Self-Duality for the Haagerup Tensor Product and Hilbert Space Factorizations

EDWARD G. EFFROS*

Department of Mathematics, University of California, Los Angeles, California 90024

AND

ZHONG-JIN RUAN*

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

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D. Blecher and V. Paulsen showed that the Haagerup tensor product \( V \otimes^h W \) for operator spaces \( V \) and \( W \) preserves inclusions. It is proved to also preserve complete quotient maps, and to be self-dual in the sense that it induces the Haagerup norm on the algebraic tensor product \( V^* \otimes W^* \). The full operator dual space \( (V \otimes^h W)^* \) is computed. It coincides with the natural operator space \( \mathcal{P}(V, W^*) \) of maps \( \varphi: V \to W^* \) which have completely bounded factorizations through Hilbert spaces (with vectors identified with row matrices). More generally, one has the natural complete isometry \( \mathcal{P}(V, \mathcal{P}(W, X)) \cong \mathcal{P}(V, \mathcal{P}(W, X)) \). Given Hilbert spaces \( H \) and \( K \) with vectors regarded as column matrices, it is shown that one may identify the operator spaces \( B(H, K) \) and \( CB(H, K) \).

1. Introduction

It has become increasingly evident that there is a far-reaching analogy between linear spaces of bounded functions and linear spaces of bounded operators. In this parallel, one replaces bounded linear maps by those that are completely bounded. Since the latter notion is defined by using the norms of matrices over the given spaces, it is necessary to shift from the category \( \mathcal{R} \) of normed vector spaces and bounded maps (the "abstract function spaces") to the category \( \mathcal{D} \) of \( L^\infty \)-matrically normed vector spaces and completely bounded maps (the "abstract operator spaces" of [16]).

The projective, injective, and (Grothendieck's) \( H \)-tensor products of normed space theory (see [10, 15]), which we denote by \( \otimes^\mathbb{H} \), \( \otimes^\mathbb{R} \), and \( \otimes^H \), have corresponding operator space analogues \( \otimes^\mathbb{C} \), \( \otimes^\mathbb{C} \), and \( \otimes^h \).
called the projective, spatial, and Haagerup tensor products, respectively [2, 6–8]. It was pointed out by Blecher, Paulsen, and Pisier (see [2]) that in one respect the theory of operator spaces is simpler than that for normed spaces. The Haagerup tensor product is associative, whereas that need not be the case for the $H$-norm. As was explained in [2], there is a natural functor $\text{Min}: \mathbb{R} \to \mathbb{D}$ (see [2, 7]) such that for any normed spaces $V$ and $W$,

$$\text{Min}(V \otimes_{\text{rt}} W) \cong \text{Min} V \otimes_{\text{c}} \text{Min} W$$

$$\text{Min}(V \otimes_{\text{rt}} W) \cong \text{Min} V \otimes_{\text{c}} \text{Min} W,$$

where $\cong$ and $\approx$ denote isometry and complete isometry, respectively. The non-associativity of the $H$-norm is related to the fact that as an operator space, the Haagerup tensor product is not in the image of the $\text{Min}$ functor.

In this paper we show that the operator spaces have a number of other remarkable properties that do not hold for normed spaces. Omitting the subscript $\mathbb{D}$ (as we shall do throughout the paper), we have the following results.

(a) The Haagerup norm is self-dual (Theorem 3.2). This result (for the finite dimensional case) was first reported to the second author by David Blecher. Our proof uses the Christensen–Sinclair theorem [4] and the fact that the Haagerup tensor product preserves both complete injections and complete quotient maps.

(b) Given Hilbert spaces $H$ and $K$, and letting $H_c$ and $K_c$ be the "column" operator spaces $B(C, H)$ and $B(C, K)$, we provide a direct proof that the operator spaces $B(H, K)$ and $CB(H_c, K_c)$ coincide (Theorem 4.1).

In particular the dual of $H_c$ is the "row" operator space $(H^*)^\sim = B(H, C)$.

An analogous result is true for $H_r$ and $K_r$, but when one considers $H_r$ with $K_r$ or $H_c$ with $K_r$, the completely bounded norm coincides with the Hilbert–Schmidt norm (Corollary 4.5).

We are indebted to David Blecher and Vern Paulsen for pointing out to us that the finite dimensional cases of these results also follow from [2, Theorem 3.7] and the subsequent discussion of $M_n^*$ (applied to rectangular matrices).

(c) Given an operator space $V$ and a Hilbert space $H$, we have (Theorem 4.3)

$$V \otimes^h H_c = V \otimes H_c, \quad H_c \otimes^h V = H_c \otimes V.$$

(d) We may identify the dual operator space $(W \otimes^h V)^*$ with the space $\Gamma_2(V, W^*)$ of linear maps $\varphi: V \to W^*$ for which there is a completely bounded factorization through a "column" Hilbert space.
(Theorem 5.3). The appropriate operator norms $\gamma_2$ are defined by using matrix versions of these diagrams.

(e) Using the associativity of the Haagerup tensor product, we obtain the more general relation

$$\Gamma_2(W \otimes V, X) = \Gamma_2(V, \Gamma_2(W, X))$$

(Corollary 5.5).

