J. Math. Anal. Appl. 387 (2012) 776–790

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Degenerate lower dimensional tori in reversible systems ☆

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ARTICLE INFO

Article history:
Received 29 August 2011
Available online 16 September 2011
Submitted by Richard M. Aron

Keywords:
Reversible systems
KAM iteration
Degenerate lower dimensional tori

ABSTRACT

In this paper we prove the persistence of lower dimensional invariant tori with prescribed frequencies and singular normal matrices in reversible systems. The normal variable is two-dimensional and the unperturbed nonlinear terms in the differential equation for this variable have a special structure.

1. Introduction and main results

We are concerned with the existence of n-dimensional invariant tori of the system:

$$\begin{cases} \dot{x} = \omega(y) + P^1(x, y, u, v), \\ \dot{y} = P^2(x, y, u, v), \\ \dot{u} = A(y)v + P^3(x, y, u, v), \\ \dot{v} = B(y)u + P^4(x, y, u, v), \end{cases} \tag{1.1}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p$ $(n \leqslant m)$, $\omega(y) = (\omega_1(y), \ldots, \omega_n(y)) \in \mathbb{R}^n$ is called frequency vector, and $y \in \mathcal{M} \subset \mathbb{R}^m$, where $\mathcal{M}$ is a bounded open domain. A and B are $p \times p$ matrices, P^1, P^2, P^3 and P^4 are small perturbations.

A dynamical system is reversible if it admits an involutive symmetry G. Let G be defined by

$$G : (x, y, u, v) \to (-x, y, -u, v).$$

Denote the vector field of dynamical system (1.1) by

$$F = \left(\omega + P^1, P^2, Av + P^3, Bu + P^4\right)^T.$$

The system (1.1) is called reversible if

$$\mathcal{D}G \cdot F = -F \circ G, \tag{1.2}$$

where $\mathcal{D}G$ is the differential of G. From (1.2), it is easy to see that system (1.1) is reversible with respect to G when

☆ The first author is supported by Scientific Research Foundation of Huaiyin Institute of Technology grant HGC0922 and the second by the National Natural Science Foundation of China (10826035), (11001048) and the Specialized Research Fund for the Doctoral Program of Higher Education for New Teachers (200802861043).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.030

$$P^1(-x, y, -u, v) = P^1(x, y, u, v), \qquad P^2(-x, y, -u, v) = -P^2(x, y, u, v),$$
$$P^3(-x, y, -u, v) = P^3(x, y, u, v), \qquad P^4(-x, y, -u, v) = -P^4(x, y, u, v).$$

A mapping is said to compatible with the involution G if Φ and G commute. Compatible transformations preserve the reversible structure, that is, they transform reversible systems into reversible systems.

If $P^j = 0$ $(j = 1, 2, 3, 4)$, then reversible system (1.1) becomes

$$\dot{x} = \omega(y), \qquad \dot{y} = 0, \qquad \dot{u} = A(y)v, \qquad \dot{v} = B(y)u. \tag{1.3}$$

The reversible system (1.3) is integrable and admits invariant tori $\mathbb{T}^n \times \{y^0\} \times \{0\} \times \{0\}$ carrying a quasi-periodic flow $x(t) = \omega(y^0)t + x_0$, $y(t) = y^0$, $u(t) = 0$, $v(t) = 0$ with the frequencies $\omega(y^0)$ for all $y^0 \in \mathcal{M}$. Some of the invariant tori can be destroyed by an arbitrarily small perturbations. Whether some invariant tori can persist under small perturbations is an important problem which has been studied for a long time [1–5,9,11–13,16–24,26].

In the special case of $p = 0$, that is, where there are no normal variables (u, v), the above result was obtained and proved by Arnold [1] and Sevryuk [16]. They proved that if $m \geqslant n$ and the unperturbed frequency map $y \to \omega(y)$ is submersive in $\mathcal{M}$, i.e., the rank of its differential is equal to n, then the majority of invariant tori survive small perturbations.

If $p > 0$, the invariant n-tori of reversible system (1.3) are said to be lower dimensional. The persistence of lower dimensional invariant tori for the reversible system (1.1) have been extensively studied under the following assumptions [2,3, 17–20]:

(i) $\mathrm{Det}(\Omega) \neq 0$, where

$$\Omega = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

(ii) Every eigenvalue of the matrix Ω is simple.

Assumption (ii) implies assumption (i). Indeed, since the matrix Ω is infinitesimally reversible, its eigenvalues come in pairs $\lambda, -\lambda$. So, if 0 is an eigenvalue of Ω, its multiplicity is even, i.e., at least 2. For clearness we write out both of them.

Later on, the results were extended by Broer et al. to the case that Ω has the multiple eigenvalues [4] or zero eigenvalues [5]. Let λ_i, $i = 1, 2, \dots, 2p$, be eigenvalues of Ω. In the papers just quoted, Broer et al. strongly required the following non-resonance conditions:

$$\langle k, \omega \rangle \neq 0, \quad \forall k \neq 0, \tag{1.4}$$
$$\sqrt{-1}\langle k, \omega \rangle - \lambda_j \neq 0, \quad \forall k \neq 0,\ j = 1, 2, \dots, 2p, \tag{1.5}$$
$$\sqrt{-1}\langle k, \omega \rangle \pm \lambda_i \pm \lambda_j \neq 0, \quad \forall k \neq 0,\ i, j = 1, 2, \dots, 2p, \tag{1.6}$$

where $k \in \mathbb{Z}^n$. (1.4) is called Diophantine condition. (1.5) and (1.6) are usually called the first Melnikov's condition and the second Melnikov's condition, respectively. These Melnikov's non-resonance conditions are very important for lower dimensional KAM tori for both Hamiltonian systems and reversible systems. But the second Melnikov's condition (1.6) is actually a technical condition. By a KAM skill Xu [26] proved a similar result without the second Melnikov's condition (1.6).

As is well known, the persisting invariant tori usually form a Cantor like family depending on parameters. A natural question is that in what way the KAM tori depend on parameters or how the KAM tori are connected together with parameters. Recently, Wang and Xu [22] obtained some result about the above question in reversible system. More recently, Wang, Xu and Zhang [23,24] obtained some results about the persistence of lower dimensional invariant tori with prescribed frequencies in reversible systems. The problem of quasi-periodic bifurcations in reversible systems is also one of hot issues for the KAM theory, and there are already some well-known results on this problem. See [6,8].

In the papers just quoted, they essentially required the condition that the matrix $A(y)$ is non-singular on $\mathcal{M}$. When the matrix A is non-singular, we can use the linear term Av to control the shift of lower-order terms from small perturbation in KAM steps and so we can completely control the shift of equilibrium point.

If the matrix $A(y)$ is singular at some point $y^0 \in \mathcal{M}$, that is $\det(A(y^0)) = 0$, the previous results cannot give any information on the persistence of the invariant torus $\mathbb{T}^n \times \{y^0\} \times \{0\} \times \{0\}$. Actually, consider the following dynamical system:

$$\dot{x} = \omega(y), \qquad \dot{y} = 0, \qquad \dot{u} = P^3(\epsilon), \qquad \dot{v} = B(y)u, \tag{1.7}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p$, $P^3 = (\epsilon, 0, \dots, 0) \in \mathbb{R}^p$. Then for all $\epsilon > 0$ the reversible system (1.7) has no invariant torus.

If $\det(A(y^0)) = 0$ with $y^0 \in \mathcal{M}$, then the invariant tori $\mathbb{T}^n \times \{y^0\} \times \{0\} \times \{0\}$ of the reversible system (1.3) are called degenerate lower dimensional tori. The purpose of this paper is to obtain some information on the persistence of the degenerate lower dimensional invariant tori for the reversible systems.

There are already some results on degenerate lower dimensional invariant tori for Hamiltonian systems [7,10,29]. But the relation between the papers [7,10,29] and the present paper is rather feeble. Therefore, we do not intend to introduce these results.

So far as we know, there is no similar result for reversible systems. Liu [9] considered the reversible systems of the following form:

$$\begin{cases} \dot{x} = \omega + P^1(x, y, u, v, \omega), \\ \dot{y} = D(\omega) + P^2(x, y, u, v, \omega), \\ \dot{u} = C(\omega)y + A(\omega)v + P^3(x, y, u, v, \omega), \\ \dot{v} = B(\omega)u + P^4(x, y, u, v, \omega), \end{cases} \tag{1.8}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$, ω is an independent parameter varying over a positive measure set $\mathcal{O} \subset \mathbb{R}^n$. Liu [9] replaced the condition $\det \Omega \neq 0$ by the condition $\operatorname{rank}(A, C) = p$ in the reversible system (1.8). A natural question is what happens when $\operatorname{rank}(C, A) < p$? In fact, the result in [9] does not hold in this case.

In this paper, we shall consider the simplest case of this degenerate problem: $p = q = 1$. If we do not want to impose further restriction on the perturbations besides the smallness and smoothness, the higher order terms of the unperturbed integrable system have to be taken into account. To be more precise, we consider the reversible system of the following form:

$$\begin{cases} \dot{x} = \omega_0 + Q(x)y + P^1(x, y, u, v), \\ \dot{y} = P^2(x, y, u, v), \\ \dot{u} = y_m^{2n_0+1} + v^2 + P^3(x, y, u, v), \\ \dot{v} = u + P^4(x, y, u, v), \end{cases} \tag{1.9}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ $(m \geqslant n+1)$, $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$, $n_0 > 0$ is a positive integer, P^1, P^2, P^3 and P^4 are small perturbations. Q is an $n \times m$ matrix. The corresponding involution G is $(x, y, u, v) \to (-x, y, -u, v)$. The purpose of this paper is to study the persistence of the degenerate lower dimensional invariant tori with given frequency ω_0.

