Journal of Symbolic Computation 45 (2010) 734–756

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Algebraic geometry codes from polyhedral divisors

Nathan Owen Ilten [a], Hendrik Süß [b,1]

[a] *Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany*

[b] *Mathematisches Institut, Brandenburgische Technische Universität Cottbus, PF 10 13 44, 03013 Cottbus, Germany*

ARTICLE INFO

Article history:
Received 5 November 2008
Accepted 6 July 2009
Available online 25 March 2010

Keywords:
AG codes
Evaluation codes
Toric varieties

ABSTRACT

A description of complete normal varieties with lower-dimensional torus action has been given by Altmann et al. (2008), generalizing the theory of toric varieties. Considering the case where the acting torus T has codimension one, we describe T-invariant Weil and Cartier divisors and provide formulae for calculating global sections, intersection numbers, and Euler characteristics. As an application, we use divisors on these so-called T-varieties to define new evaluation codes called T-codes. We find estimates on their minimum distance using intersection theory. This generalizes the theory of toric codes and combines it with AG codes on curves. As the simplest application of our general techniques we look at codes on ruled surfaces coming from decomposable vector bundles. Already this construction gives codes that are better than the related product code. Further examples show that we can improve these codes by constructing more sophisticated T-varieties. These results suggest looking further for good codes on T-varieties.

1. Introduction

An important class of linear codes is the class of algebraic geometry Codes, introduced by Goppa (1981). These codes arise by evaluating global sections of a line bundle on a curve over $\mathbb{F}_q$ at a number of $\mathbb{F}_q$-rational points; good estimates on the dimension and minimum distance of such codes can be obtained by using the theorem of Riemann and Roch. Such codes have been generalized to higher-dimensional varieties. It is however often difficult to obtain non-trivial estimates on the parameters of

E-mail addresses: nilten@cs.uchicago.edu (N.O. Ilten), suess@math.tu-cottbus.de (H. Süß).
URL: http://people.cs.uchicago.edu/~nilten/ (N.O. Ilten).
[1] Tel.: +49 355693043; fax: +49 355693042.

0747-7171/$ – see front matter
doi:10.1016/j.jsc.2010.03.008

such codes. One class of varieties where non-trivial estimates have been made is that of toric varieties, which one can describe combinatorially.

Toric varieties have been generalized in Altmann and Hausen (2006) and Altmann et al. (2008) to so-called T-varieties, which are normal varieties X admitting an effective m-dimensional torus action. For $m = \dim X$ we are in the case of toric varieties, but in general m is supposed to be smaller than the dimension of X. T-varieties can then be described by a variety Y of dimension $\dim X - m$ along with combinatorial data called a divisorial fan. If the acting torus has codimension one, Y is then a curve. The aim of this paper is to analyze certain evaluation codes on such varieties; we shall call these codes T-codes.

In short, a T-code over $\mathbb{F}_q$ is constructed from:

- a curve Y over $\mathbb{F}_q$;
- a so-called *divisorial polytope* (cf. Definition 15), essentially a concave function $h^* : \Box_h \to \mathrm{Div}_{\mathbb{Q}} Y$ where $\Box_h$ is a polytope with vertices in some lattice $M \cong \mathbb{Z}^m$ and h^* satisfies some additional conditions;
- and a set $\mathcal{P} = \{P_1, \dots, P_l\}$ of $\mathbb{F}_q$-rational points on Y.

Assuming that the support of $h^*(u)$ is disjoint from $\mathcal{P}$ for each $u \in \Box_h \cap M$, we can define the T-code $\mathcal{C}(Y, h^*, \mathcal{P})$ as the sum of a number of product codes:

$$\mathcal{C}(Y, h^*, \mathcal{P}) := \sum_{u \in \Box_h \cap M} \mathcal{C}_u \otimes \mathcal{C}(Y, h^*(u), \mathcal{P})$$

where $\mathcal{C}_u$ is the $[(q-1)^m, 1, (q-1)^m]$ code generated by $(t^u)_{t \in (\mathbb{F}_q^*)^m}$ and $\mathcal{C}(Y, h^*(u), \mathcal{P})$ is the AG code corresponding to the curve Y, divisor $h^*(u)$, and point set $\mathcal{P}$. By interpreting $\mathcal{C}(Y, h^*, \mathcal{P})$ as the image under a linear map of the Riemann–Roch space of a divisor on a T-variety, we are able to give non-trivial estimates for the dimension k and minimum distance d of this code.

We begin in Section 2 by recalling the basic theory of T-varieties. We then proceed to describe divisors and intersection theory on T-varieties in Section 3. In particular, we describe all T-invariant Cartier and Weil divisors combinatorially, calculate the global sections of a T-invariant Cartier divisor, and determine exactly when a T-Cartier divisor is (semi-)ample. Furthermore, we provide formulae for calculating intersection numbers and for the Euler characteristic of a line bundle. The theory of this section is analogous to that of divisors on toric varieties and is essential for estimating the parameters of the evaluation codes we construct.

In Section 4, we define T-codes and show how to estimate the dimension and minimum distance, providing upper and lower bounds for both parameters. We give special attention to the case of two-dimensional T-varieties, where we provide a better lower bound for the minimum distance.

Finally, we provide a number of examples in Section 5. We first consider T-codes coming from those ruled surfaces corresponding to a rank-two decomposable vector bundle. In particular, we show that some of these codes have better parameters than those estimated for the product of a Reed–Solomon code and a one-point Goppa code. In a second example, we show how one can use the Hasse–Weil bound to improve the lower bound on the minimum distance. This example also shows that there are better T-codes than those coming from ruled surfaces. In a final example, we describe a T-code over $\mathbb{F}_7$ whose parameters are as good as any known linear code.

2. The theory of *T*-varieties

First we recall some facts and notations from convex geometry. Here, N always is a lattice and $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual. The associated $\mathbb{Q}$-vector spaces $N \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}$ are denoted by $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ respectively. Let $\sigma \subset N_{\mathbb{Q}}$ be a pointed convex polyhedral cone. A polyhedron Δ which can be written as a Minkowski sum $\Delta = \pi + \sigma$ of σ and a compact polyhedron π is said to have σ as its tail cone.

With respect to Minkowski addition the polyhedra with tail cone σ form a semigroup which we denote by $\mathrm{Pol}_\sigma^+(N)$. Note that $\sigma \in \mathrm{Pol}_\sigma^+(N)$ is the neutral element of this semigroup and that $\emptyset$ is by definition also an element of $\mathrm{Pol}_\sigma^+(N)$.

A polyhedral divisor with tail cone σ on a normal variety Y is a formal finite sum

$$\mathcal{D} = \sum_D \Delta_D \otimes D,$$

where D runs over all prime divisors on Y and $\Delta_D \in \mathrm{Pol}_\sigma^+$. Here, finite means that only finitely many coefficients differ from the tail cone.

We may evaluate a polyhedral divisor for every element $u \in \sigma^\vee \cap M$ via

$$\mathcal{D}(u) := \sum_D \min_{v \in \Delta_D} \langle u, v \rangle D$$

in order to obtain an ordinary divisor on $\operatorname{Loc}\mathcal{D}$. Here, $\operatorname{Loc}\mathcal{D} := Y \setminus \left(\bigcup_{\Delta_D=\emptyset} D\right)$ denotes the locus of $\mathcal{D}$.

Definition 1. A polyhedral divisor $\mathcal{D}$ is called *Cartier* if every evaluation $\mathcal{D}(u)$, $u \in \sigma^\vee \cap M$, is Cartier.

To a Cartier polyhedral divisor we associate an M-graded k-algebra sheaf and consequently an affine scheme over $\operatorname{Loc}\mathcal{D}$ admitting a T^M-action:

$$\tilde{X} := \tilde{\mathrm{X}}(\mathcal{D}) := \mathbf{Spec}_{\operatorname{Loc}\mathcal{D}} \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u)).$$

From Altmann and Hausen (2006), we know that this construction gives a normal variety of dimension $\dim N + \dim Y$ admitting a torus action of T^N with $\operatorname{Loc}\mathcal{D}$ as its good quotient.

Moreover, for every affine normal variety X there exists a polyhedral divisor $\mathcal{D}$ such that $X = \operatorname{Spec}\Gamma(\tilde{\mathrm{X}}(\mathcal{D}), \mathcal{O}_{\tilde{\mathrm{X}}(\mathcal{D})})$. X and $\tilde{X}$ coincide if $\operatorname{Loc}\mathcal{D}$ is affine. In this case $\operatorname{Loc}\mathcal{D}$ equals the categorical quotient of $\tilde{X} = X$.

Definition 2. Let $\mathcal{D} = \sum_D \Delta_D \otimes D$, $\mathcal{D}' = \sum_D \Delta'_D \otimes D$ be two polyhedral divisors on Y.

(1) We write $\mathcal{D}' \subset \mathcal{D}$ if $\Delta'_D \subset \Delta_D$ holds for every prime divisor D.
(2) We define the intersection of polyhedral divisors

$$\mathcal{D} \cap \mathcal{D}' := \sum_D (\Delta'_D \cap \Delta_D) \otimes D.$$

(3) We define the degree of a polyhedral divisor

$$\deg \mathcal{D} := \sum_D \Delta_D.$$

(4) For a (not necessarily closed) point $y \in Y$ we define the fibre polyhedron $\Delta_y := \mathcal{D}_y := \sum_{y \in D} \Delta_D$.
(5) We call $\mathcal{D}'$ a *face* of $\mathcal{D}$ and write $\mathcal{D}' \prec \mathcal{D}$ if $\mathcal{D}'_y$ is a face of $\mathcal{D}_y$ for every $y \in Y$.

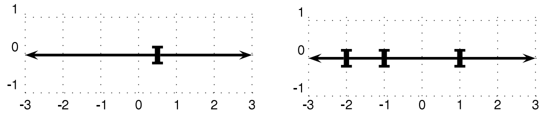
Assume that $\mathcal{D}' \subset \mathcal{D}$. This implies that

$$\bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}'(u)) \hookleftarrow \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u))$$

and we get a dominant morphism $\tilde{\mathrm{X}}(\mathcal{D}') \to \tilde{\mathrm{X}}(\mathcal{D})$.

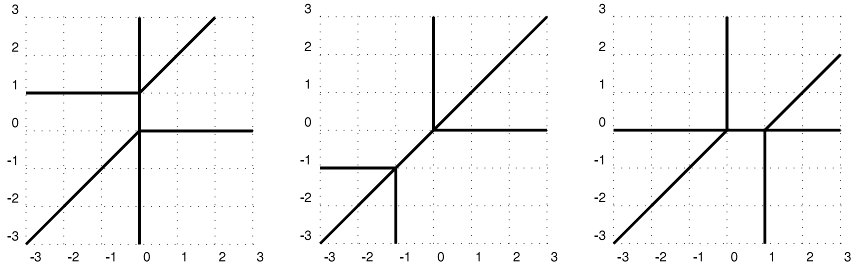
Proposition 3 (*Altmann et al. (2008), Proposition 3.4, Remark 3.5*)**.** *This morphism defines an open embedding if and only if $\mathcal{D}' \prec \mathcal{D}$ holds.*

Now we define the global analogue of a polyhedral divisor. The step from the affine to the complete case is reflected by the replacement of the polyhedra by complete polyhedral subdivisions. For every polyhedron in such a subdivision we get a corresponding tail cone. We will refer to the set of all tail cones as the tail fan of the subdivision.



(a) Ξ_{Q_1}. (b) Ξ_{Q_2}.

Fig. 1. The fansy divisor of a surface.



(a) Ξ_0. (b) Ξ_∞. (c) Ξ_1.

Fig. 2. The fansy divisor of a threefold.

Definition 4. Consider a smooth projective curve Y. A *fansy divisor* is a formal finite sum

$$\Xi = \sum_{P \in Y} \Xi_P \otimes Z$$

such that:

(1) Ξ_P are polyhedral subdivisions covering $N_{\mathbb{Q}}$ and sharing a common tail fan;
(2) Finite means here that for all but finitely many points, Ξ_P equals the tail fan.

Consider a finite set of polyhedral divisors $\mathcal{S}$, such that $\mathcal{D} \succ \mathcal{D}' \cap \mathcal{D} \prec \mathcal{D}'$ for every pair $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$. Assume furthermore that their polyhedral coefficients $\mathcal{D}_P$ form the subdivisions Ξ_P of a fansy divisor.

From such a set we may construct a scheme $\tilde{\mathrm{X}}(\Xi)$ by gluing $\mathrm{X}(\mathcal{D})$s via

$$\tilde{\mathrm{X}}(\mathcal{D}) \leftarrow \tilde{\mathrm{X}}(\mathcal{D} \cap \mathcal{D}') \rightarrow \tilde{\mathrm{X}}(\mathcal{D}').$$

Note that we had to check the cocycle condition; this is done in Altmann et al. (2008, Theorem 5.3). From Theorem 7.5 ibid. we know that we get a complete variety this way.