(f) The functor $\text{Min}: \mathfrak{H} \to \mathfrak{O}$ carries the 2-summing norm $\pi_2$ of Banach space theory to the column Hilbert space factorization norm $\gamma_2$ (Corollary 5.8). For general operator spaces, what appears to be the correct analogue of the 2-summing norm coincides with the operator Hilbert space factorization norm (see the discussion after Corollary 5.8—the corresponding statement is false for Banach spaces).

2. THE THREE TENSOR PRODUCTS

If $V$ is a linear space of bounded operators on a Hilbert space $H$, then each of the matrix spaces $M_n(V)$ has a canonical norm induced by the inclusion

$$M_n(V) \subseteq B(H^n, H^n).$$

If one is given matrices $v \in M_m(V)$, $w \in M_n(W)$, and scalar matrices $x \in M_{p,m}$, $\beta \in M_{m,p}$, then the matrices $v \oplus w \in M_{m+n}(V)$, $\alpha \beta \in M_{p}(W)$ satisfy

$$\|v \oplus w\| = \max \{ \|v\|, \|w\| \},$$

$$\|\alpha \beta\| \leq \|x\| \|v\| \|\beta\|.$$  (2.1, 2.2)

Given a vector space $V$, a system of operator norms for $V$ is the assignment of a norm to each of the matrix spaces $M_n(V)$ which satisfies (2.1) and (2.2). Given a linear map $\varphi: V \to W$, we define $\varphi_n: M_n(V) \to M_n(W)$ by $\varphi_n([v_{ij}]) = [\varphi(v_{ij})]$. If we let

$$\|\varphi\|_{cb} = \sup \{ \|\varphi_n\| : n \in \mathbb{N} \},$$

$\varphi$ is completely bounded if $\|\varphi\|_{cb} < \infty$. $\varphi$ is completely isometric if all of the $\varphi_n$ are isometries, and a complete quotient map if for each $n$, $\varphi_n$ maps the open unit ball of $M_n(V)$ onto that of $M_n(W)$.

The Representation Theorem for Operator Spaces states that if $V$ is a
vector space with a system of operator norms, then it is completely iso-
metric to a subspace of $B(H)$ for some Hilbert space $H$ \[16\]. One of the
first applications of this result is that if $V_0$ is a closed subspace of $V$, then
$V_1 = V/V_0$ is an operator space \[16\]. It is easy to see that the quotient map
$V \to V_1$ is a complete quotient map, and that all such maps essentially arise
in this manner.

The above discussion readily generalizes to the space $\mathbb{M}_{m,n}(V)$ of $m \times n$
rectangular matrices over $V$. One may embed this space in a space of
square matrices by inserting zero matrix entries. In this manner a system
of operator norms on $V$ determines unique norms on each of these spaces.
We identify the space $V^n$ of $n$-tuples

$$v = (v_i) = (v_1, \ldots, v_n)$$

with the space $\mathbb{M}_{n,1}(V)$ of column matrices

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$ 

In order to keep track of our various matrix identifications, it is often con-
venient to consider the vector space $\mathbb{M}_{I,J}(V)$ of matrices $[v_{ij}]_{i \in I, j \in J}$
indexed by finite sets $I, J$. We let $\mathbb{M}_{I}(V) = \mathbb{M}_{I,I}(V)$. We identify a matrix of
matrices

$$[[v_{ik,jk}]]_{i \in I, j \in J} \in \mathbb{M}_{I',I}(\mathbb{M}_{I,J}(V))$$

with the matrix

$$[v_{(i,k),(j,l)}] \in \mathbb{M}_{I \times I', J \times J'}(V),$$

i.e., we “ignore the internal brackets.” Given bijections $f: I \to I'$, $g: J \to J'$,
we have a corresponding identification $\mathbb{M}_{I,J}(V) \cong \mathbb{M}_{I',J'}(V)$. For example, if
$I$ and $J$ have $p$ and $q$ elements, respectively, we identify $\mathbb{M}_{I,J}(V)$ with
$\mathbb{M}_{pq}(V)$ (see, e.g., the definition below for $v \otimes w$). We occasionally consider
infinitely indexed matrices over an operator space which are “bounded” in
the sense that there is a uniform bound for the finite submatrices (see \[9\]
for a careful discussion).

Given an operator space $V$, a $C^*$-algebra $A$ containing $V$, and Hilbert
spaces $H$ and $K$, any complete contraction $\varphi: V \to B(H, K)$ may be written
in the form

$$\varphi(v) = S\pi(v)R,$$

where $\pi: A \to B(L)$ is a representation of $A$ on a Hilbert space $L$, and $R$
and $S$ are “bridging” maps in a diagram of contractions

$$H \xrightarrow{R} L \xrightarrow{S} K$$
(see [13, Theorem 7.4]—this is a consequence of the Arveson–Wittstock Hahn–Banach theorem [1, 18]). We refer to (2.3) as a dilation representation for \( \phi \). We often do not specify the containing C*-algebra \( A \), referring to \( \pi \) as simply a “representation of \( V \).” For a state on a C*-algebra, this corresponds to the usual GNS representation.

