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Counting curves on elliptic ruled surface

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1. Introduction

In this paper, we present some calculation of the Gromov–Witten invariants of $S^2 \times T^2$. Since the symplectic Gromov–Witten invariants in fact only depend on the deformation class of symplectic forms and we have shown in [10] that there is a unique deformation class on $S^2 \times T^2$, we merely need to compute the Gromov–Witten invariants for some specific symplectic structure. We will actually pick some Kahler structures in the computation.

Let (M,ω) be a symplectic $S^2 \times T^2$ and $[S^2]$ and $[T^2]$ be the homology classes represented by $S^2 \times pt$ and $pt \times T^2$, respectively, and pair positively with the symplectic form ω. Denote the homology class $l[S^2]+d[T^2]$ by $A_{l,d}$ and we simply write $A_{1,d}$ as A_d. Our first result is about the embedded genus one curves of the sequence of classes $A_{l,1}$. More precisely, let us define $N_1(A_{l,1})$ as the number of embedded genus 1 curves in the class $A_{1,d}$ and passing through $2l$ points.

Theorem 1. *The Gromov–Witten invariants* $N_1(A_{l,1})=2$.

The second result is about the general nodal curves of the sequence of classes A_d. Denote $N_g(A_d)$ (or $n_g(A_d)$) the number of genus g curves in class A_d, passing through $g+1$ points (or passing through g points and intersecting two circles which generate the first integral homology). It is more illuminating to assemble them into generating functions. To that end, we recall that the quasimodular form G_2 is defined by

$$G_2 = -\frac{1}{24} + \sum_{k=1}^{\infty} \sigma_k q_k$$

where $\sigma_k = \sum_{d|k} d$ is the partition function.

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Theorem 2. *The Gromov–Witten invariants $N_g(A_d)$ and $n_g(A_d)$ are given by the generating functions,*

$$\sum_{d=0}^{\infty} N_g(A_d)q^d = (g+1)\big(DG_2(q)\big)^g,$$
$$\sum_{d=0}^{\infty} n_g(A_d)q^d = \big(DG_2(q)\big)^g,$$

where D is the differential operator $q\frac{d}{dq}$.

Theorem 1 is proved by viewing M as a nontrivial holomorphic S^2 bundle (see also [13]). Actually $N_1(A_{1,d})$ are just the Gromov invariants appeared in [18] (in [6], it was shown that the Gromov–Taubes invariants can be obtained from the Ruan–Tian invariants [15]). Therefore Theorem 1 follows immediately from Taubes's celebrated theorem 'SW $=$ Gr' [19] and our wall crossing formula of the Seiberg–Witten invariants [9]. However, we think the direct counting presented here is still interesting. Theorem 2 is proved by viewing M as a trivial product holomorphic bundle, similar to the approach in [2].

These invariants are enumerative. Thus it is natural to compare with Göttsche's beautiful conjectural functions of the number of curves on algebraic surfaces [4,8].

The organization of this note is as follows. In Section 2 we define the Gromov–Witten invariants of symplectic four manifolds. In Section 3 we prove Theorems 1 and 2.

2. Stable maps and Gromov–Witten invariants

Let M be a closed symplectic four manifold with the symplectic form ω and a compatible almost complex structure J. A (g,k) prestable map is a tuple $(f, \Sigma, x_1, \ldots, x_k)$, where $\Sigma = \bigcup \Sigma_i$ is a connected projective curve of genus g with at worst ordinary double points as singularities, x_i are distinct smooth points on Σ, f is a continuous map from Σ to M, and pseudo-holomorphic on each Σ_i. A prestable map is in class $A \in H_2(M;\mathbb{Z})$ if $f_*[\Sigma] = A$.

f is called stable if the automorphism group is finite. Two stable maps are equivalent if there is a biholomorphism $\sigma : \Sigma \to \Sigma'$ such that $\sigma(x_i) = x_i'$ and $f' = f \circ \sigma$.

We denote the equivalence classes of (g,k) stable maps in class A by $\mathcal{M}_{g,k}(A, M, \omega, J)$. There is an evaluation map

$$ev : \mathcal{M}_{g,k}(A, M, \omega, J) \to M^k,$$
$$(f, \Sigma, x_1, \ldots, x_k) \to (f(x_1), \ldots, f(x_k)),$$

which is crucial to the definition of the Gromov–Witten invariants. Li and Tian [12] (see also [3,14,16]) construct a virtual fundamental cycle $[\mathcal{M}_{g,k}(A, M, \omega, J)]^{vir}$ (in fact since four manifolds are positive, for generic J, Ruan and Tian construct actual fundamental cycles in [15]) which have real dimension

$$d_{g,k}(A) = -2K_\omega \cdot A + 2k - 2(1-g),$$

where K_ω is the symplectic canonical class.

