JOURNAL OF ALGEBRA 44, 339–362 (1977)

Conjugacy Classes in Linear Groups

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Communicated by Walter Feit

Received April 27, 1975

Let G belong to one of the three families of complex classical linear groups or to one of the seven families of corresponding real forms Let L denote its Lie algebra. We give a simple and effective method for finding all conjugacy classes of G and all orbits of G in L. We also describe the splitting of classes and orbits when G is replaced by a normal subgroup. We discuss the situation for other fields.

INTRODUCTION

Let G be a complex or real linear Lie group. Let L denote its linear Lie algebra so that G acts naturally on L by conjugation. When G is simple there arise two important and closely connected problems: (i) describe all conjugacy classes in G, and (ii) describe all orbits in L under the action of G. In this article we give a complete solution to these problems when G belongs to any of the nonexceptional series of simple groups. A description of the groups G is given in Section 1.

There has been extensive previous work on the above problems and on related variations. For those groups leaving invariant a bilinear or Hermitian form this work begins with Weierstrass [1], Kronecker [2], and Frobenius [3], then during the first half of this century continues with a series of papers by Williamson [4] as well as work by many others (see [5] for references). More recently we particularly note the results of Zassenhaus [6], Wall [7], Cikunov [8], Springer and Steinberg [9], and Milnor [10].

With the exception of Wall the above authors always deal with classical linear groups over a commutative field (usually arbitrary and of odd or zero characteristic). Thus they do not include those families of real Lie groups which may be described as classical groups over a quaternionic vector space. Wall allows noncommutative fields, but there are still certain difficulties in using his methods when the field is noncommutative.

* The authors wish to thank the U. S. Army Research Office for support through Grant DA-ARO-G168.

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ISSN 0021-8693

Our approach differs from the previous work in several respects. We always work over a complex vector space and consider the real Lie groups as fixed point sets of a certain involutary automorphism of their coresponding complex group. This allows us to give a unified and simultaneous treatment for all the cases. In particular, the quaternionic groups are handled as easily as the real groups. We make constant use of the unique decomposition of any linear mapping into a sum of commuting semisimple and nilpotent parts. By this technique we can quickly reduce questions about conjugacy to the corresponding question for the semisimple part, a useful simplification. In (2.1) and (2.2) we introduce the idea of a "type" and an "indecomposable type." For example, if $G \simeq GL(n, \mathbb{C})$, an indecomposable type is just an abstraction for the usual Jordan matrix with some $\zeta \in \mathbb{C}$ on its diagonal and 1's on its superdiagonal. Although elementary, we find that this concept of "types" results in a concise and convenient language for stating results and proofs

The main results are contained in the four propositions in (2.4) and the theorem in (2 2) In (2 7) we use these results to solve problem (ii) above. In Section 3 we give the minor modifications necessary to solve problem (i). In (2 8) we give examples which hopefully demonstrate the ease with which our results may be used. Our methods also provide an effective algorithm for finding in which orbit (or class) a given element of L (or G) lies. This could be interpreted as a generalization of the Jordan-normal form for $GL(n, \mathbb{C})$, although there is little point in writing down "normal-form" matrices in general. In Section 4 we describe the splitting of orbits or classes when G is replaced by a normal subgroup.

Our results in (2.1)–(2.6) hold for classical groups over more general (commutative) fields. In Section 5 we briefly fiscuss this. Our aim is merely to show that problems (i) and (ii) can be reduced to a purely field theoretical question. Since this case is fully discussed in, for example, [8, 9] and especially [10] we omit proofs.

Appendix 1 collects some information on involutary automorphisms Appendix 2 contains some elementary and repetitive calculations connected with Table II in (2 7). Our arguments involve only simple linear algebra and we only need to quote one result, Sylvester's theorem on the signature (see for example, [11]).

1. Notation

(1.1) *The Complex Classical Groups.* Let V be a complex vector space of finite dimension. Let $GL(V)$ denote the general linear group on V.

Let τ be a nondegenerate symmetric or alternating bilinear form on V. For $g \in GL(V)$ define τ^g by $\tau^g(u, v) = \tau(gu, gv)$ for $u, v \in V$. The isometry group of τ is $\{g \in GL(V) \mid \tau^g = \tau\}$ and is denoted by $O(V, \tau)$ or $Sp(V, \tau)$ depending on whether τ is symmetric or alternating

(1.2) *The Real Forms.* Let G denote one of the groups in (1.1). All of its real forms may be described as subgroups $G_\sigma = \{g \in G \mid g^\sigma = g\}$, where σ is an automorphism of order 2 of G and is either (i) an anti-linear map of V onto V such that $g^\sigma = \sigma^{-1} g \sigma$ or (ii) an antilinear map of V onto V^* such that $g^\sigma = \sigma^{-1} g_*^{-1} \sigma$, if $G = GL(V)$. Here V^* denotes the dual space of V and $g_* \in GL(V^*)$ is defined by $g_* u_*(v) = u_*(gv)$ for $u_* \in V^*$, $v \in V$ Antilinear means $\sigma(\alpha u + \beta v) = \bar{\alpha}\sigma u + \bar{\beta}\sigma v$ for $\alpha, \beta \in \mathbb{C}$

For a given G we say that σ_1 and σ_2 are equivalent if for some $k \in G$ both σ_1 and $k^{-1}\sigma_2 k$ ($k_*\sigma_2 k$ in case (ii)) induce the same automorphism of G. If σ_1 and σ_2 are equivalent then $G_{\sigma_1} \simeq G_{\sigma_2}$.

Replacing σ by an equivalent choice we may suppose in case (i) that $\sigma^2 = \pm 1$ and $\tau(\sigma u, \sigma v) = \overline{\tau(u, v)}$ and in case (ii) that $\sigma u(v) = \overline{\sigma v(u)}$. In this latter case we put $\tau_*(u, v) = \sigma u(v)$ and from now on will use the Hermitian form τ_* instead of the corresponding σ.

Using the Gram–Schmidt algorithm the equivalence classes for σ and τ_* are easily described. The calculations are well known so we only sketch them in Appendix 1. We summarize the results in Table I Column 1 gives the complex

TABLE I

$GL(V)$	$GL(V, \sigma_+)$		$GL(n, \mathbb{R})$
	$GL(V, \sigma_-)$	$n =$ even	$U^*(n)$
	$GL(V, \tau_*^{(p)})$	$0 \leqslant p \leqslant (n/2)$	$U(n-p, p)$
$O(V, \tau)$	$O(V, \tau, \sigma_+^{(p)})$	$0 \leqslant p \leqslant (n/2)$	$O(n-p, p)$
	$O(V, \tau, \sigma_-)$	$n =$ even	$O^*(n)$
$Sp(V, \tau)$	$Sp(V, \tau, \sigma_+)$		$Sp(n, \mathbb{R})$
	$Sp(V, \tau, \sigma_-^{(p)})$	$0 \leqslant p \leqslant (n/4)$	$Sp(n-p, p)$

group G. Column 2 gives our notation for the possible G_σ. The class representatives σ_+, σ_-,. are described in Appendix 1. The subscript $\pm$ indicates the sign of σ^2 in case (1). The superscript p in $\tau_*^{(p)}$, $\sigma_+^{(p)}$, $\sigma_-^{(p)}$ is the index of certain complex-, real-, quaternion-valued forms which are defined in Appendix 1 and are naturally associated with G_σ In particular G_σ is compact for the cases where $p = 0$. In column 3 we give the notation of Helgason [12, Chap. IX] for the corresponding matrix groups. We let $n = \dim V$ and note that for $G = Sp(V, \tau)$, n is always even.

Thus, besides the three complex families, there are seven families of real forms. Among the latter the three families of case (i) with $\sigma^2 = +1$ may be described as real linear groups on V_σ^+ (see Appendix 1) while the three families with $\sigma^2 = -1$ may be described as quaternionic linear groups on V_σ^-.

In those families where the index p occurs, if the particular value of p is not relevant, we often omit it and just write $GL(V, \tau_*)$, $O(V, \tau, \sigma_+)$, etc.

(1.3) *The Lie Algebra.* For $GL(V)$ its Lie algebra, as represented on V, is $\mathrm{End}(V)$. For $O(V, \tau)$ or $Sp(V, \tau)$ the corresponding Lie algebra is $\{A \in \mathrm{End}(V) \mid \tau(Au, v) + \tau(u, Av) = 0, \text{ all } u, v \in V\}$. For the unitary groups $GL(V, \tau_*)$ is obtained by replacing τ by τ_* above. For the real forms in case (ι) there is the additional condition $\sigma A = A\sigma$.

(1.4) *Notation and Conventions.* In Sections 2 and 3 we find it convenient to introduce a generic symbol $G(V, \sigma, \tau)$ to denote any one of the groups from the 10 families defined in (1.1) or (1 2). Thus either σ or τ or both may not actually occur in the definition of the group Furthermore τ may also denote τ_*, as defined in (1 2), and in this case σ is absent. We let $L(V, \sigma, \tau)$ denote the Lie algebra of $G(V, \sigma, \tau)$.

In Sections 2 and 3 statements are formulated with the assumption that both σ and τ occur in the definition of the group $G(V, \sigma, \tau)$. To adapt the definitions and proofs to the other cases it is only necessary omit any irrelevant statements

If $W \subseteq V$ is a subspace let $W^{\perp} = \{v \in V \mid \tau(v, w) = 0 \text{ all } w \in W\}$. We often allow a symbol, such as $\sigma, \tau, \ldots$ to denote both an object defined on V and also, by restriction, the corresponding object defined on W. If $u, v \cdots \in V$ let $\langle u, v, \ldots \rangle$ denote their span.

2. Main Results

(2 1) *Types.* Using the notation of (1.4) let $A \in L(V, \sigma, \tau)$. We require a notion of equivalence among pairs of the form (A, V). Let $A' \in L(V', \sigma', \tau')$ then we write $(A, V) \sim (A', V')$ if there exists an isomorphism ϕ of V onto V' such that $\phi A = A'\phi$, $\phi\sigma = \sigma'\phi$ and $\tau = \tau'^{\phi}$, i.e., $\tau(u, v) = \tau'(\phi u, \phi v)$ Note that ϕ defines an isomorphism $G(V, \sigma, \tau) \simeq G(V', \sigma', \tau')$.