To state our results, we first give some definitions and notations.

Let $f(x_1, \ldots, x_n)$ be a continuous function with period 2π in every x_i, $i = 1, 2, \ldots, n$, denote the average of f by

$$[f] = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(x)\, dx_1 \cdots dx_n. \tag{1.10}$$

Let

$$D(s, r) = \bigl\{(x, y, u, v) \in (\mathbb{C}/2\pi\mathbb{Z})^n \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C} \bigm| |\operatorname{Im} x| \leqslant s,\ |y| \leqslant r,\ |u| \leqslant r,\ |v| \leqslant r\bigr\}.$$

If $f(x, y, u, v)$ is analytic on $D(s, r)$, expanding f as Fourier series with respect to x, we have

$$f(x, y, u, v) = \sum_{k \in \mathbb{Z}^n} f_k(y, u, v) e^{\sqrt{-1}\langle k, x\rangle}. \tag{1.11}$$

Since $f_k(y, u, v)$ in (1.11) are analytic in y, u, v around the origin, we have

$$f_k(y, u, v) = \sum_{l \in \mathbb{Z}_+^m, i, j \in \mathbb{Z}_+} f_{klij} y^l u^i v^j.$$

Define

$$\|f\|_{D(s,r)} = \sum_{k \in \mathbb{Z}^n} |\mathbf{M} f_k|_r\, e^{s|k|},$$

where $\mathbf{M} f_k(y, u, v) = \sum_{l,i,j} |f_{klij}| y^l u^i v^j$ and $|\mathbf{M} f_k|_r$ denotes the sup-norm of $\mathbf{M} f_k$ over the domain $D(s, r)$. The following theorem is the main result of this paper.

Theorem 1.1. *Consider the reversible system* (1.9)*. Suppose the perturbation terms P^j $(1 \leqslant j \leqslant 4)$ and $Q(x)$ are real analytic in (x, y, u, v) on $D(s, r)$ and ω_0 satisfies the Diophantine condition*:

$$\bigl|\langle k, \omega_0\rangle\bigr| \geqslant \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \tag{1.12}$$

where $\alpha > 0$ and $\tau > n - 1$ are some constants. Let $Q(x) = (Q_1^0(x), Q_2^0(x))$, where $Q_1^0(x)$ and $Q_2^0(x)$ are $n \times n$ and $n \times (m - n)$ matrices, respectively. We assume that $Q_2^0(x) \equiv 0$ and the average $[Q_1^0]$ of $Q_1^0(x)$ is non-singular. Then there exists a positive constant $\epsilon > 0$, such that if

$$\bigl\|P^j\bigr\|_{D(s,r)} \leqslant \epsilon \quad (j = 1, 2, 3, 4),$$

then the reversible system (1.9) *has an invariant n-torus with ω_0 as the frequency, i.e., the torus persists under small perturbations.*

Remark 1. The previous results cannot provide any information on the persistence of the degenerate lower dimensional invariant tori for the reversible system (1.9). Our result shows that the reversible system (1.9) has a torus with the frequencies ω_0 if P^j $(j = 1, 2, 3, 4)$ are sufficiently small.

Although the paper [5] by Broer et al. allowed Ω to have zero eigenvalues, the results of [5] cannot be applied to the reversible system (1.9). Actually, Broer et al. replaced the condition $\det \Omega \neq 0$ by the condition $\ker \operatorname{ad} N \cap B^+ = \{0\}$ where (in the framework of our paper) $N = \omega_0\partial_x + Av\partial_u + Bu\partial_v$, $\operatorname{ad} N$ is the corresponding adjoint operator in the Lie algebra of vector fields, and B^+ is the space of constant vector fields X with zero y component such that $\mathcal{D}G \cdot X = X \circ G$. For system (1.9), not only the non-degeneracy condition $\det \Omega \neq 0$ is not met but even the condition $\operatorname{ad} N \cap B^+ = \{0\}$ is violated. Liu [9] replaced the condition $\det \Omega \neq 0$ by the condition $\operatorname{rank}(A, C) = p$ in the reversible system (1.8). For system (1.9), the non-degeneracy condition $\operatorname{rank}(A, C) = p$ is also violated.

Remark 2. Consider the following reversible system:

$$\dot{x} = \omega_0 + Q(x)y + P^1, \qquad \dot{y} = P^2, \qquad \dot{u} = v^3 + P^3, \qquad \dot{v} = u + P^4, \tag{1.13}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ $(m \geqslant n)$. Noth that all eigenvalues of the normal matrix $\Omega = \begin{pmatrix} 0 & v^2 \\ 1 & 0 \end{pmatrix}$ in system (1.13) have nonzero real parts if $0 \neq v \in \mathbb{R}$. Then $(u, v) = (0, 0)$ is called a hyperbolic-type point. By the similar method as in [28,29], the authors believe that the reversible system (1.13) also has an invariant torus with ω_0 as the frequency. This is one of subjects of future work.

Remark 3. If $P^j = 0$ $(j = 1, 2, 3, 4)$, then reversible system (1.9) becomes

$$\dot{x} = \omega_0 + Q(x)y, \qquad \dot{y} = 0, \qquad \dot{u} = y_m^{2n_0+1} + v^2, \qquad \dot{v} = u, \tag{1.14}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ $(m \geqslant n + 1)$. The system (1.14) admits an $(m - n)$-parameter continuous family of invariant n-tori which labeled by $y_{n+1}, y_{n+2}, \ldots, y_{m-1}, v$ (with $y_m = -v^{2/(2n+1)}$). Theorem 1.1 shows that the reversible system (1.9) also admits an $(m - n)$-parameter Cantor family of invariant n-tori with frequency vector ω_0.

Remark 4. Noth that the eigenvalues of the normal matrix $\Omega = \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}$ in system (1.14) are nonzero pure imaginary for all $v < 0$. Then $(u, v) = (0, 0)$ is called an elliptic-type point. So the methods in [28,29] cannot be applied to our problem.

In the proof of our result, by the assumption that $\det[Q_1^0] \neq 0$, we can use the first n components of y to remove shifts of tangential frequencies. Moreover, we can also remove the shifts of normal frequencies by the higher order term v^2. Then we keep both tangential frequencies ω_0 and normal frequencies fixed in our KAM steps, so Melnikov's non-resonance conditions always hold without deleting any parameter. The higher order term $y_m^{2n_0+1}$ is used to control the shift of equilibrium point.

2. Proof of the theorems

The previous method cannot be applied to the degenerate case, so we must develop some KAM technique for our problem. At first we introduce some parameters and change the reversible system (1.9) to a parameterized system and this idea is used in [14] for Hamiltonian systems.

Let $y_m = \xi + y_m^+$. Then the reversible system (1.9) becomes

$$\begin{cases} \dot{x} = \omega_0 + Q(x)y + f^1(x, y, u, v, \xi), \\ \dot{y} = f^2(x, y, u, v, \xi), \\ \dot{u} = N(\xi) + C(\xi)y + \langle y, E(\xi)y\rangle + v^2 + f^3(x, y, u, v, \xi), \\ \dot{v} = u + f^4(x, y, u, v, \xi), \end{cases} \tag{2.1}$$

where

$$N(\xi) = \xi^{2n_0+1}, \qquad C(\xi) = \big(0, 0, \ldots, 0, (2n_0 + 1)\xi^{2n_0}\big),$$

$$E(\xi) = \operatorname{diag}\big(0, \ldots, 0, (2n_0 + 1)n_0\xi^{2n_0-1}\big),$$

$$f^i = P^i(x, y_1, y_2, \ldots, y_{m-1}, y_m + \xi, u, v), \quad i = 1, 2, 4,$$

$$f^3 = P^3(x, y_1, y_2, \ldots, y_{m-1}, y_m + \xi, u, v) + \big((y_m + \xi)^{2n_0+1} - \xi^{2n_0+1} - C(\xi)y - \langle y, E(\xi)y\rangle\big),$$

and $\xi \in \Pi = [-\delta, \delta] \subset \mathbb{R}$ is regarded as a parameter. Note that we have used y_m instead of the new variable y_m^+ in the transformed equations for simplicity.

Now we consider the parameterized reversible system (2.1). Let $\Pi_d = \{\xi \in \mathbb{C} \mid \operatorname{dist}(\xi, \Pi) \leqslant d\}$. We choose $d = \delta = \epsilon^{\frac{1}{2n_0+4}}$ and $r = \delta^2 = \epsilon^{\frac{1}{n_0+2}}$. Then it is easy to see that $f^i(x, y, u, v, \xi)$ $(i = 1, 2, 3, 4)$ are analytic on $D(s, r) \times \Pi_d$ and $\|f^i\|_{D(s,r)\times\Pi_d} \leqslant c\epsilon$ $(i = 1, 2, 3, 4)$. If $f^i = 0$ $(i = 1, 2, 3, 4)$, then reversible system (2.1) becomes

$$\dot{x} = \omega_0 + Q(x)y, \qquad \dot{y} = 0, \qquad \dot{u} = N(\xi) + C(\xi)y + \langle y, E(\xi)y\rangle + v^2, \qquad \dot{v} = u. \tag{2.2}$$

For $\xi = 0$ the reversible system (2.2) has a lower dimensional invariant torus with ω_0 as frequency. Now we want to prove that if f^i $(i = 1, 2, 3, 4)$ are sufficiently small, there exists sufficiently small parameters $\xi_* \in \Pi$ such that at $\xi = \xi_*$, the reversible system (2.1) also has a lower dimensional invariant torus with ω_0 as frequency.

We use the Herman method to prove Theorem 1.1. The Herman method is a well-known KAM technique that introduces an artificial external parameter to make the unperturbed system highly non-degenerate. This method has been used in [2, 19,20,23,27].

