On Generalized Spectral Functions, the Parametrization of Block Hankel and Block Jacobi Matrices, and Some Root Location Problems

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ABSTRACT

The parametrization of a strongly regular block Hankel matrix in terms of certain block entries of an appropriately chosen sequence of block inverses is established. This leads to a new recipe for solving the generalized spectral problem for Jacobi matrices. A generalized spectral function is introduced and, together with the parametrization referred to above, is used to investigate the root location of certain orthogonal matrix polynomials. The matrix analogues of two classical stability tests are discussed.

1. INTRODUCTION

In the present article we shall discuss some connections between block Hankel and block Jacobi matrices which give, in particular, a new recipe for solving the generalized inverse spectral problem (GISP) for Jacobi matrices. In this problem one is given a sequence of $p \times p$ matrices

$$h_0 = I_p, h_1, \ldots \tag{1.1}$$

such that for every $k = 0, 1, \ldots$ the block Hankel matrix

$$H_k = \left[h_{i+j}\right], \qquad i, j = 0, 1, \ldots, k, \tag{1.2}$$

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is invertible, and the objective is to find a block Jacobi matrix of the form

$$L = \begin{bmatrix} B_0 & A_0 & & 0 \\ I_p & B_1 & A_1 & \\ & I_p & B_2 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{1.3}$$

which has the sequence (1.1) as moments in the sense that

$$(L^k)_{00} = h_k, \qquad k = 0, 1, \dots . \tag{1.4}$$

In the *scalar* case $p = 1$, formulas for the entries in a *symmetric* Jacobi matrix in terms of its moments are well known; see, e.g., [1]. These formulas will also be derived below as a special case of the method developed herein; see (3.29). For the symmetric *block* Jacobi matrices the inverse spectral problem was extensively studied in [3]. An explicit solution for the (nonsymmetric) block matrix case was presented recently in [11]; see Procedure 2 in our paper. We propose a new set of formulas (3.20) and (3.21) for a solution of the GISP which seem to be a more natural generalization of the classical scalar case and, at the same time, require significantly less computational effort. The latter is important, since the solution of the GISP is a key step in the integration of matrix Toda-like nonlinear equations by the inverse spectral problem method; see, e.g., [4, 11].

Three-term recursions (i.e., Jacobi matrices) corresponding to strongly regular Hankel matrices appear in the theory of systems, continued fractions, Padé approximation, and many other fields (see, e.g., [10], [13], and [5]). We remark that the coefficients of these recursions can be computed within a general framework of solving the appropriate GISP.

The paper is organized as follows. In Section 2 we shall introduce in full generality the notation of the generalized spectral function (GSF) and then review some facts about orthogonal matrix polynomials. This machinery will be used in Section 3 to establish a three-term recursion for the block Hankel case and to solve the GISP for block Jacobi matrices. In this section we shall also establish a useful parametrization of a strongly regular block Hankel matrix in terms of certain block entries of an appropriately chosen sequence of block inverses. Then we shall discuss the connections between our results and the classical theory of symmetric Jacobi matrices, and conclude the section with other algorithms which provide a solution of the GISP. In Section 4 we shall turn to the Hermitian case and apply the theory developed earlier, to stability problems for matrix polynomials. By completely elementary means, we shall investigate the root location of matrix polynomials which satisfy a three-term recursion; see Theorem 4.3 below. This theorem will allow us to obtain some sufficient conditions for a matrix polynomial to be stable. In the scalar case they reduce to the well-known Routh-Hurwitz and Liénard-Chipart stability tests. Related results can be found in [6], [8], and [15–19].

Some words on notation: The symbols $\mathbb{C}$, $\mathbb{C}^{p\times p}$, and $\mathbb{R}$ will denote the complex numbers, the space of complex $p \times p$ matrices, and the real numbers, respectively, whereas $\mathbb{C}^p$ is short for $\mathbb{C}^{p\times 1}$; $\mathbb{C}_+$ [$\mathbb{C}_-$] denotes the open upper [lower] half plane, whereas $\overline{\mathbb{C}}_+$ [$\overline{\mathbb{C}}_-$] denotes the closed upper [lower] half plane. The symbol I_n designates the $n \times n$ identity matrix. If A is a matrix, then A^* denotes its adjoint with respect to the standard inner product, and

$$\mathfrak{I}(A) \overset{\text{def}}{\equiv} \frac{A - A^*}{2i}, \qquad \mathfrak{R}(A) \overset{\text{def}}{\equiv} \frac{A + A^*}{2}.$$

For a matrix B with $p \times p$ block entries the symbol $(B)_{st}$ or $\{B\}_{st}$ stands for the st block entry. If $S = [s_0 \ \ s_1 \ \ \cdots \ \ s_n]$ is a block row vector, then it is convenient to denote the ith block entry s_k by $(S)_{0k}$ and its block transpose by $S^{b\tau}$, i.e.,

$$S^{b\tau} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix}.$$

The diagonal and block diagonal matrices will be denoted as

$$\operatorname{diag}(X_0, X_1, \dots, X_n).$$

If $A(\lambda)$ is a matrix-valued function, then $A^{\#}(\lambda) \overset{\text{def}}{\equiv} A(\lambda^*)^*$.

2. PRELIMINARIES

In this section we review some basic facts about matrix polynomials which are orthogonal with respect to a generalized spectral function and then recall some formulas for block triangular factorization. In the next section this will be used to reparametrize a sufficiently regular block Hankel matrix, in terms of the "corner" and "next to corner" block entries of an appropriately chosen sequence of block inverses.

A doubly indexed sequence

$$q_{ij}, \qquad i,j = 0,1,\ldots,$$

of elements from $\mathbb{C}^{p\times p}$ is said to be strongly regular if the block matrix

$$Q_k = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0k} \\ q_{10} & q_{11} & \cdots & q_{1k} \\ \cdot & \cdot & & \cdot \\ q_{k0} & q_{k1} & \cdots & q_{kk} \end{bmatrix} \tag{2.1}$$

is invertible for $k = 0,1,\ldots$.

We shall denote Q_k^{-1} by Γ_k and the ij block entry of Γ_k by $\gamma_{ij}^{(k)}$. Thus

$$\Gamma_k = \left[\gamma_{ij}^{(k)}\right], \qquad i,j = 0,1,\ldots,k,$$

where $\gamma_{ij}^{(k)} \in \mathbb{C}^{p\times p}$.

If Q_k is invertible for $k = 0,1,\ldots,n$, the matrix Q_n will be called nondegenerate.

LEMMA 2.1. *Let Q_n be a nondegenerate block matrix. Then $\gamma_{kk}^{(k)}$ is an invertible $p \times p$ matrix for every $k = 0,1,\ldots,n$.*

Proof. See, e.g., Lemma 3.1 of [6]. ■

Let $\mathscr{P}$ denote the set of polynomials in λ with $p \times p$ matrix coefficients. With any strongly regular sequence q_{ij} we shall associate two sets of polynomials

$$R_k(\lambda) = \sum_{j=0}^{k} \gamma_{kj}^{(k)}\lambda^j \tag{2.2a}$$

and

$$C_k(\lambda) = \sum_{j=0}^{k} \gamma_{jk}^{(k)}\lambda^j \tag{2.2b}$$

based on the last block row and the last block column of the corresponding inverses $\Gamma_k = Q_k^{-1}$ respectively, with invertible top coefficients $\gamma_{kk}^{(k)}$, $k = 0, 1, \ldots$. Consider also the mapping

$$\sigma : \mathscr{P} \times \mathscr{P} \to \mathbb{C}^{p \times p},$$

which is described by the following rule:

$$\sigma\left\{\sum_{i=0}^{n} \alpha_i \lambda^i, \sum_{j=0}^{m} \beta_j \lambda^j\right\} = \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_j^* q_{ji} \alpha_i, \tag{2.3}$$

where $n, m = 0, 1, \ldots$; α_i and β_j in $\mathbb{C}^{p \times p}$, $i = 0, \ldots, n$; $j = 0, \ldots, m$. We shall refer to this mapping as the generalized spectral function (GSF) corresponding to the sequence q_{ij}.

Note that from (2.3) the basic properties of the GSF are immediate:

$$\sigma\{\lambda^n I_p, \lambda^m I_p\} = q_{mn}, \qquad n, m = 0, 1, \ldots; \tag{2.4}$$

$$\sigma\left\{\sum_{i=1}^{2} X_i(\lambda) T_i, \sum_{j=1}^{2} Y_j(\lambda) S_j\right\} = \sum_{i=1}^{2} \sum_{j=1}^{2} S_j^* \cdot \sigma\{X_i(\lambda), Y_j(\lambda)\} \cdot T_i, \tag{2.5}$$

where T_i, S_i are in $\mathbb{C}^{p \times p}$, and $X_i(\lambda)$, $Y_i(\lambda)$ are in $\mathscr{P}$, $i = 1, 2$.

We proceed with the following elementary

LEMMA 2.2. *Let σ be a GSF associated with a strongly regular sequence q_{ij}, and let $C_k(\lambda)$, $R_k(\lambda)$ be the polynomials* (2.2). *Then*

$$\sigma\{\lambda^k I_p, R_n^{\#}\} = \delta_{kn} I_p, \tag{2.6}$$

$$\sigma\{C_n(\lambda), \lambda^k I_p\} = \delta_{kn} I_p \tag{2.7}$$

for $n = 0, 1, \ldots, k = 0, 1, \ldots, n$, and

$$\sigma\{C_j(\lambda), R_i^{\#}(\lambda)\} = \delta_{ij} \cdot \gamma_{jj}^{(j)}, \qquad i, j = 0, 1, \ldots, \tag{2.8}$$

where δ_{st} stands for the Kronecker delta.

