset\big)\big)_R,$$

and suppose that the bijection (4.4) maps $W\{k\}$ to

$$\big(W_1\{k\}, W_2\{k\}\big) \in \mathbf{LR}^{\mu+(1^k)}_{\sigma\,\eta} \times \mathbf{LR}^{\nu+(1^k)}_{\tau\,\eta}$$

for some $\eta \in \mathscr{P}$.

Since W is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ $(i \geqslant 1)$, we may assume that $(U, V) = (H^{>0}_\sigma, E^{>0}_\tau(n))$ for a sufficiently large $n > k$ (see (4.2)). As a $\mathfrak{gl}_{[n]}$-crystal element, (U, V) is a highest weight element, and $\varsigma_n^p(U, V) = (H^{>0}_\zeta, \emptyset)$, where $p \geqslant \tau_1$ and $\zeta = \sigma + (p - \tau_n, \ldots, p - \tau_1)$ (see [14, Section 4.1] for the definition of the map ς_n). This also implies that $S = H^{>0}_\mu$. By [26, Lemma 7.6], we have

$$\big(\emptyset, \big(W\{k\} \downarrow n\big)^\vee\big) = \varsigma_n^{-p}\big[\big(\varsigma_n^p\big(\emptyset, T\{k\}\big) \to \big(S\{k\}, \emptyset\big)\big)_R\big] \tag{5.3}$$

where $(W\{k\} \downarrow n)$ is the tableau obtained from $W\{k\}$ by replacing $-i$ with $n - i + 1$ (see also the proof of [14, Lemma 4.8]). Since $S\{k\} = H^{>0}_{\mu+(1^k)}$, we have $(\varsigma_n^p(\emptyset, T\{k\}) \to (S\{k\}, \emptyset))_R = \varsigma_n^p(\emptyset, T\{k\})$ and hence $(W\{k\} \downarrow n)^\vee = T\{k\}$. Similarly, we have $(W \downarrow n)^\vee = T$.

Now, it is straightforward to check that

$$W\{k\} = \begin{matrix} 1 \\ \vdots \\ k \end{matrix} * \kappa_k(W) = H_{(1^k)} * \kappa_k(W).$$

This implies that

$$W_1\{k\} = W_1 * \Sigma_k,$$
$$W_2\{k\} = W_2 * \Sigma'_k,$$

where Σ_k and Σ'_k are vertical strips of shape $(\mu + (1^k))/\mu$ and $(\nu + (1^k))/\nu$ filled with $1, \ldots, k$ from top to bottom, respectively. Now, we have

$$\begin{aligned}\tilde{U} * W_1\{k\} = \tilde{U} * W_1 * \Sigma_k &\leftrightsquigarrow H_\lambda * X * \Sigma_k \quad (\text{switching } \tilde{U} \text{ and } W_1)\\ &\leftrightsquigarrow H_{\lambda+(1^k)} * \kappa_k(X) \quad (\text{switching } X \text{ and } \Sigma_k),\\ (\tilde{V})^\vee * W_2\{k\} = (\tilde{V})^\vee * W_2 * \Sigma'_k &\leftrightsquigarrow H_\lambda * Y * \Sigma'_k \quad \bigl(\text{switching } (\tilde{V})^\vee \text{ and } W_2\bigr)\\ &\leftrightsquigarrow H_{\lambda+(1^k)} * \kappa_k(Y) \quad \bigl(\text{switching } Y \text{ and } \Sigma'_k\bigr).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\psi_{\mu+(1^k),\nu+(1^k)}\bigl(\iota^{\omega_k}_{\omega_\mu,\omega_\nu}(S \otimes T)\bigr) &= \psi_{\mu+(1^k),\nu+(1^k)}\bigl(S\{k\} \otimes T\{k\}\bigr)\\ &= \kappa_k(Y)^\vee \otimes \kappa_k(X)\\ &= \kappa_k^\vee \otimes \kappa_k\bigl(\psi_{\mu,\nu}(S \otimes T)\bigr). \qquad \square\end{aligned}$$

5.2. Let $\mathcal{M}$ be the set of $\mathbb{N} \times \mathbb{N}$ matrices $A = (a_{ij})$ such that $a_{ij} \in \mathbb{Z}_{\geqslant 0}$ and $\sum_{i,j\geqslant 1} a_{ij} < \infty$. Let $A = (a_{ij}) \in \mathcal{M}$ be given. For $i \geqslant 1$, the i-th row $A_i = (a_{ij})_{j\geqslant 1}$ is naturally identified with a unique semistandard tableau in $\mathcal{B}_{(m_i)}$, where $m_i = \sum_{j\geqslant 1} a_{ij}$ and $\mathrm{wt}(A_i) = \sum_{j\geqslant 1} a_{ij}\epsilon_j$. Hence A can be viewed as an element in $\mathcal{B}_{(m_1)} \otimes \cdots \otimes \mathcal{B}_{(m_r)}$ for some $r \geqslant 0$. This defines a $\mathfrak{gl}_{>0}$-crystal structure on $\mathcal{M}$. Now, we put

$$\tilde{\mathcal{M}} = \mathcal{M}^\vee \times \mathcal{M}, \tag{5.4}$$

which can be viewed as a tensor product of $\mathfrak{gl}_{>0}$-crystals. Let $\mathcal{P} = \bigoplus_{i\geqslant 1} \mathbb{Z}\epsilon_i$ be the integral weight lattice for $\mathfrak{gl}_{>0}$. For $\omega \in \mathcal{P}$, let

$$\tilde{\mathcal{M}}_\omega = \bigl\{\bigl(M^\vee, N\bigr) \in \tilde{\mathcal{M}} \bigm| \mathrm{wt}\bigl(N^t\bigr) - \mathrm{wt}\bigl(M^t\bigr) = \omega\bigr\}.$$

Here A^t denotes the transpose of $A \in \mathcal{M}$. Then $\tilde{\mathcal{M}}_\omega$ is a subcrystal of $\tilde{\mathcal{M}}$. Now, we can state the main result in this section.

Theorem 5.5. *For $\omega \in \mathcal{P}$, we have*

$$\tilde{\mathcal{M}}_\omega \simeq \mathbf{B}(\infty) \otimes T_\omega \otimes \mathbf{B}(-\infty).$$

Proof. Let $\mu, \nu \in \mathscr{P}$ be such that $\omega = \omega_\mu - \omega_\nu$. Suppose that $\psi_{\mu,\nu}(S \otimes T) = Y^\vee \otimes X$ for $S \otimes T \in \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee$, where $\psi_{\mu,\nu}$ is the isomorphism in Proposition 5.1. Let $M = (m_{ij})$ (resp. $N = (n_{ij})$) be the unique matrix in $\mathcal{M}$ such that the i-th row of M (resp. N) is $\mathfrak{gl}_{>0}$-equivalent to the i-th row of Y (resp. X). Since $\sum_{j\geqslant 1} m_{ij}$ (resp. $\sum_{j\geqslant 1} n_{ij}$) is equal to y_i (resp. x_i) the number of dots or boxes in the i-th row of Y (resp. X) for $i \geqslant 1$ and $\omega = \sum_{i\geqslant 1}(x_i - y_i)\epsilon_i$ by Proposition 5.1, we have $\mathrm{wt}(N^t) - \mathrm{wt}(M^t) = \omega$. Then we define

$$\iota'_{\mu,\nu} : \mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee \to \tilde{\mathcal{M}}_\omega$$

by $\iota'_{\mu,\nu}(S \otimes T) = (M^\vee, N)$. By Proposition 5.1, it is easy to see that $\iota'_{\mu,\nu}$ is a strict embedding and

$$\tilde{\mathcal{M}}_\omega = \bigcup_{\substack{\mu,\nu\in\mathscr{P}\\ \omega_\mu-\omega_\nu=\omega}} \mathrm{Im}\, \iota'_{\mu,\nu}.$$