Now we introduce an artificial external parameter and consider the following reversible system:

$$\begin{cases} \dot{x} = \omega_0 + Q(x)y + f^1(x, y, u, v, \xi), \\ \dot{y} = f^2(x, y, u, v, \xi), \\ \dot{u} = N(\xi, \gamma) + C(\xi)y + \langle y, E(\xi)y\rangle + v^2 + f^3(x, y, u, v, \xi), \\ \dot{v} = u + f^4(x, y, u, v, \xi), \end{cases} \tag{2.3}$$

where $N(\xi, \gamma) = \xi^{2n_0+1} + \gamma$, $\gamma \in \mathbb{R}$ is an artificial external parameter. The reversible system (2.1) corresponds to the reversible system (2.3) with $\gamma = 0$. We will give a KAM theorem for the reversible system (2.3) with parameters (ξ, γ) and then prove Theorem 1.1.

Define

$$B(\Gamma, d) = \{\gamma \in \mathbb{C} \mid \operatorname{dist}(\gamma, \Gamma) \leqslant d\}$$

the complex d-neighborhood of Γ in 1-dimensional complex space $\mathbb{C}$. Let $\mu = \max_{\xi\in\Pi} |\xi^{2n_0+1}| = \delta^{2n_0+1}$ and $M = \Pi_d \times B(0, 2\mu + 1)$. If $f(x, y, u, v, \xi, \gamma)$ is analytic on $D(s, r) \times M$, then

$$f(x, y, u, v, \xi, \gamma) = \sum_{k\in\mathbb{Z}^n, l\in\mathbb{Z}_+^m, i, j\in\mathbb{Z}_+} f_{klij}(\xi, \gamma) y^l u^i v^j e^{\sqrt{-1}\langle k,x\rangle}.$$

Define

$$\|f\|_{D(s,r)\times M} = \sum_{k\in\mathbb{Z}^n} |\mathbf{M} f_k|_{D(s,r)\times M}\, e^{s|k|},$$

where $\mathbf{M} f_k(y, u, v, \xi, \gamma) = \sum_{l,i,j} |f_{klij}(\xi, \gamma)||y^l u^i v^j|$ and $|\mathbf{M} f_k|_{D(s,r)\times M}$ denotes the sup-norm of $\mathbf{M} f_k$ over the domain $D(s, r) \times M$. Let $f = (f^1, f^2, f^3, f^4)$ be a vector field depending on x, y, u, v, ξ and γ. Define a weighted norm by

$$|||f|||_{D(s,r)\times M} = \frac{1}{r}\|f^1\|_{D(s,r)\times M} + \sum_{i=2}^{4} \frac{1}{r^2}\|f^i\|_{D(s,r)\times M}.$$

Then we have the following theorem.

Theorem 2.1. *Consider the reversible system* (2.3)*, where $Q(x)$ is the same as in Theorem* 1.1*. Suppose that the frequency vector ω_0 satisfies the Diophantine condition*:

$$|\langle k, \omega_0\rangle| \geqslant \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n\backslash\{0\}, \tag{2.4}$$

where $\alpha > 0$ and $\tau > n - 1$ are some constants. Then, there exists an $\epsilon > 0$, such that if

$$\|f^j\|_{D(s,r)\times M} \leqslant \epsilon, \quad j = 1, 2, 3, 4,$$

then in M we have a C^∞-smooth curve

$$\Gamma_*: \quad \gamma = \gamma_*(\xi), \quad \xi \in \Pi,$$

which is determined by the equation

$$\xi^{2n_0+1} + \gamma + \hat{N}_*(\xi, \gamma) = 0, \tag{2.5}$$

where $\hat{N}_*(\xi, \gamma)$ is a C^∞-smooth function on M with

$$|\hat{N}_*(\xi, \gamma)| \leqslant c\epsilon/r \quad \text{and} \quad |\hat{N}_{*\xi}(\xi, \gamma)| + |\hat{N}_{0*\gamma}(\xi, \gamma)| \leqslant \frac{1}{2}.$$

Moreover, we have a parameterized family of compatible transformations

$$\Phi_*(\cdot,\cdot,\cdot,\cdot,\xi,\gamma): D\left(\frac{s}{2},\frac{r}{2}\right) \to D(s,r), \quad (\xi,\gamma)\in\Gamma_*,$$

where Φ_* is analytic in (x,y,u,v) on $D(\frac{s}{2},\frac{r}{2})$ and C^∞-smooth in ξ,γ on Γ_*, such that for each $(\xi,\gamma)\in\Gamma_*$, the compatible transformation $\Phi_*(\cdot,\cdot,\cdot,\cdot,\xi,\gamma)$ transforms the reversible system (2.3) *into*

$$\dot{x}=\omega_0+f_*^1, \qquad \dot{y}=f_*^2, \qquad \dot{u}=C_*y+f_*^3, \qquad \dot{v}=B_*u+f_*^4, \tag{2.6}$$

where f_*^j satisfy $f_*^j(x,0,0,0)=0$ $(j=1,2,3,4)$. Hence, the reversible system (2.3) *has an invariant torus $\Phi_*(\mathbb{T}^n,0,0,0,\xi,\gamma)$ with the frequencies ω_0.*

By Theorem 2.1, we can easily get Theorem 1.1. In fact, by (2.5) and the implicit function theorem we have

$$\gamma_*(\xi)=-\xi^{2n_0+1}+\hat{\gamma}(\xi), \quad \xi\in\Pi.$$

Moreover, if ϵ is sufficiently small, we have $|\hat{\gamma}|\leqslant c\epsilon/r\leqslant c\delta^{2n_0+2}$ for all $\xi\in\Pi$. It follows that $\gamma_*(\pm\delta)=\mp\delta^{2n_0+1}+\hat{\gamma}_0(\pm\delta)$ must have different sign if $\delta>0$ is sufficiently small. Thus there exists $\xi_*\in\Pi$ such that $\gamma_*(\xi_*)=0$. Hence, by the compatible transformation $\Phi_*(\cdot,\cdot,\cdot,\cdot,\xi_*,\gamma_*(\xi_*))=\Phi_*(\cdot,\cdot,\cdot,\cdot,\xi_*,0)$, the reversible system (2.3) is changed into (2.6). Therefore, the reversible system (2.1) has a lower dimensional invariant torus with ω_0 as frequency at $\xi=\xi_*$. This completes the proof of Theorem 1.1.

Now it remains to prove Theorem 2.1. In the following, we will use the KAM iteration to prove Theorem 2.1. In the proof of this theorem, we can remove the shifts of tangential frequencies ω_0 by a small translation of components of y in KAM steps. The existence of such translation of coordinates can be guaranteed by the condition that $\det[Q_1^0]\neq 0$. Moreover, we can also remove the shifts of normal frequencies by a small translation of v. The existence of such translation of coordinates can be guaranteed by the higher order term v^2. Then we keep both tangential frequencies ω_0 and normal frequencies fixed in our KAM steps, so Melnikov's non-resonance conditions always hold without deleting any parameter.

2.1. KAM step

In this section, we give the details of one KAM step. To simplify notations, in what follows, the quantities without subscripts refer to those at the j-th step, while the quantities with subscript "+" denote the corresponding ones at the $(j+1)$-th step. We will use the same notation c to indicate different constants, which are independent of the iteration process.

Suppose at the j-th step, the reversible system is written as

$$\begin{cases} \dot{x}=\omega_0+Q(x,\xi,\gamma)y+W_1(x,\xi,\gamma)u+W_2(x,\xi,\gamma)v+f^1(x,y,u,v,\xi,\gamma), \\ \dot{y}=\displaystyle\sum_{|l|+i+j=2} B_{lij}^1(x,\xi,\gamma)y^lu^iv^j+f^2(x,y,u,v,\xi,\gamma), \\ \dot{u}=N(\xi,\gamma)+C(\xi,\gamma)y+\displaystyle\sum_{|l|+i+j=2} B_{lij}^2(x,\xi,\gamma)y^lu^iv^j+f^3(x,y,u,v,\xi,\gamma), \\ \dot{v}=B(\xi,\gamma)u+\displaystyle\sum_{|l|+i+j=2} B_{lij}^3(x,\xi,\gamma)y^lu^iv^j+f^4(x,y,u,v,\xi,\gamma), \end{cases} \tag{2.7}$$

where $(x,y,u,v,\xi,\gamma)\in D(s,r)\times M$, $N(\xi,\gamma)=\xi^{2n_0+1}+\gamma+\hat{N}(\xi,\gamma)$. Let $\mathcal{Q}=(Q,W_1,W_2)$, $\mathcal{N}=(0_{1\times m},N(\xi,\gamma),0)^T$ with $0_{1\times m}$ being a $1\times m$ zero matrix. Set

$$\mathcal{B}(x)z^2 \triangleq \sum_{|l|+i+j=2}\begin{pmatrix} B_{lij}^1(x) \\ B_{lij}^2(x) \\ B_{lij}^3(x)\end{pmatrix} y^lu^iv^j, \qquad \mathcal{A}=\begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & B & 0\end{pmatrix}, \qquad g=\begin{pmatrix} f^2 \\ f^3 \\ f^4\end{pmatrix}.$$

Then system (2.7) is written as

$$\dot{x}=\omega+\mathcal{Q}(x)z+f^1(x,z), \qquad \dot{z}=\mathcal{N}+\mathcal{A}z+\mathcal{B}(x)z^2+g(x,z). \tag{2.8}$$

Note that in the above equations the parameters (ξ,γ) are implied. We summarize one KAM step in the following lemma.