The Gromov–Witten invariants are defined by pulling back cohomology classes on M^k via the evaluation map *ev*. More precisely, given q circles $\gamma_1, \ldots, \gamma_q$ in M with Poincaré duals $\hat{\gamma}_1, \ldots, \hat{\gamma}_q$, define the Gromov–Witten invariant

$$\Psi_{g,k}(A, \gamma_1, \ldots, \gamma_q) = \int_{[\mathcal{M}_{g,k}(A,M,\omega,J)]^{vir}} ev^*\big([M]^p \times \hat{\gamma}_1 \times \cdots \times \hat{\gamma}_q\big),$$

where $[M]$ is the fundamental cohomology class and $p = k - q$. Given p points $y_1, \ldots, y_p$, $\Psi_{g,k}(A, \gamma_1, \ldots, \gamma_q)$ counts the number of (g,k)-stable maps in class A such that $f(x_i) = y_i,\ 1 \leqslant i \leqslant p$, and $f(x_{p+j}) \in \Gamma_j,\ 1 \leqslant j \leqslant q$.

Three kinds of Gromov–Witten invariants are of particular interest. Given A and g, introduce

$$k_{g,A} = -K_\omega \cdot A - (1-g),$$
$$l_{g,A} = k_{g,A} - b_1(M)/2.$$

Define

$$N_g(A) = \Psi_{g,k_{g,A}}(A),$$
$$n_g(A) = \Psi_{g,l_{g,A}}(A, \gamma_1, \ldots, \gamma_{b_1}),$$

where $\gamma_1, \ldots, \gamma_{b_1}$ is an integral basis of $H_1(M;\mathbb{Z})/Tor$. In case both of them are enumerative, $N_g(A)$ counts the number of genus g curves in class A and passing through $k_{g,A}$ number of points, while $n_g(A)$, by a result in [2], counts the number of genus g curves in a fixed linear system and passing through $l_{g,A}$ number of points.

To introduce the third interesting invariant $N_{g(A)}(A)$, we need to define $g(A)$ the genus of A,

$$g(A) = \frac{K_\omega \cdot A + A \cdot A}{2} + 1.$$

By the adjunction formula [13], it is the maximal genus of any curve representing A, and any curve with such genus must be embedded. It is shown in [6] that $N_{g(A)}(A)$ is the same as Taubes's Gromov invariant in [18].

We want to remark that Gromov–Witten invariants actually count the number of maps instead of just the image curves as in traditional algebraic geometry. Though these two ways of counting often coincide as is the case in the present paper, the Gromov–Witten invariants do differ from the enumerative invariants sometimes and tend to be easier to calculate. They have richer structures like the composition law [15] and the fascinating Virasoro constraints.

3. Enumeration of curves

In this section we will prove Theorems 1 and 2. For $M = S^2 \times T^2$ with a symplectic form ω, $[S^2]$ and $[T^2]$ are the two positive homology classes representing the factors. Denote the Poincaré dual of $[T^2]$ still by $[T^2]$, the symplectic canonical class is then simply

$$K_\omega = -2\big[T^2\big].$$

Therefore easy computations show that

$$\begin{aligned} g(A_{l,d}) &= (d-1)l+1, \\ k_{d(l-1)+1,A_{l,d}} &= (d+1)l, \\ k_{g,A_d} &= g+1. \end{aligned}$$

In broader terms, our strategy of proving Theorems 1 and 2 is similar. To calculate the Gromov–Witten invariants, we have the freedom to choose any compatible almost complex structure and any configuration of points and circles on the symplectic four manifold. We will choose some very special integrable complex structure (and some natural Kahler form which we will not make explicit) such that M is holomorphically fibered by rational curves. And we pick the points and the circles carefully so that the domain curves of the maps are forced to be degenerate and simple and thus make the final enumeration a fairly easy task.

We start with Theorem 1.