For $k \geqslant 1$, we have $\iota'_{\mu,\nu} = \iota'_{\mu+(1^k),\nu+(1^k)} \circ \iota^{\omega_k}_{\omega_\mu,\omega_\nu}$ by Proposition 5.4. Using induction, we have

$$\iota'_{\mu,\nu} = \iota'_{\mu+\xi,\nu+\xi} \circ \iota^{\omega_\xi}_{\omega_\mu,\omega_\nu} \quad (\xi \in \mathscr{P}).$$

Therefore, by (2.3), it follows that $\tilde{\mathcal{M}}_\omega \simeq \mathbf{B}(\infty) \otimes T_\omega \otimes \mathbf{B}(-\infty)$. $\square$

Corollary 5.6. *As a* $\mathfrak{gl}_{>0}$*-crystal, we have*

$$\mathbf{B}\big(\tilde{U}_q(\mathfrak{gl}_{>0})\big) \simeq \tilde{\mathcal{M}}.$$

Proof. It follows from $\tilde{\mathcal{M}} = \bigsqcup_{\omega \in \mathcal{P}} \tilde{\mathcal{M}}_\omega$. □

For $A \in \mathcal{M}$ and $i \geqslant 1$, we also define

$$\tilde{e}_i^t A = \big(\tilde{e}_i A^t\big)^t, \qquad \tilde{f}_i^t A = \big(\tilde{f}_i A^t\big)^t. \tag{5.5}$$

Then $\mathcal{M}$ has another $\mathfrak{gl}_{>0}$-crystal structure with respect to $\tilde{e}_i^t$, $\tilde{f}_i^t$ and wt^t, where $\mathrm{wt}^t(A) = \mathrm{wt}(A^t)$. By [4], $\mathcal{M}$ is a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystal, that is, $\tilde{e}_i, \tilde{f}_i$ on $\mathcal{M} \cup \{\mathbf{0}\}$ commute with $\tilde{e}_j^t, \tilde{f}_j^t$ for $i, j \geqslant 1$, and so is the tensor product $\tilde{\mathcal{M}} = \mathcal{M}^\vee \times \mathcal{M}$. Now we have the following Peter–Weyl type decomposition.

Corollary 5.7. *As a* $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$*-bicrystal, we have*

$$\mathbf{B}\big(\tilde{U}_q(\mathfrak{gl}_{>0})\big) \simeq \bigsqcup_{\mu,\nu \in \mathscr{P}} \mathcal{B}_{\mu,\nu} \times \mathcal{B}_{\mu,\nu}.$$

Proof. Note that the usual RSK correspondence gives an isomorphism of $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystals $\mathcal{M} \simeq \bigsqcup_{\lambda \in \mathscr{P}} \mathcal{B}_\lambda \times \mathcal{B}_\lambda$ [4]. We assume that $\tilde{e}_i, \tilde{f}_i$ act on the first component, and $\tilde{e}_j^t, \tilde{f}_j^t$ act on the second component. The decomposition of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$ follows from Proposition 4.7. □

6. Extremal weight crystals of type A_∞

In this section, we describe the tensor product of $\mathfrak{gl}_\infty$-crystals $\mathbf{B}(\Lambda) \otimes \mathbf{B}(-\Lambda')$ for $\Lambda, \Lambda' \in P^+$ in terms of extremal weight crystals.

6.1. For a skew Young diagram λ/μ, we put

$$\mathbf{B}_{\lambda/\mu} = SST_{\mathbb{Z}}(\lambda/\mu), \tag{6.1}$$

and we identify $\mathbf{B}_{\lambda/\mu}^\vee$ with $SST_{\mathbb{Z}^\vee}((\lambda/\mu)^\vee)$. Note that for $\mu \in \mathscr{P}$, $\mathbf{B}_\mu$ has neither a highest weight nor lowest weight element. It is shown in [15] that for $\mu, \nu, \sigma, \tau \in \mathscr{P}$, $\mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ is connected, $\mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee \simeq \mathbf{B}_\nu^\vee \otimes \mathbf{B}_\mu$, and $\mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee \simeq \mathbf{B}_\sigma \otimes \mathbf{B}_\tau^\vee$ if and only if $(\mu, \nu) = (\sigma, \tau)$. Put

$$\mathbf{B}_{\mu,\nu} = \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee. \tag{6.2}$$

Note that $\mathbf{B}_{\mu,\nu}$ can be viewed as a limit of $\mathbf{B}_{\mu,\nu}^{>r}$ $(r \to -\infty)$ since $\mathbf{B}_{\mu,\nu}^{>r} \simeq (\mathbf{B}_\nu^{>r})^\vee \otimes \mathbf{B}_\mu^{>r}$.

For $n \geqslant 1$, let $\mathbb{Z}_+^n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n\}$ be the set of generalized partitions of length n. For $\lambda \in \mathbb{Z}_+^n$, we put

$$\Lambda_\lambda = \Lambda_{\lambda_1} + \cdots + \Lambda_{\lambda_n} \in P_n^+.$$

Theorem 6.1. *(See Theorem 4.6 in [15].) For* $\Lambda \in P_n$ $(n \geqslant 0)$*, there exist unique* $\lambda \in \mathbb{Z}_+^n$ *and* $\mu, \nu \in \mathscr{P}$ *such that*

$$\mathbf{B}(\Lambda) \simeq \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda_\lambda).$$

Here we assume that $\Lambda_\lambda = 0$ *when* $n = 0$*.*

Note that $\{\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda) \mid \Lambda \in P^+,\ \mu, \nu \in \mathscr{P}\}$ forms a complete list of extremal weight crystals of non-negative level up to isomorphism.

6.2. For intervals I, J in $\mathbb{Z}$, let $M_{I,J}$ be the set of $I \times J$ matrices $A = (a_{ij})$ with $a_{ij} \in \{0, 1\}$. We denote by A_i the i-th row of A for $i \in I$.