Lemma 2.2. *Let us consider the above reversible system* (2.8) *with*

$$|||f|||_{D(s,r)\times M}\leqslant\epsilon=\alpha^6E\rho^{6\tau+2}.$$

We assume ω_0 satisfying the Diophantine condition (2.4) and $\hat{N}(p)$ satisfying

$$\left|\hat{N}_\xi(\xi,\gamma)\right| + \left|\hat{N}_\gamma(\xi,\gamma)\right| \leqslant \frac{1}{2}, \quad \forall(\xi,\gamma) \in M. \tag{2.9}$$

Moreover, the equation

$$N(\xi,\gamma) = \xi^{2n_0+1} + \gamma + \hat{N}(\xi,\gamma) = 0,$$

defines implicitly an analytic curve

$$\Gamma: \quad \gamma = \gamma(\xi): \quad \xi \in \Pi_d \to \gamma(\xi) \in B(0, 2\mu+1),$$

such that $\Gamma = \{(\xi,\gamma(\xi)) \mid \xi \in \Pi_d\} \subset M$. Let $\sigma = E^{\frac{4}{5}} r$ we have

$$U(\Gamma,\sigma) = \left\{(\xi,\gamma') \in \Pi_d \times \mathbb{C} \,\middle|\, \left|\gamma' - \gamma(\xi)\right| \leqslant \sigma\right\} \subset M.$$

Let $\eta = E^{\frac{1}{5}}$, $d_+ = d - \frac{\sigma}{2}$, $s_+ = s - 10\rho$, $\rho_+ = \frac{\rho}{2}$, $r_+ = \eta r$, $E_+ = cE^{\frac{6}{5}}$, $\epsilon_+ = \alpha^6 E_+ \rho_+^{6\tau+2}$. Assume that

$$\max\left\{r, E^{\frac{1}{5}}\right\} \leqslant \alpha^6 \rho^{6\tau+2}, \qquad |B|_M > \frac{1}{2}, \qquad \max\left\{\|\mathcal{B} - \mathcal{B}_0\|_{D(s,r)\times M}, \left\|\mathcal{Q}(x) - \mathcal{Q}_0\right\|_{D(s,r)\times M}\right\} \leqslant E_0^{\frac{2}{3}}, \tag{2.10}$$

where E_0 is a parameter that can be chosen sufficiently small, $\mathcal{B}_0(x)z^2 = (0, (2n_0+1)n_0\xi^{2n_0-1}y_m^2 + v^2, 0)^T$, $[\mathcal{Q}_0] = [Q_1^0, 0] \in \mathbb{R}^{n\times(m+2)}$ with $Q_1^0 \in \mathbb{R}^{n\times n}$ being a non-singular matrix. Then, there exists

$$M_+ = \left\{(\xi,\gamma') \in \Pi_{d_+} \times \mathbb{C} \,\middle|\, \xi \in \Pi_{d_+},\ (\xi,\gamma) \in \Gamma,\ \left|\gamma' - \gamma(\xi)\right| \leqslant \frac{\sigma}{2}\right\} \subset M, \tag{2.11}$$

such that for any $(\xi,\gamma) \in W_+$, there exists a compatible transformation $\Phi(\cdot,\cdot,\xi,\gamma): D(s_+,r_+) \to D(s,r)$ which changes the reversible system (2.8) to

$$\dot{x} = \omega + \mathcal{Q}_+(x)z + f_+^1(x,z), \qquad \dot{z} = \mathcal{N}_+ + \mathcal{A}_+ z + \mathcal{B}_+(x)z^2 + g_+(x,z), \tag{2.12}$$

where $\mathcal{N} = (0_{1\times m}, N_+(\xi,\gamma), 0)^T$ with $N_+(\xi,\gamma) = N(\xi,\gamma) + \Delta\hat{N}(\xi,\gamma)$. Moreover, we have the following conclusions:

(i) *The compatible transformation Φ satisfies*

$$\left\|\Xi(\Phi - id)\right\|_{D(s_+,r_+)\times M_+} \leqslant cE, \tag{2.13}$$

$$\left\|\Xi(\mathcal{D}\Phi - I_{n+m+2})\Xi^{-1}\right\|_{D(s_+,r_+)\times M_+} \leqslant cE, \tag{2.14}$$

where $\Xi = \operatorname{diag}(I_n, \frac{1}{r}I_m, \frac{1}{r}, \frac{1}{r})$. Here I_m denotes the $m\times m$ identity matrix.

(ii) *The new perturbation term $f_+ = (f_+^1, f_+^2, f_+^3, f_+^4)$ satisfies*

$$|||f_+|||_{D(s_+,r_+)\times M_+} \leqslant \epsilon_+ = \alpha^6 \rho_+^{6\tau+2} E_+. \tag{2.15}$$

(iii) *$\Delta\hat{N}(p)$, $\mathcal{Q}_+$, $\mathcal{B}_+$ and $\mathcal{A}_+$ satisfy*

$$\left|\Delta\hat{N}(\xi,\gamma)\right| \leqslant c\epsilon r, \quad \forall(\xi,\gamma) \in M, \tag{2.16}$$

$$\left|\Delta\hat{N}_\xi(\xi,\gamma)\right| + \left|\Delta\hat{N}_\gamma(\xi,\gamma)\right| \leqslant \frac{c\epsilon r}{\sigma} \leqslant cE^{\frac{1}{5}}, \quad \forall(\xi,\gamma) \in M_+, \tag{2.17}$$

$$\max\left\{\|\mathcal{Q}_+ - \mathcal{Q}\|_{D(s_+,r_+)\times M_+}, \|\mathcal{B}_+ - \mathcal{B}\|_{D(s_+,r_+)\times M_+}, |\mathcal{A}_+ - \mathcal{A}|_{M_+}\right\} \leqslant cE. \tag{2.18}$$

(iv) *The equation*

$$\gamma + \xi^{2n_0+1} + \hat{N}_+(\xi,\gamma) = \gamma + \xi^{2n_0+1} + \hat{N}(\xi,\gamma) + \Delta\hat{N}(\xi,\gamma) = 0,$$

defines implicitly an analytic curve

$$\Gamma_+: \quad \gamma_+ = \gamma_+(\xi): \quad \xi \in \Pi_{d_+} \to \gamma_+(\xi) \in B(0, 2\mu+1),$$

satisfying

$$\left|\gamma_+(\xi) - \gamma(\xi)\right| \leqslant c\epsilon r \leqslant \frac{\sigma}{4} \tag{2.19}$$

and

$$\Gamma_+ := \left\{(\xi,\gamma_+(\xi)) \,\middle|\, \xi \in \Pi_{d_+}\right\} \subset M_+. \tag{2.20}$$

If

$$\sigma_+ \leqslant \frac{\sigma}{4}, \tag{2.21}$$

then we have $U(\Gamma_+, \sigma_+) \subset M_+$.

We divide the proof of Lemma 2.2 into the following several parts.

A. Constructing compatible transformation. In the following, we will construct a compatible transformation Φ which changes the reversible system (2.7) into (2.12). Let $\Phi : (x_+, y_+, u_+, v_+) \to (x, y, u, v)$ be defined by

$$\begin{cases} x = x_+ + h(x_+), \\ y = y_+ + a_1(x_+) + b_{11}(x_+)y_+ + b_{12}(x_+)u_+ + b_{13}(x_+)v_+, \\ u = u_+ + a_2(x_+) + b_{21}(x_+)y_+ + b_{22}(x_+)u_+ + b_{23}(x_+)v_+, \\ v = v_+ + a_3(x_+) + b_{31}(x_+)y_+ + b_{32}(x_+)u_+ + b_{33}(x_+)v_+. \end{cases} \tag{2.22}$$

Denote $z = (y, u, v)^T$ and $z_+ = (y_+, u_+, v_+)^T$. Φ is written in a more compact form:

$$x = x_+ + h(x_+), \qquad z = z_+ + a(x_+) + b(x_+)z_+, \tag{2.23}$$

where

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Let $S = \operatorname{diag}(I_m, -1, 1)$. It is easy to see that Φ is compatible with the involution G if and only if

$$h(-x) = -h(x), \qquad Sa(-x) = a(x), \qquad Sb(-x)S = b(x). \tag{2.24}$$

Under the transformation Φ the system (2.8) is changed into

$$\begin{cases} \dot{x} = \omega_0 + \big(I_n + \mathcal{D}_x h(x)\big)^{-1}\big(-\partial_{\omega_0} h + f^1(x, 0) + \mathcal{Q}(x)a(x)\big) \\ \qquad + \big(I_n + \mathcal{D}_x h(x)\big)^{-1}\big(\mathcal{Q}(x+h)\big(1 + b(x)\big) + f^1_z(x, 0)\big)z + f^1_+(x, z), \\ \dot{z} = \mathcal{N}_+ + \mathcal{A}_+ z + \big(I_{m+2} + b(x)\big)^{-1}\big(-\partial_{\omega_0} a + \mathcal{A}a(x) + g(x, 0) - \hat{\mathcal{N}}\big) \\ \qquad + \big(I_{m+2} + b(x)\big)^{-1}\big(-\partial_{\omega_0} b + \mathcal{A}b(x) - b(x)\mathcal{A} + g_z(x, 0) + 2\mathcal{B}(x)a(x) - \partial_{\mathcal{Q}} a(x) - \hat{\mathcal{A}}\big)z \\ \qquad + \big(I_{m+2} + b(x)\big)^{-1}\left(\mathcal{B}(x+h)(1 + b(x))^2 + \frac{1}{2}\partial_{zz} g(x, 0) - \partial_{\mathcal{Q}_+} b(x)\right)z^2 + g_+(x, z), \end{cases} \tag{2.25}$$