This variety is uniquely determined by the underlying fansy divisor. Different sets $\mathcal{S}$ correspond to different open coverings. Therefore, we may denote the resulting variety by $\tilde{\mathrm{X}}(\Xi)$.

Theorem 5.6 in Altmann et al. (2008) tell us that for every normal T-variety X with $\dim X = \dim T + 1$ we may find a fansy divisor Ξ and a proper birational map $\tilde{\mathrm{X}}(\Xi) \rightarrow X$. If X has categorical quotient of the expected dimension this morphism turns out to be the identity.

Remark 5. For a fansy divisor Ξ and an open covering $\{U_i\}_{i \in I}$ of Y we can find a set $\mathcal{S}$ as above, such that for every $\mathcal{D} \in \mathcal{S}$ there is a $i \in I$ such that $\mathrm{Loc}\, \mathcal{D} = U_i$.

Example 6. Let Y be a smooth projective curve and $Q_1, Q_2 \in Y$ two points. We consider the fansy divisor Ξ given by the coefficients in Fig. 1. $\tilde{\mathrm{X}}(\Xi)$ is a complete surface with one-dimensional torus action.

Example 7. We consider the fansy divisor on $\mathbb{P}^1$ given by the coefficients in Fig. 2. $\tilde{\mathrm{X}}(\Xi)$ is a complete (singular) threefold with two-dimensional torus action.

3. Divisors and intersection theory on *T*-varieties

From now on we shall only consider torus actions of codimension one; we will study them via fansy divisors.

3.1. Cartier divisors

Let $\Sigma \subset N_{\mathbb{Q}}$ be a complete polyhedral subdivision of N consisting of tailed polyhedra. We consider continuous functions $h : |\Sigma| \to \mathbb{Q}$ which are affine on every polyhedron in Σ. Let $\Delta \in \Sigma$ be a polyhedron with tail cone δ. Then h induces a linear function h_0^Δ on $\delta = \operatorname{tail}\Delta$ by defining $h_0^\Delta(v) := h(P+v) - h(P)$ for some $P \in \Delta$. We call h_0^Δ the linear part of $h|_\Delta$.

Definition 8. An (integral) *support function* on a polyhedral subdivision Σ is a piecewise affine function as above with integer slope and integer translation. To be precise: for $v \in |\Sigma|$ and $k \in \mathbb{N}$ such that kv is a lattice point we have $kh(v) \in \mathbb{Z}$. The group of support functions on Σ is denoted by SF_Σ.

Let Ξ be a fansy divisor on Y. We consider $\mathrm{SF}(\Xi)$, the group of formal sums $\sum_{P\in Y} h_P P$ with the following conditions.

(1) $h_P \in \mathrm{SF}_{\Xi_P}$ a support function of the P-slice of Ξ.
(2) all h_P have the same linear part h_0.
(3) h_P differs from h_0 for only finitely many points $P \in Y$.
We refer to this fact by calling this sum finite and we omit those summands which equal h_0.

Definition 9. A support function $h \in \mathrm{SF}(\Xi)$ is called principal if $h(v) = \langle u, v\rangle + D$, with $u \in M$ and D is a principal divisor on Y. By $h(v) = \langle u, v\rangle + D$ we mean that $h_P(v) = \langle u, v\rangle + a_P$, where $D = \sum_P a_P P$.

If $h = \sum h_P P \in \mathrm{SF}(\Xi)$ we consider a covering $\{Y_i\}$ of Y such that P is a principal divisor on the Y_i for every $P \in Y$ with $h_P \neq h_0$, and such that every Y_i contains at most one of these points.

We may find a set $\mathcal{S}$ as above which is compatible with this covering and induces Ξ. Now we choose a $\mathcal{D} \in \mathcal{S}$ with $\operatorname{Loc}\mathcal{D} = Y_i$ and $h_P \neq h_0$. h_P is an affine function on every polyhedron in Ξ_P so we get $-h_P|_{\mathcal{D}_P}(v) = \langle v, u\rangle + a$ for some $u \in M$ and $a \in \mathbb{Z}$. Assume that $\operatorname{div}(f) = aP$ on Y_i; then $f \cdot \chi^u \in K(\tilde{\mathrm{X}}(\mathcal{D}))^T$ defines a T-invariant principal divisor $H_{\mathcal{D}}$ on $\tilde{\mathrm{X}}(\mathcal{D})$. These principal divisors fit together to a Cartier divisor D_h on $\tilde{\mathrm{X}}(\Xi)$. Here $K(\tilde{\mathrm{X}}(\mathcal{D}))^T := \bigoplus_{u\in M} K(Y)\cdot\chi^u \supset \Gamma(\tilde{\mathrm{X}}(\mathcal{D}))$ denotes the ring of invariant rational functions on $\tilde{\mathrm{X}}(\mathcal{D})$. In this way the group of integral support functions on Ξ corresponds to that of invariant Cartier divisors on $\tilde{\mathrm{X}}(\Xi)$.

3.2. Weil divisors

In general there are two types of T-invariant prime divisors, namely

(1) those which consist of orbit closures of dimension $\dim T$; and
(2) those which consist of orbit closures of dimension $\dim T - 1$.

Proposition 10. *If $\mathcal{D}$ is a polyhedral divisor on a curve with tail cone σ, there are one-to-one correspondences*

(1) *between prime divisors of type 1 and pairs (P, v) with P a point on Y and v a vertex of Δ_P; and*
(2) *between prime divisors of type 2 and rays ρ of σ with* $\deg\mathcal{D} \cap \rho = \emptyset$.

Proof. Consider the quotient map $\pi : \tilde{\mathrm{X}} \to \operatorname{Loc}\mathcal{D}$. In Altmann and Hausen (2006) the orbit structure of the fibres of π is described. Thus, we know that faces $F \prec \mathcal{D}_y$ correspond to T-invariant subvarieties of codimension $\dim(F)$ in $\pi_y := \pi^{-1}(y)$. The correspondences follow by using this for closed points and the generic point, respectively. □

Remark 11. We may also describe the ideals of prime divisors in terms of polyhedral divisors:

(1) For prime divisors of type 1 corresponding to a vertex (P, v), the ideal is given by

$$I_{P,v} = \bigoplus_{u\in\sigma^\vee} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) \cap \{f \mid \operatorname{ord}_P(f) > \langle v, u\rangle\}.$$

(2) For prime divisors of type 2, the corresponding ideal is generated by all multidegrees which are not orthogonal to ρ:

$$I_\rho = \bigoplus_{u\in\sigma^\vee\setminus\rho^\perp} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))).$$

Proposition 12. *Let $h = \sum_P h_P$ correspond to the Cartier divisor D_h on $\tilde{X}(\mathcal{D})$. The corresponding Weil divisor is given by*

$$-\sum_\rho h_0(n_\rho)\rho - \sum_{(P,v)} \mu(v)h_P(v)(P, v),$$

where $\mu(v)$ is the smallest integer $k \geq 1$ such that $k \cdot v$ is a lattice point. This lattice point is a multiple of the primitive lattice vector n_v: $\mu(v)v = \varepsilon(v)n_v$.

Proof. This is a local statement, so we will pass to a sufficiently small invariant open affine set which meets a particular prime divisor. If we translate this to our combinatorial language and we consider a prime divisor corresponding to (P, v) or ρ then we have to choose a polyhedral divisor $\mathcal{D}' \prec \mathcal{D} \in \mathcal{S}$ such that v is also a vertex of $\mathcal{D}'_P$ or ρ is a ray in tail $\mathcal{D}'$, respectively.

So we restrict ourselves to the following two (affine) cases:

(1) $\mathcal{D}$ is a polyhedral divisor with tail cone $\sigma = 0$ and a single point $\Delta_P = \{v\} \subset N$ as the only nontrivial coefficient. Moreover, Y is affine and factorial. In particular, P is a prime divisor with (local) parameter t_P.
(2) $\mathcal{D}$ is the trivial polyhedral divisor with one-dimensional tail cone ρ over an affine locus Y.

In the first case we may choose $\mathbb{Z}$-basis $e_1, \ldots, e_m$ of N with $e_1 = n_v$. Consider the dual basis $e_1^*, \ldots, e_m^*$. By definition $\varepsilon(v)$ and $\mu(v)$ are coprime so we will find $a, b \in \mathbb{Z}$ such that $a\mu(v)+b\varepsilon(v) = 1$. In this situation $y := t_P^a\chi^{be_1^*}$ is irreducible in

$$\Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_Y)[y, t_P^{\pm\varepsilon(v)}\chi^{\mp\mu(v)e_1^*}, \chi^{\pm e_2^*}, \ldots, \chi^{\pm e_m^*}]$$

and defines the prime divisor (P, v). We consider an element $t_P^\alpha\chi^u$ with $u = \sum_i \lambda_i e_i^*$. The y-order of $t_P^\alpha\chi^u$ is

$$\varepsilon(v)\lambda_1 + \mu(v)\alpha = \mu(v)(\langle u, v\rangle + \alpha),$$

because $t_P^\alpha\chi^u = y^{\varepsilon(v)\lambda_1+\mu(v)\alpha}(t_P^{-\varepsilon(v)}\chi^{\mu(v)e_1^*})^{\lambda_1 a+b\alpha}$, and $(t_P^{-\varepsilon(v)}\chi^{\mu(v)e_1^*})$ is a unit.

In the second case we choose a $\mathbb{Z}$-basis $e_1, \ldots, e_m$ of N with $e_1 = n_\rho$. We once again consider the dual basis $e_1^*, \ldots, e_m^*$. In this situation

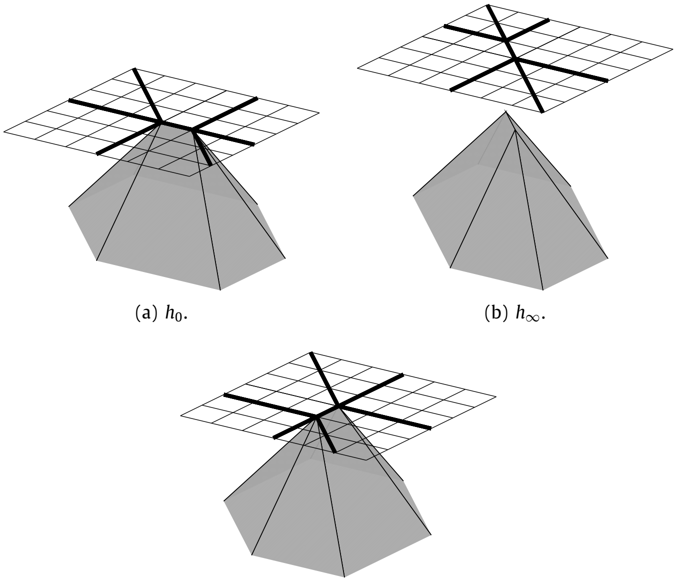
$$\Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_Y)[\chi^{e_1^*}, \chi^{\pm e_2^*}, \ldots, \chi^{\pm e_m^*}].$$

Now $(\chi^{e_1^*})$ defines the prime divisor ρ on X. For a principal divisor $f \cdot \chi^u$, the $\chi^{e_1^*}$-order equals the e_1^*-component of u; i.e., $\langle u, n_\rho\rangle$. □

Example 13. For our threefold example we consider D_h where h_0, h_∞, h_1 are given by the tropical polynomials

$$h_0 = 0\odot x^{(-1,0)}\oplus 0\odot x^{(-1,1)}\oplus 0\odot x^{(0,1)}\oplus 0\odot x^{(1,0)}\oplus 1\odot x^{(1,-1)}\oplus 1\odot x^{(0,-1)}$$
$$h_\infty = (-2)\odot x^{(-1,0)}\oplus(-2)\odot x^{(-1,1)}\oplus(-1)\odot x^{(0,1)}\oplus(-1)\odot x^{(1,0)}\oplus(-2)\odot x^{(1,-1)}\oplus(-2)\odot x^{(0,-1)}$$
$$h_1 = 1\odot x^{(-1,0)}\oplus 1\odot x^{(-1,1)}\oplus 0\odot x^{(0,1)}\oplus 0\odot x^{(1,0)}\oplus 0\odot x^{(1,-1)}\oplus 0\odot x^{(0,-1)}$$

where we are using the tropical semi-ring with operations $\oplus = \min$, $\odot = +$. These support functions are pictured in Fig. 3. The Weil divisor corresponding to D_h is $\sum_\rho D_\rho + 2D_{(\infty,0)} + 2D_{(\infty,(-1,-1))}$. This is the anti-canonical divisor of $X := \tilde{X}(\Xi)$ (Petersen and Süß, 2008).