Suppose that $A \in M_{I,J}$ is given. For $j \in J^\circ$ and $i \in I$, we define

$$\tilde{e}_j A_i = \begin{cases} A_i + E_{i\,j} - E_{i\,j+1}, & \text{if } (a_{i\,j}, a_{i\,j+1}) = (0,1), \\ \mathbf{0}, & \text{otherwise}, \end{cases} \tag{6.3}$$

$$\tilde{f}_j A_i = \begin{cases} A_i - E_{i\,j} + E_{i\,j+1}, & \text{if } (a_{i\,j}, a_{i\,j+1}) = (1,0), \\ \mathbf{0}, & \text{otherwise}. \end{cases} \tag{6.4}$$

Then we can regard A_i as an element of a regular $\mathfrak{gl}_{\{j,j+1\}}$-crystal with weight $a_{i\,j}\epsilon_j + a_{i\,j+1}\epsilon_{j+1}$. So we have $\varepsilon_j(A_i) = \max\{k \mid \tilde{e}_j^k A_i \neq \mathbf{0}\} \in \{0,1\}$ and $\varphi_j(A_i) = \max\{k \mid \tilde{f}_j^k A_i \neq \mathbf{0}\} \in \{0,1\}$. We say that A is *row j-admissible* if there exist $L, L' \in I$ ($L < L'$) such that (1) $\varphi_j(A_i) \neq 1$ for all $i < L$, and (2) $\varepsilon_j(A_i) \neq 1$ for all $i > L'$. Note that if I is finite, then A is row j-admissible for all $j \in J^\circ$. Suppose that A is row j-admissible. Then we can define $\tilde{x}_j A$ ($x = e, f$) by regarding A as $\cdots \otimes A_{i-1} \otimes A_i \otimes A_{i+1} \otimes \cdots$ (by abuse of notation) and applying tensor product rule of crystal or *signature rule* [13]. Note that when I is infinite, A cannot be viewed as an element of a $\mathfrak{gl}_{\{j,j+1\}}$-crystal in general since the $\mathfrak{gl}_{\{j,j+1\}}$-weight of A is not well defined in a natural way. But, $\tilde{x}_j A$ is still well defined since A is row j-admissible (see also [15, Section 3.1]).

Let $\rho : M_{I,J} \to M_{-J,I}$ be a bijection given by $\rho(A) = (a'_{-j\,i}) \in M_{-J,I}$ with $a'_{-j\,i} = a_{ij}$, where $-J = \{-j \mid j \in J\}$. For $i \in I^\circ$, we say that A is *column i-admissible* if $\rho(A)$ is row i-admissible. If A is column i-admissible, then we define

$$\tilde{E}_i(A) = \rho^{-1}\big(\tilde{e}_i \rho(A)\big), \qquad \tilde{F}_i(A) = \rho^{-1}\big(\tilde{f}_i \rho(A)\big). \tag{6.5}$$

If A is both row j-admissible and column i-admissible for some $i \in I^\circ$ and $j \in J^\circ$, then

$$\tilde{x}_j \tilde{X}_i A = \tilde{X}_i \tilde{x}_j A, \tag{6.6}$$

where $x = e, f$ and $X = E, F$ [15, Lemma 3.2].

For convenience, let us say that A *is row admissible* (resp. *column admissible*) if A is row j-admissible (resp. column i-admissible) for all $j \in J^\circ$ (resp. $i \in I^\circ$). Suppose that A is row admissible and column i-admissible for some $i \in I^\circ$. Then both A and $\tilde{X}_i A$ generate the same J°-colored oriented graph with respect to $\tilde{e}_j$ and $\tilde{f}_j$ for $j \in J^\circ$ whenever $\tilde{X}_i A \neq \mathbf{0}$ ($X = E, F$) [15, Lemma 3.3]. A similar fact holds when A is column admissible and row j-admissible for some $j \in J^\circ$.

If I and J are finite, then $M_{I,J}$ is a $(\mathfrak{gl}_I, \mathfrak{gl}_J)$-bicrystal, where the $\mathfrak{gl}_I$-weight (resp. $\mathfrak{gl}_J$-weight) of $A = (a_{ij}) \in M_{I,J}$ is given by $\sum_{i\in I}(\sum_{j\in J} a_{ij})\epsilon_i$ (resp. $\sum_{j\in J}(\sum_{i\in I} a_{ij})\epsilon_j$). Note that $M_{I,J}$ is a regular $\mathfrak{gl}_J$-crystal (resp. $\mathfrak{gl}_I$-crystal) with respect to $\tilde{e}_j$, $\tilde{f}_j$ for $j \in J^\circ$ (resp. $\tilde{E}_i$, $\tilde{F}_i$ for $i \in I^\circ$).

6.3. For $n \geqslant 1$, let $\mathcal{E}^n$ be the subset of $M_{[n],\mathbb{Z}}$ consisting of matrices $A = (a_{ij})$ such that $\sum_{i,j} a_{ij} < \infty$. It is clear that A is row admissible for $A \in \mathcal{E}^n$. If we define $\mathrm{wt}(A) = \sum_{j\in\mathbb{Z}}(\sum_{i\in[n]} a_{ij})\epsilon_j$, then $\mathcal{E}^n$ is a regular $\mathfrak{gl}_\infty$-crystal with respect to $\tilde{e}_j, \tilde{f}_j$ ($j \in \mathbb{Z}$) and wt. For $r \in \mathbb{Z}$ and $\lambda \in \mathscr{P}$ with $\lambda_1 \leqslant n$, let $A_\lambda^*(r) = (a_{ij}) \in \mathcal{E}^n$ ($* = \circ, \diamond$) be such that for $i \in [n]$ and $j \in \mathbb{Z}$

$$\begin{aligned} a_{ij}^\circ = 1 \quad &\Longleftrightarrow \quad 1 + r \leqslant j \leqslant \lambda'_{n-i+1} + r, \\ a_{ij}^\diamond = 1 \quad &\Longleftrightarrow \quad r - \lambda'_{n-i+1} + 1 \leqslant j \leqslant r. \end{aligned} \tag{6.7}$$

Then $C(A_\lambda^*(r)) \simeq \mathbf{B}_\lambda$ ($* = \circ, \diamond$) (see (3.10) in [15]).

For $n \geqslant 1$, let $\mathcal{F}^n$ be the set of matrices $A = (a_{ij})$ in $M_{[n],\mathbb{Z}}$ such that for each $i \in [n]$, $a_{ij} = 1$ if $j \ll 0$ and $a_{ij} = 0$ if $j \gg 0$. Note that A is row admissible for $A \in \mathcal{F}^n$. If we define $\mathrm{wt}(A) = n\Lambda_0 + \sum_{j>0}(\sum_{i\in[n]} a_{ij})\epsilon_j + \sum_{j\leqslant 0}(\sum_{i\in[n]}(a_{ij} - 1))\epsilon_j$, then $\mathcal{F}^n$ is a regular $\mathfrak{gl}_\infty$-crystal with respect to $\tilde{e}_j, \tilde{f}_j$ ($j \in \mathbb{Z}$) and wt. For $\lambda \in \mathbb{Z}_+^n$, let $A_\lambda = (a_{ij}) \in \mathcal{F}^n$ be such that for $i \in [n]$ and $j \in \mathbb{Z}$

$$a_{ij} = 1 \quad \Longleftrightarrow \quad j \leqslant \lambda_{n-i+1}. \tag{6.8}$$

Then $C(A_\lambda) \simeq \mathbf{B}(\Lambda_\lambda)$ (see (3.17) in [15]).

On the other hand, for $A = (a_{ij}) \in \mathcal{E}^n$ or $\mathcal{F}^n$, A is column admissible. Hence, $\tilde{E}_i$, $\tilde{F}_i$ ($i \in [n]^\circ$) are well defined on A, and they commute with $\tilde{e}_j$, $\tilde{f}_j$ ($j \in \mathbb{Z}$).

For $A = (a_{ij}) \in \mathcal{E}^n$ or $\mathcal{F}^n$, we will identify its dual $\mathfrak{gl}_\infty$-crystal element $A^\vee \in (\mathcal{E}^n)^\vee$ or $(\mathcal{F}^n)^\vee$ with the matrix $(a^\vee_{ij}) \in M_{[n],\mathbb{Z}}$ where $a^\vee_{ij} = 1 - a_{n-i\,j}$, since $A^\vee$ and $(a^\vee_{ij})$ generate the same $\mathbb{Z}$-colored graph with respect to $\tilde{e}_j$, $\tilde{f}_j$ ($j \in \mathbb{Z}$).