It is clear that $\sim$ defines an equivalence relation. An equivalence class for $\sim$ is called a *type*. If Δ denotes a type and $(A, V) \in \Delta$ put $\dim \Delta = \dim V$.

The motivation for introducing types comes from the following result, the proof of which is a consequence of the definitions.

Proposition 1 *Let $A, B \in L(V, \sigma, \tau)$ then there exists a $g \in G(V, \sigma, \tau)$, such that $g^{-1}Ag = B$, if and only if (A, V) and (B, V) belong to the same type.*

(2 2) *Indecomposable Types.* Let $A \in L(V, \sigma, \tau)$ and let Δ denote the type containing (A, V). Suppose $V = W_1 + W_2$ is a sum of proper, disjoint, A-invariant, σ-invariant, and orthogonal subspaces. Since the restriction of τ to each W_i is nondegenerate the groups $G(W_i, \sigma, \tau)$ are well defined and so, by restriction, $A \in L(W_i, \sigma, \tau)$. Let Δ_i denote the type containing (A, W_i) for $i = 1, 2$. Then we write $\Delta = \Delta_1 + \Delta_2$.

The type Δ is called *indecomposable* if it cannot be written as the sum of two or more types. For any type Δ we can write $\Delta = \Delta_1 + \cdots + \Delta_s$, where all Δ_i are indecomposable. We have

THEOREM. *The decomposition $\Delta = \Delta_1 + \cdots + \Delta_s$ into indecomposable types is unique.*

The proof is in (2.4) but depends on results whose proofs are in (2.6). As a corollary to this theorem note that if $\Delta_1 + \Delta_2 = \Delta_1 + \Delta_3$ then $\Delta_2 = \Delta_3$.

(2.3) *Semisimple Types.* Let $A \in L(V, \sigma, \tau)$ then we can, in a unique way, write $A = S + N$, where $S, N \in L(V, \sigma, \tau)$, S is semisimple, N is nilpotent and $SN = NS$.

Suppose m is a nonnegative integer such that $N^m \neq 0$ and $N^{m+1} = 0$. We call m the *height* of the pair (A, V). Let (A, V) belong to the type Δ. Clearly m is an invariant of Δ which we call its height and denote by $ht\,\Delta$.

Let $K = \operatorname{Ker} N^m$ then $K \supseteq NV$. If $K = NV$ we say that the pair (A, V) is *uniform*. Since equivalent pairs are either both uniform or not we may speak of uniform types.

If $ht\,\Delta = 0$ we say that Δ is a *semisimple* type. Note that a semisimple type is uniform.

Let Δ be uniform and $m = ht\,\Delta$. If $(A, V) \in \Delta$ put $\bar{V} = V/NV$ and for $v \in V$ put $\bar{v} = v + NV$. Define $\bar{A}, \bar{\sigma}, \bar{\tau}$ on $\bar{V}$ by $\bar{A}\,\bar{v} = \overline{Av}$, $\bar{\sigma}\,\bar{v} = \overline{\sigma v}$, and $\bar{\tau}(\bar{u}, \bar{v}) = \tau(u, N^m v)$. Since τ is nondegenerate on V and (A, V) is uniform hence $\bar{\tau}$ is nondegenerate on $\bar{V}$. Thus $G(\bar{V}, \bar{\sigma}, \bar{\tau})$ is well defined. Let $\bar{\Delta}$ denote the type containing $(\bar{A}, \bar{V})$. $\bar{\Delta}$ is semisimple and is uniquely determined by Δ.

Note that Δ is semisimple if and only if $\Delta = \bar{\Delta}$. Observe that the bar notation used above has no connection with complex conjugation in $\mathbb{C}$.

(2.4) *Proof of the Theorem.* In (2.6) we prove the following four results. Δ denotes a type.

PROPOSITION 2. *If Δ is uniform it is uniquely determined by $ht\,\Delta$ and $\bar{\Delta}$.*

PROPOSITION 3. *If Δ is indecomposable then Δ is uniform and $\bar{\Delta}$ is indecomposable.*

PROPOSITION 4. *If Δ is not uniform there exist unique types Δ_1 and Δ_0 such that $\Delta = \Delta_1 + \Delta_0$ with Δ_1 uniform, $ht\,\Delta_1 = ht\,\Delta$ and $ht\,\Delta_0 < ht\,\Delta$.*

PROPOSITION 5. *If Δ is semisimple then its decomposition into indecomposable types is unique.*

Proof of the Theorem of (2.2). Let Δ denote any type. By Proposition 4 we can write in a unique way

$$\Delta = \Delta^{(m)} + \Delta^{(m')} + \cdots + \Delta^{(k)} + \cdots,$$

where $m > m' > \cdots > k > \cdots$ and each $\Delta^{(k)}$ is a uniform type of height k

Let $\Delta^{(k)} = \Delta_1 + \cdots + \Delta_t$, where all Δ_i are indecomposable Using Proposition 3, $\overline{\Delta^{(k)}} = \overline{\Delta_1} + \cdots + \overline{\Delta_t}$ and each $\overline{\Delta_i}$ is indecomposable. By Proposition 5 this decomposition of $\overline{\Delta^{(k)}}$ is unique. Hence, by Proposition 2 the decomposition of $\Delta^{(k)}$ is unique Q.E.D.

(2.5) *Some Preliminary Lemmas* The following results are needed in (2.6). They do not involve τ Let A, S, N be elements of $L(V, \sigma, \tau)$ and, as in (2.3), let $A = S + N$.

LEMMA 1. *Suppose (S, V) is a pair where S is semisimple. Let U be any proper subspace of V which is S-invariant and σ-invariant. Then there exists a complement to U in V which is also S-invariant and σ-invariant.*

Proof. Since S is semisimple we may choose W to be an S-invariant complement to U in V. If $\sigma W = W$ we are done. If $\sigma W \neq W$ we can write, for all $w \in W$

$$\sigma w = \sigma_1 w + \sigma_2 w,$$

where σ_1, σ_2 are antilinear maps of W into W and of W into U, respectively.

Suppose $\sigma^2 = \epsilon 1$, where $\epsilon = \pm 1$ then $\sigma_1{}^2 = \epsilon 1$ and $\sigma_2 \sigma_1 + \sigma \sigma_2 = 0$ Let W' be the image of W under the isomorphism

$$w \to w' = (1 - (1/2\epsilon)\, \sigma\sigma_2)w, \qquad w \in W.$$

Then $\sigma w' = (\sigma_1 w)'$ and so W' is σ-invariant. W' is S-invariant and is a complement to U in V. Q.E.D.

LEMMA 2. *Suppose (A, V) is a uniform pair of height m. Then there exists an S-invariant and σ-invariant subspace H such that $V = H + NH + \cdots + N^m H$ is a sum of mutually disjoint subspaces Furthermore* $\dim N^i H = \dim H$ *for* $0 \leqslant i \leqslant m$

Proof. NV is S-invariant and σ-invariant so by Lemma 1 we may find an S-invariant and σ-invariant subspace H which is a complement to NV in V. Hence, since height $= m$, we have $V = H + NH + \cdots + N^m H$

If $N^i H \cap N^j H \neq 0$ for some $i < j$ then for some nonzero $h_1, h_2 \in H$ we have $N^i(h_1 + N^{j-i} h_2) = 0$. Hence $h_1 + N^{j-i} h_2 \in \operatorname{Ker} N^m$ and since (A, V) is uniform and $j - i > 0$ therefore $h_1 \in \operatorname{Ker} N^m$. So $h_1 = 0$, a contradiction. Q.E.D.

LEMMA 3. *Let (A, V) be a pair of height m. Let U be a proper A-invariant and σ-invariant subspace of V. Suppose (A, U) is uniform and $U \not\subseteq$ Ker N^m. Then there exists a complement to U in V which is A-invariant and σ-invariant.*

Proof. Use induction on m. If $m = 0$ then, by Lemma 1 we are done. So suppose $m \geqslant 1$. Put $K = \operatorname{Ker} N^m$.

By Lemma 2 there exists a subspace H of U which is S-invariant and σ-invariant and a complement to $K \cap U$ in U and, using $U \not\subseteq K$, such that $U = H \dotplus NH \dotplus \cdots \dotplus N^m H$ Note that $U \cap K = NU$ and also that (A, NU) is uniform.

First suppose $U + K = V$. Since the height of the pair (A, K) is $m - 1$ and since $U \cap K \not\subseteq \operatorname{Ker} N^{m-1}$ we may, by induction, find a subspace Z which is A-invariant, σ-invariant, and a complement to $U \cap K$ in K. Hence Z is also a complement to U in K and we are done.

Now suppose $U + K \neq V$. By Lemma 1 we may choose a subspace F which is S-invariant, σ-invariant and a complement to $U + K$ in V. Put $W = F + NF + \cdots + N^m F$ and observe that W is A-invariant, σ-invariant and (A, W) is uniform of height m. If $U \cap W \neq 0$ then for some nonzero $h \in H$ we have $N^i h + u \in W$, where $u \in N^{i+1} U$ and $i \leqslant m$ Hence $N^m h \in W$ and since $h \neq 0$ there exists a nonzero $f \in F$ such that $N^m(h - f) = 0$. This implies $f \in U + K$, a contradiction, and hence $U \cap W = 0$. Put $U_1 = U + W$ If $U_1 = V$ we are done. If $U_1 \neq V$ then since $U_1 + K = V$ we are done by the result of the previous paragraph Q.E.D

(2.6) *Proofs of the Propositions.* We use the notation of (2.1)–(2.3). In particular, if $(A, V) \in \Delta$ then $A = S + N$ is as in (2.3), and $m = ht\,\Delta$

Proof of Proposition 2. Since Δ is uniform, by Lemma 2, we may choose a complement E to NV such that E is S-invariant and σ-invariant Then $V = E \dotplus NE \dotplus \cdots \dotplus N^m E$ is a sum of disjoint subspaces

For $0 \leqslant j \leqslant m$ define the bilinear forms τ_j on E by $\tau_j(u, v) = \tau(u, N^j v)$ for $u, v \in E$. Let $\hat{\tau}_j$ denote the linear map from E to E^* corresponding to τ_j, i.e., $\hat{\tau}_j u(v) = \tau_j(u, v)$. Since $NV = \operatorname{Ker} N^m$ τ_m is nondegenerate on E and hence $\hat{\tau}_m$ is an isomorphism of E onto E^*

Suppose for some $0 \leqslant k \leqslant m - 1$ that $\hat{\tau}_k \neq 0$ while, if $k \neq m - 1$, $\hat{\tau}_{k+i} = 0$ for $i = 1, \ldots, m - k - 1$. Then let E' denote the image of E under the isomorphism $e \to e' = (1 - N^{m-k}\rho)e$ for $e \in E$, where $\rho \in \operatorname{End}(E)$ is defined by

$$\rho = (-1)^{m-k}/2\ \hat{\tau}_m^{-1}\hat{\tau}_k .$$

Since $S_*\hat{\tau}_j = -\hat{\tau}_j S$ we see that E' is S-invariant Define σ_* as an antilinear map of V^* onto itself by $\sigma_* u_*(v) = \overline{u_*(\sigma v)}$. The condition $\tau^\sigma = \bar{\tau}$ becomes $\sigma_* \hat{\tau}_j \sigma = \hat{\tau}_j$ and hence E' is σ-invariant E' is a complement to NV in V. The forms τ_j can be defined on V. Since $\tau_{k+i} \equiv 0$ on E for $i = 1, \ldots, m - k - 1$ the same result holds on E'.