(a) h_0. (b) h_∞.

(c) h_1.

Fig. 3. Support functions for a T-threefold.

3.3. Global sections

For a support function h on X we may consider the M-graded vector space of global sections of D_h

$$L(D_h) = \bigoplus_{u \in M} L(D_h)_u := \Gamma(X, \mathcal{O}(D_h)).$$

The *weight set* of $L(D_h)$ is defined as the set $\{u \in M \mid L(D)_u \neq 0\}$. For a Cartier divisor given by $h \in \text{T-CaDiv}(\Xi)$ we will bound its weight set by a polyhedron as well as describe the graded module structure of $L(D)$.

Consider a support function $h = \sum_P h_P P$ with linear part h_0. We define its associated polytope

$$\Box_h := \Box_{h_0} := \{u \in M_{\mathbb{Q}} \mid \langle u, v \rangle \geq h_0(v) \; \forall_{v \in N}\}$$

and associate a dual function $h^* : \Box_h \to \operatorname{Div}_{\mathbb{Q}} Y$ via

$$h^*(u) := \sum_P h_P^*(u)P := \sum_P \min_{\text{vert}}(u - h_P)P,$$

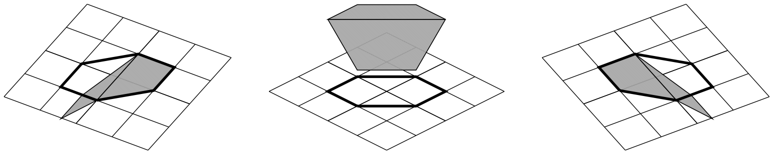
where $\min_{\text{vert}}(u - h_P)$ denotes the minimal value of $u - h_P$ on the vertices of Ξ_P.

Remark 14. Let h be a concave support function. Every affine piece of h_P corresponds to a pair $(u, -a_u) \subset M \times \mathbb{Z}$. h_P^* is defined to be the coarsest concave piecewise affine function with $h_P^*(u) = a_u$.

We can reformulate this in terms of the tropical semi-ring with operation $\oplus = \min, \odot = +$. We might think of the h_P as given by tropical polynomials $\bigoplus_{w \in I}(-a_w) \odot x^w$; then $\Box_h = \operatorname{conv}(I)$ and $h_P^*(w) = a_w$, i.e., $\Gamma_{h_P^*}$ is the reflected lower Newton boundary of the tropical polynomial for h_P.

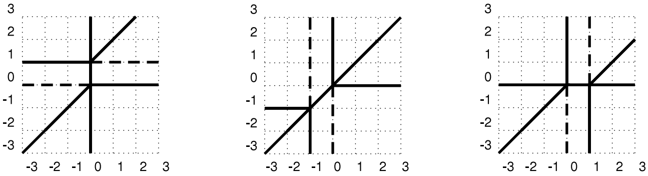
Definition 15. A *divisorial polytope* h^* is a pair consisting of an ordinary polytope $\Box_h \subset M_{\mathbb{Q}}$ and a concave piecewise affine function $h^* : \Box_h \to \operatorname{Div}_{\mathbb{Q}} Y$ such that

(1) $\deg h^*(u) \geq 0$ for all vertices u of $\Box_h$, and
(2) some multiple of $h^*(u)$ is principal in the case of $\deg h^*(u) = 0$ for a vertex u.
(3) $\Box_h$ is a lattice polytope as is $\operatorname{conv}(\Gamma_{h_P^*})$ for each $P \in Y$.



(a) h_0^*. (b) h_∞^*. (c) h_1^*.

Fig. 4. h^* for a T-threefold.



(a) Ξ'_0. (b) Ξ'_∞. (c) Ξ'_1.

Fig. 5. A refined polyhedral divisor.

Let $\Box_g, \Box_h \in M_{\mathbb{Q}}$ be polytopes. For any concave piecewise affine functions $g^* : \Box_g \to \mathrm{Div}_{\mathbb{Q}} Y$ and $h^* : \Box_h \to \mathrm{Div}_{\mathbb{Q}} Y$ we define their sum $g^* + h^*$ to be the piecewise affine concave function on $\Box_g + \Box_h$ given by

$$(g_P^* + h_P^*)(u) = \max\{h_P^*(w) + g_P^*(w') \mid u = w + w'\}.$$

Remark 16. For $g, h \in SF(\Xi)$, one easily checks that

$$\Box_g + \Box_h \subset \Box_{g+h}$$

and that

$$g_P^*(u) + h_P^*(u) \leq (g + h)_P^*(u)$$

for all $P \in Y$ and all $u \in \Box_g + \Box_h$. Furthermore, if h_P and g_P are convex, they correspond to tropical polynomials f, f'. It follows then that $(g + h)_P$ corresponds to $f \odot f'$. Its reflected lower Newton boundary is exactly the graph of $(g + h)_P^*$; thus the equality

$$(g + h)_P^* = g_P^* + h_P^*$$

holds.

To a divisorial polytope h^* we might associate a fansy divisor Ξ and support function h on Ξ such that h^* corresponds to h in the way given above. Indeed, to every h_P^* we can associate a tropical polynomial $f := \bigoplus_{(u,a_u)}(-a_u)\odot x^u$, where (u, a_u) runs over the vertices of $\Gamma_{(h_P^*)}$. This polynomial induces via evaluation a piecewise affine function and a polyhedral subdivision Ξ_P of N.

Remark 17. If we remove condition 3 from the definition of a divisorial polytope (Definition 15), the association in the above paragraph gives us a $\mathbb{Q}$-Cartier divisor.

For every fansy divisor there exists a smooth refinement, i.e. a fansy divisor Ξ' such that every Ξ'_P is a refinement of Ξ_P and $\tilde{X}(\Xi')$ is smooth (Süß, 2008). Every support function h on Ξ is obviously also a support function on Ξ'. Thus, for a given divisorial polytope h^* we might always consider a smooth fansy divisor Ξ and a support function h on it such that the associated dual function equals h^*.

Example 18. We now revisit our threefold example. Fig. 4 shows a sketch of h^*. We show a refinement of the fansy divisor in Fig. 5 which gives a smooth threefold.

Proposition 19. *Let $h \in SF(\Xi)$ be a Cartier divisor with linear part h_0. Then*

(1) *the weight set of $L(D_h)$ is a subset of $\Box_h$; and*
(2) *for $u \in \Box_h$ we have*

$$L(D_h)_u = \Gamma(Y, \mathcal{O}(h^*(u))).$$

Proof. By the definition of $\mathcal{O}(D_h)$ we have

$$\Gamma(X, \mathcal{O}(D_h))^T = \left\{ \chi^u f \mid \operatorname{div}(\chi^u f) - \sum_\rho h_0(n_\rho)\rho - \sum_{(P,v)} \mu(v) h_P(v)(P, v) \geq 0 \right\}.$$

But $\operatorname{div}(\chi^u f) = \sum_\rho \langle u, n_\rho\rangle \rho + \sum_{(P,v)} \mu(v)(\langle u, v\rangle + \operatorname{ord}_P(f))(P, v)$, so for $\chi^u f \in L(h)$ we get the following bounds:

(1) $\langle u, n_\rho\rangle \geq h_0(n_\rho)\ \forall_\rho$
(2) $\operatorname{ord}_P(f) + \langle u, v\rangle \geq h_P(v)\ \forall_{(P,v)}$.

The first implies that $u \in \Box_h \cap M$, and the second that $\operatorname{ord}_P(f) + (u - h_P)(v) \geq 0\ \forall\ (P, v)$. □

For a cone $\sigma \in \Xi_0^{(n)}$ of maximal dimension in the tail fan and a $P \in Y$ we get exactly one polyhedron $\Delta_P^\sigma \in \Xi_P$ having tail σ. For a given concave support function $h = \sum h_P P$, we have

$$h_P|\Delta_P^\sigma = \langle \cdot, u^h(\sigma)\rangle + a_P^h(\sigma).$$

The constant part gives rise to a divisor on Y:

$$h|_\sigma(0) := \sum_P a_P^h(\sigma) P.$$

Proposition 20. *A T-Cartier divisor $h = \sum h_P P \in$ T-CaDiv(Ξ) is (semi-)ample if and only if all h_P are strictly concave (concave) and $-h|_\sigma(0)$ is (semi-)ample for all tail cones σ, i.e.,* $\deg -h|_\sigma(0) = -\sum_P a_P^h(\sigma) > 0$ *(or a multiple of $-h|_\sigma(0)$ is principal).*

Proof. We first prove that semi-ampleness follows from the above criteria. Because h is (strictly) concave the same is true for h_0. This implies that the $u^h(\sigma)$ are exactly the vertices of $\Box_h$ and $h^*(u^h(\sigma)) = h|_\sigma(0)$.

The semi-ampleness for $h^*(u)$, $u \in \Box_h \cap M$ follows from the semi-ampleness at the vertices. Indeed if D, D' are semi-ample divisors on Y this is also true for $D + \lambda(D' - D)$ with $0 \leq \lambda \leq 1$.

Every vertex (u, a_u) of $\Gamma_{h_P^*}$ corresponds to an affine piece of h_P of the form $\langle u, \cdot\rangle - a_u$. If we let f be such that $\operatorname{div}(f) = a_u P$ on Loc $\mathcal{D}$ for some $\mathcal{D} \in \mathcal{S}$, we then have $D_h|_{\tilde{X}(\mathcal{D})} = \operatorname{div}(f^{-1}\chi^{-u})$ (see 3.1). A point $(u, a_u) \in M \times \mathbb{Z}$ is a vertex of h^* exactly if (ku, ka_u) is a vertex of $(k \cdot h)^*$. Hence, after passing to a suitable multiple of h we may assume, that $h^*(u)$ is base-point free with f being a global section which generates $\mathcal{O}(h^*(u))$ on Loc $\mathcal{D}$. Thus $f\chi^u$ is a global section of $\mathcal{O}(D_h)$ which generates $\mathcal{O}(D_h)|_{\tilde{X}(\mathcal{D})}$.

To show the other direction, i.e. that semi-ampleness implies the above criteria, assume that h_P is not concave. Then this is true also for every multiple of $\ell \cdot h_P$ and hence there is an affine piece $\langle u, \cdot\rangle - a_u$ of ℓh_P such that $a_u > (\ell h_P)^*(u)$. This means there is no global section $f\chi^u$ such that $\operatorname{div}(f) = a_u P$. But this contradicts the base-point freeness of $D_{\ell h}$ and hence the semi-ampleness of D_h.

To get the statement for ampleness note that a support function h on a polyhedral subdivision is strictly concave if and only if for every support function h' there is a $k \gg 0$ such that $h' + kh$ is concave. □

Corollary 21. $\tilde{X}(\Xi)$ *is projective if and only if all Ξ_P are regular subdivisions, i.e. admit a strictly convex support function.*

Remark 22. We see from Proposition 20 that for $h \in SF(\Xi)$, if the T-invariant divisor D_h is semi-ample, the corresponding dual function h^* is in fact a divisorial polytope. Conversely, if h^* is a divisorial polytope, the associated divisor on the associated T-variety is semi-ample.

3.4. Intersection numbers

Definition 23. For a divisorial polytope h^* we define its *volume* to be

$$\operatorname{vol} h^* := \sum_P \int_{\Box_h} h_P^* \operatorname{vol}_{M_\mathbb{R}}.$$

For divisorial polytopes $h_1^*, \dots, h_k^*$ we define their *mixed volume* by

$$V(h_1^*, \dots, h_k^*) := \sum_{i=1}^{k} (-1)^{i-1} \sum_{1 \le j_1 \le \cdots j_i \le k} \mathrm{vol}(h_{j_1}^* + \cdots + h_{j_i}^*).$$

Proposition 24. *Assume that on X Kodaira's vanishing Theorem holds.*

(1) *If D_h is semi-ample, for the self-intersection number we get*
$$(D_h)^{(m+1)} = (m+1)! \,\mathrm{vol}\, h^*.$$

(2) *Let $h_1, \dots, h_{m+1}$ define semi-ample divisors D_i on $X(\mathcal{Z})$. Then*
$$(D_1 \cdots D_{m+1}) = (m+1)! V(h_1^*, \dots, h_{m+1}^*).$$

Proof. If we apply (1) to every sum of divisors from $D_1, \dots, D_{m+1}$ we get (2) by the multi-linearity and symmetry of intersection numbers.