Proof of Theorem 1. Consider the complex structure coming from the projectivization of $L \oplus \mathcal{O}$ over an elliptic curve T, where $\mathcal{O}$ is the trivial holomorphic line bundle and L a nontrivial holomorphic line bundle with degree zero. The two line bundles L and $\mathcal{O}$ give rise to two disjoint sections of the projective bundle, which all represent the class $[T]$ since the two line bundles are topologically trivial. Since there are no other line subbundles, these two tori T_1 and T_2 are the only two curves in the class $[T^2]$. By the adjunction formula, there are no multiplicity one curves in the classes $d[T^2]$ for any d greater than one. So any connected curve representing $[T^2]$ or its multiples has either T_1 or T_2 as its image.

Now let us consider the classes $A_{l,1}$. Since $g_{l,1} = 1$ and $k_{1,A_{l,1}} = 2l$, we will count genus one curves with $2l$ marked points mapping to $2l$ specified points $y_1, \ldots, y_{2l}$ on M. We have freedom to pick where these points y_i are and we will make a convenient choice to make the calculations simpler. First we pick l distinct fibres $S_1, \ldots, S_l$, then we take 2 distinct points on each fiber.

Let $(f, \Sigma, x_1, \ldots, x_{l(d+1)})$ be such a stable map in class $A_{l,1}$. The first observation is that with the selection of y_i as above all the chosen fibers $S_1, \ldots, S_l$ have to be contained in $f(\Sigma)$. If a fiber is not in the image, then it intersects $f(\Sigma)$ at one point since $[S^2] \cdot A_{l,1} = 1$ and each point of intersection contributes positively. But each fiber S_i intersects the image in at least two points y_{2i-1}, y_{2i}, so it has to be part of the image. Clearly, the domain Σ must include l disjoint rational curves $\Sigma_1, \ldots, \Sigma_l$, each of which has 2 marked points, and is embedded with images S_i.

The other components of Σ have no marked points. Since they represent $[T^2]$ under f, there can be only one component which we denote by C, and as argued above $f(C)$ has to be either T_1 or T_2. Since Σ is of genus one, C must be of genus one as well, and must intersect each rational component exactly once. C is mapped isomorphically to either T_1 or T_2, so the number of stable maps is exactly two and the proof of Theorem 1 is complete. □

Now we count curves with arbitrary number of nodes in classes A_d.

Proof of Theorem 2. Fix an integrable product complex structure $P^1 \times T$ where T is an elliptic curve. We will consider it as an elliptic surface.

First we compute $N_g(A_d)$. We pick $k_{g,A_d} = g+1$ points $y_1, \ldots, y_{g+1}$, such that no two of them lie in the same fiber or in the same section.

Let $(f, \Sigma, x_1, \ldots, x_{g+1})$ be a $(g, g+1)$ stable map in class A_d and $f(x_i) = y_i$, $1 \leqslant i \leqslant g+1$. Since the projection from M to P^1 is holomorphic and there is no degree one holomorphic map from a smooth connected positive genus projective curve to a rational curve, we conclude that the image of f consists of a single section curve and a number of fiber curves. Since the points y_i are in different fibers and in different sections pairwisely, it is easy to see that the image of f has to consist of exactly g fibres and a section, each containing one of the y_i. Since the domain Σ has arithmetic genus g, this is possible only if Σ has $g+1$ components, g of which have genus one and one is a rational curve, and each component contains a marked point. Furthermore, f restricted to each genus one component f_i is a covering of a fibre and the rational component is mapped isomorphically to the section.

Let us first assume that y_{g+1} lies in the section curve. Label the genus one components Σ_i, $1 \leqslant i \leqslant g$ such that x_i is contained in Σ_i. Suppose the covering degree of f on Σ_i is α_i, then $\sum_{i=1}^{g} \alpha_i = d$. Given a fixed elliptic curve, it is well known that there are precisely $\sigma_k = \sum_{m|k} m$ number of elliptic curves which admit degree d covering to the given curve. Since the marked point x_i can be any of the preimage of $f_i^{-1}(y_i)$, there are $\prod_{i=1}^{g} \alpha_i \sigma(\alpha_i)$ of $(g, g+1)$ stable maps in class A_d for which the image of the rational component contains y_{g+1} and f is a degree α_i covering of the genus one component Σ_i for $i = 1, \ldots, g$.