where $\mathcal{N}_+ = \mathcal{N} + \hat{\mathcal{N}}$, $\mathcal{A}_+ = \mathcal{A} + \hat{\mathcal{A}}$ with $\hat{\mathcal{A}}$ and $\hat{\mathcal{N}}$ being determined later, $f^1_z(x, 0) = \frac{\partial f^1}{\partial z}|_{z=0}$, $g_z(x, 0) = \frac{\partial g}{\partial z}|_{z=0}$, $\partial_{zz} g(x, 0) = \frac{\partial^2 g}{\partial z^2}|_{z=0}$, $\partial_{\omega_0} h = \sum_{k \in \mathbb{Z}^n} \sqrt{-1}\langle k, \omega_0\rangle h_k e^{\sqrt{-1}\langle k, x\rangle}$, $\partial_{\omega_0} a$, $\partial_{\omega_0} b$, $\partial_{\mathcal{Q}} a$ and $\partial_{\mathcal{Q}_+} b$ are defined similarly. Moreover, we have

$$f^1_+(x, z) = \big(I_n + \mathcal{D}_x h(x)\big)^{-1}\big(f^1 \circ \Phi(x, z) - f^1(x, 0) - \big\langle f^1_z(x, 0), z\big\rangle + \big(\mathcal{Q}(x+h) - \mathcal{Q}(x)\big)a(x)\big), \tag{2.26}$$

$$\begin{aligned} g_+(x, z) = {} & \big(I_{m+2} + b(x)\big)^{-1}\bigg(g \circ \Phi(x, z) - g(x, 0) - \big\langle g_z(x, 0), z\big\rangle - \frac{1}{2}\big\langle \partial_{zz} g(x, 0)z, z\big\rangle \\ & + \mathcal{B}(x+h)a^2(x) - b(x)\mathcal{N}_+ - \partial_{f^1_+}\big(a(x) + b(x)z\big) - \partial_{\hat{\mathcal{Q}}} b(x)\bigg), \end{aligned} \tag{2.27}$$

with $\hat{\mathcal{Q}}(x) = \mathcal{Q}_+ - \mathcal{Q}$. Note that we have used (x, z) instead of the new variables (x_+, z_+) in the transformed equations for simplicity.

Let

$$\mathcal{Q}_+(x) = \big(I_n + \partial_x h(x)\big)^{-1}\big(\mathcal{Q}(x+h)\big(1 + b(x)\big) + f^1_z(x, 0)\big), \tag{2.28}$$

$$\mathcal{B}_+(x) = \big(I_{m+2} + b(x)\big)^{-1}\left(\mathcal{B}(x+h)\big(1 + b(x)\big)^2 + \frac{1}{2}\partial_{zz} g(x, 0) - \partial_{\mathcal{Q}_+} b(x)\right). \tag{2.29}$$

If we can find $h(x)$, $a(x)$ and $b(x)$ such that

$$\partial_{\omega_0} h(x) = f^1(x, 0) + \mathcal{Q}(x)a(x), \tag{2.30}$$

$$\partial_{\omega_0} a(x) - \mathcal{A}a(x) = g(x, 0) - \hat{\mathcal{N}}, \tag{2.31}$$

$$\partial_{\omega_0} b(x) - \mathcal{A}b(x) + b(x)\mathcal{A} = g_z(x, 0) - \partial_{\mathcal{Q}(x)} a(x) - 2\mathcal{B}(x)a(x) - \hat{\mathcal{A}}. \tag{2.32}$$

Then the system (2.8) becomes

$$\dot{x} = \omega + \mathcal{Q}_+(x) + f_+^1(x,z), \qquad \dot{z} = \mathcal{N}_+ + \mathcal{A}_+ z + \mathcal{B}_+(x)z^2 + g_+(x,z) \tag{2.33}$$

with f_+^1 and g_+ being much smaller perturbations than before.

B. Solving linear homological equations. In the following, we solve the linear homological equations (2.30)–(2.32).

We first solve Eq. (2.31). Since the system (2.7) is reversible, we have $[f^2(\cdot,0)] = 0$ and $[f^4(\cdot,0)] = 0$. Let $a = (a_1, a_2, a_3)^T$ and $\hat{\mathcal{N}} = (0_{1\times m}, \hat{N}, 0)^T$. Then the linear homological equation (2.31) becomes

$$\partial_{\omega_0} a_1(x) = f^2(x,0,0,0), \tag{2.34}$$

$$\partial_{\omega_0} a_2(x) = f^3(x,0,0,0) + Ca_1(x) - \hat{N}, \tag{2.35}$$

$$\partial_{\omega_0} a_3(x) = f^4(x,0,0,0) + Ba_2(x). \tag{2.36}$$

Let

$$a_1(x) = \sum_{k\in\mathbb{Z}^n} a_1^k e^{\sqrt{-1}\langle k,x\rangle}.$$

Note that $[f^2(\cdot,0)] = 0$ and $\|f^2(x,0,0,0)\|_{D(s,r)\times M} \leqslant \epsilon r^2$. By Lemma A.1 of Appendix A, Eq. (2.34) is solvable. Moreover, we have

$$\|a_1 - a_1^0\|_{D(s-\rho,r)\times M} = \Big\| \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_1^k e^{\sqrt{-1}\langle k,x\rangle} \Big\|_{s-\rho} \leqslant \frac{c\epsilon r^2}{\alpha\rho^\tau}, \tag{2.37}$$

where a_1^0 is determined later.

Let $\hat{N} = [f^3(\cdot,0)] + Ca_1^0$. By Lemma A.1 of Appendix A, it follows that Eq. (2.35) has a unique solution $a_2(x)$ with

$$[a_2] = 0 \quad \text{and} \quad \|a_2\|_{s-2\rho} \leqslant \frac{c\epsilon r^2}{\alpha^2\rho^{2\tau}}. \tag{2.38}$$

Let

$$a_3(x) = \sum_{k\in\mathbb{Z}^n} a_3^k e^{\sqrt{-1}\langle k,x\rangle}.$$

Note that $[f^4(\cdot,0)] = 0$ and $[a_2] = 0$. By Lemma A.1 of Appendix A, Eq. (2.36) is also solvable. Moreover, we have

$$\|a_3 - a_3^0\|_{D(s-3\rho,r)\times M} = \Big\| \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_3^k e^{\sqrt{-1}\langle k,x\rangle} \Big\|_{s-\rho} \leqslant \frac{c\epsilon r^2}{\alpha^3\rho^{3\tau}}, \tag{2.39}$$

where $a_3^0 = [a_3]$ is determined later.

Next we choose suitable a_1^0 and a_3^0 to remove the shifts of normal frequencies and normal frequencies.

Since the system (2.7) is reversible, we have

$$\big[g_z(\cdot,0) - \partial_x a(\cdot)\mathcal{Q}(\cdot) - 2\mathcal{B}(\cdot)a(\cdot)\big] = \begin{pmatrix} 0 & H^\epsilon_{m\times 1} & 0 \\ D^\epsilon_{1\times m} & 0 & E^\epsilon \\ 0 & F^\epsilon & 0 \end{pmatrix}.$$

By (2.10), (2.37), (2.38) and (2.39), it is easy to see that E^ϵ has the following form:

$$E^\epsilon = -2\big[B^2_{002}\big]a_3^0 + e_1 a_1^0 + e_2,$$

where $|[B^2_{002}]|_M > \frac{1}{2}$, $|e_1|_M \leqslant E_0^{\frac{2}{3}}$ and $|e_1|_M \leqslant \max\{\frac{c\epsilon r^2}{\alpha^3\rho^{3\tau+1}}, \epsilon r\} \leqslant c\epsilon r$.

Let

$$\hat{\mathcal{A}} = \begin{pmatrix} 0 & 0 & 0 \\ D^\epsilon_{1\times m} & 0 & 0 \\ 0 & F^\epsilon & 0 \end{pmatrix} \quad \text{and} \quad [b] = \begin{pmatrix} 0 & 0 & [b_{13}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.40}$$

with $[b_{13}] = \frac{1}{B}H^\epsilon$. Then we have

$$-\mathcal{A}[b]+[b]\mathcal{A}=\begin{pmatrix}0 & H^{\epsilon} & 0\\ 0 & 0 & -C[b_{13}]\\ 0 & 0 & 0\end{pmatrix}. \tag{2.41}$$

Noting that the system (2.7) is reversible, so H^{ϵ} does not contain the variables a_1^0 and a_3^0. By (2.10) and it is easy to see that $|[b_{13}]|_M \leqslant 2|H^{\epsilon}|_M \leqslant c\epsilon r$.

If we can find a_1^0 and a_3^0 such that

$$\big[f^1(\cdot,0)+\mathcal{Q}(\cdot)a(\cdot)\big]=0, \tag{2.42}$$

$$E_{\epsilon}+C[b_{13}]=-2\big[B_{002}^2\big]a_3^0+e_1a_1^0+e_2+C[b_{13}]=0. \tag{2.43}$$

Then the homological equations (2.30) and (2.32) are also solvable.