On E^* the transpose of ρ is $\rho_* = \frac{1}{2}\hat{\tau}_k \hat{\tau}_m^{-1}$. Hence $\hat{\tau}_k = \rho_* \hat{\tau}_m + (-1)^{m-k}\hat{\tau}_m \rho$ on E This is equivalent to $\tau_k \equiv 0$ on E'. Hence, by repeating this procedure, at most m times, we will obtain a subspace F which is S-invariant, σ-invariant, a complement to NV in V and such that $\tau_j \equiv 0$ on F for $j = 0, \ldots, m - 1$.

We may suppose that $S \in L(F, \sigma, \tau_m)$ and hence have the pair (S, F). There is a natural isomorphism of F onto $\bar{V} = V/NV$ given by $f \to f + NV$ It defines the equivalence $(S, F) \sim (\bar{A}, \bar{V})$ and hence $(S, F) \in \bar{\Delta}$.

Now suppose Δ' is another uniform type and $ht\,\Delta' = ht\,\Delta$ and $\bar{\Delta}' = \bar{\Delta}$ Let $(A', V') \in \Delta'$ and let $S', N', F', \ldots$ be analogs of $S, N, F, \ldots$. Hence $(S', F') \in \bar{\Delta}$. Let ϕ denote an isomorphism of F onto F' which gives the equivalence $(S, F) \sim (S', F')$. Extend ϕ to V by putting $\phi(Nv) = N'\phi v$ for $v \in V$. Since $ht\,\Delta' = ht\,\Delta$ and since Lemma 2 applies to both V and V' we see that ϕ is an isomorphism of V onto V' Clearly $A'\phi = \phi A$ and $\sigma'\phi = \phi\sigma$ If $v_1, v_2 \in V$ we may write, in a unique way, $v_i = \sum_{r=0}^{m} N^r f_{i,r}$, where $f_{i,r} \in F$. Thus $\phi v_i = \sum_r N'^r \phi f_{i,r}$. Now by the property of F we have $\tau(v_1, v_2) = \sum_{r+r'=m} (-1)^r \tau_m(f_{1,r}, f_{2,r'})$ and since $\tau_m = \tau_m'^{\phi}$ on F we see that $\tau = \tau'^{\phi}$ on V Hence $(A, V) \sim (A', V')$ and so $\Delta = \Delta'$. Q.E.D

Proof of Proposition 3. Proposition 2 implies that Δ is uniform. If $\bar{\Delta}$ is decomposable suppose that $\bar{\Delta} = \Delta_1' + \Delta_2'$ and let Δ_1 and Δ_2 denote the unique uniform types of height equal to $ht\,\Delta$ satisfying $\bar{\Delta}_i = \Delta_i'$ for $i = 1, 2$.

Let $(A, V) \in \Delta$ and let F denote the subspace constructed in the proof of Proposition 2. Since $(S, F) \sim (\bar{A}, \bar{V})$ we may suppose that $F = F_1 + F_2$ is a decomposition corresponding to $\bar{\Delta} = \Delta_1' + \Delta_2'$. For $i = 1, 2$ put $W_i = F_i + NF_i + \cdots + N^m F_i$, where $m = ht\,\Delta$. Then $W_1 + W_2 = V$ and by the properties of F we see that W_i is A-invariant, σ-invariant, and orthogonal. Arguing as in the last paragraph of the proof of Lemma 3 we get $W_1 \cap W_2 = 0$. Hence $\Delta = \Delta_1 + \Delta_2$, a contradiction. Q E D.

Proof of Proposition 4. Let $K = \operatorname{Ker} N^m$, where $m = ht\,\Delta$. By Lemma 1 we can choose an S-invariant and σ-invariant subspace E which is a complement to K in V. Put $Y = E + NE + \cdots + N^m E$ Then Y is uniform and $Y \not\subseteq K$.

If τ is not involved in the definition of $G(V, \sigma, \tau)$ then by Lemma 3 we can find an A-invariant, σ-invariant subspace Z which is a complement to Y in V. If Δ_1 denotes the type of (A, Y) and Δ_0 that of (A, Z) then $\Delta = \Delta_1 + \Delta_0$ and $ht\,\Delta_1 = ht\,\Delta$. Since $Z \subseteq K$ we have $ht\,\Delta_0 < ht\,\Delta$.

If τ is involved in $G(V, \sigma, \tau)$ we first show that τ is nondegenerate on Y. Suppose that $\tau(x, y) = 0$ for some $x \in Y$ and all $y \in Y$. If $x \neq 0$ write $x = N^i e + x_1$ for some nonzero $e \in E$ and some $x_1 \in N^{i+1} Y$. Thus $\tau(e, N^m f) = 0$ for all $f \in E$. Now any $v \in V$ can be written $v = f + k$ for some $f \in E$, $k \in K$. Thus $\tau(e, N^m v) = 0$ all $v \in V$. Since $\tau(e, N^m v) = (-1)^m \tau(N^m e, v)$ and τ is nondegenerate on V we have $e \in E \cap K = 0$, a contradiction. Thus τ is nondegenerate on Y. Let Δ_1 denote the type of (A, Y) Δ_1 is uniform and $ht\,\Delta_1 = ht\,\Delta$. Put $Z = Y^{\perp}$ and let Δ_0 denote the type of (A, Z). Thus $\Delta = \Delta_1 + \Delta_0$. If $z \in Z$ then arguing as above $\tau(z, N^m v) = 0$ for all $v \in V$ and so $N^m z = 0$. Thus $Z \subseteq K$ and so $ht\,\Delta_0 < ht\,\Delta$.

By repeating this procedure for Δ_0 we obtain a decomposition $\Delta = \Delta_1 + \Delta_2 + \cdots$ where each Δ_i is uniform and $ht\,\Delta_1 > ht\,\Delta_2 > \cdots$. Suppose $\Delta = \Delta_1' + \Delta_2' + \cdots$ is another such decomposition.

We first show that $\Delta_1' = \Delta_1$ Suppose $V = Y' + Z'$ with $(A, Y') \in \Delta_1'$ and $Z' \subseteq K$. Then $\bar{Y}' = Y'/NY'$ is isomorphic to V/K and hence to $\bar{Y} = Y/NY$. This yields $(\bar{A}, \bar{Y}') \sim (\bar{A}, \bar{Y})$. By Proposition 2 we get $(A, Y') \sim (A, Y)$ and so $\Delta_1' = \Delta_1$.

By induction we may suppose $\Delta_i' = \Delta_i$ for $i = 1, 2, \ldots, s-1$ To complete the proof we must show that $\Delta_s' = \Delta_s$. Suppose $V = W + U = W' + U'$, where $(A, W) \sim (A, W') \in \Delta_1 + \cdots + \Delta_{s-1}$ and $(A, U) \in \Delta_s + \cdots$ while $(A, U') \in \Delta_s' + \cdots$ Let $m_s = ht\, \Delta_s = ht\, \Delta_s'$ and put $V_s = V/\mathrm{Ker}\, N^{m_s}$ We use W_s, U_s, etc, to denote the images in V_s of W, U, etc. Define the nonsingular form τ_s on V_s by $\tau_s(e_s, f_s) = \tau(e, N^{m_s}f)$ for $e, f \in V$. Since Δ_s', Δ_s are uniform, by Proposition 2, it suffices to prove that $(A_s, U_s') \sim (A_s, U_s)$(where A_s denotes the action of A induced on V_s). If $u' \in U'$ we have the unique decomposition $u' = w + u$, where $w \in W$ and $u \in U$. Hence in V_s we have $u_s' = w_s + u_s$. Define $\psi u_s' = u_s$; we now show that ψ gives the required equivalence.

Suppose $U_s' \cap W_s \neq 0$ Then, since $ht\, \Delta_{s-1} > m_s$ we can find a nonzero $e_s \in W_s$ such that $N_s e_s \in U_s'$. Now $e_s = w_s' + u_s'$ for some $w_s' \in W_s'$, $u_s' \in U_s'$ and, since $N_s u_s' = 0$, we have $N_s e_s \in W_s'$. Since $W_s' \cap U_s' = 0$ this is a contradiction. Hence $U_s' \cap W_s = 0$ and so ψ is an isomorphism of U_s' onto U_s.

Clearly $\psi A_s = A_s \psi$ and $\psi\sigma = \sigma\psi$ so it only remains to show that

$$\tau_s(u_s', v_s') = \tau_s(\psi u_s', \psi v_s') \qquad \text{for} \quad u_s', v_s' \in U_s'.$$

Since W_s and U_s are orthogonal we need only show that $\tau_s((1-\psi)\, u_s', (1-\psi)\, v_s') = 0$ Now $(1-\psi)\, u_s' \in W_s \cap \mathrm{Ker}\, N_s$ and so $(1-\psi)\ u_s' = N_s e_s$ for some $e_s \in W_s$. Combining this with $N_s(1-\psi)\, v_s' = (1-\psi)\, N_s v_s' = 0$ we have the required result The proof is complete. Q.E.D.

Proof of Proposition 5. We need some notation. If $(S, V) \in \Delta$, where S is semisimple, let eig Δ denote the set of eigenvalues of S on V with multiplicities counted. If τ is involved in $G(V, \sigma, \tau)$ we introduce in Appendix 2 certain bilinear forms θ on $V_\sigma^{\pm}$. Let sig Δ denote the signature of θ

We need two results from Appendix 2.