6.4. Let m, n be non-negative integers with $m \geqslant n$. In the rest of this section, we fix $\mu \in \mathbb{Z}^m_+$ and $\nu \in \mathbb{Z}^n_+$. We assume that $\mathbf{B}(\Lambda_\mu) = C(A_\mu) \subset \mathcal{F}^m$, $\mathbf{B}(-\Lambda_\nu) = C((A_\nu)^\vee) \subset (\mathcal{F}^n)^\vee$, and hence

$$\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) \subset \mathcal{F}^m \otimes \left(\mathcal{F}^n\right)^\vee.$$

We also assume that $\mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ is a subset of $M_{[m+n],\mathbb{Z}}$ consisting of A such that $A_{[m],\mathbb{Z}} \in \mathcal{F}^m$ and $A_{m+[n],\mathbb{Z}} \in (\mathcal{F}^n)^\vee$. Here $A_{I',J'}$ denotes the $I' \times J'$-submatrix of $A \in M_{I,J}$ for intervals $I' \subset I$, $J' \subset J$, and $m + [n] = \{m+1, \ldots, m+n\}$.

By [15, Proposition 4.5], $\mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ is a disjoint union of extremal weight $\mathfrak{gl}_\infty$-crystals of level $m - n$, and hence so is $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$. We will describe the multiplicity of each extremal weight crystal appearing in $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$.

For $r \in \mathbb{Z}$, we define $\mathbf{B}^{>r}(\mu,\nu)$ to be the set of $A = (a_{ij}) \in \mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) \subset M_{[m+n],\mathbb{Z}}$ such that

$$a_{ij} = \begin{cases} 1, & \text{for } i \in [m] \text{ and } j \leqslant r, \\ 0, & \text{for } i \in m+[n] \text{ and } j \leqslant r. \end{cases}$$

We have

$$\mathbf{B}^{>r}(\mu,\nu) \subset \mathbf{B}^{>r-1}(\mu,\nu),$$
$$\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) = \bigcup_{r \in \mathbb{Z}} \mathbf{B}^{>r}(\mu,\nu).$$

Choose $r < \min\{\mu_m, \nu_n\}$ so that $\mu - (r^m) = (\mu_i - r)_{1 \leqslant i \leqslant m}$ and $\nu - (r^n) = (\nu_i - r)_{1 \leqslant i \leqslant n}$ are partitions. Note that

(1) $\mathbf{B}^{>r}(\mu,\nu) \neq \emptyset$ since $A_\mu \otimes (A_\nu)^\vee \in \mathbf{B}^{>r}(\mu,\nu)$,
(2) A_μ (resp. $(A_\nu)^\vee$) is $\mathfrak{gl}_{>r}$-equivalent to $H^{>r}_{(\mu-(r^m))'}$ (resp. $(H^{>r}_{(\nu-(r^n))'})^\vee$),
(3) for $A \in \mathbf{B}^{>r}(\mu,\nu)$, $A_{[m],\mathbb{Z}}$ (resp. $A_{m+[n],\mathbb{Z}}$) is connected to A_μ (resp. $(A_\nu)^\vee$) under $\tilde{e}_j$, $\tilde{f}_j$ for $j \in [r+1,\infty)$.

Hence, as a $\mathfrak{gl}_{>r}$-crystal,

$$\mathbf{B}^{>r}(\mu,\nu) \simeq \mathbf{B}^{>r}_{(\mu-(r^m))'} \otimes \left(\mathbf{B}^{>r}_{(\nu-(r^n))'}\right)^\vee. \tag{6.9}$$

Now, let $A \in \mathbf{B}^{>r}(\mu,\nu)$ be given and $C^{>r}(A)$ the connected component in $\mathbf{B}^{>r}(\mu,\nu)$ including A as a $\mathfrak{gl}_{>r}$-crystal. By (6.9) and Corollary 4.6, we have

$$C^{>r}(A) \simeq \mathbf{B}^{>r}_{\sigma,\tau}$$

for some $\sigma, \tau \in \mathscr{P}$ with $\sigma_1 \leqslant m$ and $\tau_1 \leqslant n$. On the other hand, consider $C(A)$ the connected component in $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$ including A as a $\mathfrak{gl}_\infty$-crystal. Then by Theorem 6.1

$$C(A) \simeq \mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi)$$

for some $\zeta, \eta \in \mathscr{P}$ and $\xi \in \mathbb{Z}^{m-n}_+$.

Lemma 6.2. *Under the above hypothesis, we have*

$$\zeta = \left(\sigma'_{m-n+1}, \ldots, \sigma'_m\right)', \qquad \eta = \tau, \qquad \xi = \left(\sigma'_1, \ldots, \sigma'_{m-n}\right) + \left(r^{m-n}\right).$$

Proof. Let A be as above. Choose $s \gg r$ so that

$$a_{ij} = \begin{cases} 0, & \text{if } i \in [m] \text{ and } j > s, \\ 1, & \text{if } i \in m + [n] \text{ and } j > s. \end{cases}$$

Considering the submatrix $A_{[m+n],[r+1,s]}$ as an element of a $(\mathfrak{gl}_{[r+1,s]}, \mathfrak{gl}_{[m+n]})$-bicrystal, A is connected to a unique matrix $A' = (a'_{ij}) \in \mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ satisfying

$$\begin{cases} a'_{ij} = a_{ij}, & \text{for } i \in [m+n] \text{ and } j \notin [r+1,s], \\ a'_{i-1\,j} = 0, & \text{if } a'_{ij} = 0 \text{ for } i \neq 1 \text{ and } j \in [r+1,s], \\ a'_{i\,j+1} = 0, & \text{if } a'_{ij} = 0 \text{ for } i \in [m+n] \text{ and } j+1 \in [r+1,s]. \end{cases}$$

Equivalently, A' is a $\mathfrak{gl}_{[r+1,s]}$-highest weight element and a $\mathfrak{gl}_{[m+n]}$-lowest weight element. Note that

(1) $\mathcal{F}^m \otimes (\mathcal{F}^n)^\vee \subset M_{[m+n],\mathbb{Z}}$ is column admissible,
(2) $(\tilde{x}_j A)_{[m+n],[r+1,s]} = \tilde{x}_j (A_{[m+n],[r+1,s]})$ for $j \in [r+1,s]^\circ$ and $x = e, f$,
(3) $(\tilde{X}_i A)_{[m+n],[r+1,s]} = \tilde{X}_i (A_{[m+n],[r+1,s]})$ for $i \in [m+n]^\circ$ and $X = E, F$.

So, we have $C(A') \simeq C(A)$ and $C^{>r}(A') \simeq C^{>r}(A)$ by (6.6). By definition of A', we have

$$C^{>r}\big(A'_{[m],\mathbb{Z}}\big) \simeq \mathbf{B}^{>r}_\alpha, \qquad C^{>r}\big(A'_{m+[n],\mathbb{Z}}\big) \simeq \big(\mathbf{B}^{>r}_\beta\big)^\vee,$$

where $\alpha = (\alpha_k)_{k\geqslant 1}$ and $\beta = (\beta_k)_{k\geqslant 1} \in \mathscr{P}$ are given by $\alpha_k = \sum_{i=1}^m a'_{i\,r+k}$ for $1 \leqslant k \leqslant s-r$ and $\beta_k = \sum_{i=1}^n (1 - a'_{m+i\ s-k+1})$ for $1 \leqslant k \leqslant s-r$. Indeed, $A'_{[m+n],[r+1,\infty)}$ is $\mathfrak{gl}_{>r}$-equivalent to $H^{>r}_\alpha \otimes E^{>r}_\beta(s-r)$ (see (4.2)), and hence $C^{>r}(A') \simeq \mathbf{B}^{>r}_{\alpha,\beta}$. This implies that $(\alpha,\beta) = (\sigma,\tau)$ since $C^{>r}(A') \simeq C^{>r}(A) \simeq \mathbf{B}^{>r}_{\sigma,\tau}$.

