Existence of Solutions in Weighted Sobolev Spaces for Some Degenerate Semilinear Elliptic Equations

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Abstract—In this paper, we study existence of solutions to a class of semilinear degenerate elliptic equations in weighted Sobolev spaces. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we prove the existence of solutions in $W^{1,2}_0(\Omega, \omega)$ for the semilinear Dirichlet problem

$$Lu(x) - \mu u(x) g_1(x) + h(u(x)) g_2(x) = f(x), \quad \text{in } \Omega,$$

$$u(x) = 0, \quad \text{in } \partial \Omega,$$

where $L$ is an elliptic operator in divergence form

$$Lu(x) = - \sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u(x)), \quad \text{with } D_j = \frac{\partial}{\partial x_j},$$

where the coefficients $a_{ij}$ are measurable, real-valued functions whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda|\xi|^2 \omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda|\xi|^2 \omega(x),$$

for all $\xi \in \mathbb{R}^n$ and almost everywhere $x \in \Omega \subset \mathbb{R}^n$ a bounded open set, $\omega$ is a weight function (that is, locally integrable and nonnegative function on $\mathbb{R}^n$), $\mu \in \mathbb{R}$, $\lambda$ and $\Lambda$ are positive constants.

The following theorem will be proved in Section 3.
THEOREM 1. Suppose that
(H1) the function $h: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded ($|h(t)| \leq M$, for all $t \in \mathbb{R}$);
(H2) $\omega \in A_2$;
(H3) $g_1/\omega \in L^\infty(\Omega)$, $g_2/\omega \in L^2(\Omega, \omega)$, and $f/\omega \in L^2(\Omega, \omega)$;
(H4) $\mu > 0$ is not an eigenvalue of the linearized problem

$$Lu(x) - \mu u(x) \omega(x) = 0, \quad \text{in } \Omega,$$
$$u(x) = 0, \quad \text{in } \partial \Omega.$$  \hfill (LP)

Then problem (P) has a solution $u \in W^{1,2}_0(\Omega, \omega)$.

SIMPLE EXAMPLE. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : |x| < 1 \text{ and } |y| < 1\}$ and $0 < a < b$. By Theorem 1, with $h(t) = e^{2-2t}$, $f(x,y) = x|y|$, $\omega(x,y) = (x^2 + y^2)^{-1/2}$, $g_1(x,y) = (x^2 + y^2)^{-1/2} \cos(xy)$, $g_2(x,y) = (x^2 + y^2)^{-1/2} \sin(xy)$, and

$$A = \begin{pmatrix} a (x^2 + y^2)^{-1/2} & 0 \\ 0 & b (x^2 + y^2)^{-1/2} \end{pmatrix},$$

the problem

$$Lu(x,y) - \mu u(x,y) g_1(x,y) + h(u(x,y)) g_2(x,y) = x|y|, \quad \text{in } \Omega,$$
$$u(x,y) = 0, \quad \text{in } \partial \Omega,$$

where

$$Lu(x,y) = -\frac{\partial}{\partial x} \left( a (x^2 + y^2)^{-1/2} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( b (x^2 + y^2)^{-1/2} \frac{\partial u}{\partial y} \right)$$

has a solution $u \in W^{1,2}_0(\Omega, \omega)$ if $\mu > 0$ is not an eigenvalue of linearized problem (LP).

2. DEFINITIONS AND BASIC RESULTS

Let $\omega$ be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that $\omega$ belongs to the Muckenhoupt class $A_p$, $1 < p < \infty$, or that $\omega$ is an $A_p$-weight, if there is a constant $C_1 = C_{p,\omega}$ such that

$$\left( \frac{1}{|B|} \int_B \omega(x) \, dx \right) \left( \frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) \, dx \right)^{p-1} \leq C_1,$$

for all balls $B$ in $\mathbb{R}^n$, where $|.|$ denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$ then $A_q \subseteq A_p$ (see [1] or [2] for more information about $A_p$-weights).

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\omega$ be a weight function. We shall denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the Banach space of all measurable functions, $f$, defined in $\Omega$ for which

$$\|f\|_{L^p(\Omega, \omega)} = \left( \int_{\Omega} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.$$

For $p \geq 1$ and $k$ a nonnegative integer, the Weighted Sobolev space $W^{k,p}(\Omega, \omega)$ is defined by

$$W^{k,p}(\Omega, \omega) = \{ u \in L^p(\Omega, \omega) : D^\alpha u \in L^p(\Omega, \omega), 1 \leq |\alpha| \leq k \},$$

with norm

$$\|u\|_{W^{k,p}(\Omega, \omega)} = \left( \int_{\Omega} |u(x)|^p \omega(x) \, dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) \, dx \right)^{1/p}. \quad (2.1)$$

If $\omega \in A_p$ then $W^{k,p}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Proposition 3.5 in [3, p.416]), and the space $W^{k,p}_0(\Omega, \omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega, \omega)} = \left( \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) \, dx \right)^{1/p}.$$

When $k = 1$ and $p = 2$ the spaces $W^{1,2}(\Omega, \omega)$ and $W^{1,2}_0(\Omega, \omega)$ are Hilbert spaces.
LEMMA 2. If $\omega \in A_2$ then $W^{1,2}_0(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega)$ is compact and

$$\|u\|_{L^2(\Omega, \omega)} \leq C_2 \|u\|_{W^{1,2}_0(\Omega, \omega)}.$$

PROOF. The proof of this lemma follows the line of Theorem 4.6 in [4, p.548].