Proof. The orthogonality relations (2.6) and (2.7) are just another form of the identity

$$Q_n\Gamma_n = \Gamma_n Q_n = I_{(n+1)p}, \qquad n = 0, 1, \ldots,$$

while the biorthogonality relations (2.8) drop out easily from (2.6) and (2.7). ■

Note that the relations (2.8) can be written in matrix form as follows:

$$\begin{bmatrix} \gamma_{00}^{(0)} & & & 0 \\ \gamma_{10}^{(1)} & \gamma_{11}^{(1)} & & \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n0}^{(n)} & \gamma_{n1}^{(n)} & \cdots & \gamma_{nn}^{(n)} \end{bmatrix}$$

$$\times \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0n} \\ q_{10} & q_{11} & \cdots & q_{1n} \\ \vdots & \vdots & & \vdots \\ q_{n0} & q_{n1} & \cdots & q_{nn} \end{bmatrix} \times \begin{bmatrix} \gamma_{00}^{(0)} & \gamma_{01}^{(1)} & \cdots & \gamma_{0n}^{(n)} \\ & \gamma_{11}^{(1)} & \cdots & \gamma_{1n}^{(n)} \\ & & \ddots & \vdots \\ 0 & & & \gamma_{nn}^{(n)} \end{bmatrix}$$

$$= \operatorname{diag}\left(\gamma_{00}^{(0)}, \gamma_{11}^{(1)}, \ldots, \gamma_{nn}^{(n)}\right).$$

Hence for any strongly regular sequence q_{ij} the following factorization formula holds:

$$\Gamma_n = \begin{bmatrix} \gamma_{00}^{(0)} & \gamma_{01}^{(1)} & \cdots & \gamma_{0n}^{(n)} \\ & \gamma_{11}^{(1)} & \cdots & \gamma_{1n}^{(n)} \\ & & \ddots & \vdots \\ 0 & & & \gamma_{nn}^{(n)} \end{bmatrix} \times \operatorname{diag}\left(\left\{\gamma_{00}^{(0)}\right\}^{-1}, \left\{\gamma_{11}^{(1)}\right\}^{-1}, \ldots, \left\{\gamma_{nn}^{(n)}\right\}^{-1}\right)$$

$$\times \begin{bmatrix} \gamma_{00}^{(0)} & & & 0 \\ \gamma_{10}^{(1)} & \gamma_{11}^{(1)} & & \\ \vdots & \vdots & \ddots & \\ \gamma_{n0}^{(n)} & \gamma(n)_{n1} & \cdots & \gamma_{nn}^{(n)} \end{bmatrix}. \tag{2.9}$$

The well-known formula (2.9) exhibits the fact that the sequence of the last columns and the last rows of all successive inverses $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ uniquely determines the whole matrix Γ_n, and hence also the initial block matrix Q_n.

3. BLOCK HANKEL MATRICES AND GENERALIZED SPECTRAL PROBLEMS

In this section we shall restrict ourselves to the particular case when all the matrices Q_n are block Hankel, i.e.,

$$q_{ij} = h_m$$

for any pair of indices $i, j \geqslant 0$ such that $i + j = m$, $m = 0, 1 \dots$. Accordingly, a sequence of $p \times p$ matrices $h_0, h_1, \dots$ for which all the block Hankel matrices

$$H_k = \left[h_{i+j}\right], \qquad i, j = 0, \dots, k \tag{3.1}$$

are invertible will be referred to as a strongly regular sequence. If, in addition, $h_0 = I_p$, it will sometimes be called a moment sequence.

For this special choice of q_{ij} the corresponding GSF has an important extra property

$$\sigma\{\lambda X(\lambda), Y(\lambda)\} = \sigma\{X(\lambda), \lambda Y(\lambda)\}, \qquad X, Y \text{ in } \mathscr{P}. \tag{3.2}$$

The polynomials $R_k(\lambda)$ and $C_k(\lambda)$ associated with this GSF satisfy the well-known three-term recursion (see, e.g., [13] for the scalar case, [10] for the block version).

THEOREM 3.1. *Let $h_0, h_1, \dots$ be a strongly regular sequence of $p \times p$ matrices, and let $R_k(\lambda), C_k(\lambda)$ be the polynomials based on the last row and the last column of H_k^{-1} respectively, $k = 0, 1, \dots$. Then the following identities hold:*

$$\lambda R_k(\lambda) = R_{k-1}(\lambda) + B_k R_k(\lambda) + A_k R_{k+1}(\lambda), \tag{3.3}$$

$$\lambda C_k(\lambda) = C_{k-1}(\lambda) + C_k(\lambda) \tilde{B}_k + C_{k+1}(\lambda) \tilde{A}_k, \tag{3.4}$$

where $C_{-1} = R_{-1} \overset{\text{def}}{\equiv} 0$ *and* $A_k, B_k, \tilde{A}_k, \tilde{B}_k$ *are specified by the formulas*

$$A_k = \tau_k \cdot \tau_{k+1}^{-1}, \qquad B_k = \tau_k\left(\tau_k^{-1}\delta_k - \tau_{k+1}^{-1}\delta_{k+1}\right)\tau_k^{-1}, \tag{3.5}$$

$$\tilde{A}_k = \tau_{k+1}^{-1} \cdot \tau_k, \qquad \tilde{B}_k = \tau_k^{-1}\left(\varepsilon_k\tau_k^{-1} - \varepsilon_{k+1}\tau_{k+1}^{-1}\right)\tau_k, \tag{3.6}$$

in which

$$\tau_k = \gamma_{kk}^{(k)}, \qquad \delta_k = \gamma_{kk-1}^{(k)}, \qquad \varepsilon_k = \gamma_{k-1,k}^{(k)}$$

for $k = 0, 1, \ldots,$ *subject to the convention that* $\delta_0 = \varepsilon_0 = 0$.

Proof. Since the top coefficient of $R_j(\lambda)$ is invertible for every $j = 0, 1, \ldots$ the matrix polynomial $\lambda R_k(\lambda)$ clearly admits a unique representation as

$$\lambda R_k(\lambda) = \sum_{j=0}^{k+1} F_j^{(k)} R_j(\lambda), \tag{3.7}$$

where $F_j^{(k)}$ are constant $p \times p$ matrices. For $k = 0$ we have

$$\lambda\gamma_{00}^{(0)} = F_0^{(0)} \cdot \gamma_{00}^{(0)} + F_1^{(0)}\left(\gamma_{10}^{(1)} + \gamma_{11}^{(1)} \cdot \lambda\right),$$

hence upon matching the coefficients of λ^1 and λ^0 we obtain

$$A_0 = F_1^{(0)} = \gamma_{00}^{(0)} \cdot \left\{\gamma_{11}^{(1)}\right\}^{-1}, \qquad B_0 = F_0^{(0)} = -\gamma_{00}^{(0)}\left\{\gamma_{11}^{(1)}\right\}^{-1}\gamma_{10}^{(1)}\left\{\gamma_{00}^{(0)}\right\}^{-1}.$$

For $k \geqslant 1$ the three-term recursion (3.3) is easily justified by the usual orthogonality arguments based on the relations (3.2) and (2.6). Then the formulas (3.5) drop out upon matching the coefficients of λ^k and λ^{k+1} in the identity

$$\lambda R_k(\lambda) = F_{k-1}^{(k)} R_{k-1}(\lambda) + F_k^{(k)} R_k(\lambda) + F_{k+1}^{(k)} R_{k+1}(\lambda),$$

while the equality $F_{k-1}^{(k)} = I_p$ is obvious from the following chain of equalities:

$$F_{k-1}^{(k)} = \sigma\left\{\lambda^{k-1}I_p, \left(F_{k-1}^{(k)}R_{k-1}(\lambda)\right)^{\#}\right\} = \sigma\left\{\lambda^{k-1}I_p, \left(\lambda R_k(\lambda)\right)^{\#}\right\}$$
$$= \sigma\left\{\lambda^k I_p, R_k^{\#}(\lambda)\right\} = I_p,$$

where we take advantage of (3.2) and (2.6) once again.

Similar arguments work for the proof of the formulas (3.4) and (3.6). ∎

Now we pass on to the problem of reparametrization of the $2n+1$ independent block entries of the nondegenerate block Hankel matrix H_n in terms of inverses $\Gamma_k = H_k^{-1}$, $k = 0, 1, \dots, n$. By (2.9), the $(n+1)^2$ block entries q_{ij}, $i, j = 0, 1, \dots, n$, of any nondegenerate matrix Q_n can be parametrized by the $(n+1)^2$ block entries

$$\begin{array}{ccccc} \gamma_{00}^{(0)}; & \gamma_{01}^{(1)}, \gamma_{11}^{(1)}, \gamma_{10}^{(1)}; & \dots; & \gamma_{0n}^{(n)}, \gamma_{1n}^{(n)}, \dots, \gamma_{n-1,n}^{(n)}, \gamma_{nn}^{(n)}, \gamma_{nn-1}^{(n)}, \dots, \gamma_{n1}^{(n)}, \gamma_{n0}^{(n)} \\ 1 \quad + & 3 & + \cdots + & 2n+1 \end{array}$$

of the corresponding inverses $\Gamma_0, \Gamma_1, \dots, \Gamma_n$. The matrix expression of the recursions (3.3) and (3.4) gives us the following set of identities:

$$\gamma_{k+1,j}^{(k+1)} = \tau_{k+1}\tau_k^{-1}\left(\gamma_{kj-1}^{(k)} - \gamma_{k-1,j}^{(k-1)} - \delta_k \cdot \tau_k^{-1}\gamma_{kj}^{(k)} + \tau_k\tau_{k+1}^{-1}\delta_{k+1}\tau_k^{-1}\gamma_{kj}^{(k)}\right), \tag{3.8a}$$

$$\gamma_{jk+1}^{(k+1)} = \left(\gamma_{j-1,k}^{(k)} - \gamma_{jk-1}^{(k-1)} - \gamma_{jk}^{(k)}\tau_k^{-1}\varepsilon_k + \gamma_{jk}^{(k)}\tau_k^{-1}\varepsilon_{k+1}\tau_{k+1}^{-1}\tau_k\right)\tau_k^{-1}\tau_{k+1}, \tag{3.8b}$$

$k = 0, 1, \dots,$ $j = 0, 1, \dots, k-1$, with the understanding that $\gamma_{st}^{(l)} = 0$ if either s or t falls outside the permitted set of values $\{0, 1, \dots, l\}$, and $\tau_k = \gamma_{kk}^{(k)}$, $\varepsilon_k = \gamma_{k-1,k}^{(k)}$, $\delta_k = \gamma_{kk-1}^{(k)}$. Apparently, these identities were first obtained in [5] during the study of the matrix Padé problem.