Let $A'' = (a''_{ij}) \in M_{[m+n],\mathbb{Z}}$ be such that

$$A''_{[n],\mathbb{Z}} = A^\circ_\zeta(r) \in \mathcal{E}^n, \qquad A''_{n+[n],\mathbb{Z}} = \big(A^\diamond_\eta(s)\big)^\vee \in \big(\mathcal{E}^n\big)^\vee, \qquad A''_{2n+[m-n],\mathbb{Z}} = A_\xi \in \mathcal{F}^{m-n},$$

where $\zeta = (\sigma'_{m-n+1}, \ldots, \sigma'_m)'$, $\eta = \tau$ and $\xi = (\sigma'_1, \ldots, \sigma'_{m-n}) + (r^{m-n})$ (see (6.7) and (6.8)). We assume that $A'' \in \mathcal{E}^n \otimes (\mathcal{E}^n)^\vee \otimes \mathcal{F}^{m-n}$. By definition, $C(A''_{[2n],\mathbb{Z}}) \simeq \mathbf{B}_{\zeta,\eta}$, $C(A''_{2n+[m-n],\mathbb{Z}}) \simeq \mathbf{B}(\Lambda_\xi)$ and hence $C(A'') \simeq \mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi)$.

For $L \ll 0 \ll L'$, we have

$$A''_{[m+n],[L,L']} = \begin{cases} X'X(A'_{[m+n],[L,L']}), & \text{if } m > n, \\ X(A'_{[m+n],[L,L']}), & \text{if } m = n, \end{cases}$$

where

$$X = \big(\tilde{F}^{\max}_n \cdots \tilde{F}^{\max}_1\big) \cdots \big(\tilde{F}^{\max}_{m+n-2} \cdots \tilde{F}^{\max}_{m-1}\big)\big(\tilde{F}^{\max}_{m+n-1} \cdots \tilde{F}^{\max}_m\big),$$
$$X' = \big(\tilde{E}^{\max}_{2n} \cdots \tilde{E}^{\max}_{m+n-1}\big) \cdots \big(\tilde{E}^{\max}_{n+2} \cdots \tilde{E}^{\max}_{m+1}\big)\big(\tilde{E}^{\max}_{n+1} \cdots \tilde{E}^{\max}_m\big).$$

Here $A'_{[m+n],[L,L']}$ and $A''_{[m+n],[L,L']}$ belong to a regular $\mathfrak{gl}_{[m+n]}$-crystal $M_{[m+n],[L,L']}$ with respect to $\tilde{E}_i$, $\tilde{F}_i$ ($i \in [m+n]^\circ$) and $\tilde{E}^{\max}_i b = \tilde{E}^{\varepsilon_i(b)}_i b$ and $\tilde{F}^{\max}_i b = \tilde{F}^{\varphi_i(b)}_i b$ for $b \in M_{[m+n],[L,L']}$. Note that

(1) A' is column admissible,
(2) $(\tilde{X}_i A')_{[m+n],[L,L']} = \tilde{X}_i(A'_{[m+n],[L,L']})$ for $i \in [m+n]^\circ$ and $X = E, F$.

Then by (6.6) we have

$$\tilde{x}_{j_1} \cdots \tilde{x}_{j_r} A' \neq \mathbf{0} \iff \tilde{x}_{j_1} \cdots \tilde{x}_{j_r} A'' \neq \mathbf{0}$$

for $r \geqslant 1$ and $j_1, \ldots, j_r \in [L, L']^\circ$, where $x = e, f$ for each j_k. Since L and L' are arbitrary and $\mathrm{wt}(A') = \mathrm{wt}(A'')$, A' is $\mathfrak{gl}_\infty$-equivalent to A''. Therefore, we have

$$C(A) \simeq C(A') \simeq C(A'') \simeq \mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi).$$

This completes the proof. □

For $\zeta, \eta \in \mathscr{P}$, $\xi \in \mathbb{Z}_+^{m-n}$ and $r \in \mathbb{Z}$, let $m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r)$ be the number of connected components C in $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$ such that

(1) $C \cap \mathbf{B}^{>r}(\mu,\nu) \neq \emptyset$,
(2) $C \simeq \mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi)$.

Corollary 6.3. *Under the above hypothesis,*

(1) *if $\xi_{m-n} < r$, then $m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r) = 0$,*
(2) *if $\xi_{m-n} \geqslant r$, then*

$$m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r) = c_{(\sigma,\eta)}^{((\mu-(r^m))',(\nu-(r^n))')},$$

where $\sigma = [(\xi - (r^{m-n})) \cup \zeta']'$.

Proof. It follows from (6.9), Lemma 6.2 and Corollary 4.6. □

The following lemma shows that $m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r)$ stabilizes as r goes to $-\infty$.

Lemma 6.4. *For $\zeta, \eta \in \mathscr{P}$ and $\xi \in \mathbb{Z}_+^{m-n}$, there exists $r_0 \in \mathbb{Z}$ such that*

$$m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r) = m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r_0),$$

for $r \leqslant r_0$.

Proof. For $r \in \mathbb{Z}$ with $r < \min\{\mu_m, \nu_n\}$, put

$$\mathcal{C}_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r) = \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{LR}_{\sigma\lambda}^{(\mu-(r^m))'} \times \mathbf{LR}_{\eta\lambda}^{(\nu-(r^n))'},$$

where $\sigma = [(\xi - (r^{m-n})) \cup \zeta']'$. Then

$$\mathcal{C}_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r-1) = \bigsqcup_{\delta \in \mathscr{P}} \mathbf{LR}_{\overline{\sigma}\,\delta}^{(\mu-(r^m))' \cup \{(m)\}} \times \mathbf{LR}_{\eta\,\delta}^{(\nu-(r^n))' \cup \{(n)\}},$$

where $\overline{\sigma} = [(\xi - (r^{m-n}) + (1^{m-n})) \cup \zeta']'$.

By Corollary 6.3, we have

$$\left|\mathcal{C}_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r)\right| = c_{(\sigma,\eta)}^{((\mu-(r^m))',(\nu-(r^n))')} = m_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r).$$

For a sufficiently small r, we define a map

$$\theta_r : \mathcal{C}_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r) \to \mathcal{C}_{(\zeta,\eta,\xi)}^{(\mu,\nu)}(r-1)$$

as follows:

Step 1. Suppose that $S_1 \in \mathbf{LR}_{\sigma\lambda}^{(\mu-(r^m))'}$ is given. Put $\ell = \xi_{m-n} - r$.