DEFINITION 3. We say that an element $u \in W^{1,2}_0(\Omega, \omega)$ is weak solution of problem (P) if

$$\int_{\Omega} \left( a_{ij}(x) D_i u(x) D_j \varphi(x) - \mu u(x) \varphi(x) g_1(x) \right) \, dx + \int_{\Omega} h(u(x)) \varphi(x) g_2(x) \, dx = \int_{\Omega} f(x) \varphi(x) \, dx,$$

for every $\varphi \in W^{1,2}_0(\Omega, \omega)$.

3. PROOF OF THEOREM 1

The basic idea is to reduce (P) to an operator equation $Bu + Nu = T$ and apply the following theorem.

THEOREM A. Let $B, N : X \to X^*$ be operators on the real separable reflexive Banach space $X$. Assume

(a) the operator $B : X \to X^*$ is linear and continuous;
(b) the operator $N : X \to X^*$ is demicontinuous and bounded;
(c) $B + N$ is asymptotically linear;
(d) for each $T \in X^*$ and for each $t \in [0, 1]$, the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S) in $X$.

If $Bu = 0$ implies $u = 0$, then for each $T \in X^*$, the equation $Bu + Nu = T$ has a solution in $X$.


REMARK 4. Let $B, N : X \to X^*$ be operators on the real separable reflexive Banach space $X$.

(i) The operator $N$ is said to be demicontinuous if

$$u_n \to u \text{ implies } Nu_n \rightharpoonup Nu, \quad \text{as } n \to \infty.$$

(ii) The operator $N$ is said to be strongly continuous if

$$u_n \to u \text{ implies } Nu_n \to Nu, \quad \text{as } n \to \infty.$$

(iii) $B + N$ is asymptotically linear if $B$ is linear and

$$\frac{\|Nu\|}{\|u\|} \to 0, \quad \text{as } \|u\| \to \infty.$$

(iv) The operator $B$ satisfies condition (S) if

$$u_n \to u \quad \text{and} \quad \lim_{n \to \infty} (Bu_n - Bu \mid u_n - u) = 0, \quad \text{implies } u_n \to u,$$

where $(f \mid x)$ denotes the value of linear functional $f$ at the point $x$.

We define the operators $B_1, B_2 : W^{1,2}_0(\Omega, \omega) \times W^{1,2}_0(\Omega, \omega) \to \mathbb{R}$ through

$$B_1(u, \varphi) = \int_{\Omega} a_{ij}(x) D_i u(x) D_j \varphi(x) \, dx - \mu \int_{\Omega} u(x) \varphi(x) g_1(x) \, dx,$$

$$B_2(u, \varphi) = \int_{\Omega} h(u(x)) \varphi(x) g_2(x) \, dx,$$
and $T : W^{1,2}_0(\Omega, \omega) \to \mathbb{R}$ through

$$T(\varphi) = \int_{\Omega} f(x) \varphi(x) \, dx.$$ 

We have that $u \in W^{1,2}_0(\Omega, \omega)$ is a solution of problem (P) if

$$B_1(u, \varphi) + B_2(u, \varphi) = T(\varphi), \quad \text{for every } \varphi \in W^{1,2}_0(\Omega, \omega).$$

**STEP 1.** By condition (1.2), we obtain $|a_{ij}(x)| \leq C_3 \omega(x)$. Using Lemma 2 and Hölder inequality, we get

$$|B_1(u, \varphi)| \leq \int_{\Omega} |a_{ij}(x)||D_i u(x)||D_j \varphi(x)| \, dx + |\mu| \int_{\Omega} |u(x)||\varphi(x)||g_1(x)| \, dx$$

$$\leq C_3 \int_{\Omega} \omega(x)||D_i u(x)||D_j \varphi(x)| \, dx + |\mu| \int_{\Omega} |u(x)||\varphi(x)||g_1(x)||\omega(x)| \, dx$$

$$\leq C_3 \left(\int_{\Omega} |D_i u(x)|^2 \omega(x) \, dx\right)^{1/2} \left(\int_{\Omega} |D_j \varphi(x)|^2 \omega(x) \, dx\right)^{1/2}$$

$$+ |\mu| \frac{\|g_1\|_{L^\infty(\Omega)}}{\omega} \left(\int_{\Omega} |u(x)|^2 \omega(x) \, dx\right)^{1/2} \left(\int_{\Omega} |\varphi(x)|^2 \omega(x) \, dx\right)^{1/2}$$

$$\leq \left(C_3 + C_2 |\mu| \frac{\|g_1\|_{L^\infty(\Omega)}}{\omega}\right) \|u\|_{W^{1,2}_0(\Omega, \omega)} \|\varphi\|_{W^{1,2}_0(\Omega, \omega)}.$$ 