It is easily seen from (3.8) that the sequence of invertible "corners"

$$\tau_0, \tau_1, \dots, \tau_n \tag{3.9}$$

together with the two sequences of their closest neighbors

$$\delta_1, \delta_2, \dots, \delta_n \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \tag{3.10}$$

allow us to compute the last column and the last row of each matrix Γ_k, $k = 0, 1, \dots, n$, and after that to recover the nondegenerate block Hankel matrix H_n. In fact, to parametrize $2n+1$ independent block entries of block Hankel matrix H_n we need only $2n+1$ properly chosen entries of the indicated inverses Γ_k, since one of the sequences in (3.10) turns out to be superfluous.

THEOREM 3.2. *Let $\tau_0, \tau_1, \ldots, \tau_n$ be invertible $p \times p$ matrices, and let $\delta_1, \delta_2, \ldots, \delta_n$ be arbitrary $p \times p$ matrices. Then there exists a unique sequence of $p \times p$ matrices*

$$h_0, h_1, \ldots, h_{2n}$$

such that

(1) *the Hankel matrix $H_n = [h_{i+j}]$, $i, j = 0, \ldots, n$ is nondegenerate, i.e., H_k is invertible for every $k = 0, \ldots, h$;*
(2) $(H_k^{-1})_{kk} = \tau_k$, $k = 0, \ldots, n$;
(3) $(H_k^{-1})_{k\,k-1} = \delta_k$, $k = 1, 2, \ldots, n$.

Proof. Given τ_0, we put $h_0 = \tau_0^{-1}$. If τ_0, τ_1, and δ_1 are known, we have to find only h_1 and h_2 in addition to $h_0 = \tau_0^{-1}$. From the well-known formula (see, e.g., (0.8) in [7]) it follows that

$$\begin{bmatrix} h_0 & h_1 \\ h_1 & h_2 \end{bmatrix}^{-1} = \begin{bmatrix} h_0^{-1} + h_0^{-1}h_1\left(h_0^{\square}\right)^{-1}h_1h_0^{-1} & -h_0^{-1}h_1\left(h_0^{\square}\right)^{-1} \\ -\left(h_0^{\square}\right)^{-1}h_1h_0^{-1} & \left(h_0^{\square}\right)^{-1} \end{bmatrix}, \quad (3.11)$$

provided both h_0 and the Schur complement

$$h_0^{\square} \overset{\text{def}}{\equiv} h_2 - h_1h_0^{-1}h_1$$

are invertible. Let us write down the system of two equations with two unknown variables h_1 and h_2

$$\left(h_2 - h_1h_0^{-1}h_1\right)^{-1} = \tau_1, \qquad -\left(h_2 - h_1h_0^{-1}h_1\right)^{-1}h_1h_0^{-1} = \delta_1, \quad (3.12)$$

where $h_0 = \tau_0^{-1}$ is already found, δ_1 and invertible τ_1 are given. This system has a unique solution

$$h_1 = -\tau_1^{-1}\delta_1\tau_0^{-1},$$

$$h_2 = \tau_1^{-1} + \tau_1^{-1}\delta_1\tau_1^{-1}\delta_1\tau_0^{-1}.$$

Thus, for $n = 1$ the assertion holds.

We proceed by induction. Assume that $\tau_0, \dots, t_k, \tau_{k+1}$ and $\delta_1, \dots, \delta_k, \delta_{k+1}$ are given and the sequence $h_0, \dots, h_{2k}$ with required properties is known. We have to find h_{2k+1} and h_{2k+2}. To this end we consider the block Hankel matrix H_{k+1} in the form

$$H_{k+1} = \begin{bmatrix} H_k & T \\ S & h_{2k+2} \end{bmatrix}$$

where H_k is a known invertible block Hankel matrix, and

$$S = [h_{k+1} \quad h_{k+2} \quad \cdots \quad h_{2k+1}] \tag{3.13}$$

and

$$T = [h_{k+1} \quad h_{k+2} \quad \cdots \quad h_{2k+1}]^{b\tau} \tag{3.14}$$

are block vectors with unknown last $p \times p$ block h_{2k+1}; the $p \times p$ matrix h_{2k+2} is to be found as well. By the same formula for the inverse of block matrix we can write

$$H_{k+1}^{-1} = \begin{bmatrix} H_k^{-1} + H_k^{-1}T\left(H_k^{\square}\right)^{-1}SH_k^{-1} & -H_k^{-1}T\left(H_k^{\square}\right)^{-1} \\ -\left(H_k^{\square}\right)^{-1}SH_k^{-1} & \left(H_k^{\square}\right)^{-1} \end{bmatrix}, \tag{3.15}$$

where $H_k^{\square} \overset{\text{def}}{\equiv} h_{2k+2} - SH_k^{-1}T$ is assumed to be invertible.

It is easy to verify that the system

$$h_{2k+2} - SH_k^{-1}T = \tau_{k+1}^{-1}, \tag{3.16a}$$

$$\left\{-\left(H_k^{\square}\right)^{-1}SH_k^{-1}\right\}_{0k} = \delta_{k+1} \tag{3.16b}$$

or, equivalently,

$$h_{2k+2} = \tau_{k+1}^{-1} + SH_k^{-1}T, \tag{3.17a}$$

$$\left(SH_k^{-1}\right)_{0k} = -\tau_{k+1}^{-1}\delta_{k+1} \tag{3.17b}$$

has a unique solution for h_{2k+1} and h_{2k+2}. Indeed, from the equation

(3.17b), bearing in mind (3.13), we obtain the following equation with unknown variable h_{2k+1}:

$$h_{k+1}\left(H_k^{-1}\right)_{0k} + h_{k+2}\left(H_k^{-1}\right)_{1k} + \cdots + h_{2k+1}\tau_k = -\tau_{k+1}^{-1}\delta_{k+1}, \quad (3.18)$$

where $\tau_k = (H_k^{-1})_{kk}$ by the inductive assumption. In virtue of the invertibility of τ_k there exists only one h_{2k+1} which satisfies (3.18), namely

$$h_{2k+1} = -\left(\tau_{k+1}^{-1}\delta_{k+1} + \sum_{j=1}^{k} h_{k+j}\left\{H_k^{-1}\right\}_{j-1,k}\right)\tau_k^{-1}.$$

Accordingly, the block vectors S and T are uniquely determined, and (3.17a) gives a unique possible value for h_{2k+2}. The rest is plain. ∎

THEOREM 3.3. *Let $\tau_0, \ldots, \tau_n$ be invertible $p \times p$ matrices and $\varepsilon_1, \ldots, \varepsilon_n$ be arbitrary $p \times p$ matrices. Then there exists a unique sequence of $p \times p$ matrices*

$$h_0, h_1, \ldots, h_{2n}$$

such that

(1) *the block Hankel matrix $H_n = [h_{i+j}]$, $i, j = 0, \ldots, n$, is nondegenerate;*

(2) $(H_k^{-1})_{kk} = \tau_k$, $k = 0, \ldots, h$;

(3) $(H_k^{-1})_{k-1,k} = \varepsilon_k$, $k = 1, \ldots, h$.

Proof. The proof is carried out in much the same way as the proof of the preceding theorem, but with the equation

$$\left\{-H_k^{-1}T\left(H_k^{\square}\right)^{-1}\right\}_{k0} = \varepsilon_{k+1}$$

instead of (3.16b). ∎

Our next objective is to find the relationship between the invertible block

Hankel matrices and the semiinfinite block Jacobi matrices of the form

$$L = \begin{bmatrix} B_0 & A_0 & & 0 \\ I_p & B_1 & A_1 & \\ & I_p & B_2 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}, \tag{3.19}$$

where A_k is invertible and B_k is arbitrary for $k = 0, 1, \dots$.

Theorem 3.1 shows that any strongly regular sequence $h_0, h_1, \dots$ generates the block Jacobi matrix (3.19) by the following recipe.

PROCEDURE 1.

1. From initial data $h_0, h_1, \dots$ construct invertible block Hankel matrices $H_k = [h_{i+j}]$, $i, j = 0, 1, \dots, k$, $k = 0, 1, \dots$.
2. Find all the last block row vectors

$$\begin{bmatrix} \gamma_{k0}^{(k)} & \gamma_{k1}^{(k)} & \cdots & \gamma_{kk}^{(k)} \end{bmatrix}$$

of matrices $\Gamma_k = H_k^{-1}$, $k = 0, 1, \dots$.
3. Take the invertible "corners" $\tau_k = \gamma_{kk}^{(k)}$ and "next-to-the-corner" block entries $\delta_k = \gamma_{k\,k-1}^{(k)}$ of the matrix Γ_k, $k = 0, 1, \dots$; $\delta_0 = 0$.
4. Compute A_k and B_k via the formulas

$$A_k = \tau_k \tau_{k+1}^{-1}, \tag{3.20}$$

$$B_k = \delta_k \tau_k^{-1} - \tau_k \tau_{k+1}^{-1} \delta_{k+1} \tau_k^{-1}, \qquad k = 0, 1, \dots . \tag{3.21}$$

If $h_0 = I_p$, then Procedure 1 gives a solution of the GISP, as the following theorem shows.

THEOREM 3.4. *Let $h_0, h_1, \dots$ be a strongly regular sequence of $p \times p$ matrices, and let L be a block Jacobi matrix of the form* (3.19) *where the entries A_j and B_j, $j = 0, 1, \dots,$ are specified by Procedure* 1. *Then all the matrices A_k are invertible, and the initial sequence may be reconstructed by the formulas*

$$h_k = h_0 \cdot (L^k)_{00}, \qquad k = 0, 1, \dots, \tag{3.22}$$

with the understanding that $(L^0)_{00} = I_p$.