Given vector spaces \( V \) and \( W \), a pairing (i.e., a bilinear map)

\[
V \times W \to \mathbb{C}: (v, w) \mapsto \langle v, w \rangle
\]
determines a “matrix pairing”

\[
\mathbb{M}_n(V) \times \mathbb{M}_n(W) \to \mathbb{M}_{nm}: (v, w) \mapsto \begin{bmatrix} \langle v_{ii}, w_{kl} \rangle \end{bmatrix}
\]

(see the index conventions above). If \( V \) is an operator space, and \( V^* \) is its Banach dual, the pairing

\[
V \times V^* \to \mathbb{C}: (v, f) \mapsto \langle v, f \rangle = f(v)
\]

and (2.4) determine a system of operator norms on \( V^* \). Thus given \( f \in \mathbb{M}_n(V^*) \), we define

\[
\| f \| = \sup \{ \| \langle v, f \rangle \| : \| v \| \leq 1, \ v \in \mathbb{M}_n(V), \ n \in \mathbb{N} \}.
\]

More generally, we define the operator norms on \( \text{CB}(V, W) \) by using the identification

\[
\mathbb{M}_n(\text{CB}(V, W)) = \text{CB}(V, \mathbb{M}_n(W)),
\]

(2.5)

where we let \( [\phi]_n(v) = [\phi(v)] \). This gives the previously defined matricial structure on \( V^* = \text{CB}(V, \mathbb{C}) \), i.e., we have

\[
\mathbb{M}_n(V^*) = \text{CB}(V, \mathbb{M}_n).
\]

(2.6)

Given operator spaces \( V \) and \( W \) and a completely bounded operator \( \phi: V \to W \), it is in particular bounded and thus determines a bounded operator \( \phi^*: W^* \to V^* \). It is easy to check that \( \| \phi^* \|_{cb} = \| \phi \|_{cb} \). Furthermore, we have that \( \phi \) is a complete isometry if and only if \( \phi^* \) is a complete quotient map, and if \( \phi \) is a complete quotient map, then \( \phi^* \) is a complete isometry.

Given \( v \in \mathbb{M}_p(V) \) and \( w \in \mathbb{M}_q(W) \), we define \( v \otimes w \in \mathbb{M}_{pq}(V \otimes W) \) by

\[
(v \otimes w)(i, k), (j, l) = v_{ij} \otimes w_{kl},
\]

(2.7)

Given \( v \in \mathbb{M}_{n,p}(V) \) and \( w \in \mathbb{M}_{p,n}(W) \), we define \( v \odot w \in \mathbb{M}_n(V \otimes W) \) by

\[
(v \odot w)_{ij} = \sum_k v_{ik} \otimes w_{kj}.
\]

(2.8)

Given operator spaces \( V \subset B(H) \), \( W \subset B(K) \), the spatial operators norms \( \| \cdot \|_{\vee} \) for \( V \otimes W \) are determined by the inclusions \( V \otimes W \subset B(H \otimes K) \).
Blecher and Paulsen proved that this is independent of the representations of $V$ and $W$ since for $u \in \mathcal{M}_n(V \otimes W)$,

$$\|u\|_\vee = \sup \{ \| \langle f \otimes g, u \rangle \| \}, \tag{2.9}$$

where the supremum is taken over all $f \in \mathcal{M}_p(V^*)$, $g \in \mathcal{M}_q(W^*)$ with $\|f\|, \|g\| \leq 1$, and $p, q \in \mathbb{N}$ arbitrary.

The projective operator norms $\| \|_\wedge$ for $V \otimes W$ are defined as follows. Given $u \in \mathcal{M}_n(V \otimes W)$, we let

$$\|u\|_\wedge = \inf \{ \|\alpha\| \|v\| \|w\| \|\beta\| \}, \tag{2.10}$$

where the infimum is taken over all decompositions $u = \alpha(v \otimes w)\beta$, with $\alpha \in \mathcal{M}_{p, q}$, $v \in \mathcal{M}_p(V)$, $w \in \mathcal{M}_p(W)$, and $\beta \in \mathcal{M}_{p', q'}$, with $p, q \in \mathbb{N}$ arbitrary.

Finally, the Haagerup operator norms $\| \|_h$ for $V \otimes W$ are defined for $u \in \mathcal{M}_n(V \otimes W)$ by

$$\|u\|_h = \inf \{ \|v\| \|w\| : u = v \odot w, u \in \mathcal{M}_n(V), w \in \mathcal{M}_n(W) \}. \tag{2.11}$$

We write $V \otimes \vee W$, $V \otimes ^\wedge W$, and $V \otimes _h W$ for the corresponding operator spaces, and $V \otimes \vee W$, $V \otimes ^\wedge W$, and $V \otimes _h W$ for their completions. In each of these cases, if one is given complete contractions $\varphi : V \to V_1$ and $\psi : W \to W_1$, one obtains a corresponding complete contraction $\varphi \otimes \psi$ of the completed tensor products. One may write elements of the completions $V \otimes ^\wedge W$ and $V \otimes _h W$ in terms of infinite matrices over $V$ and $W$ (see [9] for the first—this will not be used below), but the situation for $V \otimes \vee W$ is more complicated.