In view of $\mathcal{Q}=(Q,W_1,W_2)$, let $Q(x)=\sum_{k\in\mathbb{Z}^n}Q^k e^{\sqrt{-1}\langle k,x\rangle}$, $W_2(x)=\sum_{k\in\mathbb{Z}^n}W_2^k e^{\sqrt{-1}\langle k,x\rangle}$ and we have

$$\big[f^1(\cdot,0)-\mathcal{Q}(\cdot)a(\cdot)\big]=[Q]a_1^0+W_2^0a_3^0+\sum_{|k|\neq 0}\big(Q^ka_{-k}^1+W_2^ka_{-k}^3\big)+\big[W_1(\cdot)a_2(\cdot)+f^1(\cdot,0)\big].$$

Let

$$a_1^0=\big(\tilde{a}_1^0,0,\dots,0\big)^T \quad \text{with } \tilde{a}_1^0\in\mathbb{R}^n, \qquad Y=\big(\tilde{a}_1^0,a_3^0\big)^T\in\mathbb{R}^{n+1}.$$

Note that we already obtained the a_k ($|k|\neq 0$) from (2.31). By (2.10), Eqs. (2.42) and (2.43) can be written as the following form:

$$(\mathcal{M}+\mathcal{M}_{\epsilon})Y=\mathcal{W}, \tag{2.44}$$

where

$$\mathcal{M}=\begin{pmatrix}Q_1^0 & 0\\ 0 & -2[B_{002}^2]\end{pmatrix}, \qquad \mathcal{M}_{\epsilon}=\begin{pmatrix}0 & W_2^0\\ e_1 & 0\end{pmatrix},$$

$$\mathcal{W}=\begin{pmatrix}-\sum_{|k|\neq 0}(Q^ka_{-k}^1+W_2^ka_{-k}^3)-[W_1(\cdot)a_2(\cdot)+f^1(\cdot,0)]\\ -e_2-C[b_{13}]\end{pmatrix}.$$

By (2.10) it is easy to see that $\mathcal{M}$ is an $(n+1)\times(n+1)$ non-singular matrix. Moreover, we have

$$\|\mathcal{M}_{\epsilon}\|\leqslant 2E_0^{\frac{2}{3}} \quad \text{and} \quad \|\mathcal{W}\|_{s-3\rho}\leqslant c\epsilon r.$$

Hence, $\mathcal{M}+\mathcal{M}_{\epsilon}$ is also a non-singular matrix if E_0 is sufficiently small. Then Eq. (2.44) is solvable. Therefore, we obtain a_1^0 and a_3^0 such that Eqs. (2.42) and (2.43) hold. Moreover, we have

$$\big|a_1^0\big|\leqslant c\epsilon r, \qquad \big|a_3^0\big|\leqslant c\epsilon r. \tag{2.45}$$

By the form of $\mathcal{A}$ and Lemma A.1 of Appendix A, the linear homological equations (2.30) and (2.32) are also solvable. Moreover, we have the following estimates:

$$\|h\|_{D(s-4\rho,r)\times M}\leqslant\frac{c\epsilon r}{\alpha\rho^{\tau}}, \qquad \|b\|_{D(s-9\rho,r)\times M}\leqslant\frac{c\epsilon r}{\alpha^6\rho^{6\tau}}. \tag{2.46}$$

Now we already obtained $h(x)$, $a(x)$ and $b(x)$ from the homological equations (2.30)–(2.32). Then the transformation Φ is defined well by (2.23). To prove that Φ is a compatible transformation, we verify the symmetry of (2.24). Since the system (2.7) is reversible, it follows that

$$Sg(-x,0)=-g(x,0), \qquad S\mathcal{A}=-\mathcal{A}S, \qquad S\hat{\mathcal{N}}=-\hat{\mathcal{N}}, \qquad Sg_z(-x,0)S=-g_z(x,0),$$

where $S=\operatorname{diag}(I_m,-1,1)$. If $a(x)$ is a solution for (2.31), then we have

$$-\partial_{\omega_0}a(-x)-\mathcal{A}a(-x)=g(-x,0)-\hat{\mathcal{N}}.$$

Multiplying the above equation by S from the left, we have

$$\partial_{\omega_0}Sa(-x)-\mathcal{A}Sa(-x)=g(x,0)-\hat{\mathcal{N}}.$$

Thus $Sa(-x)$ is also a solution for (2.31). Then $\tilde{a}(x)=\frac{1}{2}(a(x)+Sa(-x))$ is also a solution for (2.31) and satisfies the second equation of (2.24). Thus, if $Sa(-x)\neq a(x)$, we can replace $a(x)$ by $\tilde{a}(x)$ so that $Sa(-x)=a(x)$. In the same way, we have $Sb(-x)S=b(x)$. Noting that $Sa(-x)=a(x)$, the symmetry of h holds obviously. Hence, Φ is a compatible transformation.

Nothing that $\epsilon = \alpha^6 E \rho^{6\tau+2}$ and $\eta = E^{\frac{1}{5}}$, we have $\frac{c\epsilon r}{\alpha\rho^\tau} < \rho$ and $\frac{c\epsilon r}{\alpha^6\rho^{6\tau}} < \eta r < \frac{r}{8}$. Let $s_+ = s - 10\rho, r_+ = \eta r$. It is easy to see that

$$\Phi : (x_+, z_+) \in D(s_+, r_+) \to (x, z) \in D(s - 9\rho, 2\eta r) \subset D(s, r).$$

Moreover, we have

$$\left\| \Xi(\Phi - id)\right\|_{D(s_+,r_+)\times M} \leqslant cE, \tag{2.47}$$

$$\left\| \Xi(\mathcal{D}\Phi - I_{n+m+2})\Xi^{-1}\right\|_{D(s_+,r_+)\times M} \leqslant cE, \tag{2.48}$$

where $\Xi = \operatorname{diag}(I_n, \frac{1}{r} I_m, \frac{1}{r}, \frac{1}{r})$.

C. Estimates of perturbation terms. For any $(\xi, \gamma') \in U(\Gamma, \delta)$, $\exists(\xi, \gamma) \in \Gamma$ such that $|\gamma' - \gamma| < \sigma$. So it follows that

$$\left|\mathcal{N}(\xi, \gamma')\right| = \left|\mathcal{N}(\xi, \gamma') - \mathcal{N}(\xi, \gamma)\right| = \left|\hat{N}(\xi, \gamma') - \hat{N}(\xi, \gamma)\right| \leqslant \frac{1}{2}\left|\gamma' - \gamma\right| < \sigma = E^{\frac{4}{5}} r. \tag{2.49}$$

Let M_+ be defined by (2.11), it follows easily that M_+ is closed. Obviously, we have

$$M_+ \subset U(\Gamma, \sigma) \subset M \quad \text{and} \quad \operatorname{dist}(M_+, \partial M) \geqslant \frac{1}{2}\sigma,$$

where ∂M is the boundary of M. Note that $\Delta\hat{N} = [f^3(\cdot, 0)] + Ca_1^0$ with $|[f^3(\cdot, 0)] + Ca_1^0|_M \leqslant c\epsilon r$. It is easy to see that (2.16) and (2.17) hold.

Set $\hat{N}_+(\xi, \gamma) = \hat{N}(\xi, \gamma) + \Delta\hat{N}(\xi, \gamma)$. By the implicit function theorem, if

$$\left|\frac{d\hat{N}}{d\gamma}(\xi, \gamma)\right| \leqslant \frac{1}{2}, \quad \forall(\xi, \gamma) \in M,$$

the equation

$$\gamma + \xi^{2n_0+1} + \hat{N}_+(\xi, \gamma) = 0,$$

defines implicitly an analytic curve

$$\Gamma_+\colon \quad \gamma_+ = \gamma_+(\xi)\colon \quad \xi \in \Pi_{d_+} \to \gamma(\xi) \in B(0, 2\mu + 1).$$

Note that γ_+ and γ satisfy

$$\gamma_+(\xi) + \xi^{2n_0+1} + \hat{N}_+\bigl(\xi, \gamma_+(\xi)\bigr) = \gamma(\xi) + \xi^{2n_0+1} + \hat{N}\bigl(\xi, \gamma(\xi)\bigr) = 0.$$

Then it is easy to see that

$$\begin{aligned} \left|\gamma_+(\xi) - \gamma(\xi)\right| &\leqslant \left|\hat{N}_+\bigl(\xi, \gamma_+(\xi)\bigr) - \hat{N}\bigl(\xi, \gamma(\xi)\bigr)\right| \\ &\leqslant \left|\hat{N}\bigl(\xi, \gamma_+(\xi)\bigr) - \hat{N}\bigl(\xi, \gamma(\xi)\bigr)\right| + \left|\Delta\hat{N}\bigl(\xi, \gamma_+(\xi)\bigr)\right| \\ &\leqslant \frac{1}{2}\left|\gamma_+(\xi) - \gamma(\xi)\right| + c\epsilon r. \end{aligned}$$

Hence, the conclusions (2.19) and (2.20) hold. By (2.21) and noting that $c\epsilon r \ll \frac{\sigma}{4}$, we have $U(\Gamma_+, \sigma_+) \subset M_+$.

From (2.10), (2.28) and (2.46) we have

$$\|\hat{\mathcal{Q}}\|_{D(s_+,r_+)\times M} = \|\mathcal{Q}_+ - \mathcal{Q}\|_{D(s_+,r_+)\times M} \leqslant \frac{c\epsilon r}{\alpha^6\rho^{6\tau+1}} \leqslant cE. \tag{2.50}$$

By the definition of $\mathcal{Q}_+$, $\mathcal{B}_+$, $\hat{\mathcal{A}}$ and (2.46), it is easy to see that (2.18) holds.

Now we first give an estimate of the new perturbation term f_+^1. By (2.28) it follows that $\|(I_n + \mathcal{D}_x h)^{-1}\|_{D_+} \leqslant 2$ if E is sufficiently small. Note that here and below we write $D_+ = D(s_+, r_+) \times M_+$ for simplicity.