REMARK 3.1. All the powers of the semiinfinite block Jacobi matrix L are well defined, since after any number of multiplications of L with itself in every row and column there are only a finite number of nonzero $p \times p$ block entries.

REMARK 3.2. To calculate $A_0, A_1, \dots, A_n$ and $B_0, B_1, \dots, B_n$ by Procedure 1 we need only $h_0, h_1, \dots, h_{2n+2}$. And vice versa, the moments $h_0, h_1, \dots, h_{2n}$, when computed by the formula (3.22), depend only on $A_0, A_1, \dots, A_{n-1}$ and $B_0, B_1, \dots, B_{n-1}$, $n = 1, 2, \dots$.

Proof. The invertibility of all the matrices A_k is obvious from (3.20) and Lemma 2.1. Let

$$R(\lambda) = \begin{bmatrix} R_0(\lambda) & R_1(\lambda) & \cdots & R_k(\lambda) & \cdots \end{bmatrix}^{b\tau} \tag{3.23}$$

denote the infinite block column vector, where the matrix polynomials $R_k(\lambda)$ are defined to have the coefficients from the last block row of $\Gamma_k = H_k^{-1}$ as in (2.2a), $k = 0, 1, \dots$. Then it readily follows from (3.3) and (3.19) that

$$LR(\lambda) = \lambda R(\lambda) \tag{3.24}$$

in the natural sense of block matrix multiplication, where we identify objects block by block. There are no difficulties of convergence, because there are finitely many nonzero terms involved in computing the entries in each $p \times p$ block on the left-hand side of (3.24).

Clearly (3.24) implies that

$$L^k R(\lambda) = \lambda^k R(\lambda), \qquad k = 1, 2, 3, \dots, \tag{3.25}$$

and hence by comparing the upper $p \times p$ blocks we have

$$(L^k)_{00} R_0(\lambda) + (L^k)_{01} R_1(\lambda) + \cdots + (L^k)_{0k} R_k(\lambda) = \lambda^k R_0(\lambda). \tag{3.26}$$

The basic properties of the GSF associated with the initial strongly regular sequence $h_0, h_1, \dots$, and the relations (2.6) imply that

$$\begin{aligned} \sigma\left\{ I_p, \left(\sum_{j=0}^{k} (L^k)_{0j} R_j(\lambda) \right)^{\#} \right\} &= \sum_{j=0}^{k} (L^k)_{0j} \sigma\{ I_p, R_j^{\#}(\lambda) \} \\ &= \sum_{j=0}^{k} (L^k)_{0j} \delta_{0j} \cdot I_p \\ &= (L^k)_{00} \end{aligned}$$

and

$$\begin{aligned}\sigma\left\{I_p, \left(\lambda^k R_0(\lambda)\right)^{\#}\right\} &= \sigma\left\{I_p, \lambda^k I_p \gamma_{00}^{(0)}\right\} \\ &= \gamma_{00}^{(0)} \sigma\left\{I_p, \lambda^k I_p\right\} \\ &= \gamma_{00}^{(0)} h_k = h_0^{-1} h_k.\end{aligned}$$

Thus from (3.26) we obtain

$$(L^k)_{00} = h_0^{-1} h_k,$$

as required. ■

At first glance it would seem that the entries in the Jacobi matrix which are generated by a strongly regular sequence $h_0, h_1, \ldots$ must be very special. In fact, the matrices A_k and B_k are completely arbitrary except that A_k must be invertible.

THEOREM 3.5. *Let L be any block Jacobi matrix of the form* (3.19) *with invertible block entries on the upper diagonal. Then the sequence*

$$g_0 = I_p, \qquad g_1 = (L^1)_{00} = B_0, \qquad g_2 = (L^2)_{00}, \ldots, \qquad g_k = (L^k)_{00}, \ldots$$

is strongly regular (i.e., it is a moment sequence). Moreover, the Jacobi matrix which is built from $g_0, g_1, \ldots$ *by Procedure* 1 *coincides with the initial matrix L.*

Proof. Starting from the given matrix L let us construct a sequence of matrix polynomials W_k, $k = 0, 1, \ldots$ by the recurrence relations

$$\lambda W_k(\lambda) = W_{k-1}(\lambda) + B_k W_k(\lambda) + A_k W_{k+1}(\lambda)$$

with the initial values $W_{-1} = 0$, $W_0 = I_p$. Clearly, every polynomial $W_k(\lambda)$ is of degree k with invertible leading coefficient t_k, where

$$t_0 = I_p \quad \text{and} \quad t_k = A_{k-1}^{-1} A_{k-2}^{-1} \cdots A_0^{-1}, \quad k = 1, 2, \ldots. \tag{3.27}$$

Let s_k be the coefficient of λ^{k-1} in the polynomial $W_k(\lambda)$, $k = 1, 2, \ldots$. It is easily seen that

$$s_{k+1} = A_k^{-1}(s_k - B_k t_k), \quad k = 1, 2, \ldots; \qquad s_1 = -A_0^{-1} B_0 t_0. \tag{3.28}$$

Theorem 3.2 shows that the two sequences t_k and s_k, $k = 0, 1, \dots$, generate a strongly regular sequence of $p \times p$ matrices $h_0 = t_0^{-1} = I_p, h_1, h_2, \dots$ such that $t_k = (H_k^{-1})_{kk}$ and $s_k = (H_k^{-1})_{k\,k-1}$. Let $\hat{L}$ be a Jacobi matrix of the form (3.19) which is obtained from $h_0, h_1, \dots$ by Procedure 1, i.e.,

$$\hat{A}_k = t_k \cdot t_{k+1}^{-1},$$

$$\hat{B}_k = t_k\left(t_k^{-1}s_k - t_{k+1}^{-1}s_{k+1}\right)t_k^{-1}, \qquad k = 0, 1, \dots, s_0 = 0.$$

From (3.27) and (3.28) it is clear now that $A_k = \hat{A}_k$ and $B_k = \hat{B}_k$, or equivalently

$$L = \hat{L}.$$

By taking advantage of Theorem 3.4 we can write

$$g_k = (L^k)_{00} = (\hat{L}^k)_{00} = h_0^{-1}h_k = I_p \cdot h_k = h_k,$$

i.e., g_k is a strongly regular sequence, since the sequence h_k has this property. ∎

Now we are going to show how the formulas (3.5) can be specialized for the solution of the inverse problem appearing in the classical spectral theory of scalar symmetric Jacobi matrices (see, e.g., [1], [3], [9]).

Let us consider the Jacobi matrix of the form (1.3) with scalar entries:

$$L = \begin{bmatrix} b_0 & a_0^2 & & & 0 \\ 1 & b_1 & a_1^2 & & \\ & 1 & b_2 & a_2^2 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

where $b_k \in \mathbb{R}$, $a_k > 0$, $k = 0, 1, \dots$. Let $h_k = (L^k)_{00}$, $k = 0, 1, \dots$, as before. The first observation is that for the matrix L and for the symmetric Jacobi matrix

$$J = \begin{bmatrix} b_0 & a_0 & & & 0 \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & a_2 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

the moment sequences are the same, i.e., $(L^k)_{00} = (J^k)_{00}$. Indeed, it is easy to verify that

$$L = \Lambda J \Lambda^{-1},$$

where $\Lambda = \operatorname{diag}(1, \lambda_0, \lambda_1, \lambda_2, \dots)$, $\lambda_k = a_0 a_1 \cdots a_k$, and $\Lambda^{-1} = \operatorname{diag}(1, \lambda_0^{-1}, \lambda_1^{-1}, \dots)$. Since $\Lambda^{-1}\Lambda = \operatorname{diag}(1, 1, 1, \dots)$ we have

$$L^k = \Lambda J^k \Lambda^{-1},$$

and hence $h_k = (L^k)_{00} = [1, 0, 0, \dots] L^k [1, 0, 0, \dots]^{b\tau} = (J^k)_{00}$. Note that all the operations are well defined (see the first remark following Theorem 3.4).

The formulas (3.5) imply that

$$a_n^2 = \tau_n \tau_{n+1}^{-1},$$

$$b_n = \tau_n^{-1}\delta_n - \tau_{n+1}^{-1}\delta_{n+1},$$

where

$$\tau_k = \left(H_k^{-1}\right)_{kk},$$

$$\delta_k = \left(H_k^{-1}\right)_{k\,k-1},$$

and

$$H_k = \left[h_{i+j}\right], \qquad i, j = 0, \dots, k.$$

In our scalar symmetric case obviously we have

$$\tau_k = \frac{\det H_{k-1}}{\det H_k} = \frac{D_{k-1}}{D_k},$$

$$\delta_k = -\frac{\Delta_{k-1}}{D_k}, \qquad k = 0, 1, \dots,$$

where $D_k = \det H_k > 0$ and

$$\Delta_{k-1} = \det \begin{bmatrix} h_0 & h_1 & \cdots & h_{k-2} & h_k \\ h_1 & h_2 & \cdots & h_{k-1} & h_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ h_{k-1} & h_k & \cdots & h_{2k-3} & h_{2k-1} \end{bmatrix},$$

$$D_{-1} = 1, \qquad \Delta_{-1} = 0, \qquad \Delta_0 = h_1.$$

Thus,

$$a_n^2 = \frac{D_{n-1}D_{n+1}}{D_n^2},$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}},$$

and we obtain the classical formulas

$$\begin{aligned} a_n &= \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \\ b_n &= \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \qquad n = 0, 1, \ldots, \end{aligned} \tag{3.29}$$

(see, e.g., [1]).

We conclude the section with a discussion of other algorithms which give a solution of the GISP.

Let $h_0 = I_p, h_1, h_2, \ldots$ be any strongly regular sequence of $p \times p$ matrices.

PROCEDURE 2 (M. Gekhtman [11]). For any fixed $n = 0, 1, \ldots$ set

$$h_1^{(0)} = h_1, \qquad h_2^{(0)} = h_2, \ldots, \qquad h_{2n+2}^{(0)} = h_{2n+2}.$$

For every $k = 0, 1, \ldots, n-1$.