To prove (1) we first recall that

$$(D_h)^{m+1} = \lim_{\nu \to \infty} \frac{(m+1)!}{\nu^{m+1}} \chi(X, \mathcal{O}(\nu D_h)),$$

but for projective $X := X(\mathcal{Z})$ and nef divisors the ranks of higher cohomology groups are asymptotically irrelevant Demailly (2001, Theorem 6.7) so we get

$$(D_h)^{m+1} = \lim_{\nu \to \infty} \frac{(m+1)!}{\nu^{m+1}} h^0(X, \mathcal{O}(\nu D_h)).$$

Note that $(\nu h)^*(u) = \nu \cdot h^*(\frac{1}{\nu} u)$. Now we can bound h^0 by

$$\sum_{u \in \nu \Box_h \cap M} \left(\deg \lfloor \nu h^* \left(\tfrac{1}{\nu} u \right) \rfloor - g(Y) + 1 \right) \le h^0(\mathcal{O}(\nu D_h)) \le \sum_{u \in \nu \Box_h \cap M} \deg \lfloor \nu h^* \left(\tfrac{1}{\nu} u \right) \rfloor + 1. \qquad (1)$$

On the one hand we have

$$\lim_{\nu \to \infty} \frac{(m+1)!}{\nu^{m+1}} \sum_{u \in \nu \Box_h \cap M} \deg \lfloor \nu h^* \left(\tfrac{1}{\nu} u \right) \rfloor = \lim_{\nu \to \infty} \frac{(m+1)!}{\nu^m} \sum_{u \in \Box_h \cap \frac{1}{\nu} M} \frac{1}{\nu} \deg \lfloor \nu h^*(u) \rfloor$$
$$= (m+1)! \int_{\Box_h} h^* \,\mathrm{vol}_{M_{\mathbb{R}}}.$$

On the other hand, for any constant c, we have

$$\lim_{\nu \to \infty} \frac{1}{\nu^{m+1}} \sum_{u \in \nu \Box_h \cap M} c = c \cdot \lim_{\nu \to \infty} \frac{\#(\nu \cdot \Box_h \cap M)}{\nu^{m+1}} = 0.$$

Thus, if we pass to the limit in (1), the term in the middle has to converge to vol h^*. □

Remark 25. The theorem allows us to compute intersection numbers in characteristic 0 as well as on T-surfaces in positive characteristic because Kodaira's vanishing theorem holds in these cases. We believe that the theorem holds as well for positive characteristic in higher dimensions; work is being done to show that the vanishing theorem holds there.

Corollary 26. *Let $h \in SF(\mathcal{Z})$ and let C be any one-cycle rationally equivalent to the intersection of Cartier divisors, each of which can be expressed as an integer linear combination of semi-ample Cartier divisors. Then $D_h \cdot C$ is equal to $D_{h+P-Q} \cdot C$ for all points $P, Q \in Y$.*

Proof. We have

$$D_{h+P-Q} \cdot C = (D_h - D_{-P} + D_{-Q}) \cdot C = D_h \cdot C - D_{-P} \cdot C + D_{-Q} \cdot C$$

so it is sufficient to show that $D_{-P} \cdot C = D_{-Q} \cdot C$. Now, D_{-P} and D_{-Q} are semi-ample, so we can apply Proposition 24. Using the fact that $\mathrm{vol}((-P)^* + \widetilde{h}^*) = \mathrm{vol}((-Q)^* + \widetilde{h}^*)$ for all $\widetilde{h} \in SF(\mathcal{Z})$ gives the desired equality. □

Example 27. We know by Proposition 20 that D_h in our threefold is ample. We have vol $h^* = 21$. Hence, X is Fano of degree 21.

3.5. Genus of curves on surfaces

Let $X = \tilde{X}(\Xi)$ be a two-dimensional T-variety and let $h \in SF(\Xi)$ be a support function on Ξ. For any curve $C \in |D_h|$, we show how to calculate the arithmetic genus $g(C)$. As a corollary, we can calculate the Euler characteristic $\chi(X, \mathcal{O}(D_h))$ if X is smooth.

Definition 28. For any $h \in SF(\Xi)$, let

$$\operatorname{int} h_P^* := \sum_{u \in \Box_h^\circ \cap M} \#\{a \in \mathbb{Z}_{\geq 0} \mid a < |h_P^*(u)|\} \cdot \frac{h_P^*(u)}{|h_P^*(u)|}$$

for each point $P \in Y$, where $\Box_h^\circ$ is the interior of $\Box_h$. Furthermore, let

$$\operatorname{int} h^* := \sum_{P \in Y} \operatorname{int} h_P^*.$$

Definition 29. For any $h \in SF(\Xi)$, let

$$\#h_P^* := \sum_{u \in \Box_h \cap M} \lfloor h_P^*(u) \rfloor$$

for any point $P \in Y$ and let

$$\#h^* := \sum_{u \in \Box_h \cap M} \deg \lfloor h^*(u) \rfloor = \sum_{Y \in P} \#h_P^*.$$

Remark 30. Note that $\operatorname{int} h_P^*$ is the number of "interior" lattice points between the graph of h_P^* and 0 counted with their signs, where lattice points in height 0 are counted as long as they are not on the boundary of $\Box_h$. Similarly, if $\#h_P^*(h) \geq 0$ for all $u \in \Box_h$, $\#h_P^*$ is the sum of the number of lattice points between the graph of $\#h_P^*$ and 0, where we count no lattice points in height 0 but all lattice points lying on the graph of h_P^*.

We will use the following lemma.

Lemma 31. *With notation as above,* $2 \cdot \operatorname{vol} h_P^* = \operatorname{int} h_P^* + \#h_P^*$ *for all* $P \in Y$*. It follows in particular that* $2 \cdot \operatorname{vol} h^* = \operatorname{int} h^* + \#h^*$.

Proof. Fix some $P \in Y$. Suppose now that $h_P^*(u) \geq 0$ for all $u \in \Box_h$ and set

$$\Delta = \operatorname{conv}\left\{\left\{(u, h_P^*(u))\right\} \cup \{(u, 0)\}\right\},$$

where $u \in \Box_h$. This is a convex polytope in $M'_{\mathbb{Q}}$, where $M' = M \times \mathbb{Z}$. Pick's theorem tells us that $2 \cdot \operatorname{vol} \Delta + 2 = \#(\Delta \cap M') + \#(\Delta^\circ \cap M')$. Now $\operatorname{vol} \Delta = \operatorname{vol} h_P^*$, $\#(\Delta \cap M) = \#h_P^* + \#(\Box_h \cap M)$, and $\#(\Delta^\circ \cap M) = \operatorname{int} h_P^* - \#(\Box_h \cap M) + 2$, so the desired equality follows. For general h_P^*, choose j such that $\widetilde{h}_P^*(u) := h_P^*(u) + j \geq 0$ for all $u \in \Box_h$. Then $2 \cdot \operatorname{vol} \widetilde{h}_P^* = \operatorname{int} \widetilde{h}_P^* + \#\widetilde{h}_P^*$ and for $j_P^*(u) := j$ we have $2 \cdot \operatorname{vol} j_P^* = \operatorname{int} j_P^* + \#j_P^*$. Since vol, int, and # are additive at least for integer-valued functions, the desired equality follows for $h_P^* = \widetilde{h}_P^* - j_P^*$. $\Box$

We are now able to prove the following proposition:

Proposition 32. *Let* $h \in SF(\Xi)$ *be any support function such that* D_h *is semi-ample. Then for* $C \in |D_h|$*, the arithmetic genus of* C *is given by*

$$g(C) = \operatorname{int} h^* + 1 + \operatorname{vol} \Box_h \cdot (g(Y) - 1),$$

where $g(Y)$ *is the genus of* Y.

Proof. Without loss of generality, we can take the curve C to equal D_h. Indeed, the arithmetic genus is invariant under rational equivalence and since $|D_h|$ is not empty, it must contain some T-invariant effective divisor. We compare the genus of C with that of a comparable curve C_0 on $X_0 := Y \times \mathbb{P}^1$ and then compute the genus of C_0 directly. To begin with, note that we can find monoidal transformations $\pi_i : X_i \to X_{i-1}$ $1 \leq i \leq k$ such that

(1) X_i is a T-variety;
(2) π_i is T-equivariant; and
(3) there is a birational T-equivariant morphism $\varphi : X_k \to X$.

This is done as follows. Let Σ be the fan $\{\mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}, \{0\}\}$ and let $\Xi_P^0 := \Sigma$ for all points $P \in Y$. Then $X_0 = \widetilde{X}(\Xi^0)$. Each morphism π_i corresponds to an additional subdivision in the fan Ξ^{i-1} at exactly one point. Thus, we keep on refining until we get a Ξ^k which is a smooth common refinement of Ξ and Ξ^0; this gives us our morphism φ. Finally, we let $\pi : X_k \to X_0$ be the composition of the π_i.

We now pull back C to $C_k := \varphi^*(C)$. Thus we now have $C_k = D_h$, where h is now considered as a support function on Ξ^k. Furthermore, this does not change the arithmetic genus; that is, $g(C) = g(C_k)$. Define now inductively $C_{i-1} = \pi_{i*}(C_i)$ for $1 \leq i \leq k$. One easily checks that $C_0 = D_{\widetilde{h}}$, where $\widetilde{h} \in SF(\Xi^0)$ is the support function given by the divisorial polytope $\widetilde{h}_P^* := \max_{u \in \Box_h} h_P^*(h)$ with $\Box_{\widetilde{h}} := \Box_h$. Note that since C is semi-ample, each C_i is semi-ample as well. We will now calculate the difference between $g(C_k)$ and $g(C_0)$.

We first consider a special case; namely, suppose that h_P^* is trivial everywhere except for at two points $Q_1 \neq Q_2$. If $Y = \mathbb{P}^1$, all the varieties X_i and X are toric. In this case, the divisor D_h can be understood in toric terms as the polytope

$$\Delta_h := \operatorname{conv} \Gamma_{h_{Q_1}^*} \cup \Gamma_{-h_{Q_2}^*}$$

and $D_{\widetilde{h}}$ corresponds to $\Delta_{\widetilde{h}}$, which is defined in a similar manner. Then

$$g(C_k) - g(C_0) = I(\Delta_h) - I(\Delta_{\widetilde{h}}),$$

where $I(\Delta)$ is the number of interior lattice points of Δ; see for example Little and Schenck (2006), prop. 5.1. But we have $I(\Delta_h) = \operatorname{int} h_{Q_1}^* + \operatorname{int} h_{Q_2}^* - \#(\Box_h^\circ \cap M)$ and a similar equation for $\widetilde{h}$, which leads to

$$g(C_k) - g(C_0) = \operatorname{int} h^* - \operatorname{int} \widetilde{h}^*. \tag{2}$$

Now, Eq. (2) actually holds in general, not just in the toric case. To see this, note that for each $1 \leq i \leq k$, $C_i = \pi_i^*(C_{i-1}) + r_i \cdot E_i$, where E_i is the exceptional divisor of π_i. Then similar to Hartshorne (1977), V.3.7 we have $g(C_i) = g(C_{i-1}) - \frac{1}{2} r_i(r_i + 1)$. Thus,

$$g(C_k) - g(C_0) = \sum_{i=1}^{k} -\frac{1}{2} r_i(r_i + 1).$$

However, for each $1 \leq i \leq k$, the integer r_i can be determined combinatorially by comparing the polyhedral subdivisions Ξ_P^i and Ξ_P^{i-1} for the single point $P \in Y$ where these fansy divisors differ. Thus, the integers r_i can be calculated exactly as if we were in the toric case, so we get

$$\sum_{i=1}^{k} -\frac{1}{2} r_i(r_i + 1) = \operatorname{int} h^* - \operatorname{int} \widetilde{h}^*.$$

Eq. (2) follows.