By (2.26) and (2.46), we have

$$\left\|f_+^1\right\|_{D_+} \leqslant \frac{c\epsilon r}{\alpha\rho^{\tau+1}} \cdot c\epsilon r + \left\|f^1 \circ \Phi(x, z) - f^1(x, 0) - \left\langle\partial_z f^1(x, 0), z\right\rangle\right\|_{D_+}. \tag{2.51}$$

Next we give an estimate of $\|f^1 \circ \Phi(x, z) - f^1(x, 0) - \langle\partial_z f^1(x, 0), z\rangle\|_{D_+}$. Obviously, we have

$$\left\|f^1 \circ \Phi(x, z) - f^1(x, 0) - \left\langle\partial_z f^1(x, 0), z\right\rangle\right\|_{D_+} = \mathcal{F}^1 + \mathcal{F}^2, \tag{2.52}$$

where

$$\mathcal{F}^1 = f^1 \circ \Phi - f^1(x, z), \qquad \mathcal{F}^2 = f^1(x, z) - f^1(x, 0) - \left\langle\partial_z f^1(x, 0), z\right\rangle.$$

By the Cauchy estimates and noting that $\epsilon = \alpha^6 E\rho^{6\tau+2}$ and $r \leqslant \alpha^6\rho^{6\tau+1}$, it follows that

$$\|\mathcal{F}^1\|_{D_+} \leqslant c\left(\frac{\epsilon r}{\rho}\cdot\frac{\epsilon r}{\alpha\rho^\tau} + \frac{\epsilon r}{r}\cdot\frac{\epsilon r}{\alpha^6\rho^{6\tau}}\right) \leqslant c\epsilon E r, \qquad \|\mathcal{F}^2\|_{D_+} \leqslant \eta^2\epsilon r.$$

In view of $\eta = E^{\frac{1}{5}}$, we have

$$\|f^1\circ\Phi(x,z) - f^1(x,0) - \langle\partial_z f^1(x,0), z\rangle\|_{D_+} \leqslant c\eta\epsilon r_+.$$

By (2.51) we have

$$\|f_+^1\|_{D_+} \leqslant c\eta\epsilon r_+.$$

Now we estimate the new perturbation term g_+. Similarly to the above estimates of f_+^1, it is easy to see that $\|(I_{m+2}+b)^{-1}\|_{D_+} \leqslant 2$ if E is sufficiently small. Combining (2.10), (2.27), (2.46), (2.49) and (2.50), it follows that

$$\|g_+\|_{D_+} \leqslant 2\left\|g\circ\Phi(x,z) - g(x,0) - \langle g_z(x,0), z\rangle - \frac{1}{2}\langle\partial_z g(x,0)z, z\rangle\right\|_{D_+} + \frac{c\epsilon r}{\alpha^6\rho^{6\tau+2}}E^{\frac{4}{5}}r + cEr\|f_+^1\|_{D_+}.$$

In the same way as (2.52), we have

$$g\circ\Phi(x,z) - g(x,0) - \langle g_z(x,0), z\rangle - \frac{1}{2}\langle\partial_z g(x,0)z, z\rangle = \mathcal{P}_1 + \mathcal{P}_2,$$

where

$$\mathcal{P}^1 = g\circ\Phi(x,z) - g(x,z), \qquad \mathcal{P}^2 = g(x,z) - g(x,0) - \langle g_z(x,0), z\rangle - \frac{1}{2}\langle\partial_z g(x,0)z, z\rangle.$$

Then, it follows that

$$\|\mathcal{P}^1\|_{D_+} \leqslant c\left(\frac{\epsilon r^2}{\rho}\cdot\frac{\epsilon r}{\alpha\rho^\tau} + \frac{\epsilon r^2}{r}\cdot\frac{\epsilon r}{\alpha^6\rho^{6\tau}}\right) \leqslant cE\epsilon r^2, \qquad \|\mathcal{P}^2\|_{D_+} \leqslant \eta^3\epsilon r^2.$$

Noting that $E^{\frac{1}{5}} \leqslant \alpha^6\rho^{6\tau+2}$, $\eta = E^{\frac{1}{5}}$ and $\|f_+^1\|_{D_+} \leqslant c\eta\epsilon r_+$, we have

$$\|g_+\|_{D_+} \leqslant c\eta^3\epsilon r \leqslant c\eta\epsilon r_+^2.$$

Therefore, we have

$$|||f_+|||_{D_+} = \frac{1}{r_+}\|f_+^1\|_{D_+} + \frac{1}{r_+^2}\|g_+\|_{D_+} \leqslant c\eta\epsilon = \alpha^6\rho_+^{6\tau+2}cE^{\frac{4}{3}} = \alpha\rho_+^{6\tau+2}E_+ = \epsilon_+,$$

where $E_+ = cE^{\frac{6}{5}}$. Thus, Lemma 2.2 is proved. □

2.2. Setting the parameters and iteration

Now we choose some suitable parameters so that the above iteration can go on infinitely. At the initial step, let $\mathcal{Q}_0(x) = (Q(x), 0, 0)$, it is easy to see that $[\mathcal{Q}_0] = [Q_1^0, 0_{n\times(m-n+2)}] \in \mathbb{R}^{n\times(m+2)}$ with $Q_1^0 \in \mathbb{R}^{n\times n}$ being a non-singular matrix. We set $\mathcal{B}_0(x)z^2 \triangleq (0, (2n_0+1)n_0\xi^{2n_0-1}y_m^2 + v^2, 0)^T$, $\mathcal{N}_0 = (0, \xi^{2n_0+1}+\gamma, 0)^T$, $f_0^1 = f^1$, $g_0 = (f^2, f^3, f^4)^T$, $C_0 = (0, 0, \dots, (2n_0+1)\xi^{2n_0})$, $B_0 = 1$, $s_0 = s$, $r_0 = \min\{\delta^2, \alpha^6(\frac{s}{48})^{6\tau+1}\}$, $d_0 = \delta$ and $E_0 = \epsilon_0/\alpha^6(\frac{s_0}{48})^{6\tau+1}$ with $\epsilon_0 = \delta^{2n_0}$.

Let

$$s_j = s_0\left(\frac{1}{2} + \left(\frac{1}{2}\right)^{j+1}\right), \qquad \rho_j = \frac{s_j - s_{j+1}}{12}, \qquad \eta_j = E_j^{\frac{1}{3}}, \qquad \sigma_j = E_j^{\frac{4}{5}}r_j,$$

$$r_{j+1} = \eta_j r_j, \qquad E_{j+1} = cE_j^{\frac{4}{3}}, \qquad d_{j+1} = d_j - \frac{1}{2}E_j, \qquad \epsilon_{j+1} = \alpha^6 E_{j+1}\rho_{j+1}^{6\tau+2}.$$

Then, it is easy to see that $s_j, r_j, \rho_j, E_j, d_j, \sigma_j, \epsilon_j$ are all well defined for $j \geqslant 0$. In the following we are going to check all assumptions in the iteration Lemma 2.2 to ensure KAM steps are valid for all $j \geqslant 0$.

We verify the assumption $r_j \leqslant \alpha^6\rho_j^{6\tau+6}$ by induction. By the choice of r_0, we have $r_0 \leqslant \alpha^6\rho_0^{6\tau+2}$. We now assume that $r_\nu \leqslant \alpha^2\rho_\nu^{6\tau+2}$ for some nonnegative integer ν. Then, if $E_0^{\frac{1}{5}} \leqslant (\frac{1}{2})^{6\tau+2}$, we have

$$r_{\nu+1} = E_\nu^{\frac{1}{5}}r_\nu \leqslant E_0^{\frac{1}{5}}\alpha^6\rho_\nu^{6\tau+2} \leqslant \alpha^2\left(\frac{\rho_\nu}{2}\right)^{2\tau+1} = \alpha^6\rho_{\nu+1}^{6\tau+2}.$$

Hence, the assumption $r_j \leqslant \alpha^6\rho_j^{6\tau+2}$ holds for all $j \geqslant 0$.

By the definitions of E_j and ρ_j, we have $E_j \leqslant (c^5 E_0)^{(\frac{6}{5})^j}$ and $\rho_j = \frac{1}{3}(\frac{1}{2})^{j+3} s_0$. Hence, the assumption $E_j^{\frac{1}{5}} \leqslant \alpha^6 \rho_j^{6\tau+2}$ holds for all $j \geqslant 0$ if E_0 is sufficiently small.

Obviously,

$$\frac{\sigma_{j+1}}{\sigma_j} = \frac{E_{j+1}^{\frac{4}{5}} r_{j+1}}{E_j^{\frac{4}{5}} r_j} = c^{\frac{4}{5}} E_{j+1}^{\frac{9}{25}} \leqslant c^{\frac{4}{5}} E_0^{\frac{9}{25}} \quad \text{for } \forall j \geqslant 0.$$

Hence, for sufficiently small E_0, the assumption $\sigma_+ \leqslant \frac{1}{4}\sigma$ holds in KAM steps.

Let $D_j = D(s_j, r_j) \times M_j$. By (2.18), if $c^5 E_0 < \frac{1}{2}$, it follows that

$$\|\mathcal{Q}_j - \mathcal{Q}_0\|_{D_j} \leqslant \sum_{\nu=0}^{j-1} \|\mathcal{Q}_{\nu+1} - \mathcal{Q}_\nu\|_{D_{\nu+1}} \leqslant \sum_{\nu=0}^{j-1} cE_\nu \leqslant \sum_{\nu=0}^{\infty} cE_\nu \leqslant cE_0 \quad \text{for } \forall j \geqslant 1.$$

Similarly, we have $\|B_j - B_0\|_{D_j} \leqslant cE_0$, $\|\mathcal{B}_j - \mathcal{B}_0\|_{D_j} \leqslant cE_0$ and $\|\mathcal{A}_j - \mathcal{A}_0\|_{D_j} \leqslant cE_0$ for all $\forall j \geqslant 1$. Then it is easy to see that the assumptions that $\|B_j\|_{D_j} > \frac{1}{2}$ $\|\mathcal{B}_j - \mathcal{B}^0\|_{D_j} \leqslant E_0^{\frac{2}{3}}$ and $\|\mathcal{Q}(x) - \mathcal{Q}_0\|_{D_j} \leqslant E_0^{\frac{2}{3}}$ hold in KAM steps if E_0 is sufficiently small.