1. Find the entries

$$B_k = h_1^{(k)}, \qquad A_k = h_2^{(k)} - \left(h_1^{(k)}\right)^2. \tag{3.30}$$

2. Compute the $p \times p$ matrices $\mathscr{D}_2^{(k)}, \mathscr{D}_3^{(k)}, \dots, \mathscr{D}_{2(n-k+1)}^{(k)}$ by the formulas

$$\mathscr{D}_j^{(k)} = \Sigma_{i_1+i_2+\cdots+i_m=j}(-1)^{m+1}h_{i_1}^{(k)}h_{i_2}^{(k)}\cdots h_{i_m}^{(k)},$$

$$j = 2, 3, \dots, 2(n-k+1).$$

3. Recalculate the sequence $h_1^{(k+1)}, h_2^{(k+1)}, \dots, h_{2(n-k)}^{(k+1)}$ via the formulas

$$h_j^{(k+1)} = \left(\mathscr{D}_2^{(k)}\right)^{-1}\mathscr{D}_{j+2}^{(k)}, \qquad j = 1, 2, \dots, 2(n-k).$$

4. Take

$$B_n = h_1^{(n)}, \qquad A_n = h_2^{(n)} - \left(h_1^{(n)}\right)^2. \tag{3.30'}$$

Let us explain that the summation in step 2 is carried over all 2^{j-1} possible nonzero "partitions with repetitions" of the number j; e.g., since $2 = 2 = 1 + 1$, we have $\mathscr{D}_2^{(k)} = h_2^{(k)} - (h_1^{(k)})^2 = A_k$ (which turns out to be invertible), $3 = 1 + 1 + 1 = 1 + 2 = 2 + 1 = 3$, and hence $\mathscr{D}_3^{(k)} = (h_1^{(k)})^3 - h_1^{(k)}h_2^{(k)} - h_2^{(k)}h_1^{(k)} + h_3^{(k)}$, and so forth.

It follows from Theorem 3.5 that for any moment sequence (1.1) the GISP has a unique solution. Thus, Procedures 1 and 2 give the same block Jacobi matrix. Note that the computational complexity of Procedure 2 is at least $O(2^{2n})$, while the following algorithm adapted from [13] (see also [10]) reduces the complexity of Procedure 1 to $O(n^2)$ operations.

THEOREM 3.6. *The block entries A_k, B_k of the block Jacobi matrix given by Procedure* 1, *as well as the coefficients $R_{kj} \overset{\text{def}}{\equiv} \gamma_{kj}^{(k)}$ of the matrix polynomials $R_k(\lambda)$, $k = 0, 1, \dots, n-1$, $j = 0, \dots, k$, may be computed recursively from the nondegenerate block Hankel matrix*

$$H_n = \left[h_{i+j}\right], \qquad i, j = 0, 1, \dots, n,$$

as follows: *Set $U_{-1} = 0$, $V_{-1} = 0$, $R_{00} = h_0^{-1}$, and then for every $k = 0, 1, \dots, n-1$:*

1. *Compute*

$$U_k = \sum_{j=0}^{k} R_{kj}h_{j+k+1}, \qquad V_k = \sum_{j=0}^{k} R_{kj}h_{j+k+2}. \tag{3.31}$$

2 *Take*

$$B_k = U_k - U_{k-1}, \qquad A_k = V_k - V_{k-1} - B_k U_k. \tag{3.32}$$

3 *Set* $R_{k,-1} = 0,\ R_{k-1,k} = 0,\ R_{k-1,k+1} = 0.$
4 *For every* $j = 0, 1, \dots, k+1$ *recalculate*

$$R_{k+1,j} = A_k^{-1}\left(R_{k,j-1} - R_{k-1,j} - B_k R_{kj}\right). \tag{3.33}$$

Proof. Fix any strongly regular sequence whose first terms coincide with given $h_0, h_1, \dots, h_{2n}$. Such a sequence exists, since H_n is nondegenerate. Let σ be the corresponding GSF.

Using the properties (2.3), (2.5), (2.6), and (3.2) of σ, and the three-term recursion (3.3) in the form

$$\lambda R_k^{\#}(\lambda) = R_{k-1}^{\#}(\lambda) + R_k^{\#}(\lambda) B_k^* + R_{k+1}^{\#}(\lambda) A_k^*,$$

it is easily seen that for every $k = 0, 1, \dots, n-1$

$$\begin{aligned} U_k &= \sum_{j=0}^{k} R_{kj} h_{j+k+1} = \sigma\left\{\lambda^{k+1} I_p, R_k^{\#}(\lambda)\right\} = \sigma\left\{\lambda^k I_p, \lambda^2 R_k^{\#}(\lambda)\right\} \\ &= \sigma\left\{\lambda^k I_p, R_{k-1}^{\#}(\lambda) + R_k^{\#}(\lambda) B_k^* + R_{k+1}^{\#}(\lambda) A_k^*\right\} \\ &= U_{k-1} + B_k, \end{aligned}$$

and similarly that

$$\begin{aligned} V_k &= \sum_{j=0}^{k} R_{kj} h_{j+k+2} = \sigma\left\{\lambda^{k+2} I_p, R_k^{\#}(\lambda)\right\} = \sigma\left\{\lambda^{k+1} I_p, \lambda R_k^{\#}(\lambda)\right\} \\ &= \sigma\left\{\lambda^{k+1} I_p, R_{k-1}^{\#}(\lambda) + R_k^{\#}(\lambda) B_k^* + R_{k+1}^{\#}(\lambda) A_k^*\right\} \\ &= V_{k-1} + B_k U_k + A_k, \end{aligned}$$

where $U_{-1} = V_{-1} = R_{-1}(\lambda) \overset{\text{def}}{\equiv} 0$. Notice that all the matrices A_k are invertible by Theorem 3.4. The rest is immediate from (3.8a). ∎

4. THE HERMITIAN CASE AND SOME ROOT LOCATION PROBLEMS

Let us assume that the $p \times p$ matrices h_k are subject to the constraint

$$h_k^* = h_k, \qquad k = 0, 1, \dots, \tag{4.1}$$

which forces all the block Hankel matrices

$$H_k = \left[h_{i+j}\right], \qquad i, j = 0, \dots, k, \tag{4.2}$$

to be Hermitian. Such a sequence $h_0, h_1, \dots$ will be called a Hermitian sequence. As before, all the block matrices (4.2) are assumed to be invertible. Therefore the parameters

$$\tau_k = \left(H_k^{-1}\right)_{kk}, \quad \delta_k = \left(H_k^{-1}\right)_{kk-1}, \quad k = 0, 1, \dots, \qquad \delta_0 \overset{\text{def}}{\equiv} 0, \tag{4.3}$$

are well defined, and moreover, by Lemma 2.1 all the $p \times p$ matrices τ_k are invertible. Notice that now the corresponding GSF enjoys the following property:

$$\sigma\{X(\lambda), Y(\lambda)\}^* = \sigma\{Y(\lambda), X(\lambda)\}, \qquad X, Y \text{ in } \mathscr{P}. \tag{4.4}$$

Theorem 3.2 shows that in the non-Hermitian case the parameters (4.3) can be completely arbitrary up to the invertibility of the τ_k. Under the additional condition (4.1), they must possess a certain symmetry.

THEOREM 4.1. *Let $h_0, h_1, \dots$ be a strongly regular sequence. Then it is Hermitian if and only if the following two conditions hold:*

(1) *The invertible matrices τ_k are Hermitian, i.e.,*

$$\tau_k^* = \tau_k. \tag{4.5}$$

(2) *One has*

$$\left(\delta_k - \tau_k \tau_{k+1}^{-1} \delta_{k+1}\right)^* = \delta_k - \tau_k \tau_{k+1}^{-1} \delta_{k+1}, \tag{4.6}$$

where $k = 0, 1, \dots,$ $\delta_0 \overset{\text{def}}{\equiv} 0$.

Proof. Let the $p \times p$ matrices τ_k and δ_k satisfy assumptions (1) and (2) of the theorem. Obviously, $h_0 = \tau_0^{-1}$ is a Hermitian matrix. Fix any $n = 1, 2, \dots$. We are going to prove that $h_1, h_2, \dots, h_{2n}$ are all Hermitian too. Let us take the $p \times p$ matrices A_k and B_k which are specified by (3.5):

$$A_k = \tau_k \cdot \tau_{k+1}^{-1}, \qquad B_k = \tau_k\left(\tau_k^{-1}\delta_k - \tau_{k+1}^{-1}\delta_{k+1}\right)\tau_k^{-1}, \tag{4.7}$$

and consider the finite block Jacobi matrix

$$L = \begin{bmatrix} B_0 & A_0 & & & 0 \\ I_p & B_1 & A_1 & & \\ & I_p & B_2 & \ddots & \\ & & \ddots & \ddots & A_{n-1} \\ 0 & & & I_p & B_n \end{bmatrix}. \tag{4.8}$$

The block diagonal matrix

$$T = \operatorname{diag}(\tau_0, \tau_1, \dots, \tau_n) \tag{4.9}$$

is clearly invertible and Hermitian. Assumption (2) implies that

$$(B_k\tau_k)^* = B_k\tau_k. \tag{4.10}$$

It is readily checked now that

$$(LT)^* = LT,$$

$$L^* = T^{-1}LT,$$

and

$$L^{*k} = T^{-1}L^kT. \tag{4.11}$$

By matching the upper left-hand-side blocks we obtain

$$(L^{*k})_{00} = \tau_0^{-1}(L^k)_{00}\tau_0. \tag{4.12}$$

Since $h_0^* = h_0$, we get

$$\{h_0(L^k)_{00}\}^* = h_0(L^k)_{00}$$

from (4.12). Then the formula (3.22) and the second remark following Theorem 3.4 yield

$$h_k^* = h_k, \qquad k = 0, 1, \dots, 2n,$$

which proves the sufficiency.