By Condition (H1) and Lemma 2, we obtain

$$|B_2(u, \varphi)| \leq \int_{\Omega} |h(u(x))||\varphi(x)||g_2(x)| \, dx$$

$$\leq M \int_{\Omega} |\varphi(x)||g_2(x)||\omega(x)| \, dx$$

$$\leq M \|\varphi\|_{L^2(\Omega, \omega)} \left\|\frac{g_2}{\omega}\right\|_{L^2(\Omega, \omega)}$$

$$\leq MC_2 \left\|\frac{g_2}{\omega}\right\|_{L^2(\Omega, \omega)} \|\varphi\|_{W^{1,2}_0(\Omega, \omega)}.$$ 

Moreover, we also have

$$|T(\varphi)| \leq \int_{\Omega} |f(x)||\varphi(x)| \, dx$$

$$= \int_{\Omega} \left(\frac{|f(x)|}{\omega(x)}\right) |\omega(x)|^{1/2} |\varphi(x)||\omega(x)|^{1/2} \, dx$$

$$\leq \left\|\frac{f}{\omega}\right\|_{L^2(\Omega, \omega)} \|\varphi\|_{L^2(\Omega, \omega)}$$

$$\leq C_2 \left\|\frac{f}{\omega}\right\|_{L^2(\Omega, \omega)} \|\varphi\|_{W^{1,2}_0(\Omega, \omega)}.$$ 

**STEP 2.** Since $W^{1,2}_0(\Omega, \omega)$ is a real Hilbert space, using the Identification Principle (Theorem 21.18 in [7, p. 254]) we set $W^{1,2}_0(\Omega, \omega) = [W^{1,2}_0(\Omega, \omega)]^*$ and $(u, v) = (u|v)$ (where $\langle.,.\rangle$ denotes the inner product on a Hilbert space).

We define the operators $B, N : W^{1,2}_0(\Omega, \omega) \to W^{1,2}_0(\Omega, \omega)$ through

$$(Bu | \varphi) = B_1(u, \varphi),$$

$$(Nu | \varphi) = B_2(u, \varphi),$$

for every $u, \varphi \in W^{1,2}_0(\Omega, \omega).$
Since $T \in [W_0^{1,2}(\Omega, \omega)]^*$, problem (P) is equivalent to operator equation

$$Bu + Nu = T, \quad u \in W_0^{1,2}(\Omega, \omega).$$

STEP 3. Using that $W_0^{1,2}(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega)$ is compact (see Lemma 2), we have that $B_1(\cdot, \cdot)$ is a regular Gårding form. In fact, by condition (1.2) we obtain

$$B_1(u, u) = \int_\Omega a_{ij}(x) D_i u(x) D_j u(x) \, dx - \mu \int_\Omega u^2(x) g_1(x) \, dx$$

$$\geq \lambda \int_\Omega |Du(x)|^2 \omega(x) \, dx - \mu \int_\Omega u^2(x) \frac{g_1(x)}{\omega(x)} \omega(x) \, dx$$

$$\geq \lambda \|u\|_{W_0^{1,2}(\Omega, \omega)}^2 - \mu \|\frac{g_1}{\omega}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega, \omega)}^2.$$

Hence, there exists a decomposition of the form $B = T_1 + T_2$, where $T_1$ and $T_2$ are bilinear and bounded, $T_1(\cdot, \cdot)$ is strongly positive and $T_2(\cdot, \cdot)$ is compact (Lemma 22.38 in [7, p. 364]). Thus, $B$ is Fredholm of index zero (see Definition 8.13 in [8, p. 365] and Theorem 21.F in [7, p. 275]) and $B$ satisfies condition (S) (see Proposition 27.12 in [6, p. 595]).

STEP 4. By (3.1), we have

$$|(Nu, \varphi)| \leq |B_2(u, \varphi)| \leq C_2 M \left\|\frac{g_2}{\omega}\right\|_{L^2(\Omega, \omega)} \left\|\frac{\varphi}{g_1}\right\|_{W_0^{1,2}(\Omega, \omega)}.$$

Hence, $\|Nu\| \leq C$, for all $u \in W_0^{1,2}(\Omega, \omega)$. Therefore,

$$\frac{\|Nu\|}{\|u\|} \rightarrow 0, \quad \text{as} \quad \|u\|_{W_0^{1,2}(\Omega, \omega)} \rightarrow \infty,$$

i.e., $B + N$ is asymptotically linear and the operator $N$ is strongly continuous (see Corollary 26.14 in [6, p. 572]).

STEP 5. For each $t \in [0, 1]$ the operator $A_t(u) = Bu + t(Nu - T)$ is a strongly continuous perturbation of the operator $B$. Hence, the operator $A_t$ satisfies condition (S) (Proposition 27.12 in [6, p. 595]).

If $\mu$ is not an eigenvalue of the linearized problem (LP), $Bu = 0$ implies $u = 0$. Therefore, by Theorem A, the operator equation $Bu + Nu = T$ has a solution $u \in W_0^{1,2}(\Omega, \omega)$, and $u$ is solution of problem (P).

REFERENCES