(i) If Δ_i is a semisimple and indecomposable type then it is uniquely determined by eig Δ_i and, if relevant, sig Δ_i.

(ii) If Δ_i and Δ_j are distinct semisimple indecomposable types then either eig $\Delta_i \cap$ eig $\Delta_j = \phi$ or eig $\Delta_i =$ eig Δ_j and sig $\Delta_i \neq$ sig Δ_j

Now suppose we have some decomposition of Δ into a sum of semisimple indecomposable types then put $\Delta = \Delta^{(1)} + \cdots + \Delta^{(t)}$, where in each $\Delta^{(i)}$ all indecomposable components have the same eigenvalue set while if $i \neq j$ eig $\Delta^{(i)} \cap$ eig $\Delta^{(j)} = \phi$. Since eig $\Delta =$ eig $\Delta^{(1)} \cup \cdots \cup$ eig $\Delta^{(t)}$ each set eig $\Delta^{(i)}$ is unique, hence if $V = W_1 + \cdots + W_t$ is a decomposition corresponding to $\Delta = \Delta^{(1)} + \cdots + \Delta^{(t)}$ the subspaces W_i are unique Thus, if relevant, each sig $\Delta^{(i)}$ is uniquely determined by Δ. By (i), (ii), and Sylvester's theorem the indecomposable components of each $\Delta^{(i)}$ are unique. Q.E.D.

(2 7) *Classification of Types.* We must first describe all the indecomposable types Suppose Δ is indecomposable and $(A, V) \in \Delta$, where $A \in L(V, \sigma, \tau)$. Let $G_\Delta = G(V, \sigma, \tau)$ be the corresponding group G_Δ is determined up to isomorphism by Δ.

By Proposition 2, Δ is uniquely determined by $ht\,\Delta = m$ and by the structure of $\bar{\Delta}$. By Proposition 3 the semisimple type $\bar{\Delta}$ is indecomposable. Note that if $\tau(u, v) = \lambda\tau(v, u)$, where $\lambda = \pm 1$ then the $\bar{\tau}$ corresponding to $\bar{\Delta}$ satisfies $\bar{\tau}(\bar{u}, \bar{v}) = \lambda(-1)^m \bar{\tau}(\bar{v}, \bar{u})$.

The description of all semisimple indecomposable types is an easy calculation. We do this in Appendix 2 and summarize the results there in Table A. This table gives our notation for the possible $\bar{\Delta}$ Note that $\zeta \in \mathbb{C}$ (and $\bar{\zeta}$ is its complex conjugate) and $\epsilon = \pm 1$. We can now write down all indecomposable types Δ. This is done in Table II below. Column 1 lists the ten possible families for G_Δ. Column 2 gives our notation for the indecomposable types: The subscript m gives the height while the other symbols describe the structure of $\bar{\Delta}$ and correspond to the notation of Table A in Appendix 2 If $\Delta = \Delta_m(\zeta, ..)$ then $\dim \Delta = (m + 1) \dim \bar{\Delta}$ and $\dim \bar{\Delta}$ =number of eigenvalues ζ,.. of $\bar{A}$ on $\bar{V}$. Our use of the same notation for types belonging to different families of groups should not cause trouble. The meaning of, $\Delta_m(\zeta, -\zeta)$, for example, should always be clear from context.

In the last column of Table II we give an integer $s \in \{0, \pm 1, \pm 2\}$ which we now define Suppose G_Δ belongs to one of the families $GL(V, \tau_*)$, $O(V, \tau, \sigma_+)$ or $Sp(V, \tau, \sigma_-)$. Let τ_*, τ_+, τ_- (see (1 2) and Appendix 1) be the related bilinear forms and let (n_+, n_-) describe the signature of this form, where $n_\pm$ are non-negative integers giving the unique number of ± 1's in this signature. We do not necessarily have $n_- \leqslant n_+$. Put $s = n_+ - n_-$. It is a simple exercise, using the construction in the proof of Proposition 2, to find (n_+, n_-) and hence s for each Δ.

It is convenient to define $\operatorname{ind} \Delta = n_-$, provided G_Δ belongs to one of the three families mentioned above. With a slight abuse of notation we call ind Δ the *index* of Δ. This definition does not require Δ to be indecomposable. Since $\dim \Delta = (n_+ + n_-)$ for the families $GL(V, \tau_*)$ and $O(V, \tau, \sigma_+)$ and $\dim \Delta = 2(n_+ + n_-)$ for $Sp(V, \tau, \sigma_-)$ the index of Δ is easily found from s and $\dim \Delta$.

For convenience, we put $\delta = (-1)^{m/2}\epsilon$ (for m even).

Now let $G = G(V, \sigma, \tau)$ be fixed and let $A \in L(V, \sigma, \tau)$ and $(A, V) \in \Delta$. In general Δ is not indecomposable so to describe it we must give its indecomposable components. First observe that if $\Delta = \Delta_1 + \Delta_2$ and $V = W_1 + W_2$ is the corresponding decomposition of V then the groups $G(W_i, \sigma, \tau)$ for $i = 1, 2$ below to the same family as G.

Let Δ_1 and Δ_2 denote any types belonging to the same family, i.e., G_{Δ_1} and G_{Δ_2} are in the same family. Suppose $(A_i, W_i) \in \Delta_i$ then construct $W = W_1 \oplus W_2$

TABLE II

$GL(V)$	$\Delta_m(\zeta)$			
$GL(V, \sigma_+)$	$\Delta_m(\zeta, \bar{\zeta})$	$\zeta \neq \bar{\zeta}$		
	$\Delta_m(\zeta)$	$\zeta = \bar{\zeta}$		
$GL(V, \sigma_-)$	$\Delta_m(\zeta, \bar{\zeta})$			
$GL(V, \tau_*)$	$\Delta_m(\zeta, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$		0
	$\Delta_m^\epsilon(\zeta)$	$\zeta = -\bar{\zeta}$	m even	δ
			m odd	0
$O(V, \tau)$	$\Delta_m(\zeta, -\zeta)$	$\zeta \neq 0$		
	$\Delta_m(0)$		m even	
	$\Delta_m(0, 0)$		m odd	
$O(V, \tau, \sigma_+)$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$		0
	$\Delta_m(\zeta, -\zeta)$	$\zeta = \bar{\zeta} \neq 0$		0
	$\Delta_m^\epsilon(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$	m even	2δ
			m odd	0
	$\Delta_m^\epsilon(0)$		m even	δ
	$\Delta_m(0, 0)$		m odd	0
$O(V, \tau, \sigma_-)$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$		
	$\Delta_m^\epsilon(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$		
	$\Delta_m(0, 0)$		m even	
	$\Delta_m^\epsilon(0, 0)$		m odd	
$Sp(V, \tau)$	$\Delta_m(\zeta, -\zeta)$	$\zeta \neq 0$		
	$\Delta_m(0, 0)$		m even	
	$\Delta_m(0)$		m odd	
$Sp(v, \tau, \sigma_+)$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$		
	$\Delta_m(\zeta, -\zeta)$	$\zeta = \bar{\zeta} \neq 0$		
	$\Delta_m^\epsilon(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$		
	$\Delta_m(0, 0)$		m even	
	$\Delta_m^\epsilon(0)$		m odd	
$Sp(V, \tau, \sigma_-)$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$		0
	$\Delta_m^\epsilon(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$	m even	δ
			m odd	0
	$\Delta_m^\epsilon(0, 0)$		m even	δ
	$\Delta_m(0, 0)$		m odd	0

and define A, σ, τ as the obvious direct sum actions of A_i, σ_i, τ_i. Then (A, W) is well defined and its type Δ is uniquely determined by Δ_1 and Δ_2. Clearly $\Delta = \Delta_1 + \Delta_2$ and $\dim \Delta = \dim \Delta_1 + \dim \Delta_2$ If relevant, we also have $\operatorname{ind} \Delta = \operatorname{ind} \Delta_1 + \operatorname{ind} \Delta_2$.

Now if $\Delta = \Delta_1 + \cdots + \Delta_s$ is the decomposition of Δ into its indecomposable components we have

$$\dim \Delta = \dim \Delta_1 + \cdots + \dim \Delta_s$$

and if relevant

$$\operatorname{ind} \Delta = \operatorname{ind} \Delta_1 + \cdots + \operatorname{ind} \Delta_s .$$

Conversely suppose $\Delta_1, \ldots, \Delta_s$ are indecomposable types belonging to the same family and satisfying the above restrictions on dimension and index. Then, using the construction in the previous paragraph, $\Delta_1 + \cdots + \Delta_s$ is a well-defined type which will contain a pair (A, V) for some $A \in L(V, \sigma, \tau)$.

As a consequence of Proposition 1 in (2.1) we thus have a complete description of all orbits of G on its Lie algebra.

(2.8) EXAMPLES. Note that if G belongs to one of the three families for which the index of a type is defined and if $(A, V) \in \Delta$, where $G = G(V, \sigma, \tau)$ and $A \in L(V, \sigma, \tau)$ then $\operatorname{ind} \Delta$ is the integer p given in Table I and described in Appendix 1.

As a first example suppose G is compact, i.e., G is one of $GL(V, \tau_*^{(0)})$, $O(V, \tau, \sigma_+^{(0)})$, or $Sp(V, \tau, \sigma_-^{(0)})$. Thus for any type Δ corresponding to G we have $\operatorname{ind} \Delta = 0$. By inspecting Table II the only indecomposable types of index 0 have height 0 and pure imaginary eigenvalues. For example, if $\dim V = 5$ and $G = O(V, \tau, \sigma_+^{(0)})$ the possible indecomposable types are $\Delta_0^+(\zeta, -\zeta)$ with $\zeta = -\bar{\zeta} \neq 0$ and $\Delta_0^+(0)$. So the possible types for G are

$$\Delta_0^+(\zeta_1, -\zeta_1) + \Delta_0^+(\zeta_2, -\zeta_2) + \Delta_0^+(0),$$
$$\Delta_0^+(\zeta, -\zeta) + 3\,\Delta_0^+(0),$$
$$5\,\Delta_0^+(0),$$

where we let 2Δ denote $\Delta + \Delta$, etc.