The spatial tensor product is associative and commutative and preserves completely isometric injections. Thus if $\varphi : V \to V_1$ and $\psi : W \to W_1$ are completely isometric injections, it is immediate that the same is true for the map $\varphi \otimes \psi : V \otimes ^\vee W \to V_1 \otimes ^\vee W_1$ (see [2]). Given a $C^*$-algebra $A$, the linear identification

$$\mathcal{M}_n(A) = \mathcal{M}_n \otimes A$$

is a *-isomorphism and thus is completely isometric. If we let $\mathcal{M}_{p, q} \subset \mathcal{M}_n$ and $V \subset A$ be completely isometric inclusions, it follows that

$$\mathcal{M}_{p, q}(V) = \mathcal{M}_{p, q} \otimes V. \tag{2.12}$$

In particular, we have that

$$\mathcal{M}_{m, n} = \mathcal{M}_{m, 1}(\mathcal{M}_{1, n}) = \mathcal{M}_{m, 1} \otimes \mathcal{M}_{1, n}. \tag{2.13}$$

The projective tensor product is associative and commutative and preserves complete quotient maps. Also, we have a complete isometry

$$CB(V, W^*) = (W \otimes V)^* \tag{2.14}$$
determined by the pairing
\[ W \otimes V \times CB(V, W^*) \rightarrow \mathbb{C}: (w \otimes v, \varphi) \rightarrow \langle w \otimes v, \varphi \rangle = \varphi(v)(w). \quad (2.15) \]

The Haagerup norm is associative, but it is not commutative. Paulsen and Smith proved that it is injective, i.e., it preserves completely isometric injections [14] (see [2] for a particularly elegant proof). It is easy to show that for any \( m, n \in \mathbb{N} \) we have a natural identification
\[ \mathcal{M}_{m,n}(V \otimes W) = \mathcal{M}_{m,1}(V) \otimes \mathcal{M}_{1,n}(W) \quad (2.16) \]
(this important observation is due to Blecher and Paulsen [2]). In particular if we let \( W = \mathbb{C} \), it follows that
\[ V \otimes \mathcal{M}_{1,n} = \mathcal{M}_{1,n}(V) = \tilde{V} \otimes \mathcal{M}_{1,n}, \quad (2.17) \]
and similarly that
\[ \mathcal{M}_{m,1} \otimes V = \mathcal{M}_{m,1} \otimes V. \quad (2.18) \]

Letting \( V = W = \mathbb{C} \), we have
\[ \mathcal{M}_{m,n} = \mathcal{M}_{m,1} \otimes \mathcal{M}_{1,n} = \mathcal{M}_{m,1} \otimes \mathcal{M}_{1,n}. \quad (2.19) \]

We consider infinite and dual versions of these results in Section 4.

Given Hilbert spaces \( H \) and \( K \), any completely contractive linear map
\[ \varphi: W \otimes V \rightarrow \mathcal{B}(H, K) \]
has a dilation representation
\[ \varphi(w, v) = T \pi_2(w) S \pi_1(v) R, \quad (2.20) \]
where we may choose \( \pi_1: A \rightarrow \mathcal{B}(L_1) \) and \( \pi_2: B \rightarrow \mathcal{B}(L_2) \) to be representations of arbitrary \( C^* \)-algebras \( A \supseteq V \) and \( B \supseteq W \) and \( R, S, T \) are “bridging” maps in a diagram of contractions
\[ H \xrightarrow{R} L_1 \xrightarrow{S} L_2 \xrightarrow{T} K. \]

Again we often omit mentioning the containing \( C^* \)-algebras \( A \) and \( B \). This result is obtained by combining Christensen and Sinclair’s theorem (which was proved for \( C^* \)-algebras) [4] with the injective property of \( \otimes^h \) [14], and the Arveson–Wittstock analogue of the Hahn–Banach theorem [1, 18].
Given operator spaces $V$ and $W$, we say that a system of operator matrix norms $\alpha$ on $V \otimes W$ is standard if the maps

$$
V \times W \to V \otimes_s W: (v, w) \mapsto v \otimes w
$$

$$
V^* \times W^* \to (V \otimes_s W)^*; (f, g) \mapsto f \otimes g
$$

are jointly completely contractive, i.e., they extend to complete contractions on the projective tensor products $V \otimes W$ and $V^* \otimes W^*$, respectively. It follows that $\alpha$ is a "cross-norm" in the sense that

$$
\|v \otimes w\| = \|v\| \|w\|
$$

(see the more sophisticated discussion in [2]). Assuming that $\alpha$ is standard, we define the dual operator norm structure $\alpha^*$ on $V^* \otimes W^*$ to be that induced by the natural map

$$
V^* \otimes W^* \to (V \otimes_s W)^*.
$$

Blecher and Paulsen [2] proved that $\otimes_s$ is the dual of $\otimes$, i.e., the natural map

$$
V^* \otimes W^* \subseteq (V \otimes W)^* = \text{CB}(W, V^*)
$$

(2.21)

is completely isometric. The converse is false (even if one of the spaces is finite dimensional—see [9]).