We now calculate $g(C_0)$. From the adjunction formula, we have

$$g(C_0) = \frac{D_{\widetilde{h}}^2 + D_{\widetilde{h}} \cdot K_0}{2} + 1$$

for K_0 a canonical divisor on X_0; see Hartshorne (1977, V.1.5). The theorem of Riemann–Roch for surfaces (Hartshorne, 1977, V.1.6) gives us

$$\chi(X_0, \mathcal{O}(D_{\widetilde{h}})) = \frac{D_{\widetilde{h}}^2 - D_{\widetilde{h}} \cdot K_0}{2} + \chi(X_0, \mathcal{O}_{X_0}).$$

Thus,

$$g(C_0) = D_{\widetilde{h}}^2 + 1 + \chi(X_0, \mathcal{O}_{X_0}) - \chi(X_0, \mathcal{O}(D_{\widetilde{h}})).$$

Now, $\chi(X_0, \mathcal{O}_{X_0}) = 1 - g(Y)$ (see Hartshorne (1977), V.2.5). Likewise, if $p : X_0 \to Y$ is the projection, we have

$$\begin{aligned}\chi\left(X_0, \mathcal{O}(D_{\widetilde{h}})\right) &= \chi\left(Y, p_*\mathcal{O}(D_{\widetilde{h}})\right)\\ &= \sum_{u \in \square_h \cap M} \chi(Y, \mathcal{O}(\widetilde{h}^*(u)))\\ &= \#\widetilde{h} + (1-g)\cdot(\mathrm{vol}\,\square_h + 1),\end{aligned}$$

where the last equation follows from the Riemann–Roch theorem for curves. We also have $D_{\widetilde{h}}^2 = 2 \cdot \mathrm{vol}\,\widetilde{h}$. Making these substitutions results in

$$\begin{aligned}g(C_0) &= 2\cdot \mathrm{vol}\,\widetilde{h} + 1 + \mathrm{vol}\,\square_h \cdot (g(Y)-1) - \#\widetilde{h}\\ &= \mathrm{int}\,\widetilde{h} + 1 + \mathrm{vol}\,\square_h \cdot (g(Y)-1),\end{aligned}$$

the second equality coming from Lemma 31. Combining this with Eq. (2) completes the proof. □

Corollary 33. *For any semi-ample T-invariant Cartier divisor D_h on a smooth T-variety X, we have*

$$\chi(X, \mathcal{O}(D_h)) = \#h^* - (g(Y)-1)\cdot \#(\square_h \cap M) = \sum_{u\in \square_h \cap M} \chi(Y, \mathcal{O}(h^*(u))).$$

Proof. Using the adjunction formula and the Riemann–Roch theorem for surfaces as in the above theorem gives us the formula

$$\chi\,(X, \mathcal{O}(D_h)) = D_h^2 + 1 + \chi(X, \mathcal{O}_X) - g(C)$$

for some $C \in |D_h|$. We can use the above proposition to calculate $g(C)$. Combining this with the facts that $D_h^2 = 2\cdot \mathrm{vol}\,h$ and $\chi(X, \mathcal{O}_X) = 1 - g(Y)$ along with Lemma 31 completes the proof of the first equality. The second equality follows directly from the theorem of Riemann–Roch for curves. □

At the end of this section we revisit our surface example and use it to illustrate the concepts we have introduced.

Example 34. We look at the Cartier divisor D_h on our surface example where h_{Q_1} and h_{Q_2} are given by the tropical polynomials $0 \oplus (-2) \odot x^4$ and $0 \oplus (-2) \odot x^2 \oplus (-1) \odot x^3 \oplus 1 \odot x^4$, respectively. One easily sees that $\square_h = [0, 4]$, and that $h^*_{Q_1}$ and $h^*_{Q_1}$ respectively correspond to the tropical polynomials $x^{1/2}$ and $x \oplus 4 \odot x^{-1} \oplus 7 \odot x^{-2}$. In other words, $h^*_{Q_1}(u) = u/2$ and

$$h^*_{Q_2}(u) = \begin{cases} u & \text{if } u \le 2\\ 4-u & \text{if } 2 \le u \le 3\\ 7-2u & \text{if } u \ge 3.\end{cases}$$

In Fig. 6 we sketch h and the corresponding divisorial polytope h^*.

We can use Proposition 12 to compute the corresponding Weil divisor: $4D_{\mathbb{Q}_{\le 0}} + 4D_{(Q_2,2)} + 7D_{(Q_2,1)}$. D_h is semi-ample, so by Proposition 24 we get $(D_h)^2 = 15$. Finally, from Proposition 32 we know that a section of D_h has genus $5 + 4 \cdot g(Y)$.

We may also start with h^* and take the dual h to construct a fansy divisor as described above. We recover Ξ this way. $X := \tilde{\mathrm{X}}(\Xi)$ is not smooth, but a refinement of the polyhedral subdivisions (see Fig. 7) gives a smooth surface X' (this is will not be proved here; see Süß (2008)). Using Corollary 33, we can calculate that $\chi(X', \mathcal{O}(D_h)) = 12 - 5 \cdot g(Y)$.

4. *T*-codes and their parameters

4.1. Construction

Let Y be a curve over $\mathbb{F}_q$ and let h^* be a divisorial polytope. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be some subset of the $\mathbb{F}_q$-rational points of Y such that for $i = 1, \ldots, l$, $h^*_{P_i}$ is affine and $h^*_{P_i}(u) \in \mathbb{Z}$ for $u \in \square_h \cap M$.

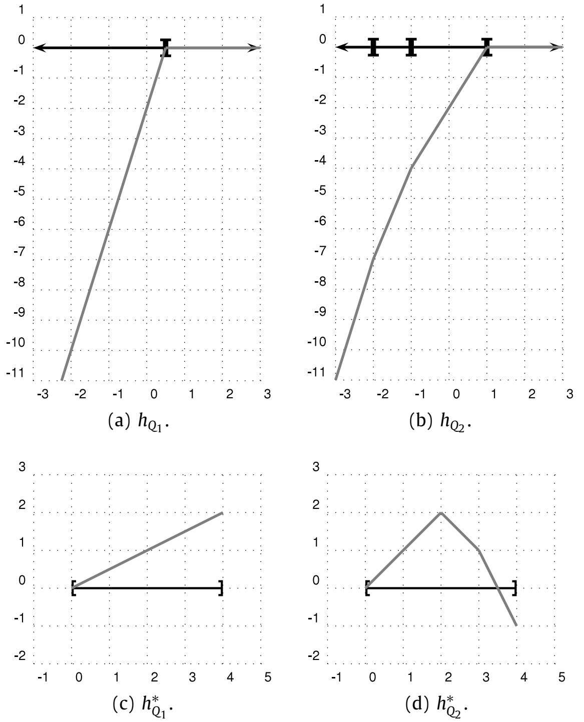


Fig. 6. h and h^* for a T-surface.

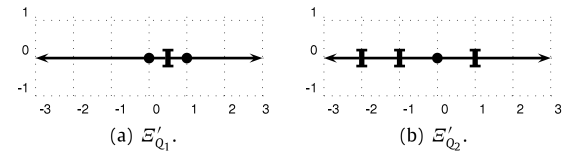


Fig. 7. A refined fansy divisor.

Let Ξ be the fansy divisor associated to h^* and let Ξ' be some minimal refinement such that $X := \tilde{X}(\Xi')$ is smooth. Note that for each point $P_i \in \mathcal{P}$, $\Xi'_{P_i} = v(P_i) + \Sigma$, for a unique lattice point $v(P_i)$ and tail fan Σ. Set $m = \dim M$. For each point P_i, let $P_i^1, \ldots, P_i^{(q-1)^m}$ be the $(q-1)^m$ $\mathbb{F}_q$-rational points on X of the open T-orbit contracting to P_i.

The support function h associated to h^* corresponds to a semi-ample T-invariant $\mathbb{F}_q$-rational Cartier divisor D_h on X. We denote the corresponding line bundle by $\mathcal{O}(D_h)$ and let $L(D_h) = \Gamma(X, \mathcal{O}(D_h))$. For each point P_i^j, fix some isomorphism $\mathcal{O}(D_h)_{P_i^j} \cong \mathbb{F}_q$. Consider the $\mathbb{F}_q$-linear map

$$\begin{aligned} \mathrm{ev} : L(D_h) &\to \mathbb{F}_q^{l(q-1)^m} \\ f &\mapsto \left(f_{P_1^1}, f_{P_1^2}, \ldots, f_{P_l^{(q-1)^m}}\right), \end{aligned}$$

where $f_{P_i^j}$ is the image of f in $\mathbb{F}_q$ following the identification with $\mathcal{O}(D_h)_{P_i^j}$. In other words, the above map evaluates the rational function f at the $l(q-1)^m$ points P_i^j $1 \le i \le l$, $1 \le j \le (q-1)^m$. The image of ev is a linear subspace of $\mathbb{F}_q^{l(q-1)^m}$ and thus a linear code of length $n = l(q-1)^m$; we denote it by $\mathcal{C}(Y, h^*, \mathcal{P})$. If $\mathcal{P}$ is maximal, we simply denote it by $\mathcal{C}(Y, h^*)$. Note that although $\mathcal{C}(Y, h^*, \mathcal{P})$ indeed depends on the way we identify $\mathcal{O}(D_h)_{P_i^j}$ with $\mathbb{F}_q$, its length n, dimension k, and its minimum distance d do not. Thus, we will always assume that some such isomorphisms are given, but will not concern ourselves further with them.

Remark 35. If $h^*_{P_i} = 0$ for $i = 1, \dots, l$, then $\mathcal{C}(Y, h^*, \mathcal{P})$ is equivalent as code to the image of the map

$$\mathrm{ev} : \bigoplus_{u \in \Box_h \cap M} \Gamma\left(\mathcal{O}(h^*(u))\right) \chi^u \to \mathbb{F}_q^{l(q-1)^m}$$

$$g\chi^u \mapsto \left(g(P_1)\chi^u(Q_1), g(P_1)\chi^u(Q_2), \dots, g(P_l)\chi^u(Q_{(q-1)^m})\right)$$

where $Q_1, \dots, Q_{(q-1)^m}$ are the $\mathbb{F}_q$-rational points of the m-dimensional torus. Thus, in this case the isomorphisms $\mathcal{O}(D_h)_{P_i^j} \cong \mathbb{F}_q$ are not only irrelevant but also unnecessary. Now let $\mathcal{C}_u$ be the $[(q-1)^m, 1, (q-1)^m]$ code generated by $(t^u)_{t \in (\mathbb{F}_q^*)^m}$ and let $\mathcal{C}(Y, h^*(u), \mathcal{P})$ be the AG code corresponding to the curve Y, divisor $h^*(u)$, and point set $\mathcal{P}$. Then as mentioned in the introduction, we can also define $\mathcal{C}(Y, h^*, \mathcal{P})$ simply as

$$\mathcal{C}(Y, h^*, \mathcal{P}) = \sum_{u \in \Box_h \cap M} \mathcal{C}_u \otimes \mathcal{C}(Y, h^*(u), \mathcal{P}).$$

4.2. Estimate on dimension

Assume that the map ev is injective. This is always the case if the bound given below for the minimum distance is larger than zero. We then have

$$k = \dim_{\mathbb{F}_q} L(D_h).$$

Using Proposition 19, we thus get

$$k = \sum_{u \in \Box_h \cap M} \dim \Gamma(Y, \mathcal{O}(h^*(u))).$$

We can approximate k using only the combinatorics of h^*. Let

$$\gamma(u) = \begin{cases} \deg\lfloor h^*(u)\rfloor + 1 - g(Y) & \text{if } \deg\lfloor h^*(u)\rfloor + 1 - g(Y) > 0 \\ 1 & \text{if } \deg\lfloor h^*(u)\rfloor + 1 - g(Y) \leq 0 \text{ and } h^*(u) \geq 0 \\ 0 & \text{if otherwise.} \end{cases}$$

Proposition 36. *If the evaluation map* ev *is injective, then*

$$\#h^* + \#(\Box_h \cap M)(1-g) \leq \sum_{u \in \Box_h \cap M} \gamma(u) \leq k \leq \#h^* + \#(\Box_h \cap M). \tag{3}$$

Furthermore,

$$k = \#h^* + \#(\Box_h \cap M)(1-g)) \tag{4}$$

if $\deg h^*(u) > 2g(Y) - 2$ *for all* $u \in \Box_h \cap M$.

Proof. The leftmost inequality in (3) follows from the definition of $\gamma(u)$. We now consider the second inequality in (3). Fix some degree $u \in \Box_h \cap M$. Then we always have $\dim \Gamma(Y, \mathcal{O}(h^*(u))) \geq 0$, and if $h^*(u)$ is effective, then $\dim \Gamma(Y, \mathcal{O}(h^*(u))) \geq 1$. Using the theorem of Riemann and Roch (see for example Hartshorne (1977)), we also have $\dim \Gamma(Y, \mathcal{O}(h^*(u))) \geq \deg h^*(u) + 1 - g$, and the inequality follows. If $\deg h^*(u) > 2g(Y) - 2$, then equality holds, so (4) follows. Finally, the right inequality in (3) follows from $\dim \Gamma(Y, \mathcal{O}(h^*(u))) \leq \deg h^*(u) + 1$. □

4.3. General lower bound on minimum distance

One strategy to produce an estimate for d is to use techniques of intersection theory, as first presented in Hansen (2001). These techniques have been applied to toric varieties; see for example

Hansen (2002) and Ruano (2007). We first consider the general case and then specialize to surfaces.[2]

Let $e_1^*, \ldots, e_m^*$ be a basis for M. For $P \in \mathcal{P}$ and $\eta_1, \ldots, \eta_{m-1} \in \mathbb{F}_q^*$, define $l(q-1)^{m-1}$ curves

$$C_{P,\eta_1,\ldots,\eta_{m-1}} := (P, v(P)) \cap V\left(\{\chi^{e_i^*} - \eta_i\}_{i=1}^{m-1}\right).$$

Each point P_i^j lies on exactly one of these curves. Furthermore, each curve $C_{P,\eta_1,\ldots,\eta_{m-1}}$ is rationally equivalent to

$$C_P := (P, v(P)) \cap V\left(\{\chi^{e_i^*}\}_{i=1}^{m-1}\right) = D_{0-P} \cdot (D_{-e_1^*})_{\geq 0} \cdot \ldots \cdot (D_{-e_{m-1}^*})_{\geq 0}$$

where the second equality follows from Proposition 12, e_i^* is considered as an element of $SF(\Xi)$, and $(D_{-e_i^*})_{\geq 0}$ is the effective part of $D_{-e_i^*}$.