A second example, for a noncompact group, is $G = Sp(V, \tau, \sigma_-^{(2)})$ with $\dim V = 8$. Thus if Δ is a type for G we have $\operatorname{ind} \Delta = 2$. We describe all "nilpotent" types, i.e., those containing a pair (A, V) with A nilpotent. The possible indecomposable types with dimension $\leqslant 8$ and index $\leqslant 2$ are found in Table II:

type	dim	ind	type	dim	ind
$\Delta_0^+(0, 0)$	2	0	$\Delta_2^+(0, 0)$	6	2
$\Delta_0^-(0, 0)$	2	1	$\Delta_2^-(0, 0)$	6	1
$\Delta_1(0, 0)$	4	1	$\Delta_3(0, 0)$	8	2 .

Hence there are six possible nilpotent types for G:

$$2\,\Delta_0^+(0, 0) + 2\,\Delta_0^-(0, 0),$$
$$\Delta_0^+(0, 0) + \Delta_0^-(0, 0) + \Delta_1(0, 0),$$
$$2\,\Delta_1(0, 0),$$
$$\Delta_0^+(0, 0) + \Delta_2^+(0, 0),$$
$$\Delta_0^-(0, 0) + \Delta_2^-(0, 0),$$
$$\Delta_3(0, 0).$$

The first type has $A = 0$, the next two have $A^2 = 0$, the next two have $A^3 = 0$ while the last has $A^4 = 0$.

From these examples it should be clear that, using Table II, the types for any particular case are easily described.

Our proofs in (2 6) give a practical algorithm for computing an explicit representative (A, V) for any type Δ. If Δ is indecomposable the structure of $(\bar{A}, \bar{V})$ is given in Appendix 2. Using the construction in the proof of Proposition 2 we can then describe (A, V). If Δ is decomposable use the direct sum construction in (2.7). For example, suppose $G = O(V, \tau, \sigma_-)$ and dim $V = 10$ and $\Delta = \Delta_1^+(0, 0) + \Delta_2^-(\zeta, -\zeta)$, where $\zeta = -\bar{\zeta} \neq 0$. Then $V = W_1 + W_2$, where $W_1 = \langle e_1, f_1, Ne_1, Nf_1\rangle$ and $W_2 = \langle e_2, f_2, Ne_2, N^2e_2, N^2f_2\rangle$ and $A = N$ and $N^2 = 0$ on W_1 while

$$Ae_2 = \zeta e_2 + Ne_2,$$
$$Af_2 = -\zeta f_2 + Nf_2,$$

and $N^3 = 0$ on W_2. We have $\sigma e_1 = f_1$, $\sigma e_2 = f_2$ and $\sigma^2 = -1$. While $\tau(e_1, Nf_1) = +1$ and $\tau(e_2, N^2e_2) = \tau(f_2, N^2f_2) = -1$ and hence $\tau(Ne_2, Ne_2) = \tau(Nf_2, Nf_2) = +1$. Other nonzero values of τ are obtained from these by using $\tau(u, v) = \tau(v, u)$. All other values of τ on the basis elements are zero. Thus A, σ, τ are explicitly described

Using the description of V_σ^- in Appendix 1 it is an elementary exercise to rewrite the above description of A, σ, τ in terms of 5×5 quaternionic matrices. Similarly for any of the σ_+ families descriptions of A, σ, τ as real matrices on V_σ^+ can easily be given.

Finally we mention the converse problem: Given (A, V), determine its type as a sum of (unique) indecomposable types. This is easily solved. First find the eigenvalues of A on V and use them to get $A = S + N$. Find m such that $N^{m+1} = 0$ but $N^m \neq 0$. Use the construction in the proof of Proposition 4 to decompose $V = W_1 + W_2 + \cdots$ such that each (A, W_i) is uniform and the heights are all distinct. For each such pair find $(\bar{A}, \bar{W})$ and use the eigenvalues of $\bar{A}$ and, if relevant, the signature of θ to describe the semisimple indecomposable types occurring in the decomposition of $(\bar{A}, \bar{W})$. By Proposition 2 we now know all the indecomposable types occurring in the decomposition of the type of (A, V).

An example of such a calculation occurs in [13] for the group $Sp(V, \tau, \sigma_+)$. However, the notation and methods are rather more cumbersome than in the present article.

3. Conjugacy Classes in the Groups

We show how the results of Section 2 allow one to describe all conjugacy classes for the groups defined in Section 1.

If $A \in G(V, \sigma, \tau)$ define equivalence of pairs (A, V) exactly as in (2 1). Then Proposition 1 holds with $A \in L(V, \sigma, \tau)$ replaced by $A \in G(V, \sigma, \tau)$.

The definitions of (2.2) and (2.3) go through with only one change: If $A \in G(V, \sigma, \tau)$ and $A = S + N_1$ as in (2.3) put $U = (1 + S^{-1}N_1)$ then U is unipotent, $A = SU = US$ and both S, U lie in $G(V, \sigma, \tau)$ Put $N = S^{-1}N_1$ and note that if τ is involved in $G(V, \sigma, \tau)$ then $\tau(Su, Sv) = \tau(u, v)$ and $\tau(Nu, v) + \tau(u, Nv) + \tau(Nu, Nv) = 0$. The last condition implies that, if $N^{m+1} = 0$, then $\tau(N^m u, v) + (-1)^m \tau(u, N^m V) = 0$ and $\tau(N^j u, N^k v) = 0$ for all $j + k > m$. These two observations allow one to adapt the proofs in (2 5) and (2.6) almost verbatim In particular the calculations in the proof of Proposition 2 all go through. Note however that in the fourth paragraph $S_* \hat{\tau}_j = -\hat{\tau}_j S$ becomes $S_*^{-1} \hat{\tau}_j = \hat{\tau}_j S$. As we will show, the classification of the semisimple indecomposable types is essentially the same. Hence the proof of Proposition 5 goes through unchanged and so the theorem holds. All statements in (2.7) hold in the group case

It only remains to consider the classification of the semisimple indecomposable types We use $\tilde{\Delta}$ to denote types corresponding to groups and Δ to denote types corresponding to their Lie algebras. Although it is easy to go through the calculations of Appendix 2 again and make the necessary changes a quicker approach is to use the Cayley transformation.

For a fixed family of groups let $\mathscr{T}$ denote the set of all semisimple indecomposable types. If $\tilde{\Delta} \in \mathscr{T}$ note that $0 \notin \operatorname{eig} \tilde{\Delta}$. For the corresponding family of Lie algebras let $\mathscr{T}_1$ denote the set of all semisimple indecomposable types Δ satisfying $0, \pm 1 \notin \operatorname{eig} \Delta$ and let $\mathscr{T}_0$ denote the set of semisimple indecomposable types Δ with $0 \in \operatorname{eig} \Delta$ From Appendix 2 we see that $|\mathscr{T}_0| \leqslant 2$.

We now define two mappings, γ_+ and γ_-, of $\mathscr{T}_1 \cup \mathscr{T}_0$ into $\mathscr{T}$ If $\Delta \in \mathscr{T}_1 \cup \mathscr{T}_0$ and $(S, W) \in \Delta$ put

$$\gamma_+ S = (1 - S)(1 + S)^{-1} \qquad \text{and} \qquad \gamma_- S = (S + 1)(S - 1)^{-1}.$$

Then $\gamma_\pm S$ is an element of the group, is semisimple, and $(\gamma_\pm S, W)$ is indecomposable. If $(\gamma_\pm S, W) \in \tilde{\Delta}$ we put $\gamma_\pm \Delta = \tilde{\Delta}$. These two maps are well defined. If $\Delta \in \mathscr{T}_1$ and $(S, W) \in \Delta$ let Δ' denote the type of $(-S^{-1}, W)$. Then $\gamma_+ \Delta = \gamma_- \Delta'$. Hence $\gamma_+ \mathscr{T}_1 = \gamma_- \mathscr{T}_1$. However, $\gamma_+ \mathscr{T}_0$, $\gamma_- \mathscr{T}_0$ and $\gamma_+ \mathscr{T}_1$ are all disjoint.

LEMMA $\tilde{\mathscr{T}} = \gamma_+\mathscr{T}_1 \cup \gamma_+\mathscr{T}_0 \cup \gamma_-\mathscr{T}_0$.

Proof First observe that if $\tilde{\Delta} \in \tilde{\mathscr{T}}$ and $\pm 1 \in \operatorname{eig} \tilde{\Delta}$ then, due to the indecomposability, if $(\tilde{S}, W) \in \tilde{\Delta}$ then $\tilde{S} = \pm I$ on W Thus for any $\tilde{\Delta} \in \tilde{\mathscr{T}}$ if $(\tilde{S}, W) \in \tilde{\Delta}$ we can define either $S = (1 - \tilde{S})(1 + \tilde{S})^{-1}$ or $S = (\tilde{S} + 1)(\tilde{S} - 1)^{-1}$. This S will be an element of the Lie algebra and if $(S, W) \in \Delta$ then $\Delta \in \mathscr{T}_1 \cup \mathscr{T}_0$. Thus we can invert γ_+ and γ_- Q E D

Thus all results for the Lie algebra carry over to the corresponding group. In particular, using the entries in Table II in (2.7), we can immediately write down all the indecomposable types for any family of groups. For example, for $Sp(V, \tau, \sigma_-)$ we would get

$$\tilde{\Delta}_m(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}), \qquad \lambda \neq \bar{\lambda}^{-1},$$

$$\tilde{\Delta}_m^{\pm}(\lambda, \lambda^{-1}), \qquad \lambda = \bar{\lambda}^{-1} \neq \pm 1,$$

$$\left.\begin{matrix} \tilde{\Delta}_m^{\pm}(1, 1) \\ \tilde{\Delta}_m^{\pm}(-1, -1) \end{matrix}\right\} \qquad m = \text{even},$$

$$\left.\begin{matrix} \tilde{\Delta}_m(1, 1) \\ \tilde{\Delta}_m(-1, -1) \end{matrix}\right\} \qquad m = \text{odd}.$$

The explicit structure of these types is found by applying the Cayley transforms γ_+ or γ_- to the corresponding semisimple type for the algebra and then constructing $\tilde{\Delta}_m$ by the methods used in the proof of Proposition 2

As an example, from the calculations in the third paragraph of (2.8) we see that if $\dim V = 8$ then the group $Sp(V, \tau, \sigma^{(2)})$ has six unipotent classes.

4. SPLITTING OVER SUBGROUPS

Let G denote one of the groups defined in (1.1) or (1.2) and let G' denote its commutator subgroup and $Z(G)$ its center. Let L denote its Lie algebra.