3. SELF-DUALITY

The following simple result is not valid for the Grothendieck $H$-norm.

**Proposition 3.1.** Given operator spaces $V$ and $W$ with closed subspaces $V_0$ and $W_0$, the corresponding map

$$
V \otimes W \to V/V_0 \otimes W/W_0
$$

is a complete quotient map.

**Proof.** Given the completion $X$ of a normed vector space $X$, and an element $x$ of $X$ with $\|x\| < 1$, the usual telescoping sum argument shows that $x = \sum_1^\infty x_n$, where $x_n \in X$ and $\sum_1^\infty \|x_n\| < 1$. It follows that if $\phi: X \to Y$ is a quotient map, i.e., $\phi$ maps the open ball of $X$ onto that of $Y$, the same is true for the continuous extension $\bar{\phi}: \bar{X} \to \bar{Y}$. Thus it suffices to prove the assertion for the non-completed tensor product $\otimes_h$. It is evident that the indicated map is completely contractive. Given $u_1 \in \mathcal{M}_{p,q}(V/V_0 \otimes_h W/W_0)$ with $\|u_1\| < 1$, we may assume that $u_1 = v_1 \odot w_1$, where $v_1 \in V/V_0$ and $w_1 \in W/W_0$ satisfy $\|v_1\|$ and $\|w_1\| < 1$. Fixing preimages $v$ and $w$ with $\|v\|$, $\|w\| < 1$, we have that $u = v \odot w$ is a preimage of $u_1$ with $\|u\| < 1$. 


Again in contrast to the Grothendieck $H$ norm, we have

**Theorem 3.2.** Suppose that $V$ and $W$ are operator spaces. Then the dual of the Haagerup operator norms for $V \otimes W$ are just the Haagerup operator norms for $V^* \otimes W^*$.

**Proof.** Suppose that $e \in \mathbb{M}_{p,q}(V^* \otimes W^*)$ satisfies $\|e\|_h < 1$. Then we may select $r \in \mathbb{N}$ and $f \in \mathbb{M}_{p,r}(V^*)$, $g \in \mathbb{M}_{r,q}(W^*)$ with $e = f \circ g$ and $\|f\|_{cb} < 1$, $\|g\|_{cb} < 1$. Since $\mathbb{M}_{p,r}(V^*) = CB(V, \mathbb{M}_{p,r})$ and $\mathbb{M}_{r,q}(W^*) = CB(W, \mathbb{M}_{r,q})$, we may let $f(v) = T_1 \pi_1(v)S_1$ and $g(w) = T_2 \pi_2(w)S_2$ be dilation representations, where $\pi_1$ (resp., $\pi_2$) is a representation of $V$ (resp., $W$) on $H_1$ (resp., $H_2$), and we have bridging diagrams of contractions

$$
\mathbb{C}^r \overset{S_1}{\rightarrow} H_1 \overset{T_1}{\rightarrow} \mathbb{C}^p,
$$

$$
\mathbb{C}^q \overset{S_2}{\rightarrow} H_2 \overset{T_2}{\rightarrow} \mathbb{C}^r.
$$

Regarding $T_j$ (resp., $S_j$) ($j = 1, 2$) as column (resp., row) matrices, we have for $v \in V$ and $w \in W$,

$$
e(v \otimes w) = \left[ e_{ij}(v \otimes w) \right]
$$

$$= \left[ \sum_k f_{ik}(v) g_{kj}(w) \right]
$$

$$= \left[ \sum_k (T_1)_k \pi_1(v)(S_1)_k (T_2)_k \pi_2(w)(S_2)_k \right]
$$

$$= T_1 \pi_1(v) S_1 T_2 \pi_2(w) S_2,
$$

hence $e$ is a complete contraction from $V \otimes_h W$ into $\mathbb{M}_{p,q}$.

Conversely suppose that $e \in \mathbb{M}_{p,q}(V^* \otimes W^*)$ determines a complete contraction from $V \otimes_h W$ into $\mathbb{M}_{p,q}$. Then we may assume that $e = [e_{ij}]$, where

$$
e_{ij} = \sum_{k=1}^{n} f_{ij}^k \otimes g_{ij}^k,
$$

$f_{ij}^k \in V^*$, $g_{ij}^k \in W^*$. Letting $V_0 \subseteq V$ and $W_0 \subseteq W$ be the intersections of the kernels of the maps $f_{ij}^k$ and $g_{ij}^k$, respectively, and letting $\rho: V \to V_1 = V/V_0$ and $\sigma: W \to W_1 = W/W_0$ be the quotient maps, we have that $f_{ij}^k = \rho^*(f_{ij}^k)$ and $g_{ij}^k = \sigma^*(g_{ij}^k)$ for suitable functions $f_{ij}^k$ and $g_{ij}^k$ on the finite dimensional spaces $V_1$ and $W_1$, respectively. Thus $e = (\rho \otimes \sigma)^*_{p,q}(\bar{e})$, where $\bar{e} = \sum f_{ij}^k \otimes g_{ij}^k$. From Proposition 3.1, $(\rho \otimes \sigma)^*_{p,q}$ is a complete isometry, hence