Assume now that $h_0, h_1, \dots$ is a Hermitian strongly regular sequence. Then $\Gamma_k = (H_k^{-1})$ is a Hermitian matrix, and hence (4.5) holds. To prove (4.6), let us consider the identity (3.3) in the form

$$\lambda R_k^{\#}(\lambda) = R_{k-1}^{\#}(\lambda) + R_k^{\#}(\lambda) B_k^* + R_{k+1}^{\#}(\lambda) A_k^*, \qquad (4.13)$$

where $k = 0, 1, \dots,$ and $R_{-1} \stackrel{\text{def}}{\equiv} 0$. Since the assumption (4.1) implies that

$$C_k(\lambda) = R_k^{\#}(\lambda), \qquad (4.14)$$

it follows from (2.5), (2.8), and (4.13) that

$$\sigma\{R_k^{\#}(\lambda), \lambda R_k^{\#}(\lambda)\} = B_k \tau_k. \qquad (4.15)$$

The properties (3.2) and (4.4) of the GSF show that the left-hand side of (4.15) is Hermitian, and therefore so is the right-hand side. The rest is plain. ∎

COROLLARY. *A block Jacobi matrix* L *of the form* (3.19) *generates a Hermitian moment sequence* $h_k = (L^k)_{00} = h_k^*$, $k = 0, 1, \dots,$ *if and only if*

(1′) $(A_0 A_1 \cdots A_j)^* = A_0 A_1 \cdots A_j$,
(2′) $B_0^* = B_0$, $(A_0 A_1 \cdots A_j B_{j+1})^* = A_0 A_1 \cdots A_j B_{j+1}$, $j = 0, 1, \dots$.

Proof. The conditions (1′) and (2′) are actually a rewritten form of (1) and (2) of Theorem 4.1, where the formulas (4.7) and $h_0 = I_p$ are taken into account. ∎

The Hermitian sequence $h_0, h_1, \dots$ is called positive definite if

$$H_k > 0, \qquad k = 0, 1, \dots . \qquad (4.16)$$

This case can be characterized as follows (compare with [10, p. 98]):

THEOREM 4.2. *A sequence $h_0, h_1, \dots$ is positive definite if and only if*

$$\tau_j > 0, \tag{4.17}$$

$$\left(\delta_k - \tau_k \tau_{k+1}^{-1} \delta_{k+1}\right)^* = \delta_k - \tau_k \tau_{k+1}^{-1} \delta_{k+1}. \tag{4.18}$$

Proof. The statement of this theorem is an immediate consequence of Theorem 4.1 and the relations (2.8), (2.9), and (4.14). ∎

Our next objective is to study the root location of some special linear combinations of the orthogonal matrix polynomials $C_k(\lambda)$. Let us recall that a complex number ω is defined to be a root of $p \times p$ matrix polynomial $X(\lambda)$ if

$$\det X(\omega) = 0.$$

THEOREM 4.3. *Let $h_0, h_1, \dots$ be a positive definite strongly regular sequence. Let*

$$W_k = \Im(P^* \tau_k Q), \qquad k = 1, 2, \dots,$$

where P and Q are $p \times p$ constant matrices such that Q is invertible, $\Im$ stands for the imaginary part of the indicated matrix, and τ_k is given by (4.3).

Then all the roots of the p × p matrix polynomial

$$X_k(\lambda) = C_{k-1}(\lambda) P + C_k(\lambda) Q \tag{4.19}$$

belong to:

(1) $\mathbb{C}_+$ $[\overline{\mathbb{C}}_+]$ if $W_k > 0$ $[W_k \geqslant 0]$;
(2) $\mathbb{C}_-$ $[\overline{\mathbb{C}}_-]$ if $W_k < 0$ $[W_k \leqslant 0]$;
(3) $\mathbb{R}$ if $W_k = 0$.

Proof. Let ω be a root of the matrix polynomial $X_k(\lambda)$. Then for some nonzero $v \in \mathbb{C}^p$

$$X_k(\omega) v = 0.$$

Consider the matrix polynomial

$$Y(\lambda) = \frac{X_k(\lambda) - X_k(\omega)}{\lambda - \omega} = \sum_{j=0}^{k-1} Y_j \lambda^j, \tag{4.20}$$

which is of degree at most $k-1$ (in λ). Then

$$\omega Y(\lambda) = \lambda Y(\lambda) - X_k(\lambda) + Z, \tag{4.21}$$

where $Z = X_k(\omega)$. Therefore

$$\begin{aligned}\omega^* \cdot \sigma\{Y(\lambda), Y(\lambda)\} &= \sigma\{Y(\lambda), \lambda Y(\lambda)\} \\ &\quad - \sigma\{Y(\lambda), X_k(\lambda)\} + \sigma\{Y(\lambda), Z\}.\end{aligned} \tag{4.22}$$

By passing to adjoints and bearing (4.4) in mind, we find

$$\begin{aligned}\omega \cdot \sigma\{Y(\lambda), Y(\lambda)\} &= \sigma\{\lambda Y(\lambda), Y(\lambda)\} - \sigma\{X_k(\lambda), Y(\lambda)\} \\ &\quad + \sigma\{Z, Y(\lambda)\}.\end{aligned} \tag{4.23}$$

It is readily seen from (3.2), (4.22), and (4.23) that

$$\begin{aligned}(\omega - \omega^*)\sigma\{Y(\lambda), Y(\lambda)\} &= \sigma\{Y(\lambda), X_k(\lambda)\} - \sigma\{X_k(\lambda), Y(\lambda)\} \\ &\quad + \sigma\{Z, Y(\lambda)\} - \sigma\{Y(\lambda), Z\}.\end{aligned} \tag{4.24}$$

Let us multiply both sides of (4.24) by v^* on the left and by v on the right. Then the last two terms containing Z vanish by (2.3), since $Zv = 0$, and we are left with

$$(\omega - \omega^*) v^* Y v = v^* X v - v^* X^* v, \tag{4.25}$$

where $X = \sigma\{Y(\lambda), X_k(\lambda)\}$ and $Y = \sigma\{Y(\lambda), Y(\lambda)\}$.

Let us figure out the right-hand side of (4.25). Using the orthogonality relation (2.6) and the basic properties of the GSF, we find

$$\begin{aligned}X = \sigma\{Y(\lambda), X_k(\lambda)\} &= \sigma\{Y(\lambda), C_{k-1}(\lambda)P + C_k(\lambda)Q\} \\ &= P^*\sigma\{Y(\lambda), C_{k-1}(\lambda)\} + Q^*\sigma\{Y(\lambda), C_k(\lambda)\} \\ &= P^*\sigma\left\{Y_{k-1}\lambda^{k-1} + \sum_{j=0}^{k-2} Y_j\lambda^j, C_{k-1}(\lambda)\right\} \\ &= P^*\sigma\{Y_{k-1}\lambda^{k-1}, C_{k-1}(\lambda)\} \\ &= P^* Y_{k-1}.\end{aligned}$$

It follows from (4.21) that $Y_{k-1}v = \tau_k Qv$, i.e.,

$$v^*Xv = v^*P^*\tau_k Qv.$$

Let us rewrite (4.25) in the form

$$\frac{\omega - \omega^*}{2i} v^*Yv = v^* \frac{P^*\tau_k Q - (P^*\tau_k Q)^*}{2i} v = v^*\Im(P^*\tau_k Q)v$$
$$= v^*W_k v. \tag{4.26}$$

By the very definition of the GSF we get

$$v^*Yv = \eta^* H_{k-1}\eta, \tag{4.27}$$

where

$$\eta^* = v^*[Y_0^* \quad Y_1^* \quad \cdots \quad Y_{k-1}^*].$$

The vector η is nonzero; otherwise we would have

$$X_k(\lambda)v = (\lambda - \omega)Y(\lambda)v = 0 \qquad \text{for every} \quad \lambda \in \mathbb{C},$$

which contradicts the assumed invertibility of the top coefficient $\tau_k Q$ of $X_k(\lambda)$. Under the conditions of the theorem, all the block Hankel matrices H_j are positive. Therefore the number $v^*Yv = \eta^* H_{k-1}\eta$ is positive too:

$$v^*Yv > 0. \tag{4.28}$$

Finally, (4.26) and (4.28) give us the following equality:

$$\frac{\omega - \omega^*}{2i} = \frac{v^*W_k v}{v^*Yv}. \tag{4.29}$$

The rest is plain. ■

REMARK 4.1. For the scalar case with a Toeplitz matrix in place of a block Hankel, similar arguments were presented in [14, p. 52].

COROLLARY. *Let $h_0, h_1, \dots$ be a positive definite strongly regular sequence. Then for every $k = 1, 2, \dots,$*

(a) *all the roots of the matrix polynomial $C_k(\lambda)$ are real;*
(b) *for $\alpha \in \mathbb{C}$ all the roots of the linear combination*

$$X_{k,\alpha}(\lambda) = C_k(\lambda) + \alpha C_{k-1}(\lambda)$$

are located on the real axis, in $\mathbb{C}_+$, or in $\mathbb{C}_-$ according as $\alpha \in \mathbb{R}$, $\alpha \in \mathbb{C}_+$, or $\alpha \in \mathbb{C}_-$;
(c) *all the roots of the matrix polynomial*

$$E_n(\lambda) = \lambda C_n(\lambda) - C_{n-1}(\lambda) + iC_n(\lambda) \tag{4.30}$$

belong to $\mathbb{C}_-$, while all the roots of

$$F_n(\lambda) = \lambda C_n(\lambda) - C_{n-1}(\lambda) - iC_n(\lambda) \tag{4.31}$$

belong to $\mathbb{C}_+$, $n = 1, 2, \dots$.

Proof. (b): It is enough to take $Q = I_p$, $P = \alpha I_p$ and to invoke the equality (4.29), where now

$$v^* W_k v = v^* \Im(P^* \tau_k Q) v = v^* \Im(\alpha) I_p v = \Im(\alpha) v^* v,$$

and $v^* Y v$ is still positive. To obtain item (a) just set $\alpha = 0$.