In this section we describe how the orbits in L under G split when G is replaced by any subgroup M lying between G' and G. Using the correspondence given in Section 3 we will also see how to describe the splitting of conjugacy classes of G on restriction to M.

(4.1) *Normal Subgroups of G.* Since the orbits in L under M are the same as under $MZ(G)$ we need only consider those M which satisfy $G'Z(G) \subseteq M \subseteq G$. For the structure of G refer to [11]. We collect in Table III the relevant results. Let $n = \dim V$ and suppose $n \geqslant 3$, then column 3 describes the quotient $G/G'Z(G)$ for those families where it is $\neq 1$.

Let G denote one of the groups in Table III If $g \in G$ then $\det g$ is a real number and we put $D(g) = \operatorname{sign}(\det g) = \pm 1$. If $g \in O(V, \tau, \sigma_+^{(p)})$ and $p \neq 0$ let

TABLE III

G	Restrictions		$G/G'Z(G)$
$GL(V, \sigma_+)$		$n =$ even	Z_2
$O(V, \tau)$		$n =$ even	Z_2
$O(V, \tau, \sigma_+^{(p)})$	$p = 0$	$n =$ even	Z_2
	$p \neq 0$	$n =$ odd	Z_2
	$p \neq 0$	$n =$ even, $p =$ odd	Z_2
	$p \neq 0$	$n =$ even, $p =$ even	$Z_2 \times Z_2$

$C(g)$ denote the spinor norm of g, see [11, II Sect. 7]. We may suppose that $C(g) = \pm 1$. Put $E(g) = D(g)\, C(g)$ Note that C, D, E all define homomorphisms of G onto $\{\pm 1\}$. We can now describe the various possibilities for M. If G is one of $GL(V, \sigma_+)$, $O(V, \tau)$, $O(V, \tau, \sigma_+^{(0)})$ or $O(V, \tau, \sigma_+^{(p)})$ with $n =$ even, $p =$ odd then $G'Z(G) = \operatorname{Ker} D$ If $p \neq 0$ and $G = O(V, \tau, \sigma_+^{(p)})$ with $n =$ odd then $G'Z(G) = \operatorname{Ker} C$ if $p =$ odd and $G'Z(G) = \operatorname{Ker} E$ if $p =$ even. Finally if both n and $p \neq 0$ are even $G'Z(G)$ is $\operatorname{Ker} D \cap \operatorname{Ker} C$.

(4.2) *The Centralizer Argument* Suppose G and L are defined on V. If $A \in L$ let Δ denote the type of (A, V). Let $C_G(A) = \{g \in G \mid gA = Ag\}$ and put $M(\Delta) = MC_G(A)$ Since G/M is abelian the subgroup $M(\Delta)$ only depends on Δ.

Now Δ represents a unique orbit in L under G and so we may speak of Δ "splitting" when G is restricted to M. Thus Δ splits if and only if $M(\Delta) \neq G$ and if k is the index of $M(\Delta)$ in G then Δ splits into k distinct types. From Table III, $k = 2$ or 4

Let $g \in C_G(A)$ and write $g = zu$, where z is semisimple, u is unipotent, and $zu = uz$ Then both $z, u \in C_G(A)$ and since u lies in the connected component of the identity of G we have $D(g) = D(z)$ and $C(g) = C(z)$ Thus to determine $M(\Delta)$ we may restrict our attention to the semisimple elements in $C_G(A)$

Let $z \in C_G(A)$ be semisimple and let $\tilde{\Delta}$ denote the type of (z, V). Write $\tilde{\Delta} = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_s$, where each $\tilde{\Delta}_i$ is the sum of all semisimple indecomposables with the same eigenvalue set while for $i \neq j$ $\tilde{\Delta}_i$ and $\tilde{\Delta}_j$ have no eigenvalues in common Let $V = U_1 + \cdots + U_s$ be the corresponding decomposition of V then each (A, U_i) is a well-defined pair and hence gives a decomposition of Δ. Suppose $(A, U_i) \in \Delta_i$ and $\Delta_i = \Delta_i' + \Delta_i'' + \cdots$, where each summand is indecomposable By considering the various possibilities for $\tilde{\Delta}_i$ and the possibilities for the $\Delta_i', \Delta_i'', \cdot$ it is a straightforward calculation to find the values of $D(z)$ and $C(z)$ on U_i. It turns out that all possible values for D, C are exhausted by that $\tilde{\Delta}_i$ on which $z = -1$ In Table IV we summarize the calculations. In column 2 we list only those indecomposable types for L for which a nonidentity contribution to either D or C may occur and in columns 3, 4 we give the contribution.

TABLE IV

G	Types		D	C
$GL(V, \sigma_+)$	$\Delta_m(\zeta)\zeta = \bar{\zeta}$	m = even	-1	
$O(V, \tau)$	$\Delta_m(0)$	m = even	-1	
$O(V, \tau, \sigma_+^{(o)})$	$\Delta_m^+(0)$	m = even	-1	
$O(V, \tau, \sigma_+^{(p)})$	$\Delta_m(\zeta, -\zeta)\zeta = \bar{\zeta}$	m = even	$+1$	-1
$p \neq 0$	$\Delta_m^+(0)$	m = even	-1	$(-1)^{m/2}$
	$\Delta_m^-(0)$	m = even	-1	$-(-1)^{m/2}$

To determine $M(\Delta)$ write Δ as a sum of indecomposable types and then refer to Table IV For example, if $\Delta = \Delta_1 + \quad + \Delta_t$, where all Δ_i are indecomposable and none of the Δ_i occur in Table IV then $M(\Delta) = M$ Now suppose $M = \operatorname{Ker} D$ and from Table IV some Δ_i contributes $D = -1$ (this means that there exists a $z \in C_G(A)$ which will equal -1 on the subspace corresponding to Δ_i and will equal $+1$ on a suitable complement), then $M(\Delta) = G$.

(4.3) EXAMPLES Let $G = GL(V, \sigma_+)$ and n = even and suppose $M = \operatorname{Ker} D$. Note that this is equivalent to considering the orbits in L under the action of $SL(V, \sigma_+) = \{g \in G \mid \det g = +1\}$. Of course, $G \simeq GL(n, \mathbb{R})$, $SL(V, \sigma_+) \simeq SL(n, \mathbb{R})$ and $L \sim$ {set of real $n \times n$ matrices} From Section 2 the orbits in L under the action of G are of the form

$$\Delta = \sum_i \Delta_{m_i}(\zeta_i, \bar{\zeta}_i) + \sum_j \Delta_{n_j}(\xi_j),$$

where $\zeta_i \neq \bar{\zeta}_i$ and $\xi_j = \bar{\xi}_j$. From Table IV we see that Δ splits under M if and only if no $\Delta_{n_j}(\xi_j)$ with n_j = even occurs in Δ If Δ splits there are two orbits under M.

As a second example let $G = O(V, \tau, \sigma_+^{(2)})$ with dim $V = 8$, i.e., $G \simeq O(6,2)$. Using Table II one finds that G has six unipotent classes, namely, (if $\tilde{\Delta}$ denotes $\tilde{\Delta}(1)$, etc)

$$\begin{array}{ll} 6\tilde{\Delta}_0^+ + 2\tilde{\Delta}_0^-, & \tilde{\Delta}_2^- + 5\tilde{\Delta}_0^+, \\ \tilde{\Delta}_2^+ + 4\tilde{\Delta}_0^+ + \tilde{\Delta}_0^-, & \tilde{\Delta}_4^+ + 3\tilde{\Delta}_0^+, \\ 2\tilde{\Delta}_2^+ + 2\tilde{\Delta}_0^+, & \tilde{\Delta}_1 + 4\Delta_0^+. \end{array}$$

The possibilities for M are G, Ker D, Ker C, Ker E, and $G' = \operatorname{Ker} D \cap \operatorname{Ker} C$. Using Table IV the only splitting occurs if $M = \operatorname{Ker} C$ or G' in which case the three classes in the right-hand column each split into two.

5. Other Fields

Let k be a perfect field of characteristic not 2 Let V be a vector space over k and τ a nondegenerate symmetric or alternating bilinear form on V Define the groups $O(V, \tau)$, $Sp(V, \tau)$ as in (1 1).

Note that, by omitting any reference to σ, all definitions in (2.1), (2 2), (2 3) carry over to the present case Since k is perfect the $S + N$ decomposition in (2.3) holds. For a practical construction of S and N, see [14]. Next, note that Propositions 1–4 remain true The proofs are unchanged except that they become considerably shorter since σ is omitted The problem is entirely with Proposition 5 and is, of course, due to the failure of Sylvester's theorem. However, we show that there is a reasonable generalization of the arguments used to prove Proposition 5 and hence of the theorem in (2 2)

Let $p \in k[t]$ be a monic irreducible polynomial over k. Suppose $p = t^{r+1} + c_1 t^r + \cdots + c_r t + c_{r+1}$ thus $\deg p = r + 1$. Define $\hat{p} \in k[t]$ by $\hat{p} = t^{r+1} + d_1 t^r + \cdots + d_{r+1}$, where $d_i = c_{r+1-i}/c_{r+1}$ with $c_0 = 1$. Then $p = \hat{p}$ if and only if the inverse of every root of p is also a root. If $p = \hat{p}$ and $\deg p = 1$ then $p = t \pm 1$, while if $\deg p \geqslant 2$ then $\deg p$ is even and $c_{r+1} = 1$

If p is irreducible and $p = \hat{p}$ define the field $k_p = k[t]/\langle p \rangle$. For $\zeta \in k_p$, let $\zeta \to \hat{\zeta}$ be the automorphism of k_p induced by fixing k and mapping t into t^{-1}. Let $f_p = \{\zeta \in k_p \mid \hat{\zeta} = \zeta\}$ and put $n_p = \{\zeta\hat{\zeta} \mid \zeta \in k_p^*\}$. Thus n_p is a subgroup of the multiplicative group f_p^*. Put $T_p = f_p^*/n_p$.

For example: If k is algebraically closed then $T_p = 1$. If k is a finite field then $T_p \simeq Z_2$ if $\deg p = 1$ and $T_p = 1$ if $\deg p \geqslant 2$. If $k = \mathbb{R}$ then $T_p \simeq Z_2$. In general T_p is not finite.