$$
\bar{e}: V_1 \otimes_h W_1 \to \mathbb{M}_{p,q}.
$$
has completely bounded norm less than one. It follows that we have the dilation representation
\[ \tilde{e}(v_1 \otimes w_1) = T\pi_1(v_1)S\pi_2(w_1)R, \]
where
\[ \mathbb{C}^q \xrightarrow{R} H_2 \xrightarrow{S} H_1 \xrightarrow{T} \mathbb{C}^p \]
is a diagram of proper contractions, and \( \pi_1: V_1 \to \mathcal{B}(H_1) \), and \( \pi_2: W_1 \to \mathcal{B}(H_2) \) are representations. Letting \( E \) and \( F \) be the projections of \( H_1 \) and \( H_2 \) onto the finite dimensional subspaces \( H'_1 = \pi_1(V_1)T^*(\mathbb{C}^p) \) and \( H'_2 = \pi_2(W_1)R(\mathbb{C}^q) \), respectively, it follows that
\[ \tilde{e}(v_1 \otimes w_1) = T\pi_1(v_1)ESF\pi_2(w_1)R. \]

Letting \( r \) be the rank of \( ESF \), we may write \( ESF = S_1S_2 \), where \( S_j \) are proper contractions in a diagram
\[ H_2 \xrightarrow{S_2} \mathbb{C}^r \xrightarrow{S_1} H_1. \]

We conclude that \( \tilde{e} = f \circ \tilde{g} \), where \( f(v_1) = T\pi_1(v_1)S_1 \) and \( \tilde{g}(w_1) = S_2\pi_2(w_1)R \) determine proper complete contractions in \( \mathbb{M}_{r,q}(V_1^*) \) and \( \mathbb{M}_{r,q}(W_1^*) \), respectively. It follows that \( e = f \circ g \), where \( f = \rho^*(f) \) and \( g = \sigma^*(\tilde{g}) \) are proper complete contractions.

The following is analogous to (2.21). The second part of the formula is proved in Section 5 (see (5.8)).

**Corollary 3.3.** We have the natural completely isometric inclusion
\[ V^* \otimes^h W^* \subset^h (V \otimes W)^* = \bar{F}_2(V, W^*). \]

We also obtain an analogue of (2.9) (a version of this using infinite rank maps was given in [14, Theorem 4.2]). Given linear maps \( g: V \to \mathbb{M}_{n,p} \) and \( h: W \to \mathbb{M}_{p,n} \), we define \( gh: V \otimes W \to \mathbb{M}_n \) by \( (gh)(v \otimes w) = g(v)h(w) \).

**Proposition 3.4.** Given operator spaces \( V \) and \( W \), the norm on \( \mathbb{M}_n(V \otimes_h W) \) is determined by
\[ \|u\| = \sup \{ \|(gh)_n(u)\| \}, \]
where the supremum is taken over all complete contractions \( g: V \to \mathbb{M}_{n,p}, \) \( h: W \to \mathbb{M}_{p,n} \), and \( p \in \mathbb{N} \).
Proof. It suffices to prove this for \( u \in V \otimes_h W \). Given \( \varepsilon > 0 \), we may select \( \varphi \in M_n(V \otimes^b W)^* \) with \( \| \varphi \| < 1 \) and \( \| \langle u, \varphi \rangle \| > \| u \| - \varepsilon \). Dilating \( \varphi \) as in (2.20) (with \( V \) and \( W \) reversed), and strongly approximating \( S \) by finite rank contractions \( S' \), the corresponding complete contractions \( f : V \otimes_h W \to M_n \) determined by

\[
f(v \otimes w) = T \pi_1(v) S' \pi_2(w) R
\]

are such that \( f(v \otimes w) \) converges in norm to \( \varphi(v \otimes w) \) for each \( v \) and \( w \). Thus we may select such an \( f \) with \( \| \langle u, f \rangle \| > \| u \| - \varepsilon \). Assuming \( S' \) is of rank \( p \), we have that \( S' = S_1 \circ S_2 \), where \( S_j \) are contractions in the diagram

\[
H_2 \xrightarrow{S_2} \mathbb{C}^p \xrightarrow{S_1} H_1.
\]

Then letting \( g(v) = T \pi_1(v) S_1 \) and \( h(w) = S_2 \pi_2(w) R \), we are done.

Another surprising aspect of the Haagerup tensor product is that for elements of the algebraic tensor product, the infimum in (2.11) is actually attained:

**Proposition 3.5.** Given operator spaces \( V \) and \( W \) and an element \( u \in M_{m}(V \otimes_h W) \), there exist an integer \( p \) and elements \( v \in M_{m}(V) \), \( w \in M_{m}(W) \) for which \( u = v \circ w \) and \( \| u \| = \| v \| \cdot \| w \| \).