Define T_1 to be the tableau in $\mathbf{LR}_{\overline{\sigma}\ \lambda\cup\{(n)\}}^{(\mu-(r^m))'\cup\{(m)\}}$, which is obtained from S_1 as follows:

(1) The entries of T_1 in the i-th row ($1 \leqslant i \leqslant \ell$) are equal to those in S_1.
(2) The entries of T_1 in the $(\ell+1)$-st row are given by

$$a_1 + 1 \leqslant a_2 + 1 \leqslant \cdots \leqslant a_n + 1,$$

where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ are the entries in the ℓ-th row in S_1.
(3) Let S_1' (resp. T_1') be the subtableau of S_1 (resp. T_1) consisting of its i-th row for $\ell < i$ (resp. $\ell + 1 < i$). Then we define

$$T_1'(p+1,q) = \begin{cases} S_1'(p,q), & \text{if } S_1'(p,q) \leqslant a_1, \\ S_1'(p,q)+1, & \text{if } S_1'(p,q) > a_1, \end{cases}$$

for (p,q) in the shape of S_1'.

Since $\ell \gg 0$, we can check that T_1' is a well-defined Littlewood–Richardson tableau.

Step 2. Let $S_2 \in \mathbf{LR}^{(\nu-(r^n))'}_{\eta\lambda}$ be given. Applying the same argument as in Step 1 (when $m = n$), we obtain $T_2 \in \mathbf{LR}^{(\nu-(r^n))' \cup \{(n)\}}_{\eta\, \lambda \cup \{(n)\}}$.

Now we define

$$\theta_r(S_1, S_2) = (T_1, T_2) \in \mathcal{C}^{(\mu,\nu)}_{(\zeta,\eta,\xi)}(r-1).$$

By construction, we observe that θ_r gives a bijection

$$\mathbf{LR}^{(\mu-(r^m))'}_{\sigma\lambda} \times \mathbf{LR}^{(\nu-(r^n))'}_{\eta\lambda} \to \mathbf{LR}^{(\mu-(r^m))' \cup \{(m)\}}_{\overline{\sigma}\ \lambda \cup \{(n)\}} \times \mathbf{LR}^{(\nu-(r^n))' \cup \{(n)\}}_{\eta\ \lambda \cup \{(n)\}}$$

for $\lambda \in \mathscr{P}$. In particular, θ_r is one-to-one. On the other hand, if r is sufficiently small (or $\ell \gg 0$), then we have $(n) \subset \delta$ for $\delta \in \mathscr{P}$ with

$$\mathbf{LR}^{(\mu-(r^m))' \cup \{(m)\}}_{\overline{\sigma}\delta} \times \mathbf{LR}^{(\nu-(r^n))' \cup \{(n)\}}_{\eta\delta} \neq \emptyset,$$

that is, $\delta = \lambda \cup \{(n)\}$ for some $\lambda \in \mathscr{P}$, which implies that θ_r is onto. Therefore, θ_r is a bijection and $m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}(r)$ stabilizes as r goes to $-\infty$. $\square$

Theorem 6.5. *Suppose that $m \geqslant n$. For $\mu \in \mathbb{Z}^m_+$ and $\nu \in \mathbb{Z}^n_+$, we have*

$$\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) \simeq \bigsqcup_{\substack{\zeta,\eta \in \mathscr{P} \\ \zeta_1, \eta_1 \leqslant n}} \left(\bigsqcup_{\xi \in \mathbb{Z}^{m-n}_+} \mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi)^{\oplus m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}} \right)$$

with

$$m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = \sum_{\lambda \in \mathscr{P}} c^{\mu+(k^m)}_{\sigma\lambda} c^{\nu+(k^n)}_{\eta'\lambda},$$

where k is a sufficiently large integer and $\sigma = (\xi + (k^{m-n})) \cup \zeta'$.

Proof. For $\zeta, \eta \in \mathscr{P}$ and $\xi \in \mathbb{Z}^{m-n}_+$, let $m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}$ be the number of connected components in $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$ isomorphic to $\mathbf{B}_{\zeta,\eta} \otimes \mathbf{B}(\Lambda_\xi)$. Then by Lemma 6.4, we have

$$m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = m^{(\mu,\nu)}_{(\zeta,\eta,\xi)}(r)$$

for some $r \in \mathbb{Z}$. By Corollary 6.3, we have

$$m^{(\mu,\nu)}_{(\zeta,\eta,\xi)} = \sum_{\lambda\in\mathscr{P}} c^{\mu+(k^m)}_{\sigma\lambda} c^{\nu+(k^n)}_{\eta'\lambda},$$

where $k = -r$ and $\sigma = (\xi + (k^{m-n})) \cup \zeta'$. □

The decomposition when $m \leqslant n$ can be obtained by taking the dual crystal of the decomposition in Theorem 6.5.

7. Combinatorial description of the level zero part of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))$

7.1. For $\mu, \nu \in \mathbb{Z}^n_+$ ($n \geqslant 1$), let us describe the decomposition of $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu)$ in a bijective way. We assume that $\mathbf{B}(\Lambda_\mu) = C(A_\mu) \subset \mathcal{F}^n$ and $\mathbf{B}(-\Lambda_\nu) = C((A_\nu)^\vee) \subset (\mathcal{F}^n)^\vee$.

Suppose that $A \in \mathbf{B}(\Lambda_\mu)$ and $A' \in \mathbf{B}(-\Lambda_\nu)$ are given. Choose $r \in \mathbb{Z}$ such that $A \otimes A' \in \mathbf{B}^{>r}(\mu,\nu)$. Let $S^{>r} \otimes T^{>r} \in \mathbf{B}^{>r}_{(\mu-(r^n))'} \otimes (\mathbf{B}^{>r}_{(\nu-(r^n))'})^\vee$ correspond to $A \otimes A'$ under (6.9). Note that the set of entries in the i-th column of $S^{>r}$ (from the right) is $\{j \mid a_{ij} = 1,\ j > r\}$, and the set of entries in the i-th column of $T^{>r}$ (from the right) is $\{j^\vee \mid a_{ij} = 0,\ j > r\}$. Now we define

$$\psi^\infty_{\mu,\nu}\big(A \otimes A'\big) = \psi^{>r}_{(\mu-(r^n))',(\nu-(r^n))'}\big(S^{>r} \otimes T^{>r}\big), \tag{7.1}$$

where $\psi^{>r}_{(\mu-(r^n))',(\nu-(r^n))'}$ denotes the isomorphism in Proposition 5.1 corresponding to $\mathfrak{gl}_{>r}$-crystals.

Proposition 7.1. *For $\mu, \nu \in \mathbb{Z}^n_+$, the map*

$$\psi^\infty_{\mu,\nu} : \mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) \to \bigsqcup_{\alpha,\beta} \mathbf{B}^\vee_\alpha \otimes \mathbf{B}_\beta$$

is an isomorphism of $\mathfrak{gl}_\infty$-crystals, where the union is over all skew Young diagrams α and β such that $\alpha = (\nu - (r^n))'/\lambda$ and $\beta = (\mu - (r^n))'/\lambda$ for some $r \leqslant \min\{\mu_n, \nu_n\}$ and $\lambda \in \mathscr{P}$.