We now describe all the indecomposable types for $O(V, \tau)$. Let Δ be any indecomposable type of height m. Δ is characterized by an irreducible $p \in k[t]$ and, if $p = \hat{p}$, an element $t \in T_p$ For a suitable group all such p may occur and, if $p = \hat{p}$, all such t may occur. We give, in Table V, a list of all possible indecomposable types for the family of groups containing $O(V, \tau)$. In brackets are the changes required for the group $Sp(V, \tau)$ Our notation is adapted from Table II (as it would appear in the group case, see the example at the end of Sect 3): $p \in k[t]$ is an irreducible polynomial, $t \in T_p$, and distinct p, t give distinct types, but $\Delta(p, \hat{p}) = \Delta(\hat{p}, p)$. The last column gives $\dim \Delta$.

TABLE V

$\Delta_m(p, \hat{p})$	$p \neq \hat{p}$	$2(m + 1) \deg p$
$\Delta_m{}^t(p)$	$p = \hat{p}$, $\deg p \neq 1$	$(m + 1) \deg p$
$\Delta_m(1, 1)$ $\Delta_m(-1, -1)$	$m =$ odd(even)	$2(m + 1)$
$\Delta_m{}^t(1)$ $\Delta_m{}^t(-1)$	$m =$ even(odd)	$(m + 1)$

For $i = 1,\ldots, s$, let $t_i \in T_p$, where p is irreducible and $p = \hat{p}$ The set $\{t_1, \ldots, t_s\}$ defines an equivalence class of s-dimensional Hermitian forms over k_p; namely, let $\zeta_i \in k_p$ be any inverse image of t_i then the form is $(\zeta_1 \hat{x}_1 y_1 + \cdots + \zeta_s \hat{x}_s y_s)$. We may call $\{t_1, \ldots, t_s\}$ the signature of this form. We have

PROPOSITION 6 *Let t_i, $u_i \in T_p$ then*

$$\Delta_m^{t_1}(p) + \cdots + \Delta_m^{t_s}(p) = \Delta_m^{u_1}(p) + \cdots + \Delta_m^{u_s}(p)$$

if and only if the forms with signatures $\{t_1, \ldots, t_s\}$ and $\{u_1, \ldots, u_s\}$ are equivalent

(If $\deg p = 1$ we understand $\Delta_m{}^t\ (t \pm 1)$ to be $\Delta_m{}^t\ (\pm 1)$). As a result of Propositions 2–4 and the arguments given for the proof in (2 4) we see that the above result describes the only way in which nonuniqueness can occur in the decomposition of an arbitrary type into a sum of indecomposables.

The proofs of the statements in the two paragraphs which precede and follow Table V are easy and since they occur as special cases of results in [10, Sect. 3] we omit them

As pointed out in [10] the restriction to perfect fields is not fundamental. It is only necessary to avoid the $S + N$ decomposition

APPENDIX 1: EQUIVALENCE CLASSES FOR σ

We use the notation of Section 1. We first discuss case (i) of (1.2). Then σ and σ' are equivalent if $\sigma' = \alpha k^{-1} \sigma k$ for some $k \in G$ and nonzero $\alpha \in \mathbb{C}$. We may suppose that $\sigma^2 = \pm 1$ and, if τ is involved, that $\tau^\sigma = \bar{\tau}$. Note that σ and $-\sigma$ are always equivalent Let $n = \dim V$.

$\sigma^2 = +1$. Let $V_\sigma{}^+ = \{v \in V \mid \sigma v = v\}$. Then $V_\sigma{}^+$ is a real n-dimensional vector space. If $G = GL(V)$ there is clearly just one equivalence class for σ

Let τ_+ denote the restriction of τ to $V_\sigma{}^+$ It is a nondegenerate real bilinear form.

If τ is alternating then τ_+ is alternating Since all such forms are equivalent we can always choose $k \in G = Sp(V, \tau)$ mapping $V_\sigma{}^+$ onto $V_{\sigma'}^+$. Hence there is a single equivalence class for σ

If τ is symmetric let $(n - p, p)$ denote the signature of τ_+. Replacing σ by $-\sigma$, if necessary, we may suppose that $0 \leqslant p \leqslant (n/2)$ Each p corresponds to a distinct equivalence class for σ.

$\sigma^2 = -1$ Let $\mathbb{H} = \{\alpha + \beta j \mid \alpha, \beta \in \mathbb{C}, j^2 = -1, \alpha j = j\bar{\alpha}\}$ denote the quaternions and let $(\alpha + \beta j)^q = \alpha - j\beta$ be an anti-involution of $\mathbb{H}$.

For $v \in V$ define $(\alpha + \beta j)\, v = \alpha v + \beta \sigma v$. Let $V_\sigma{}^-$ denote the set V considered as a vector space of dimension $n/2$ over $\mathbb{H}$. Again, for $G = GL(V)$, there is just one equivalence class for σ.

Define τ_- as a nondegenerate $\mathbb{H}$-valued form on V_σ^- by $\tau_-(u, v) = \tau(u, v) + \tau(u, \sigma v) j$ Then $\tau_-(\lambda u, \mu u) = \lambda \tau_-(u, v) \mu^q$ for $\lambda, \mu \in \mathbb{H}$

If τ is symmetric then $\tau_-(u, v) = \tau_-(v, u)^q$. In this case all such forms are equivalent. To see this use the usual Gram–Schmidt algorithm to find a basis $\{e_i\}$ of V_σ^- such that $\tau_-(e_i, e_j) = \delta_{ij}$. Hence there is a single equivalence class for σ.

If τ is alternating then $\tau_-(u, v) = -\tau_-(v, u)^q$ A basis can be found so that $\tau_-(e_i, e_j) = \pm\delta_{ij} j$ Let $(n/2 - p, p)$ denote the resulting signature of τ_-. Replacing σ by $-\sigma$, if necessary, we have $0 \leqslant p \leqslant n/4$ Each value of p corresponds to a distinct equivalence class for σ (see [11, I, Sect. 8] for details).

τ_* case. In case (ii) of (1.2) the equivalence of σ and σ' becomes $\sigma' = \alpha k_* \sigma k$, nonzero $\alpha \in \mathbb{C}$, $k \in GL(V)$. We may suppose $\sigma u(v) = \overline{\sigma v(u)}$ all $u, v \in V$. Let $(n - p, p)$ denote the signature of the corresponding Hermitian form $\tau_*(u, v) = \sigma u(v)$. As above, we may suppose that $0 \leqslant p \leqslant n/2$. Distinct values of p correspond to distinct equivalence classes for σ.

APPENDIX 2: The Semisimple Indecomposable Types

Let (S, W) denote a pair belonging to a semisimple and indecomposable type. Since S is semisimple its eigenvectors span W. Let G denote one of the complex groups defined in (1.). We first discuss these groups.

$G = GL(V)$. Let $Se = \zeta e$ some $\zeta \in C$ and non-zero $e \in W$ Clearly $\langle e \rangle = W$. Denote the type of (S, W) by $\Delta(\zeta)$.

$G = O(V, \tau)$ If $S \neq 0$ let $Se = \zeta e$ and $\zeta \neq 0$ and choose $f \in W$ such that f is an eigenvector of S and $\tau(e, f) = 1$. Since $S \in L(W, \sigma, \tau)$ we have $Sf = -\zeta f$ and also $\tau(e, e) = \tau(f, f) = 0$ Thus dim $\langle e, f \rangle = 2$ and τ is nondegenerate on $\langle e, f \rangle$. Hence $W = \langle e, f \rangle$. Denote this type by $\Delta(\zeta, -\zeta)$

If $S = 0$ choose $e \in W$ such that $\tau(e, e) = 1$. Thus $W = \langle e \rangle$ Denote this type by $\Delta(0)$

$G = Sp(V, \tau)$ Let $Se = \zeta e$ and choose f such that $\tau(e, f) = 1$ As above, find $Sf = -\zeta f$, and dim $\langle e, f \rangle = 2$ since τ is alternating. Thus $W = \langle e, f \rangle$. Denote this type by $\Delta(\zeta, -\zeta)$ Context will distinguish it from type $\Delta(\zeta, -\zeta)$ for $O(V, \tau)$.

Now let G_σ denote one of the case (i) real forms of G It is straightforward to discuss these cases one by one, just as for the complex groups, but we prefer to give a more general discussion.

Let Δ denote a semisimple indecomposable type for G_σ. If $(S, W) \in \Delta$ then, by omitting σ, this pair may be considered as belonging to a type for G. Let Δ^c denote this type. It is semisimple and hence is a sum of types, described above. In the following discussion we use a superscript "c" to distinguish types for G from those for G_σ

Suppose Δ_1^c is an indecomposable component of Δ^c. Let $(S, W_1) \in \Delta_1^c$. Since $\tau^\sigma = \bar{\tau}$ the pair $(S, \sigma W_1)$ is well defined. Denote its type by $\sigma \Delta_1^c$. Note that either $\sigma W_1 = W_1$ or $\sigma W_1 \cap W_1 = 0$ Then since $\sigma^2 = \pm 1$ and Δ is indecomposable we have three possible cases:

(a) $\Delta^c = \Delta_1^c + \sigma \Delta_1^c$ and $\Delta_1^c \neq \sigma \Delta_1^c$,

(b) $\Delta^c = \Delta_1^c + \sigma \Delta_1^c$ and $\Delta_1^c = \sigma \Delta_1^c$,

(c) $\Delta^c = \Delta_1^c$ and $\Delta_1^c = \sigma \Delta_1^c$.

Let eig Δ_1^c denote the set of eigenvalues of S on W_1.

LEMMA A 1 *Case* (a) *occurs if and only if* $\overline{\text{eig } \Delta_1^c} \neq \text{eig } \Delta_1^c$.

Proof. Observe that, in the above classification for G, each indecomposable type is characterized by the set of eigenvalues. Q E.D.

LEMMA A.2. *Suppose* $S \neq 0$ *then case* (b) *occurs if and only if* $\sigma^2 = -1$ *and all elements of* eig Δ_1^c *are real.*

Proof By Lemma A.1 we may suppose $\sigma \Delta_1^c = \Delta_1^c$. First let dim $\Delta_1^c = 1$. Let $Se = \zeta e$ then $\bar{\zeta} = \zeta \neq 0$. If $\sigma^2 = +1$ and dim $\langle e, \sigma e\rangle = 2$ put $e_1 = e + \sigma e$. Then $\langle e_1\rangle$ is S-invariant and hence $(S, \langle e_1\rangle) \in \Delta$, a contradiction, since Δ is indecomposable Hence dim $\langle e, \sigma e\rangle = 1$ and so this is in case (c). If $\sigma^2 = -1$ then necessarily dim $\langle e, \sigma e\rangle = 2$ and so we have case (b).