Proof. Owing to (2.16), it suffices to consider the case \( n = 1 \). Given

\[
u = \sum_{k=1}^{p} v_k \otimes w_k,
\]

we have that \( u \in V_0 \otimes W_0 \), where \( V_0 \) and \( W_0 \) are the linear spans of the \( v_k \) and \( w_k \), respectively. Since \( V_0 \otimes W_0 \subset V \otimes_h W \) is completely isometric, we may initially assume that \( V \) and \( W \) are finite dimensional. The argument in [2, Theorem 3.4] shows that if

\[
u = v \circ w = \sum_{j=1}^{p} v_j \otimes w_j,
\]

then

\[
u = v' \circ w' = \sum_{j=1}^{q} v'_j \otimes w'_j,
\]

where the elements \( w'_j \) are linearly independent, and \( \| v' \| \cdot \| w' \| \leq \| v \| \cdot \| w \| \). Thus if \( m \) is the dimension of \( W \), we conclude that

\[
\| u \|_h = \inf \{ \| v \| \cdot \| w \| : u = v \circ w, \text{ where } v \in M_{1m}(V), w \in M_{m1}(W) \}.
\]

Using the compactness of the unit balls of the finite dimensional Banach spaces \( M_{1m}(V) \) and \( M_{m1}(W) \), we may find norm convergent sequences \( v_k \in M_{1m}(V) \), \( w_k \in M_{m1}(W) \) with \( u = v_k \circ w_k \) and \( \| v_k \| \cdot \| w_k \| \to \| u \|_h \). Letting \( v_k \to v \) and \( w_k \to w \), it follows that \( u = v \circ w \) and

\[
\| u \|_h = \| v \| \cdot \| w \|.
\]
Given a Hilbert space $H$, we use the identification $H \cong B(\mathbb{C}, H)$ to determine the column operator structure on $H$, and let $H_c$ denote the corresponding operator space. Similarly we define the row operator space $H_r$ by using the identification $H \cong B(H^*, \mathbb{C})$.

It is important to distinguish these structures. By definition, $\mathbb{M}_{p,q}(H_c) = B(\mathbb{C}^q, H^p)$, whereas $\mathbb{M}_{p,q}(H_r) = B(\mathbb{C}^{*q}, \mathbb{C}^p)$. For the case $H = \mathbb{C}$, we have a canonical identification of $\mathbb{C}$ and $\mathbb{C}^*$ as Banach spaces. We may identify $\mathbb{C}_c$ and $\mathbb{C}_r$ as operator spaces since $\mathbb{M}_{p,q}(\mathbb{C}_c) = B(\mathbb{C}^q, \mathbb{C}^p) = \mathbb{M}_{p,q}(\mathbb{C}_r)$, but there is no way to do this for $H = \mathbb{C}^n$ ($n > 1$). Given $\xi_1, ..., \xi_n \in H$, then the row matrix

$$\xi = [\xi_1 \cdots \xi_n] \in \mathbb{M}_{1,n}(H_r) = B(H^{*n}, \mathbb{C})$$

has norm

$$\|\xi\| = \left(\sum \|\xi_j\|^2\right)^{1/2}. \quad (4.1)$$

On the other hand if we regard $\xi$ as an element of

$$\mathbb{M}_{1,n}(H_c) = B(\mathbb{C}^n, H),$$

$\xi$ maps the canonical basis vectors $e_j$ into the vectors $\xi_j$. In particular, if we assume the $\xi_j$ are orthogonal, then

$$\|\xi\| = \sup\{\|\xi_j\|\}. \quad (4.2)$$

Both (4.1) and (4.2) extend easily to infinite families of vectors.

**Theorem 4.1.** Given Hilbert spaces $H$ and $K$, any bounded operator $T: H \rightarrow K$ is completely bounded, and the identity map $B(H, K) \rightarrow CB(H_c, K_c)$ is a complete isometry.

**Proof.** We must show that if

$$T = [T_{ij}] \in \mathbb{M}_n B(H, K) = B(H^n, K^n),$$

then defining $\tilde{T}: H_c \rightarrow \mathbb{M}_n(K_c)$ by $\tilde{T}(\xi) = [T_{ij}(\xi)]$, we have that $\|\tilde{T}\|_{cb} = \|T\|$, and that any $\varphi \in CB(H_c, \mathbb{M}_n(K_c))$ has the form $\tilde{T}$.

Given $\eta = (\eta_1, ..., \eta_n) \in H^n$ with $\|\eta\| \leq 1$, we let $e_1, ..., e_n$ be a basis for the space spanned by the $\eta_j$, and we let $\eta_j = \sum_k c_{jk} e_k$. Then

$$1 \geq \|\eta\|^2 = \sum_j \|\eta_j\|^2 = \sum_{jk} |c_{jk}|^2,$$
and thus the \( np \times 1 \) column matrix \( c = (c_1, ..., c_p) \), with \( c_j = (c_{1j}, c_{2j}, ..., c_{nj}) \) satisfies \( \|c\| \leq 1 \). From (4.2) the matrix

\[
e = \begin{bmatrix} e_1 \cdots e_p \end{bmatrix} \in \mathbb{M}_{1,p}(H) = B(C^p, H)
\]
satisfies \( \|e\| = 1 \). Noting that

\[
(\tilde{T})_{1,p}(e) = [\tilde{T}(e_1) \cdots \tilde{T}(e_p)] = [[T_{ij}(e_1)] \cdots [T_{ij}(e_p)]] = [T_{ij}(e_k)]
\]
is an \( n \times np \) matrix (with indices \( i, (j, k) \)), we have

\[
T(\eta) = \left( \sum_j T_{ij} \eta_j \right)
\]

\[
= \left( \sum_{jk} T_{ij}(e_k)c_{jk} \right)
\]

\[
= (\tilde{T})_{1,p}(e)c,
\]

and thus \( \|T\| \leq \|\tilde{T}\|_{cb} \).