Proof. First, we will show that $\psi^\infty_{\mu,\nu}(A \otimes A')$ does not depend on the choice of r. Keeping the above notations, suppose that

$$\big(U^{>r}, V^{>r}\big) = \big(\big(\emptyset, T^{>r}\big) \to \big(S^{>r}, \emptyset\big)\big) \in \mathbf{B}^{>r}_{\sigma,\tau},$$
$$W^{>r} = \big(\big(\emptyset, T^{>r}\big) \to \big(S^{>r}, \emptyset\big)\big)_R \in \mathcal{C}^{((\mu-(r^n))',(\nu-(r^n))')}_{(\sigma,\tau)},$$

for some $\sigma, \tau \in \mathscr{P}$. By Proposition 4.7, there exist unique $\tilde{U}^{>r} \in \mathbf{B}^{>r}_\sigma$ and $\tilde{V}^{>r} \in (\mathbf{B}^{>r}_\tau)^\vee$ such that $\tilde{V}^{>r} \otimes \tilde{U}^{>r} \equiv (U^{>r}, V^{>r})$. Suppose that the bijection (4.4) maps $W^{>r}$ to

$$\big(W^{>r}_1, W^{>r}_2\big) \in \mathbf{LR}^{(\mu-(r^n))'}_{\sigma\lambda} \times \mathbf{LR}^{(\nu-(r^n))'}_{\tau\lambda}$$

for some $\lambda \in \mathscr{P}$. Then by definition of $\psi^{>r}_{(\mu-(r^n))',(\nu-(r^n))'}$, we have

$$\psi^\infty_{\mu,\nu}\big(A \otimes A'\big) = Y^\vee \otimes X \in \mathbf{B}^\vee_{(\nu-(r^n))'/\lambda} \otimes \mathbf{B}_{(\mu-(r^n))'/\lambda},$$

where

$$\jmath(X) = \tilde{U}^{>r}, \qquad \jmath(X)_R = W^{>r}_1,$$
$$\jmath(Y)^\vee = \tilde{V}^{>r}, \qquad \jmath(Y)_R = W^{>r}_2.$$

Now, suppose that

$$S^{>r-1} \otimes T^{>r-1} \in \mathbf{B}^{>r-1}_{(\mu-(r^n))'\cup\{(n)\}} \otimes \big(\mathbf{B}^{>r-1}_{(\nu-(r^n))'\cup\{(n)\}}\big)^\vee$$

is $\mathfrak{gl}_{>r-1}$-equivalent to $A \otimes A'$. Then

$$S^{>r-1} = (\underbrace{r \cdots r}_{n}) * S^{>r}, \qquad T^{>r-1} = T^{>r} * \big(\underbrace{r^\vee \cdots r^\vee}_{n}\big),$$

and

$$\big(\big(\emptyset, T^{>r-1}\big) \to \big(S^{>r-1}, \emptyset\big)\big) = \big(\big(\emptyset, T^{>r}\big) \to \big(S^{>r}, \emptyset\big)\big) = \big(U^{>r}, V^{>r}\big).$$

Hence we have $(U^{>r-1}, V^{>r-1}) = (U^{>r}, V^{>r})$.

Suppose that $W^{>r} = W_+^{>r} * W_-^{>r}$, where $W_+^{>r}$ (resp. $W_-^{>r}$) is the subtableau of $W^{>r}$ consisting of positive (resp. negative) entries. By definition of the insertion, it is straightforward to check that

(1) $W_-^{>r-1} = W_-^{>r}$,
(2) $W_+^{>r-1} = (\underbrace{\sigma_n' + 1 \cdots \sigma_1' + 1}_{n}) * W_+^{>r}[1]$,

where $W_+^{>r}[1]$ is the tableau obtained from $W_+^{>r}$ by increasing each entry by 1. Since $\imath(W_+^{>r-1}) = W_1^{>r-1}$, we have

$$W_1^{>r-1} = \Sigma_n * W_1^{>r}[1],$$

where Σ_n is the horizontal strip of shape $\sigma \cup \{(n)\}/\sigma$ filled with 1, and $W_1^{>r}[1]$ is the tableau obtained from $W_1^{>r}$ by increasing each entry by 1. Here, we assume that the shape of $W_1^{>r}$ is $(\mu - (r^n))' \cup \{(n)\}/\sigma \cup \{(n)\}$. Now, we have

$$\begin{aligned} \tilde{U}^{>r-1} * W_1^{>r-1} &= \tilde{U}^{>r} * \Sigma_n * W_1^{>r}[1] \\ &\leftrightsquigarrow (\underbrace{1 \cdots 1}_{n}) * \tilde{U}^{>r} * W_1^{>r}[1] \quad \big(\text{switching } \tilde{U}^{>r} \text{ and } \Sigma_n\big) \\ &\leftrightsquigarrow (\underbrace{1 \cdots 1}_{n}) * H_\lambda[1] * X \quad \big(\text{switching } \tilde{U}^{>r} \text{ and } W_1^{>r}[1]\big) \\ &= H_{\lambda \cup \{(n)\}} * X. \end{aligned}$$

This implies that X does not depend on r. Similarly, we have

$$W_2^{>r-1} = \Sigma_n' * W_2^{>r}[1],$$

where Σ_n' is the horizontal strip of shape $\tau \cup \{(n)\}/\tau$ filled with 1, and

$$\begin{aligned} \big(\tilde{V}^{>r-1}\big)^\vee * W_2^{>r-1} &= \big(\tilde{V}^{>r}\big)^\vee * \Sigma_n' * W_2^{>r}[1] \\ &\leftrightsquigarrow (\underbrace{1 \cdots 1}_{n}) * \big(\tilde{V}^{>r}\big)^\vee * W_2^{>r}[1] \quad \big(\text{switching } \big(\tilde{V}^{>r}\big)^\vee \text{ and } \Sigma_n'\big) \\ &\leftrightsquigarrow (\underbrace{1 \cdots 1}_{n}) * H_\lambda[1] * Y \quad \big(\text{switching } \big(\tilde{V}^{>r}\big)^\vee \text{ and } W_2^{>r}[1]\big) \\ &= H_{\lambda \cup \{(n)\}} * Y. \end{aligned}$$

This also implies that Y does not depend on r. Therefore, $\psi_{\mu,\nu}^\infty$ is well defined.

Since $\psi_{\mu,\nu}^\infty$ is a bijection and commutes with $\tilde{e}_k$ and $\tilde{f}_k$ ($k \in \mathbb{Z}$) by construction, it is an isomorphism of $\mathfrak{gl}_\infty$-crystals. □

Example 7.2. Let $\mu = (2, 2, 1)$ and $\nu = (3, 2, 1)$. Consider

$$A = \begin{array}{c|ccccccccccc} & & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \cdots & \bullet & \bullet & \bullet & \bullet & \cdot & \bullet & \cdot & \cdot & \cdot & \cdots \\ 2 & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \bullet & \cdot & \cdot & \cdots \\ 3 & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdots \end{array} \in \mathbf{B}(\Lambda_\mu) \subset \mathcal{F}^3,$$

$$A' = \begin{array}{c|ccccccccccc} & & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \cdots \\ 2 & \cdots & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \bullet & \bullet & \cdots \\ 3 & \cdots & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \bullet & \bullet & \cdots \end{array} \in \mathbf{B}(-\Lambda_\nu) \subset (\mathcal{F}^3)^\vee,$$

where $\bullet$ and $\cdot$ denote 1 and 0 in a matrix, respectively. Then $A \otimes A' \in \mathbf{B}^{>0}(\mu, \nu)$. Suppose that A (resp. A') is $\mathfrak{gl}_{>0}$-equivalent to $S^{>0} \in \mathbf{B}^{>0}_{(3,2)}$ (resp. $T^{>0} \in (\mathbf{B}^{>0}_{(3,2,1)})^\vee$). Then $S^{>0} = S$ and $T^{>0} = T$, where S and T are the tableaux in Example 4.5. Hence, by Example 5.2 we have

$$\psi^\infty_{\mu,\nu}\big(A \otimes A'\big) = \begin{array}{ccc} & & 4^\vee \\ & 1^\vee & \bullet \\ 2^\vee & \bullet & \bullet \end{array} \otimes \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \end{array}.$$