Fix some section $s \in L(D_h)$; this corresponds to an effective divisor $(s)_0 = D_h + (s)$. By $Z(s)$ we denote the number of points P_i^j such that $s_{P_i^j} = 0$. Equivalently, $Z(s)$ is the number of points P_i^j contained in the support of $(s)_0$. Thus, one has the following lower bound for the minimum distance:

$$d \geq l(q-1)^m - \max_{s \in L(D_h)} Z(s).$$

Let $(s)_0$ vanish on exactly λ of the curves $\{C_{P,\eta_1,\ldots,\eta_{m-1}}\}$. Following (Hansen, 2001) and setting $C = C_P$ for some $P \in \mathcal{P}$ we then have

$$Z(s) \leq \lambda(q-1) + (l-\lambda)D_h \cdot C \tag{5}$$

since $(s)_0 \sim D_h$ and it follows from Corollary 26 that $D_h \cdot C = D_h \cdot C_{P_i} = D_h \cdot C_{P_i,\eta_1,\ldots,\eta_{m-1}}$ for all $1 \leq i \leq l$. Assuming that Kodaira's vanishing theorem holds on X, we can use Proposition 24 to calculate $D_h \cdot C$.

We now bound λ in a method similar to Ruano (2007). For the divisorial polytope $h^* : \Box_h \to \mathrm{Div}_{\mathbb{Q}} Y$ let $\mathrm{pr}(\Box_h)$ be the projection of $\Box_h$ to $M/\mathbb{Z}e_m^*$ and define $\mathrm{pr}(h^*) : \mathrm{pr}(\Box_h) \to \mathrm{Div}_{\mathbb{Q}}(Y)$ by

$$\mathrm{pr}(h^*)_P(u) = \max_{(u,u_m) \in \Box_h \cap M} h_P^*((u, u_m)).$$

One easily checks that $\mathrm{pr}(h^*)$ is a divisorial polytope. Assume that $\Box_h \subset \widetilde{u} + \{u \in M | 0 \leq u_i \leq q-2\}$ for some $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_m) \in M$. This also then holds for $\mathrm{pr}(\Box_h)$. We can write

$$s = \chi^{\widetilde{u}_m e_m^*} \cdot \left(s_0 + s_1 \chi^{e_m^*} + s_{q-2} \chi^{(q-2)e_m^*}\right)$$

where $s_i \in K(Y)(\chi^{u_1}, \ldots, \chi^{u_{m-1}})$. In fact, one easily checks that $s_i \in L(D_{\mathrm{pr}(h)})$, where $D_{\mathrm{pr}(h)}$ is the T-invariant Cartier divisor on the m-dimensional T-variety $X_{\mathrm{pr}(h^*)}$ over Y both determined by $\mathrm{pr}(h^*)$. If we restrict $s \cdot \chi^{-\widetilde{u}_m e_m^*}$ to some curve $C_{P,\eta_1,\ldots,\eta_{m-1}}$ we get a polynomial $\bar{s} = \bar{s}_0 + \bar{s}_1 \chi^{e_m^*} + \bar{s}_{q-2} \chi^{(q-2)e_m^*} \in \mathbb{F}_q[\chi^{e_m}]$ of degree less than or equal to $q-2$. If $C_{P,\eta_1,\ldots,\eta_{m-1}}$ is a curve where s vanishes, then $\bar{s}$ has $q-1$ zeros, so $\bar{s} \equiv 0$ and $\bar{s_i} = 0$ for $0 \leq i \leq q-2$. Thus the section $s_i \in L(D_{\mathrm{pr}(h)})$ vanishes on the point of $X_{\mathrm{pr}(h^*)}$ corresponding to the tuple $(P, \eta_1, \ldots, \eta_{m-1})$. It follows that

$$\lambda \leq \max_{t \in L(D_{\mathrm{pr}(h)})} Z(t).$$

Thus, we can recursively bound λ until $\dim(X) = 2$.

4.4. *Lower bound on minimum distance for* $\dim(X) = 2$

We can provide a much better bound for $Z(s)$ when X is a surface. Consider a global section s of $\mathcal{O}(D_h)$ as before such that $(s)_0$ vanishes on exactly λ of the curves $\{C_{P_i}\}$, say $C_{Q_1}, \ldots, C_{Q_\lambda}$ where the Q_i

[2] A more recent strategy to estimate d for toric surface codes involves bounding the number of irreducible components of a section and then applying the Hasse–Weil bound, see for example Little and Schenck (2006) and Soprunov and Soprunova (2008). The first author is currently working on applying this strategy to T-codes, (Ilten, in preparation).

are distinct points in $\mathcal{P}$. Thus, $s \in L(D_{\widetilde{h}})$, where $\widetilde{h} = h + \sum_{i=1}^{\lambda} Q_i$. Since $\widetilde{h}$ and $\sum_{i=1}^{\lambda}(-Q_i)$ are concave, it follows that $h^* = \widetilde{h}^* + (\sum_{i=1}^{\lambda}(-Q_i))^*$. In particular, we have that

$$\deg \widetilde{h}^*(u) = \deg h^*(u) - \lambda.$$

Thus, s can only have support in the weights $u \in \Box_{(h,\lambda)}$, where

$$\Box_{(h,\lambda)} = \left\{u \in \Box_h \cap M \mid \deg\lfloor h^*(u)\rfloor \geq \lambda\right\}.$$

It follows immediately that

$$\lambda \leq \max_{u\in\Box_h\cap M} \deg\lfloor h^*(u)\rfloor := \lambda_0.$$

Having found a good bound for λ, we now try to improve on the upper bound for $Z(s)$ in equation Eq. (5). By choosing a generator we can identify the lattice N with $\mathbb{Z}$. Then $\sigma_- := \mathbb{Q}_{\leq 0}$ and $\sigma_+ := \mathbb{Q}_{\geq 0}$ are the two rays in Σ. Each of these rays corresponds to a T-invariant divisor. Let μ_- and μ_+ respectively be the coefficients of the prime divisors σ_- and σ_+ in $(s)_0$. We want to find a lower bound for the sum $\mu_- + \mu_+$. This is easy if s has support only in a single weight u, say $s = f \cdot \chi^u$. In this case, (s) is T-invariant corresponding to the support function $-u - \operatorname{div}(f)$ and thus $\mu_- + \mu_+ = -h_0(-1) - h_0(1)$ using Proposition 12.

Let $u_{\min}$ and $u_{\max}$ be respectively the smallest and the largest weights in which s has non-trivial support and let $\nu = u_{\max} - u_{\min}$. Note that we can bound ν by

$$\nu \leq \nu(\lambda) := \max \Box_{(h,\lambda)} - \min \Box_{(h,\lambda)}.$$

Let $\mathcal{S}$ be some set of polyhedral divisors corresponding to some open covering of X and consider some polyhedral divisor $\mathcal{D} \in \mathcal{S}$. Now, the divisor σ_- or σ_+ is contained in $\widetilde{X}(\mathcal{D})$ if and only if $\mathcal{D}$ has respectively σ_- or σ_+ as tail cone. If the tail cone of $\mathcal{D}$ is σ_+, we can write

$$s = \chi^{u_{\min}} f^{-1} \cdot (s_0 + s_1\chi + \cdots + s_\nu \chi^\nu)$$

with $f, s_0, \ldots, s_\nu \in \mathcal{O}(\operatorname{Loc}\mathcal{D})$ and so (s) is the sum of some effective divisor and the T-invariant principal divisor $(f^{-1} \cdot \chi^{u_{\min}})$. Thus, using Proposition 12, we have $\mu_+ \geq -h_0(1) + u_{\min}$. On the other hand, if the tail cone of $\mathcal{D}$ is σ_-, we can write

$$s = \chi^{u_{\max}} f^{-1} \cdot (s_0\chi^{-\nu} + s_1\chi^{-\nu+1} + \cdots + s_\nu)$$

with $f, s_0, \ldots, s_\nu \in \mathcal{O}(\operatorname{Loc}\mathcal{D})$. Thus, using Proposition 12 again, we have $\mu_- \geq -h_0(-1) - u_{\max}$. Combining these two inequalities gives us

$$\mu_- + \mu_+ \geq \operatorname{vol}\Box_h - \nu \geq \operatorname{vol}\Box_h - \nu(\lambda),$$

where we use the easily checked fact that $-h_0(-1) - h_0(1) = \operatorname{vol}\Box_h$.

Now, each curve C_P intersects with σ_+ in one point; similarly, C_P and σ_- intersect in some other point. Neither of these points is one of the points P_i^j at which we are evaluating our section s. This means that for each of the $l-\lambda$ curves where we calculate the number of zeros of $(s)_0$ using intersection numbers, we have counted at least $\mu_- + \mu_+$ too many points. Furthermore, we can use Proposition 24 to calculate that $D_h \cdot C = \operatorname{vol}\Box_h$. Thus, we can improve Eq. (5) to

$$Z(s) \leq \lambda(q-1) + (l-\lambda)\nu(\lambda).$$

Summing up the results obtained here leads to the following.

Proposition 37. *Let $\mathcal{C}(Y, h^*, \mathcal{P})$ be a code on a two-dimensional T-variety. Then the minimum distance of this code is bounded from below by*

$$d \geq \min_{0\leq\lambda\leq\lambda_0} \left[(l-\lambda)(q-1-\nu(\lambda))\right].$$

Remark 38. In the literature concerning toric surface codes, the estimate for the minimum distance often contains a term involving the self-intersection number of one of the curves C_P. In our case, this term does not help since $C_P^2 = 0$, which can be easily seen using Proposition 24. However, the correction we make using μ_+ and μ_- has a similar effect.

4.5. Upper bound on minimum distance

A simple upper bound on the minimum distance of a toric code is given in Ruano (2007). We adapt this to the case of T-varieties. This then gives us a way of testing if the lower bound on minimum distance attained above is sharp.

Proposition 39. *Let $f \in K(Y)$ be such that $f \cdot \chi^u \in L(D_h)$ for all $u \in B \cap M$, where B is lattice isomorphic to a lattice hyper-rectangle with side lengths $r_1, \dots, r_m$, $r_i \leq q-1$. Furthermore, suppose that f vanishes at r_0 of the points $P_i \in \mathcal{P}$. Then*

$$d \leq (l - r_0) \cdot \left((q-1)^m + \sum_{j=1}^{m} (-1)^j \sum_{i_1 < \cdots < i_j} r_{i_1} \cdots r_{i_j} (q-1)^{m-j} \right). \qquad (6)$$

In particular, for $m = 1$ we have $d \leq l(q-1) - r_1 l - r_0(q-1) + r_0 r_1$.

Proof. Choose a basis $e_1^*, \dots, e_m^*$ of the lattice M such that $B = \widetilde{u} + \prod_{i=1}^m [0, r_i]$. Let $\mathbb{F}_q^* = \{\eta_1, \dots, \eta_{q-1}\}$. Now consider the rational function

$$f' := f \cdot \chi^{\widetilde{u}} \cdot \prod_{i=1}^{m} \prod_{j=1}^{r_i} (\chi^{e_i^*} - \eta_j).$$

One easily checks that $f' \in L(D_h)$. On the other hand, using inclusion–exclusion one sees that for each point $P_i \in Y$, f' vanishes on

$$\sum_{j=1}^{m} (-1)^{j+1} \sum_{i_1 < \cdots < i_j} r_{i_1} \cdots r_{i_j} (q-1)^{m-j}$$

rational points of the open T-orbit contracting to P_i. The function f' vanishes entirely on r_0 of these orbits, each of which has $(q-1)^m$ relevant points. Using inclusion–exclusion again and subtracting the total number of points on which f' vanishes from the length $n = l(q-1)^m$ yields the desired result. □

As a consequence of the above proposition we get the following corollary.