Now let dim $\Delta_1^c = 2$. Suppose $Se = \zeta e$, $Sf = -\zeta f$ and $\tau(e, f) = 1$. If $\sigma^2 = +1$ and $\bar{\zeta} = \zeta$ put $e_1 = e + \sigma e$, $f_1 = f + \sigma f$ while if $\bar{\zeta} = -\zeta$ put $e_1 = e + \gamma\sigma f$, $f_1 = f + \gamma\sigma e$, where $\gamma = 1$ if τ is symmetric and $\gamma = i$ if τ is alternating. If $\sigma^2 = -1$ and $\bar{\zeta} = -\zeta$ put $e_1 = e + i\gamma\sigma f$, $f_1 = f - i\gamma\sigma e$. Then in each case $(S, \langle e_1, f_1\rangle) \in \Delta$ and so this is case (c). If $\sigma^2 = -1$ and $\bar{\zeta} = \zeta$ then dim$\langle e, \sigma e\rangle = 2$ and so the eigenvalue ζ occurs with multiplicity 2 in W. Since $\zeta \neq 0$ this implies that we have case (b). Q.E.D.

It is possible for distinct types Δ, Δ' for G_σ to give the same type $\Delta^c = \Delta'^c$ for G The next result shows that this can only happen in case (c).

LEMMA A 3 *In cases* (a) *and* (b) *if* Δ, Δ' *are types for* G_σ *and* $\Delta^c = \Delta'^c$ *then* $\Delta = \Delta'$.

Proof. With the above notation we have $W = W_1 + \sigma W_1$, a disjoint, orthogonal sum. Suppose $(S', W') \in \Delta'$ and let $W' = W_1' + \sigma' W_1'$ be the corresponding decomposition. We may suppose $(S', W_1') \in \Delta_1^c$.

Let ϕ be any isomorphism of W_1 onto W_1' which gives the equivalence $(S, W_1) \sim (S', W_1')$. Then define the isomorphism $\hat{\phi}$ of W onto W' by putting

$$\hat{\phi}(w_1 + \sigma w_2) = \phi w_1 + \sigma' \phi w_2 \qquad \text{for} \quad w_1, w_2 \in W_1.$$

Then $\hat{\phi}\sigma = \sigma'\hat{\phi}$ since we may suppose $\sigma^2 = \sigma'^2 = \pm 1$. The other conditions for equivalence are immediate. Thus $\Delta = \Delta'$. Q E D.

Now consider case (c) when $S \neq 0$ First suppose $\bar{\zeta} = \zeta$ then by Lemma A 2, $\sigma^2 = +1$. Suppose $\dim \Delta = 2$ then choose e_1, f_1 spanning W_σ (see Appendix 1) and such that $Se_1 = \zeta e_1$, $Sf_1 = -\zeta f_1$. Then $\tau(e_1, e_1) = \tau(f_1, f_1) = 0$ and without restriction we may choose e_1, f_1 to satisfy $\tau(e_1, f_1) = 1$. Now let $(S', W') \in \Delta'$, where $\Delta^c = \Delta'^c$. Then we may choose similar e_i', f_i' in $W_{\sigma'}'$ Put $\phi e_1 = e_1'$, $\phi f_1 = f_1'$, this yields $(S, W) \sim (S', W')$ and hence $\Delta = \Delta'$. The case $\dim \Delta = 1$ is similar.

Next, still in case (c) and $S \neq 0$, suppose $\bar{\zeta} = -\zeta$. Then $\dim \Delta = 2$. If $\sigma^2 = +1$ define the symmetric, nondegenerate and real bilinear form θ_+ on W_σ^+ as follows. If τ is symmetric $\theta_+ = \tau_+$ (see Appendix 1), and if τ is alternating $\theta_+(u, v) = \tau_+(u, Sv)$ for $u, v \in W_\sigma^+$. We can choose e_1, f_1 spanning W_σ^+ such that $Se_1 = i\zeta f_1$, $Sf_1 = -i\zeta e_1$ and from $\tau_+(Su, v) + \tau_+(u, Sv) = 0$ we have $\theta_+(e_1, f_1) = 0$ while $\theta_+(e_1, e_1) = \theta_+(f_1, f_1)$. Without restrictions we can choose e_1, f_1 so that $\theta_+(e_1, e_1) = \pm 1$ Let Δ^+ and Δ^- denote types for G_σ giving the two possible cases $\theta_+(e_1, e_1) = +1$ or -1. Arguing as in the previous paragraph one sees that $\Delta^+ \neq \Delta^-$ while if Δ' is any type for G_σ with $\Delta'^c = (\Delta^\pm)^c$ then $\Delta' = \Delta^+$ or $\Delta' = \Delta^-$

Continuing case (c) with $S \neq 0$, we now suppose $\sigma^2 = -1$. As in Appendix 1 consider W as one-dimensional over $\mathbb{H}$. Define the Skew-Hermitian form θ_- on W_σ^- as follows Put $\theta_- = \tau_-$ if τ is alternating and $\theta_-(u, v) = \tau_-(u, Sv)$ if τ is symmetric. Now choose e_1 so that $Se_1 = \zeta e_1$. From $\tau_-(Se_1, e_1) + \tau_-(e_1, Se_1) = 0$ we may suppose e_1 chosen so that $\theta_-(e_1, e_1) = \pm j$. As in the previous paragraph there are two distinct types Δ^+, Δ^- for G_σ with $(\Delta^+)^c = (\Delta^-)^c$

Finally suppose $S = 0$ Then we have exactly the situation discussed in Appendix 1 For $G = GL(V)$ and $\sigma^2 = +1$ there is one type of case (c) and for $\sigma^2 = -1$ it is case (b) For $G = O(V, \tau)$ if $\sigma^2 = +1$ there are two types Δ^+, Δ^- with $(\Delta^+)^c = (\Delta^-)^c = \Delta(0)$ while if $\sigma^2 = -1$ it is case (b). For $G = Sp(V, \tau)$ if $\sigma^2 = +1$ there is one type of case (c) if $\sigma^2 = -1$ there are two types.

In the unitary case, i.e , case (ii) of (1.2), a direct calculation quickly gives the semi-simple indecomposable types Let $Se = \zeta e$ and suppose $\zeta \neq -\bar{\zeta}$ Choose f an eigenvalue of S such that $\tau_*(e, f) = \sigma e(f) = 1$. Then we have $\tau_*(e, e) = \tau_*(f, f) = 0$ and $Sf = -\bar{\zeta} f$ Hence $\dim \langle e, f\rangle = 2$ and so $W = \langle e, f\rangle$. Denote this type by $\Delta(\zeta, -\bar{\zeta})$. If $\zeta = -\bar{\zeta}$ and $\tau_*(e, e) \neq 0$ renormalize e to get $\tau_*(e, e) = \pm 1$. The two signs give distinct types $\Delta^\pm(\zeta)$. If $\tau_*(e, e) = 0$ choose eigenvector f such that $\tau_*(e, f) = 1$ then a suitable linear combination e_1 of e and f gives $\tau_*(e_1, e_1) \neq 0$ and $Se_1 = \zeta e_1$ so $W = \langle e_1\rangle$, a contradiction.

These results are summarized in the following table. The notation for an indecomposable type Δ for G_σ is the same as for Δ^c in case (c), but with superscripts $\pm$ as defined above. In cases (a) and (b) an obvious concatenation is used, for example, if $\Delta^c = \Delta(\zeta) + \Delta(\bar{\zeta})$ for $G = GL(V)$ we use the notation $\Delta(\zeta, \bar{\zeta})$ for

TABLE A

σ^2	τ	Type	Conditions
		$\Delta(\zeta)$	
$+1$		$\Delta(\zeta, \bar{\zeta})$	$\zeta \neq \bar{\zeta}$
		$\Delta(\zeta)$	$\zeta = \bar{\zeta}$
-1		$\Delta(\zeta, \bar{\zeta})$	
	alt	$\Delta(\zeta, \zeta)$	
$+1$	alt	$\Delta(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$
		$\Delta(\zeta, -\zeta)$	$\zeta = \bar{\zeta}$
		$\Delta^{\pm}(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$
-1	alt	$\Delta(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$
		$\Delta^{\pm}(\zeta, -\zeta)$	$\zeta = -\bar{\zeta}$
	sym	$\Delta(\zeta, -\zeta)$	$\zeta \neq 0$
		$\Delta(0)$	
$+1$	sym	$\Delta(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$
		$\Delta(\zeta, -\zeta)$	$\zeta = \bar{\zeta} \neq 0$
		$\Delta^{\pm}(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$
		$\Delta^{\pm}(0)$	
-1	sym	$\Delta(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$
		$\Delta^{\pm}(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$
		$\Delta(0, 0)$	
	*	$\Delta(\zeta, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$
		$\Delta^{\pm}(\zeta)$	$\zeta = -\bar{\zeta}$

Δ. Note that the ordering of the eigenvalues in $\Delta(\zeta, . .)$ is unimportant, for example, $\Delta(\zeta, \bar{\zeta}) = \Delta(\bar{\zeta}, \zeta)$. Columns 1 and 2 give σ^2 and the symmetry of τ, a blank denotes that σ or τ (or both) not involved in this case.

Since the forms $\theta_{\pm}$ are referred to several times in Section 2 we summarize their definition:

$$\theta_+(u, v) = \tau_+(u, v) \quad \text{if } \tau \text{ is symmetric,}$$
$$\qquad\qquad = \tau_+(u, Sv) \quad \text{if } S \neq 0 \text{ and } \tau \text{ is alternating,}$$

where $u, v \in V_\sigma^+$.

$$\theta_-(u, v) = \tau_-(u, v) \quad \text{if } \tau \text{ is alternating,}$$
$$\qquad\qquad = \tau_-(u, Sv) \quad \text{if } S \neq 0 \text{ and } \tau \text{ is symmetric,}$$

where $u, v \in V_\sigma^-$. Note that $\theta_\pm$ are well defined even if (S, V) is not indecomposable. Thus if $(S, V) \in \Delta$ with Δ semisimple by Sylvester's theorem the signature of θ is uniquely determined by Δ. We denote this signature by sig Δ.

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