7.2. Let us give an explicit description of $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ for $\Lambda \in P_0$. For this, we define an analogue of (5.2) for $\mathfrak{gl}_\infty$-crystals. Suppose that $\mu \in \mathbb{Z}^n_+$ is given. For $k \in \mathbb{Z}$, let $\mu \cup \{(k)\}$ be the generalized partition in $\mathbb{Z}^{n+1}_+$ given by rearranging $\mu_1, \ldots, \mu_n$ and k. For $r \leqslant \mu_n$, we assume that the columns in $(\mu - (r^n))' \in \mathscr{P}$ are enumerated by $1, 2, \ldots$ from the left, and the rows are enumerated by $r+1, r+2, \ldots$ from the top, or we identify $(\mu - (r^n))'$ with $\{(i, j) \mid r+1 \leqslant i \leqslant \mu_j,\ 1 \leqslant j \leqslant n\} \subset \mathbb{Z} \times \mathbb{Z}$. For a skew Young diagram $\alpha = (\mu - (r^n))'/\lambda$ and $S \in \mathbf{B}_\alpha$, we also denote by $S(i, j)$ the entry in S located in the i-th row and the j-th column.

For $k \in \mathbb{Z}$, we define $\kappa_k : SST_{\mathbb{Z}}(\alpha) \to SST_{\mathbb{Z}}(\kappa_k(\alpha))$, where

$$\kappa_k(\alpha) = \big(\big(\mu \cup \big\{(k)\big\}\big) - \big(r^{n+1}\big)\big)' / \big(\lambda + \big(1^{k-r}\big)\big)$$

and $\kappa_k(S) = S'$ is given by $S'(i, j) = S(i, j)$ if $i > k$, and $S(i, j-1)$ if $i \leqslant k$. We put $\kappa^\vee_k = \vee \circ \kappa_k \circ \vee$. Here, if $k < r$, then we assume that $\alpha = (\mu - (s^n))'/\lambda \cup \{(n^{r-s})\}$ for some $s \leqslant k$.

By applying the arguments in Proposition 5.4 to Proposition 7.1 with a little modification, we obtain the following.

Proposition 7.3. *For $\mu, \nu \in \mathbb{Z}^n_+$ and $k \in \mathbb{Z}$, we have the following commutative diagram of $\mathfrak{gl}_\infty$-crystal morphisms.*

$$\begin{array}{ccc} \mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(-\Lambda_\nu) & \xrightarrow{\iota^{\Lambda_k}_{\Lambda_\mu, \Lambda_\nu}} & \mathbf{B}(\Lambda_\mu + \Lambda_k) \otimes \mathbf{B}(-\Lambda_k - \Lambda_\nu) \\ \Big\downarrow {\scriptstyle \psi^\infty_{\mu,\nu}} & & \Big\downarrow {\scriptstyle \psi^\infty_{\mu \cup \{(k)\}, \nu \cup \{(k)\}}} \\ \bigsqcup_{\alpha,\beta} \mathbf{B}^\vee_\alpha \otimes \mathbf{B}_\beta & \xrightarrow{\kappa^\vee_k \otimes \kappa_k} & \bigsqcup_{\gamma,\delta} \mathbf{B}^\vee_\gamma \otimes \mathbf{B}_\delta \end{array}$$

Let $\mathbf{M}$ be the set of $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{ij})$ such that $a_{ij} \in \mathbb{Z}_{\geqslant 0}$ and $\sum_{i,j \in \mathbb{Z}} a_{ij} < \infty$. As in Section 5.2, we have a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal structure on $\mathbf{M}$ with respect to $\tilde{e}_i, \tilde{f}_i$ and $\tilde{e}^t_j, \tilde{f}^t_j$ for $i, j \in \mathbb{Z}$. Now, we put

$$\begin{aligned} &\tilde{\mathbf{M}} = \mathbf{M}^\vee \times \mathbf{M}, \\ &\tilde{\mathbf{M}}_\Lambda = \big\{\big(M^\vee, N\big) \in \tilde{\mathbf{M}} \bigm| \mathrm{wt}\big(N^t\big) - \mathrm{wt}\big(M^t\big) = \Lambda\big\} \quad (\Lambda \in P_0). \end{aligned} \tag{7.2}$$

Note that $\tilde{\mathbf{M}}$ can be viewed as a tensor product of $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystals and $\tilde{\mathbf{M}}_\Lambda$ is a subcrystal of $\tilde{\mathbf{M}}$ with respect to $\tilde{e}_i, \tilde{f}_i$. By Proposition 7.3, we have the following combinatorial realization, which is our second main result. The proof is almost the same as in Theorem 5.5.

Theorem 7.4. *For $\Lambda \in P_0$, we have*

$$\tilde{\mathbf{M}}_\Lambda \simeq \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty).$$

Let $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))_0 = \bigsqcup_{\Lambda \in P_0} \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ be the level zero part of $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))$. Since $\tilde{\mathbf{M}} = \bigsqcup_{\Lambda \in P_0} \tilde{\mathbf{M}}_\Lambda$ and $\mathbf{M} \simeq \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{B}_\lambda \times \mathbf{B}_\lambda$ as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal, we obtain the following immediately.

Corollary 7.5. *As a $\mathfrak{gl}_\infty$-crystal, we have*

$$\mathbf{B}\big(\tilde{U}_q(\mathfrak{gl}_\infty)\big)_0 \simeq \tilde{\mathbf{M}}.$$

Corollary 7.6. *As a $(\mathfrak{gl}_\infty, \mathfrak{gl}_\infty)$-bicrystal, we have*

$$\mathbf{B}\big(\tilde{U}_q(\mathfrak{gl}_\infty)\big)_0 \simeq \bigsqcup_{\mu, \nu \in \mathscr{P}} \mathbf{B}_{\mu,\nu} \times \mathbf{B}_{\mu,\nu}.$$

In [1], Beck and Nakajima proved a Kashiwara's conjecture [12] on the Peter–Weyl type decomposition of the level zero part of $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ for an affine Kac–Moody algebra $\mathfrak{g}$ of finite rank, where the crystal structure induced from the involution $*$ on $\tilde{U}_q(\mathfrak{g})$ gives a bicrystal structure on $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ together with usual $\tilde{e}_i$, $\tilde{f}_i$. The second crystal structure on $\mathbf{B}(\tilde{U}_q(\mathfrak{g}))$ is usually known as $*$-crystal structure [10], say $\tilde{e}_i^*$ and $\tilde{f}_i^*$. Based on some computation, we give the following conjecture.

Conjecture 7.7. *The crystal structure on $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_{>0}))$ and $\mathbf{B}(\tilde{U}_q(\mathfrak{gl}_\infty))_0$ with respect to $\tilde{e}_i^t$ and $\tilde{f}_i^t$ is compatible with the dual of the $*$-crystal structure with respect to $\tilde{e}_i^*$ and $\tilde{f}_i^*$. That is, $\tilde{e}_i^t = \tilde{f}_i^*$ and $\tilde{f}_i^t = \tilde{e}_i^*$ for all i.*

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