Corollary 40. *Let $B \subset \square_h$ be lattice isomorphic to a lattice hyper-rectangle with side lengths $r_1, \dots, r_m$, $r_i \leq q-1$. Furthermore, for each $Q_j \in Y(\mathbb{F}_q)$ let $c_j \in \mathbb{Z}$ be such that $h_{Q_j}^*(u) \geq c_j$ for all $u \in \square_h \cap M$. If $\sum c_j \geq g(Y)$, inequality (6) then holds for $r_0 := \left(\sum c_j\right) - g(Y)$.*

Proof. Using the above proposition, we just need to find $f \in K(Y)$ such that $f \cdot \chi^u \in L(D_h)$ for all $u \in B \cap M$ and such that f vanishes at r_0 of the points $P_i \in \mathcal{P}$. Now, for any r_0 points $P_1, \dots, P_{r_0} \in Y(\mathbb{F}_q)$, the divisor $D := \sum c_j Q_j$ on Y has a global section f which vanishes on all r_0 points. Indeed, by the Riemann–Roch theorem,

$$\dim L\left(D - \sum_{i=1}^{r_0} P_i\right) = \left(\sum c_j\right) - r_0 + 1 - g(Y) = 1.$$

Now, since $h_{Q_j}^*(u) \geq c_j$ for all $u \in \square_h \cap M$, $L(D) \subset L(h^*(u))$ for all $u \in B \cap M$ and we have found f as desired. □

Remark 41. In the case of a toric code, the above corollary gives exactly the upper bound of Ruano (2007).

5. Examples

5.1. Ruled surfaces from decomposable vector bundles

Codes on ruled surfaces, or equivalently $\mathbb{P}^1$-bundles over a curve Y, were first considered in Hansen (2001), where formulae for n and k and a lower bound for d are given; global sections of some line

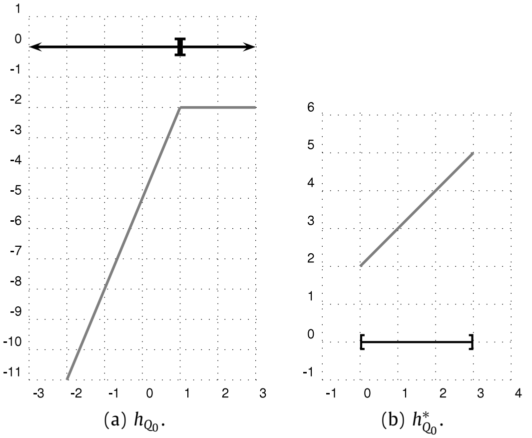


Fig. 8. *h* and h^* for a simple ruled surface.

bundle on X are evaluated at all $\mathbb{F}_q$-rational points. This was then applied in Lomont (2003) to surfaces of the form $X = \mathbf{Proj}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-e))$. Assuming that the lower bound attained for d there is sharp, the resulting codes are never better than a product code coming from a Reed–Solomon code and a Goppa code. However, by restricting the points at which we evaluate to a smaller set, better codes can be found. Indeed, consider the case $Y = \mathbb{P}^1$, $e > 0$, where the resulting surface is the Hirzebruch surface $\mathcal{H}_e$, a toric variety. Codes obtained by evaluation on the points of the torus were considered in Hansen (2002), with parameters considerably better than those of product codes. We wish to generalize this to bundles over curves of higher genus.

Consider the rank-two locally free sheaf

$$\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}\left(\sum_{Q_i \in \mathbb{F}_q(Y)} \alpha_i Q_i\right)$$

for $\alpha_i \in \mathbb{Z}$ and set $X = \mathbf{Proj}(\mathcal{E})$. Any ruled surface coming from a decomposable vector bundle is isomorphic to such a X. Furthermore, X can easily be described as a T-variety. Let $\Sigma \subset \mathbb{Q}$ be the fan consisting of the cones $\mathbb{Q}_{\leq 0}$, $\mathbb{Q}_{\geq 0}$, and $\{0\}$, and let Ξ be the fansy divisor with $\Xi_{Q_i} = \alpha_i + \Sigma$. Then one can easily confirm that $X = \tilde{X}(\Xi)$. We set $\alpha = \sum \alpha_i$.

Consider now any semi-ample T-invariant Cartier divisor D_h on X. Then h_0 is of the form

$$h_0(v) = \begin{cases} u_{\max} \cdot v & \text{if } v \leq 0 \\ u_{\min} \cdot v & \text{if } v \geq 0 \end{cases}$$

for some $u_{\min}, u_{\max} \in \mathbb{Z}$ with $a := u_{\max} - u_{\min} \geq 0$. It follows that $\Box_h = [u_{\min}, u_{\max}]$. Furthermore, for each $Q_i \in \mathbb{F}_q(Y)$, h_{Q_i} is of the form $h_{Q_i}(v) = h_0(v - \alpha_i) - b_i$ for some $b_i \in \mathbb{Z}$. Thus $h^*_{Q_i}(u) = \alpha_i \cdot u + b_i$. It follows that $\deg h^*(u) = \alpha \cdot u + b$.

As an example, by setting $u_{\min} = 0$, $u_{\max} = 3$, $\alpha_0 = 1$, $b_0 = 2$, and all other possible parameters to 0, we get the ruled surface with h and h^* as pictured in Fig. 8.

We now consider the code $\mathcal{C}(Y, h^*, \mathcal{P})$ for any set $\mathcal{P}$ of $\mathbb{F}_q$-rational points on Y; note that h^*_P is affine and integer-valued on lattice points for any point $P \in \mathcal{P}$ as required. Set $l = \#\mathcal{P}$. For the sake of simplicity we shall assume that $u_{\min} = 0$, $\alpha > 0$ and $\alpha_i, b_i \geq 0$. This ensures that h is in fact semi-ample, i.e. that h^* is a divisorial polytope. One easily confirms that $\lambda_0 = b + a \cdot \alpha$ and that

$$\nu(\lambda) = \begin{cases} a & \text{if } \lambda \leq b \\ \lfloor a - \frac{\lambda - b}{\alpha} \rfloor & \text{if } \lambda \geq b. \end{cases}$$

Using Proposition 37 we then have that

$$d \geq \min\left\{(l - b - a \cdot \alpha)(q - 1)), (l - b)(q - 1 - a)\right\}.$$

We can then use Corollary 40 to bound d from above. Indeed, for $t \in \mathbb{Z}$, $0 \leq t \leq a$ we have that $h^*_{Q_j}(u) \geq b_j + \alpha_j t$ for all $t \leq u \leq a$. Using the particular cases $t = 0$ and $t = a$ results in the bound

$$d \leq \min\{(l - b - a \cdot \alpha + g(Y))(q-1)), (l - b + g(Y))(q - 1 - a)\}.$$

Thus, we have upper and lower bounds for d differing by at most $g(Y) \cdot (q-1)$.

We now use Proposition 36 to find a lower bound for k. We always have

$$k \geq (a+1)(b + 1 + \alpha \cdot a/2 - g(Y))$$

where equality holds if $b > g(Y) - 2$. Suppose now that $b \leq g(Y)$; set $c = \lceil (g(Y) - b)/\alpha \rceil$. Now $h^*(u)$ is effective for every $u \in \Box_h \cap M$, so we can improve the bound on k to

$$k \geq (a + 1 - c)\left(b + 1 + \frac{\alpha}{2}(c + a) - g(Y)\right) + c. \tag{7}$$

Note that equality holds if $g(Y) \leq 1$.

Remark 42. In the case $Y = \mathbb{P}^1$ and $\alpha_i, b_i = 0$ for all points Q_i with the exception of some point Q_0, X is the Hirzebruch surface $\mathcal{H}_\alpha$. If we set $\mathcal{P} = \mathbb{F}_q^*$, we recover the results of Hansen (2002). Note that the curves we use to cover the points of the torus are perpendicular to those used by Hansen. In our case, these curves have self-intersection zero, but the adjustment we make with μ_- and μ_+ compensates for this.

We now compare these codes to product codes coming from a length $q-1$ Reed–Solomon code and a Goppa code. A Reed–Solomon code has parameters $[q-1, k_1, d_1]$ with $d_1 = q - k_1$ and $k_1 \leq q - 1$. Assume $\tau \in \mathbb{N}$ with $\tau > g(Y) - 1$. Then the Goppa code on Y gotten by evaluating a divisor D of degree τ at l rational points has parameters $[l, k_2, d_2]$ with $k_2 \geq \tau - g(Y) + 1$ and $d_2 \geq l - \tau$; see for example Pless et al. (1998, Vol. I, Chapter 10). The resulting product code $\mathcal{C}_{prod}$ has parameters $[l(q-1), k_1k_2, d_1d_2]$. For the product code we thus have the estimates

$$k_{prod} \geq k_{est} := k_1(\tau - g(Y) + 1),$$
$$d_{prod} \geq d_{est} := (q - k_1)(l - \tau).$$

We can then show the following.

Proposition 43. *Fix some curve Y and assume that $l \geq q + g(Y) - 1$. Using notation as above, we can find h^* and $\mathcal{P}$ as above such that the estimated parameters for $\mathcal{C}(Y, h^*, \mathcal{P})$ are better than those for $\mathcal{C}_{prod}$. Specifically, we show that*

$$k_{est} \leq (a+1)(b + 1 + \alpha \cdot a/2 - g(Y)), \tag{8}$$
$$d_{est} < \min\{(l - b - a \cdot \alpha)(q-1)), (l - b)(q - 1 - a)\}. \tag{9}$$

Proof. First, suppose that $\tau \geq (k_1 - 1)$. We then set $a = k_1 - 1$ and choose some $\alpha \in \mathbb{N}$ such that $\alpha(k_1 - 1) \leq 2\tau$ and $\alpha(k_1 - 1)$ is divisible by two. Choose $b_i \geq 0$ such that $b = \tau - \alpha(k_1 - 1)/2$. Choose any set $\mathcal{P}$ consisting of l points. Equality in (8) follows immediately, and a quick calculation shows that (9) holds as well.

Suppose instead that $\tau < (k_1 - 1)$. Set $\widetilde{k}_1 = \tau - (g(Y) - 1)$ and $\widetilde{\tau} = k_1 + (g(Y) - 1)$. Consider then the product code $\widetilde{\mathcal{C}}_{prod}$ obtained as a product of the $\widetilde{k}_1$-dimensional Reed–Solomon code and the Goppa code corresponding to the divisor $\widetilde{\tau} Q_0$. Then one easily confirms that the estimated minimum distance and dimension for $\widetilde{\mathcal{C}}_{prod}$ are greater than or equal to those of $\mathcal{C}_{prod}$ and that $\widetilde{\tau} \geq (\widetilde{k}_1 - 1)$. Thus, we reduce to the first case above. □

5.2. A code on an elliptic curve

The following example illustrates techniques that can be used to refine our estimate for minimum distance. It also demonstrates that there are T-codes with better parameters than the those estimated in the previous example. Before we begin, we first note the following lemma.

Lemma 44. *Let D_h be a T-invariant divisor on $\tilde{X}(\Xi)$, and let s be a section such that $(s)_0$ is not irreducible. Then we can find functions $h_1, h_2 \in SF(\Xi)$ and $s_1 \in L(D_{h_1})$, $s_1 \in L(D_{h_1})$ such that*

(1) $D_h = D_{h_1} + D_{h_2}$;
(2) $(s) = (s_1) + (s_2)$; *and*
(3) D_{h_i} *is not rationally equivalent to* 0 *for* $i = 1, 2$.

Proof. Since $(s)_0$ is not irreducible, we can write it as the sum of two nontrivial effective divisors $(s)_0 = C_1 + C_2$. Since the Picard group is generated by T-invariant divisors, we can find $h'_1, h'_2 \in SF(\Xi)$ such that $C_i = D_{h'_i} + (s'_i)$ for some $s'_i \in L(D_{h'_i})$, $i = 1, 2$. We thus have

$$D_h + (s) = D_{h'_1} + (s'_1) + D_{h'_2} + (s'_2).$$

Now set $s_1 := s'_1$, $h_1 := h'_1$, and $s_2 := s/s_1$, and let h_2 be the support function corresponding to the T-invariant divisor $D_{h'_2} + (s'_2) - (s_2)$. These support functions and sections clearly fulfill the desired conditions. □

We now return to the divisor on the T-surface considered in Example 34. For Y either $\mathbb{P}^1$ or elliptic, we have already noted that D_h is semi-ample; this is the same as saying that h^* is a divisorial polytope. Now, if $Y = \mathbb{P}^1$ and $Q_1 = 0$, $Q_2 = \infty$, the T-variety associated to h^* is in fact toric, and h^* corresponds to the polytope in $\mathbb{Z}^2$ given by $\mathrm{conv}\{(0, 0), (2, -2), (3, -1), (4, 1), (4, 2)\}$. Let $\mathcal{P} = Y \setminus \{Q_1, Q_2\}$; the example of $\mathcal{C}(\mathbb{P}^1, h^*, \mathcal{P})$ is considered in Soprunov and Soprunova (2008), where it is shown using the Hasse–Weil bound that $d \geq (q-1)^2 - 3(q-1) - 2\sqrt{2} + 1$ for all $q \geq 19$. We now calculate the parameters d and k for $\mathcal{C}(Y, h^*, \mathcal{P})$ in the case that Y is an elliptic curve.