Denote the g tuple $(\alpha_1, \ldots, \alpha_g)$ by $\bar{d}$ and write $|\bar{\alpha}|$ for $\sum_i \alpha_i$. Taking into account that any of the y_i can lie in the section curve, it is evident that the total number of $(g, g+1)$ stable maps in class A_d and sending x_i to y_i is given by

$$(g+1) \sum_{\bar{d},\, |\bar{\alpha}|=d} \prod_{i=1}^{g} \alpha_i \sigma(\alpha_i).$$

By resummation, we obtain the final formula

$$\begin{aligned}
\sum_{d=0}^{\infty} N_g(A_d) q^d &= \sum_{d=0}^{\infty} q^d (g+1) \sum_{\bar{d},\, |\bar{\alpha}|=d} \prod_{i=1}^{g} \alpha_i \sigma(\alpha_i) \\
&= (g+1) \left(\sum_{\alpha=1}^{\infty} \alpha \sigma(\alpha) q^{\alpha} \right)^g = (g+1)\big(DG_2(q)\big)^g.
\end{aligned}$$

To prove the second formula, consider two oriented loops of the form $\gamma_1 = s_1 \times S^1 \times u_1$ and $\gamma_2 = s_2 \times t_2 \times S^1$, and $l_{g,A_d} = g$ points $y_1, \ldots, y_g$ such that

(1) s_1 and s_2 are different points on P^1.
(2) y_i does not lie in the section $S \times t_2 \times u_1$ or the fibers $s_1 \times T$ and $s_2 \times T$ for $i = 1, \ldots, g$.
(3) no two of the y_i lie in the same fiber or in the same section.

As in the preceding discussion, the image of a stable map in class A_d has to consist of a section and some fibers. We require that the image intersect the g points $y_1, \dots, y_g$ and the two loops γ_1 and γ_2. A simple observation is that there is no fiber intersecting both γ_1 and γ_2 and the only section intersecting both γ_1 and γ_2 is $S \times t_2 \times u_1$. From this observation and the properties 1, 2 and 3, we conclude that the number of fibers in the image is no less than g, and the number is g only when the section is $S \times t_2 \times u_1$. This forces the domain to have at least $g + 1$ components, one of which is rational. Since the arithmetic genus of the domain is g, there are precisely g genus one components. The distribution of the $g + 2$ marked points is easy to determine: one on each genus one component, two on the rational component.

An argument identical to the one before gives

$$n_g(A_d)q^d = \sum_{\bar{d},\, |\bar{\alpha}|=d} \prod_{i=1}^{g} \alpha_i \sigma(\alpha_i),$$

and the final formula

$$\sum_{d=0}^{\infty} n_g(A_d)q^d = \bigl(DG_2(q)\bigr)^g.$$

The proof of Theorem 2 is complete. □

We can show that our invariants are actually enumerative. Göttsche [4] made a very appealing conjecture about the generating function of the numbers of nodal curves in sufficiently ample linear systems on algebraic surfaces (now proved in [8]). The linear systems in our paper are not ample. Nevertheless, we think it is still interesting to compare our generating function with Göttsche's conjectural function.

In Göttsche's notation, $m_g(d + 1, -2)$ is the number of the nodal genus g curves in a fixed linear system of the class A_d passing through g points. Then

$$n_g(A_d) = m_g(d + 1, -2).$$

Gottsche's conjectural function reads

$$\sum_{d=0}^{\infty} m_g(d + 1, -2)q^d = B_2(q)^{-2}\bigl(D^2 G_2(q)\bigr)\bigl(DG_2(q)\bigr)^g,$$

where B_2 is an explicit power series.

Our generating function differs by a factor $B_2(q)^{-2} D^2 G_2(q)$. It is interesting to observe the generating function of $E(1)$ in [1] differs from Göttsche's conjectural generating function by a similar factor $(B_2(q)^{-2} D^2 G_2(q))^{1/2}$. In fact, for the elliptic surfaces $E(n)$, using the parametrized Gromov–Witten invariants (see [1,2,7,11]) with a ball of real dimension $2p_g$ as the base, it should be possible to show that the genus g generating function of the classes A_d is given by

$$q^{n/2} \Delta(q)^{-n/2}\bigl(DG_2(q)\bigr)^g,$$

where $\Delta(q) = q\prod_{m=1}^{\infty}(1 - q^m)^{24}$ is a modular form of weight 24. Again, the difference from Göttsche's formula is a similar factor $(B_2(q)^{-2} D^2 G_2(q))^{n/2}$.

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