By iteration we have $\hat{N}_j = \sum_{i=0}^{j-1} \Delta \hat{N}_i$. Combining with estimates for $\Delta \hat{N}_j$, we have that

$$\left|\hat{N}_{j\xi}(\xi, \gamma)\right| + \left|\hat{N}_{j\gamma}(\xi, \gamma)\right| \leqslant \sum_{i=0}^{j-1} cE_j^{\frac{1}{5}} \leqslant \sum_{i=0}^{\infty} cE_j^{\frac{1}{5}} \leqslant cE_0^{\frac{1}{5}}.$$

Hence, if E_0 is sufficiently small, the assumption $|\hat{N}_{j\xi}(\xi, \gamma)| + |\hat{N}_{j\gamma}(\xi, \gamma)| \leqslant \frac{1}{2}$ hold in KAM steps.

2.3. Convergence of iteration

Let $M_0 = \Pi_{d_0} \times B(0, 2\mu + 1)$ and $D_0 = D(s_0, r_0) \times M_0$. By the iteration lemma, we have a sequence of closed sets $\{M_j\}$ with $M_{j+1} \subset M_j$, and a sequence of compatible transformations $\{\Phi_j\}$ such that for each $(\xi, \gamma) \in M_{j+1}$, $\Phi_j(\cdot, \cdot; \xi, \gamma) : D(s_{j+1}, r_{j+1}) \to D(s_j, r_j)$. Moreover, we have the following estimates

$$\left\|\Xi_j(\Phi_j - id)\right\|_{D_{j+1}} \leqslant cE_j \quad \text{and} \quad \left\|\Xi_j(D\Phi_j - I_{n+m+2})\Xi_j^{-1}\right\|_{D_{j+1}} \leqslant cE_j,$$

where $D_j = D(s_j, r_j) \times M_j$. Let $\Phi^j = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{j-1}$ with $\Phi^0 = id$. In the same way as in [14,15], it follows that

$$\left\|\Xi_0 D\Phi^j \Xi_j^{-1}\right\|_{D_j} \leqslant \prod_{i=0}^{j-1}(1 + cE_i) \leqslant \prod_{i=0}^{\infty}(1 + cE_i) \leqslant 2$$

if E_0 is sufficiently small. So, we have

$$\left\|\Xi_0\big(\Phi^j - \Phi^{j-1}\big)\right\|_{D_j} \leqslant cE_{j-1} \quad \text{and} \quad \left\|\Xi_0 D\big(\Phi^j - \Phi^{j-1}\big)\right\|_{D_j} \leqslant cE_{j-1}.$$

Let $M_* = \bigcap_{j \geqslant 0} M_j$, $D_* = D(\frac{s}{2}, 0) \times M_*$ and $\Phi_* = \lim_{j\to\infty} \Phi^j$. Thus we have

$$\left\|\Xi_0(\Phi_* - id)\right\|_{D_*} \leqslant cE_0 \quad \text{and} \quad \left\|\Xi_0 D(\Phi_* - id)\right\|_{D_*} \leqslant cE_0.$$

Since Φ^j is affine in (y, u, v), we have the convergence of Φ^j to Φ_* on $D(\frac{s}{2}, \frac{r}{2}) \times M_*$.

Now we consider the convergence of $\hat{N}_j$. By the KAM step we have

$$\left|\hat{N}_{j+1}(\xi, \gamma) - \hat{N}_j\right| \leqslant |\Delta \hat{N}_j| \leqslant c\epsilon_j r_j \to 0, \quad \text{as } j \to \infty, \text{ for all } (\xi, \gamma) \in M_*.$$

So we have $\hat{N}_* = \lim_{j\to\infty} \hat{N}_j$ on M_*. Moreover, we have

$$\left|\hat{N}_*(\xi, \gamma)\right| \leqslant c\epsilon_0 r_0 \leqslant c\epsilon/r \quad \text{and} \quad \left|\hat{N}_{*\xi}(\xi, \gamma)\right| + \left|\hat{N}_{*\gamma}(\xi, \gamma)\right| \leqslant \frac{1}{2}.$$

Let $d_* = d_0 - \frac{1}{2}\sum_{j=0}^{\infty} \sigma_j = d_0 - \frac{1}{2}\sum_{j=0}^{\infty} E_j^{\frac{4}{5}} r_j$. It follows that $d_* > d_0 - E_0^{\frac{4}{5}} r_0$. Note that $E_0^{\frac{4}{5}} r_0 = O(\delta^{2+\frac{8}{5}n_0}) \ll d_0 = \delta$ if δ is sufficiently small. Then we have $d_* > \frac{1}{3} d_0$. Thus $\Pi_{d_*} \subset \bigcap_{j \geqslant 0} \Pi_{d_j}$. By (2.19) and $\epsilon_j \to 0$ as $j \to 0$, it is easy to see that $\{\gamma_j(\xi)\}$ and also convergent on Π_{d_*}. In fact, by the iteration lemma, for $i > j$

$$\left|\gamma_i(\xi) - \gamma_j(\xi)\right| \leqslant \sum_{l=j}^{i-1} \frac{\sigma_l}{4} \leqslant \frac{\sigma_j}{2}.$$

Let $\gamma_j(\xi) \to \gamma_*(\xi)$, $\xi \in \Pi_{d_*}$. Since $\Gamma_j = \{(\xi, \gamma_j(\xi)) \mid \xi \in \Pi_{d_j}\} \subset M_j$ and γ_j are all analytic on Π_{d_*}, so it is limit $\gamma_*(\xi)$.

Let $i \to \infty$ and we have

$$\left|\gamma_*(\xi) - \gamma_j(\xi)\right| \leqslant \frac{\sigma_j}{2}.$$

This implies that $\varGamma_* = \{(\xi, \gamma_*(\xi)) \mid \xi \in \varPi_{d_*}\} \subset M_j$. So $\varGamma_* \subset M_* = \bigcap_{j \geqslant 0} M_j$. Obviously, for $(\xi, \gamma) \in \varGamma_*$ we have

$$\xi^{2n_0+1} + \gamma + \hat{N}_*(\xi, \gamma) = 0. \tag{2.53}$$

By (2.18) and in view of $E_j \to 0$ as $j \to \infty$, we have $\mathcal{Q}_j \to \mathcal{Q}_*$, $\mathcal{B}_j \to \mathcal{B}_*$ and $\mathcal{A}_j \to \mathcal{A}_*$ as $j \to \infty$. Then it is easy to see that for all $(\xi, \gamma) \in \varGamma_*$, $\varPhi_*$ transforms the reversible system (2.7) into the following form:

$$\dot{x} = \omega_0 + \mathcal{Q}_*(x) + f^1_\infty(x, z), \qquad \dot{z} = \mathcal{A}_* z + \mathcal{B}_* z^2 + g_\infty(x, z) \tag{2.54}$$

where

$$\mathcal{A}_* = \begin{pmatrix} 0 & 0 & 0 \\ C_* & 0 & 0 \\ 0 & B_* & 0 \end{pmatrix}, \qquad \mathcal{N}_* = \begin{pmatrix} 0_{m\times 1} \\ N_*, \\ 0 \end{pmatrix},$$

with $N_*(\xi, \gamma) = N_0(\xi) + \hat{N}_*(\xi, \gamma)$. Set $f^1_* = Q_* y + \mathcal{Q}_*(x) z + f^1_\infty(x, y, z)$ and $(f^2_*, f^3_*, f^4_*)^T = \mathcal{B}_* z^2 + g_\infty(x, z)$. Noting that $\epsilon_j \to 0$ as $j \to \infty$, it is easy to see that $f^j_*(x, 0, 0, 0) = 0$ $(j = 1, 2, 3, 4)$.

In some way as in [22] we can prove that $\hat{N}_*$ and $\varPhi_*$ are C^∞ in (ξ, γ) on M_* in the Whitney's sense. By Whitney's extension theorem [25], we can extend $\hat{N}_*$ and $\varPhi_*$ to be C^∞-smooth on $M = \varPi_d \times B(0, 2\mu + 1)$, but it only makes sense on M_* for our problem. This completes the proof of Theorem 2.2.

Appendix A

In this section we formulate a lemma which have been used in the previous section.

Let $\mathcal{U}_s$ denote the space of all real analytic functions $f(x)$ defined in the complex domain $D(s) = \{x \mid |\mathrm{Im}\, x| \leqslant s\}$; that is

$$\mathcal{U}_s = \left\{ f(x) \;\middle|\; f(x) = \sum_{k \in \mathbb{Z}^n} f_k e^{\sqrt{-1}\langle k, x\rangle},\ \|f\|_s < \infty \right\}.$$

Let

$$\mathcal{U}^0_s = \left\{ f(x) \;\middle|\; f(x) \in \mathcal{B}_s,\ [f] = 0 \right\}.$$

Lemma A.1. *Suppose that ω_0 satisfies the Diophantine condition $|\langle k, \omega_0\rangle| \geqslant \frac{\alpha}{|k|^\tau}$, $\forall k \in \mathbb{Z}^n \backslash \{0\}$. Then the equation*

$$\partial_{\omega_0} h(x) = g(x), \quad g(x) \in \mathcal{U}^0_s,$$

has a unique solution $h(x) \in \bigcup_{0<\rho<s} \mathcal{U}^0_{s-\rho}$ with

$$\|h\|_{s-\rho} \leqslant \frac{c}{\alpha \rho^\tau} \|g\|_s, \quad 0 < \rho < s,$$

where the constant c depends only on n and τ.

For this lemma, we refer to Lemma 1 in [14].

Acknowledgments

The authors are very grateful to the referees for their helpful suggestions for this revised version.

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