(c): Using the three-term recursion (3.4) and relation (4.14), let us represent (4.30) in the form (4.19):

$$E_n(\lambda) = C_n(\lambda) P + C_{n+1} Q,$$

where

$$P = B_n^* + iI_p, \qquad Q = A_n^*,$$

and A_k, B_k are given by (4.7). A simple calculation based on (4.7) and (4.10) shows that in this case

$$W_{n+1} = \Im(P^* \tau_{n+1} Q) = -\tau_n, \tag{4.32}$$

where τ_n is positive definite by Theorem 4.2. Assertion (2) of Theorem 4.3 gives the desired result for the polynomial $E_n(\lambda)$. Similar arguments work for $F_n(\lambda)$ too.

REMARK 4.2. In [15] the effect of perturbing a self-adjoint matrix polynomial by the addition of a matrix polynomial of lower degree was studied. In this context one can interpret the results (a), (b) as follows: the admissible perturbations $\alpha C_{k-1}(\lambda)$ for nonreal α have the effect of shifting all the real roots of $C_k(\lambda)$ off the real axis. Notice that in general the matrix polynomials $C_k(\lambda)$ are not self-adjoint.

REMARK 4.3. The matrix polynomials $E_n(\lambda)$ and $F_n(\lambda)$ were introduced and extensively studied in [6]. In particular, that paper contains a much more general result on the root distribution of these polynomials under essentially weaker assumptions: the Hermitian block Hankel matrix H_n is not restricted to be positive definite, or strongly regular.

In the rest of the section we indicate some applications of Theorem 4.3 to stability problems for matrix polynomials; see Chapter 13 of [17] for a nice exposition of the related *scalar* results. Recall that a matrix polynomial is called stable if all its roots belong to the open left half plane.

THEOREM 4.4. *Let A, B, C, and D be $p \times p$ matrices such that*

$$A > 0, \qquad \Re(B) > 0, \qquad F \overset{\text{def}}{\equiv} C - DB^{-1}A > 0, \qquad DB^{-1}F > 0. \tag{4.33}$$

Then the matrix polynomial

$$M_3(\lambda) = A\lambda^3 + B\lambda^2 + C\lambda + D, \tag{4.34}$$

is stable.

Proof. Let us consider the block Hankel matrix H_3 which is generated by the parameters

$$\tau_0 > 0, \qquad \tau_1 > 0, \qquad \tau_2 > 0, \qquad \tau_3 > 0, \qquad \delta_1 = \delta_2 = \delta_3 = 0,$$

as in Theorem 3.2. It follows from Theorem 4.2 that $H_3 > 0$. The three-term recursion (3.4) has now the following form:

$$\lambda C_k(\lambda) = C_{k-1}(\lambda) + C_{k+1}(\lambda)\tau_{k+1}^{-1}\tau_k, \qquad C_{-1} \overset{\text{def}}{\equiv} 0,$$

which allows us to compute the corresponding polynomials:

$$C_0(\lambda) = \tau_0, \qquad C_1(\lambda) = \tau_1\lambda, \qquad C_2(\lambda) = \tau_2\lambda^2 - \tau_0\tau_1^{-1}\tau_2,$$

and

$$C_3(\lambda) = \tau_3\lambda^3 - \left(\tau_0\tau_1^{-1}\tau_2 + \tau_1\right)\tau_2^{-1}\tau_3\lambda.$$

Let us apply now Theorem 4.3 with

$$\tau_3 = I_p, \qquad \tau_2 = A, \qquad \tau_1 = C - DB^{-1}A, \qquad \tau_0 = DB^{-1}(C - DB^{-1}A),$$

and

$$Q = \tau_3^{-1}\tau_2 = A, \qquad P = i\tau_2^{-1}B = iA^{-1}B. \tag{4.35}$$

Since

$$W_3 = \Im(P^*\tau_3 Q) = \Im(-iB^*) = -\Re(B) < 0, \tag{4.36}$$

all the roots of the matrix polynomial

$$\begin{aligned} X_3(\lambda) &= C_2(\lambda)P + C_3(\lambda)Q \\ &= A\lambda^3 + iB\lambda^2 - C\lambda - iD \end{aligned}$$

belong to $\mathbb{C}_-$. To complete the proof it is enough to observe that

$$M_3(\lambda) = iX_3(i\lambda). \qquad \blacksquare \tag{4.37}$$

In much the same way one can find the sufficient stability conditions for the matrix polynomials of higher degree.

EXAMPLE. Let $p \times p$ matrices A, B, C, D, and E be such that

$$A > 0, \qquad \Re(B) > 0, \qquad C - A > E > 0, \qquad D = B. \tag{4.38}$$

Then the matrix polynomial

$$M_4(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E \tag{4.39}$$

is stable.

To check the stability it is enough to set

$$\tau_4 = I_p, \qquad \tau_3 = A, \qquad \tau_2 = C - A, \qquad \tau_1 = C - A - E,$$
$$\tau_0 = E - E(C-A)^{-1}E,$$

to take $\delta_k = 0$, $k = 1, 2, 3, 4$, and to proceed as in the proof of Theorem 4.4 with $P = iA^{-1}B$ and $Q = A$. Notice that $\tau_0 > 0$, since the assumption $C - A > E$ yields $E^{-1} > (C - A)^{-1}$.

One can obtain the famous Routh-Hurwitz stability test for *scalar* polynomials by using the technique developed above. We shall not pursue this direction here.

Our next objective is to give an independent characterization of the matrix polynomials appearing in Theorem 4.3. This will help to establish some other sufficient conditions for a matrix polynomial to be stable. First, let us consider any nondegenerate Hermitian block Hankel matrix H_k. The corresponding $p \times p$ matrix polynomials $C_{k-1}(\lambda)$ and $C_k(\lambda)$ have the following properties:

(1) The top coefficient τ_k is an invertible Hermitian matrix.

(2) The polynomials are left coprime, i.e., if for some $x \in \mathbb{C}^p$, $\lambda_0 \in \mathbb{C}$ we have $x^* C_k(\lambda_0) = 0$ and $x^* C_{k-1}(\lambda_0) = 0$ simultaneously, then $x = 0$.

(3) They meet the identity

$$C_k(\lambda)\tau_k^{-1}C_{k-1}^{\#}(\lambda) = C_{k-1}(\lambda)\tau_k^{-1}C_k^{\#}(\lambda), \qquad \lambda \in \mathbb{C}, \tag{4.40}$$

i.e., a rational matrix function $W(\lambda) = \tau_k C_k^{-1}(\lambda) C_{k-1}(\lambda)$ is self-adjoint on the real axis:

$$(W(\lambda))^* = W(\lambda), \qquad \lambda \in \mathbb{R}. \tag{4.41}$$

Item (1) is plain from Theorem 4.2, whereas (2) is readily checked by invoking the three-term recursion (3.4), and (3) can be extracted from Theorem 3.1 in [6].

The identity (4.40) shows that the matrix-valued function

$$\Delta(z, y) \overset{\text{def}}{\equiv} \frac{C_k(z)\tau_k^{-1}C_{k-1}^{\#}(y) - C_{k-1}(z)\tau_k^{-1}C_k^{\#}(y)}{z - y}$$

is a polynomial in the scalar variables z and y. Let us write

$$\Delta(z, y) = \sum_{i,j=0}^{k-1} z^i b_{ij} y^j, \qquad b_{ij} \in \mathbb{C}^{p\times p}. \tag{4.42}$$

The $kp \times kp$ matrix

$$B = [b_{ij}], \qquad i, j = 0, 1, \dots, k-1,$$

which consists of $p \times p$ block entries b_{ij}, will be referred to as the (generalized) Bézoutian associated with the quadruple

$$\left\{C_k(\lambda)\tau_k^{-1}, C_{k-1}(\lambda), \left(C_k(\lambda)\tau_k^{-1}\right)^{\#}, C_{k-1}^{\#}(\lambda)\right\}.$$

This definition of the Bézoutian B was introduced by Anderson and Jury, who also proved the congruence of B to a block Hankel matrix which is based on the initial rational function $W(\lambda)$; see Lemma 2.3 in [2]. However, there is yet a closer connection between Bézoutians and Hankel matrices. Namely, if a generalized Bézoutian is invertible, then its inverse itself turns out to be a block Hankel matrix; see, e.g., [21], and [12] for more details. We explore this connection in order to prove the following

THEOREM 4.5. *Let $U(\lambda)$ and $V(\lambda)$ be any pair of left coprime $p \times p$ matrix polynomials of degrees*

$$\deg V(\lambda) < \deg U(\lambda) = k, \tag{4.43}$$

such that the top coefficient U_k of $U(\lambda)$ is an invertible Hermitian matrix, and

$$U(\lambda)U_k^{-1}V^{\#}(\lambda) = V(\lambda)U_k^{-1}U^{\#}(\lambda), \qquad \lambda \in \mathbb{C}. \tag{4.44}$$

Then there exists a sequence $h_0, h_1, \dots, h_{2k}$ of $p \times p$ Hermitian matrices such that:

(1) *The block Hankel matrices $H_{k-1} = [h_{i+j}]$, $i, j = 0, 1, \dots, k-1$, and $H_k = [h_{i+j}]$, $i, j = 0, 1, \dots, k$, are both invertible.*

(2) *The matrix polynomials $C_{k-1}(\lambda)$ and $C_k(\lambda)$ based on the last block column of H_{k-1}^{-1} and H_k^{-1} coincide with $V(\lambda)$ and $U(\lambda)$ respectively.*

Proof. Set

$$R(\lambda) \stackrel{\text{def}}{\equiv} \lambda U(\lambda) - V(\lambda), \qquad S(\lambda) \stackrel{\text{def}}{\equiv} U(\lambda) U_k^{-1}. \tag{4.45}$$

Since (4.44) and (4.45) imply that

$$R(\lambda) S^{\#}(\lambda) = S(\lambda) R^{\#}(\lambda),$$

the Bézoutian of the quadruple $\{R(\lambda), S(\lambda), R^{\#}(\lambda), S^{\#}(\lambda)\}$ is well defined:

$$\Delta(z, y) = \sum_{i,j=0}^{k} z^i b_{ij} y^j, \qquad b_{ij} \in \mathbb{C}^{p \times p}, \tag{4.46}$$

where

$$\Delta(z, y) \stackrel{\text{def}}{\equiv} \frac{R(z) S^{\#}(y) - S(z) R^{\#}(y)}{z - y}. \tag{4.47}$$

Obviously, the matrix polynomial $R(\lambda)$ is of degree $k + 1$, its top coefficient U_k is invertible, and the pair $(R(\lambda), S(\lambda))$ is left coprime. Therefore by Theorem 1.2 of [21], the $(k + 1)p \times (k + 1)p$ Bézoutian matrix

$$B_k = B\{R, S; R^{\#}, S^{\#}\} \tag{4.48}$$

is invertible, and moreover, its inverse is a block Hankel matrix with $p \times p$ block entries. We denote this block Hankel matrix by H_k.