In calculating k, note that $\deg h^*(u) > 0$ for $u > 0$. Thus, in these degrees we have that $\dim L(D_h)_u = \deg h^*(u)$. On the other hand, $h^*(0) = 0$, which is effective, so $\dim L(D_h)_0 = 1$. Adding everything up, we get that $k = 8$.

Proposition 39 gives us an easy upper bound for d. If we set $f := 1$, we have that $f \cdot \chi^u \in L(D_h)$ for $u \in 0, 1, 2, 3$. Indeed, $h^*(u)$ is effective in these degrees. Thus, it follows that $d \leq l(q-1) - 3l$.

We now bound d from below. One easily checks that $\lambda_0 = 3$. Likewise, one can easily calculate that $\nu(0) = 4$, $\nu(1) = 3$, $\nu(2) = 1$, and $\nu(3) = 0$. Now consider some section s such that $\lambda = 1$. We claim that we actually must have that $\nu \leq 2$. The section s cannot have support in weight 0 since $\deg h^*(0) - 1 = -1$. Furthermore, s cannot have support in weight 1. Indeed, $\Gamma(Y, \mathcal{O}(Q_2 - P)) = 0$ for any point $P \neq Q_2$, since $Y \neq \mathbb{P}^1$. It follows that for any section s with $\lambda \neq 0$ or with $\lambda = 0$ and $\nu < 4$ we have $Z(s) \leq \lambda(q-1) + l(3-\lambda)$; if we assume that $l \geq q-1$, it follows that $Z(s) \leq 3l$.

Now consider some section s such that $\lambda = 0$ and $\nu = 4$; we will show that under certain assumptions we also have $Z(s) \leq 3l$. First, suppose that $(s)_0$ is irreducible. Then using the Hasse–Weil bound for singular curves as stated in Aubry and Perret (1996), we see that the number $\#(s)_0(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points on $(s)_0$ is bounded above by

$$\#(s)_0(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$$

where $g := g((s)_0)$ is the arithmetic genus of $(s)_0$. Note that this only depends on the divisor D_h and not on s. Now, if we require that

$$q \geq \left(\frac{g + \sqrt{g^2 + 8}}{2}\right)^2,$$

it follows that

$$Z(s) \leq q + 1 + 2g\sqrt{q} \leq (q-1)3.$$

In our case, it follows from Proposition 32 that $g = 9$ so the required bound on q is $q \geq 89$.

Suppose on the other hand that $(s)_0$ is not irreducible. Let $h_1, h_2 \in SF(\Xi)$ be support functions and $s_i \in L(D_i)$ $i = 1, 2$ sections as in Lemma 44, ordered such that $\mathrm{vol}\,\square_{h_1} \leq \mathrm{vol}\,\square_{h_2}$. It easily follows that $\nu(s) = \nu(s_1) + \nu(s_2)$ and by Remark 16 we have $h^* \geq h_1^* + h_2^*$. Now if s_1 only has support in a single degree, $(s_1)_0$ is T-invariant. Thus we have $Z(s_1) = 0$ and $Z(s) = Z(s_2)$. Indeed, since $\lambda = 0$, $(s_1)_0$ cannot contain one of the curves C_P covering the points of evaluation, and all other T-invariant prime divisors do not contain any points of evaluation. Now note that $h_2^* \leq h^* + (f)$ for some $f \in K(Y)$. Thus, $g((s_2)_0) \leq g((s)_0)$ and if $(s_2)_0$ is irreducible, the above argument with the Hasse–Weil bound gives the desired bound. If not, we replace h and s by h_2 and s_2 and repeat the process until we have an irreducible section and thus the desired bound, or have sections s'_1 and s'_2 both with support in multiple weights.

We have now reduced to the situation where $h' \in SF(\Xi)$ with $s' \in L(D_{h'})$, $h'^* \leq h^* + (f)$, for this s' we have $\nu = 4$, and h' and s' admit a decomposition into h'_1, h'_2 and s'_1, s'_2 such as in Lemma 44 such that both sections have support in multiple weights. We show that this is impossible. We first note that since $\nu = 4$, s'_i must have support in the largest and smallest weights of $\Box_{h'_i}$, which we call $u_i^{\max}$ and $u_i^{\min}$, respectively. Furthermore, by adjusting with T-invariant principal divisors we can assume that $(f) = 0$, $u_i^{\min} = 0$, and $h_i'^*(0) = 0$. We then have $(h'_i)^*_{Q_1}(u_i^{\max}) < 2$ for $i = 1, 2$. Indeed, we must have

$$(h'_1)^*_{Q_1}(u_1) + (h'_2)^*_{Q_1}(u_2) < 2$$

for $u_1 \in \Box_{h'_1}$ and $u_2 \in \Box_{h'_2} \setminus \{u_2^{\max}\}$. The claim follows for $i = 1$ by setting $u_2 = 0$; for $i = 2$ we just switch the indices. Now, for at least one $i \in 1, 2$ we must also have $(h'_i)^*_{Q_2}(u_i^{\max}) < 0$. Indeed, this follows from

$$(h'_1)^*_{Q_2}(u_1^{\max}) + (h'_2)^*_{Q_2}(u_2^{\max}) \leq -1.$$

For this i,

$$L(D_{h'_i})_{u_i^{\max}} = \Gamma\left(Y, \mathcal{O}(h_i'^*(u_i^{\max}))\right) \subset \Gamma\left(Y, \mathcal{O}(Q_1 - Q_2)\right) = 0.$$

This is however impossible, since we had already concluded that s'_i has support in weight $u_i^{\max}$.

We have thus shown that a section $s \in L(D_h)$ with $\lambda = 0$ is either irreducible, in which case we can bound the number of rational points on it using the Hasse–Weil bound, or it can be decomposed into T-invariant components and some remaining section, which either is irreducible or which has support in weights differing by at most 3. Thus, if we require that $q \geq 89$ and $l \geq q - 1$, we see that for any section $s \in L(D_h)$, $Z(s) \leq 3l$. Since our upper bound already states that $d \leq l(q-1) - 3l$, we find that in fact

$$d = l(q-1) - 3l.$$

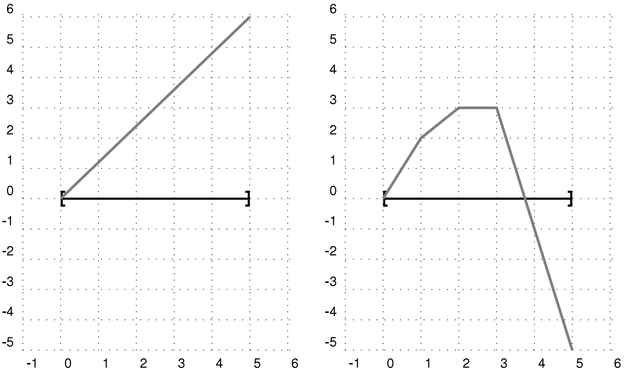
This marks an improvement over the estimates for any of the T-codes considered in the previous example. Indeed, to get the desired estimated minimum distance we would have to require $b = 0$ and $a \leq 3$. Using Eq. (7), one easily checks that the dimension of the resulting code is smaller than 8.

5.3. A computational example

We are able to provide a T-code over $\mathbb{F}_7$ with parameters [66, 19, 30], which is as good as the best known code (see Grassl (2007)). We set $Y = V(zy^2 + 6x^3 + 4z^3) \subset \mathbb{P}^2_{\mathbb{F}_7}$ and consider the divisorial polytope given in Fig. 9. Fixing two $\mathbb{F}_q$-ration points Q_1 and Q_2 we can compute a generator matrix of $C = \mathcal{C}(Y, h^*)$ using Macaulay 2 (Grayson and Stillman, 2008) and the `toriccodes` package (Ilten, 2008). We can then compute the minimal distance using Magma (Bosma et al., 1997).

It is easy to see that the length and dimension of C are always respectively 66 and 19. However, the minimum distance can be either 29 or 30, depending on the choice of Q_1 and Q_2. For example, setting $Q_1 = (1 : 2 : 1)$, $Q_2 = (1 : 5 : 1)$ results in a minimum distance of 30, whereas $Q_1 = (1 : 2 : 1)$, $Q_2 = (0 : 1 : 1)$ results in a minimum distance of 29. In fact, the automorphism group of Y divides the set of all pairs of rational points on Y into two equally large subsets; using pairs in one subset results in a minimum distance of 30, whereas pairs from the other subset result in a minimum distance of 29.

We can also use Proposition 39 to easily show that $d \leq 30$. Indeed, it is not difficult to find a section $f \in \Gamma(Y, \mathcal{O}(3Q_1 + 3Q_2))$ vanishing at 6 distinct points of $Y(\mathbb{F}_q) \setminus \{Q_1, Q_2\}$. Thus, $f \in L(D_h)_3$ and we get $d \leq 66 - 6 \cdot 6 = 30$. On the other hand, Proposition 37 guarantees only a minimum distance of 11, which is a rather bad estimate. However, consider instead the divisorial polytope $\tilde{h}^*$ acquired



(a) $h^*_{Q_1}$. (b) $h^*_{Q_2}$.

Fig. 9. A divisorial polytope defining a $[66, 19, 30]_7$ code.

by restricting h^* to the weight polytope $[0, 3]$, and the corresponding code $\tilde{C} = \mathcal{C}(Y, \tilde{h}^*)$. In this case, $L(D_h)_3 = L(D_{\tilde{h}})_3$, so we once again have $d(\tilde{C}) \leq 30$ by Proposition 39. Proposition 37 now also guarantees a minimum distance of 30. Thus, we have found a subcode $\tilde{C} \subset C$ which has minimum distance 30. The computer calculation mentioned above means that this code can be expanded to C without lowering the minimum distance.

References

Altmann, K., Hausen, J., 2006. Polyhedral divisors and algebraic torus actions. Math. Ann. 334 (3), 557–607.
Altmann, K., Hausen, J., Süß, H., 2008. Gluing affine torus actions via divisorial fans. Transform. Groups 13 (2), 215–242.
Aubry, Y., Perret, M., 1996. A Weil theorem for singular curves. In: Arithmetic, Geometry and Coding Theory (Luminy, 1993). de Gruyter, Berlin, pp. 1–7.
Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I. The user language. J. Symbolic Comput. 24 (3–4), 235–265 Computational Algebra and Number Theory (London, 1993).
Demailly, J.-P., 2001. Multiplier ideal sheaves and analytic methods in algebraic geometry. In: School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000). In: ICTP Lect. Notes. Abdus Salam Int. Cent. Theoret. Phys., vol. 6. Trieste, pp. 1–148.
Goppa, V.D., 1981. Codes on algebraic curves. Dokl. Akad. Nauk SSSR 259, 1289–1290 (in Russian).
Grassl, M., 2007. Bounds on the minimum distance of linear codes. Available at http://www.codetables.de (accessed 12.09.08).
Grayson, D.R., Stillman, M.E., 2008. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.
Hansen, J.P., 2002. Toric varieties Hirzebruch surfaces and error-correcting codes. Appl. Algebra Engrg. Comm. Comput. 13 (4), 289–300.
Hansen, S.H., 2001. Error-correcting codes from higher-dimensional varieties. Finite Fields Appl. 7 (4), 531–552.
Hartshorne, R., 1977. Algebraic Geometry. In: Graduate Texts in Mathematics, vol. 52. Springer-Verlag, New York.
Ilten, N., 2008. `toriccodes`, a Macaulay 2 package for toric- and *T*-codes. Available at http://people.cs.uchicago.edu/~nilten/m2.html.
Ilten, N., 2010. *T*-codes and Minkowski decompositions of divisorial polytopes (in preparation).
Little, J., Schenck, H., 2006. Toric surface codes and Minkowski sums. SIAM J. Discrete Math. 20 (4), 999–1014 (electronic).
Lomont, C., 2003. Error correcting codes on algebraic surfaces. Ph.D. Thesis, Purdue University. arXiv:math/0309123v1.
Petersen, L., Süß, H., 2008. Torus invariant divisors. arXiv:math/0811.0517v1.
Pless, V.S., Huffman, W.C., Brualdi, R.A. (Eds.), 1998. Handbook of Coding Theory, Vol. I (Chapter 10), II. North-Holland, Amsterdam.
Ruano, D., 2007. On the parameters of *r*-dimensional toric codes. Finite Fields Appl. 13 (4), 962–976.
Soprunov, I., Soprunova, J., 2008. Toric surface codes and Minkowski length of polygons. arXiv:0802.2088v1.
Süß, H., 2008. Canonical divisors on *T*-varieties. arXiv:math/0811.0626v1.