It follows from (4.47) that

$$(\Delta(z^*, y^*))^* = \Delta(y, z).$$

Consequently, $b_{ij}^* = b_{ji}$, i.e., both B_k and H_k are Hermitian matrices.

Using (4.45) let us rewrite (4.46) and (4.47) in the following form:

$$U(z) U_k^{-1} U^{\#}(y) + \frac{U(z) U_k^{-1} V^{\#}(y) - V(z) U_k^{-1} U^{\#}(y)}{z - y} = \sum_{i,j=0}^{k} z^i b_{ij} y^j. \tag{4.49}$$

Since U_k is assumed to be Hermitian, it is easy to see now that the last block column of the block matrix $B_k = (H_k)^{-1}$ is built from the coefficients of the matrix polynomial $U(\lambda)$, and hence

$$U(\lambda) = C_k(\lambda).$$

It follows from (4.49) that

$$b_{ij} = b_{ik}(b_{kk})^{-1}b_{kj} + c_{ij} \qquad \text{for} \quad i,j = 0,1,\dots,k-1, \tag{4.50}$$

where $b_{kk} = U_k$, and $c_{ij} \in \mathbb{C}^{p\times p}$ come from the self-evident identity

$$\frac{U(z)U_k^{-1}V^{\#}(y) - V(z)U_k^{-1}U^{\#}(y)}{z-y} = \sum_{i,j=0}^{k-1} z^i c_{ij} y^j. \tag{4.51}$$

Therefore, Lemma 3.1 in [6] serves to justify the following identification:

$$(H_{k-1})^{-1} = [c_{ij}], \qquad i,j = 0,1,\dots,k-1, \tag{4.52}$$

where H_{k-1} is a $kp \times kp$ submatrix of the block Hankel matrix $H_k = B_k^{-1}$. To complete the proof it remains only to verify that $V(\lambda)$ coincides with $C_{k-1}(\lambda)$. But this is plain from (4.51) and (4.52). ∎

REMARK 4.4. The general results of [6] show that the block Hankel matrix H_k appearing in Theorem 4.5 is unique. Therefore the formula (4.49) can be used to determine the inverse of a Hermitian block Hankel matrix in terms of the solutions of two block equations; see [20], wherein similar problems are discussed. For the precise formulation of the statement along with an independent proof we refer to [12].

COROLLARY. *Two $p \times p$ matrix polynomials $V(\lambda)$ and $U(\lambda)$ can be represented as $C_{k-1}(\lambda)$ and $C_k(\lambda)$ for some positive definite $(k+1)p \times (k+1)p$ block Hankel matrix H_k if and only if:*

(1) $\deg V(\lambda) = k-1$ *and* $\deg U(\lambda) = k$.
(2) *The top coefficient U_k is positive definite.*
(3) *The identity* (4.44) *holds*:

$$U(\lambda)U_k^{-1}V^{\#}(\lambda) = V(\lambda)U_k^{-1}U^{\#}(\lambda).$$

(4) *The Bézoutian*

$$B_{k-1} = B\{S, V; S^{\#}, V^{\#}\}$$

is positive definite, where $S(\lambda) \overset{\text{def}}{\equiv} U(\lambda)U_k^{-1}$.

Proof. Let assumptions (1)–(4) hold. Since the Bézoutian B_{k-1} is invertible, the pair of matrix polynomials $U(\lambda)$ and $V(\lambda)$ is left coprime. They generate two invertible Hermitian block Hankel matrices

$$H_{k-1} = \left[h_{i+j}\right], \quad i, j = 0, 1, \dots, k-1, \quad \text{and} \quad H_k = \left[h_{i+j}\right],$$
$$i, j = 0, 1, \dots, k,$$

as in Theorem 4.5. Under assumption (4), it follows from (4.51) and (4.52) that H_{k-1} is positive definite. Therefore, in view of Theorem 4.2, assumption (2) forces H_k to be positive definite too.

The "only if" part is plain. ■

We have characterized the matrix polynomials to which Theorem 4.3 is applicable. This will help to derive the theorem which to some extent can be regarded as the matrix generalization of the well-known Liénard-Chipart stability test; see [16] for close results.

With this in mind, let us consider a $p \times p$ matrix polynomial

$$A(\lambda) = \sum_{j=0}^{k} A_j \lambda^j, \qquad \det A_k \neq 0. \tag{4.53}$$

We define

$$H(\lambda) = \sum_{j=0}^{l} A_{2j} \lambda^j, \qquad G(\lambda) = \sum_{j=0}^{l} A_{2j+1} \lambda^j,$$

where

$$l = \begin{cases} k/2 & \text{if } k \text{ is even,} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

The obvious representation

$$A(\lambda) = H(\lambda^2) + \lambda G(\lambda^2) \tag{4.54}$$

will be referred to as the LC splitting of $A(\lambda)$; see, e.g., [17, p. 470]. Next, we introduce

$$U(\lambda) = \begin{cases} H(-\lambda^2) & \text{if } k \text{ is even,} \\ \lambda G(-\lambda^2) & \text{if } k \text{ is odd,} \end{cases} \tag{4.55}$$

$$V(\lambda) = \begin{cases} \lambda G(-\lambda^2) & \text{if } k \text{ is even,} \\ -H(-\lambda^2) & \text{if } k \text{ is odd,} \end{cases} \tag{4.56}$$

and notice that

$$A(i\lambda) = c(U(\lambda) + iV(\lambda)), \tag{4.57}$$

where c is a scalar constant. Now, we are ready to prove the following stability result.

THEOREM 4.6. *Let the representation* (4.54) *be the* LC *splitting of a monic matrix polynomial* (4.53). *Let* $U(\lambda)$ *and* $V(\lambda)$, *which are defined as in* (4.55) *and* (4.56), *meet the condition* (4.44). *Assume further that the Bézoutian of the quadruple* $\{U(\lambda), V(\lambda), U^{\#}(\lambda), V^{\#}(\lambda)\}$ *is positive definite. Then the matrix polynomial* $A(\lambda)$ *is stable.*

Proof. Since $A(\lambda)$ is monic, we have $U_k = I_p$ and

$$S(\lambda) \overset{\text{def}}{\equiv} U(\lambda)U_k^{-1} = U(\lambda).$$

By the corollary following Theorem 4.5, the matrix polynomials $U(\lambda)$ and $V(\lambda)$ can be regarded as $C_k(\lambda)$ and $C_{k-1}(\lambda)$ respectively, for some choice of a positive definite block Hankel matrix H_k.

Upon letting

$$Q = I_p \quad \text{and} \quad P = iI_p$$

we find that

$$W_k = \Im(P^*\tau_k Q) = \Im(-iI_p) = -I_p < 0. \tag{4.58}$$

By Theorem 4.3, all the roots of the matrix polynomial

$$X_k(\lambda) = iV(\lambda) + U(\lambda)$$

belong to $\mathbb{C}_-$. Therefore, the formula (4.57) exhibits $A(\lambda)$ as a stable matrix polynomial. ■

We conclude the paper with a generalization of Theorem 4.3 to the case when the Hermitian block Hankel matrix H_n is no longer presumed to be positive definite:

THEOREM 4.7. *Let*

$$H_n = [h_{i+j}], \qquad i, j = 0, 1, \ldots, n,$$

be a Hermitian block Hankel matrix with $p \times p$ *block entries* ($n \geqslant 1$), *such that both* H_n *and*

$$H_{n-1} = [h_{i+j}], \qquad i, j = 0, 1, \ldots, n-1,$$

are invertible. Let

$$W_n = \Im(P^* \tau_n Q), \tag{4.59}$$

where P *and* Q *are* $p \times p$ *constant matrices and* $\tau_n = (H_n^{-1})_{nn}$. *Consider the* $p \times p$ *matrix polynomial*

$$X_n(\lambda) = C_{n-1}(\lambda) P + C_n(\lambda) Q, \tag{4.60}$$

and suppose that W_n *is a definite matrix.*

Then $X_n(\lambda)$ has no real roots, and

$$\delta_{\pm}(X_n) = \begin{cases} \mu_{\pm}(H_{n-1}) & \text{if} \quad W_n > 0, \\ \mu_{\mp}(H_{n-1}) & \text{if} \quad W_n < 0, \end{cases} \tag{4.61}$$

where $\delta_{\pm}(X_n)$ denotes the number of roots of X_n (counting algebraic multiplicities) in $\mathbb{C}_{\pm}$, and $\mu_+(H_{n-1})$ [$\mu_-(H_{n-1})$] stands for the number of positive [negative] eigenvalues of the Hermitian matrix H_{n-1}.

Proof. The desired result can be obtained from Theorem 1.1 in [8], using the identification of the inverse $(H_{n-1})^{-1}$ as the generalized Bézoutian associated with

$$\left\{C_n(\lambda)\tau_n^{-1}, C_{n-1}(\lambda); \left(C_n(\lambda)\tau_n^{-1}\right)^{\#}, \left(C_{n-1}(\lambda)\right)^{\#}\right\}$$

[compare with (4.51) and (4.52)], and taking into account Lemma 3.2 in [6]. ∎

A detailed proof of Theorem 4.7 and its application to improving the stability results discussed earlier will be presented elsewhere.

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