Given

\[
\xi = [\xi_{kl}] \in \mathbb{M}_p(H) = B(C^p, H^p),
\]

with \( \|\xi\| < 1 \), we let \( \xi_k \) be the row matrix \( [\xi_{k1} \cdots \xi_{kp}] \). We have that

\[
\tilde{T}_p(\xi) \in \mathbb{M}_{p}(\mathbb{M}_n(K)) = B(C^{np}, K^{np}).
\]

Given \( \alpha = (\alpha_1, ..., \alpha_p) \in (C^n)^p \) with \( \|\alpha\| \leq 1 \), we let \( \alpha_l = (\alpha_{l1}, ..., \alpha_{ln}) \in C^n \). The rearranged vector \( \tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_n) \in (C^n)^n \) with \( \tilde{\alpha}_j = (\alpha_{jl}, ..., \alpha_{jp}) \) also satisfies \( \|\tilde{\alpha}\| = \|\alpha\| \leq 1 \). For each \( k \) and \( j \), \( \xi_k \tilde{\alpha}_j \) is an element of \( H \), hence \( \xi_k \tilde{\alpha} = (\xi_k \tilde{\alpha}_j)_{1 \leq j \leq n} \) is a vector in \( H^n \), and \( \xi \tilde{\alpha} = (\xi_k \tilde{\alpha})_{1 \leq k \leq p} \) is a vector in \((H^n)^p\). We conclude that

\[
\tilde{T}_p(\xi)(\alpha) = \left( \sum_i \tilde{T}(\xi_{ki})\alpha_i \right)
\]

\[
= \left( \sum_{jl} T_{ij}(\xi_{ki})\alpha_{jl} \right)
\]

\[
= \left( \sum_j T_{ij} \left( \sum_i \xi_{ki}\alpha_{jl} \right) \right)
\]

\[
= \left( \sum_j T_{ij}(\xi_k \tilde{\alpha}_j) \right)
\]

\[
= (T(\xi_k \tilde{\alpha}))
\]

\[
= T(p)(\xi \tilde{\alpha}),
\]

where \( T(p) = T \oplus \cdots \oplus T \), and thus \( \|\tilde{T}_p\| \leq \|T(p)\| = \|T\| \), and \( \|\tilde{T}\|_{cb} \leq \|T\| \).
Finally, if we are given $\varphi = [\varphi_{ij}] \in \text{CB}(H_c, \mathbb{M}_n(K_c))$, each component $\varphi_{ij} \in \text{CB}(H_c, K_c)$ is in particular an element of $B(H, K)$. Thus letting $T_{ij} = \varphi_{ij}$, we have that $\varphi = \hat{T}$, and we are done.

**Corollary 4.2.** Given Hilbert spaces $H$ and $K$, we have the complete isometry

$$B(H, K) \cong (H_c \otimes (K_c)^*)^*,$$

and in particular

$$(H_c)^* \cong (H^*)_c.$$

**Proof:** From (2.14) and Theorem 4.1,

$$(H_c \otimes (K_c)^*)^* \cong \text{CB}(H_c, K_c) = B(H, K),$$

hence in particular,

$$(H_c)^* \cong B(H, C) = (H^*)_c.$$

From the second relation, we have that

$$\mathbb{M}_{1,n} = (\mathbb{M}_{n,1})^*.$$  \hspace{0.5cm} (4.3)

Since $\mathbb{M}_{m,n} = B(C^m, C^n)$ is finite dimensional, taking the dual of the first relation (and using the fact that the map $V \to V^{**}$ is completely isometric), we conclude that

$$\mathbb{M}_{n}^* \cong \mathbb{M}_{1,n} \otimes \mathbb{M}_{1,n}^*.$$  \hspace{0.5cm} (4.4)

Turning to row spaces, we have that the adjoint map determines a complete isometry of $\text{CB}(H_c^*, K_c^*)$ with $\text{CB}(K_c, H_c)$. It follows that we have the complete isometries

$$\text{CB}(H_r, K_r) = \text{CB}((H^*_c)^*, (K^*_c)^*) \cong \text{CB}((K^*_c)_c, (H^*_c)_c) = B(K^*, H^*).$$

(We are indebted to D. Blecher for correcting an earlier calculation of $\text{CB}(H_r, K_r)$.) The map $B(H, K) \to \text{CB}(H_r, K_c)$ is more subtle. It is evident from (4.1) and (4.2) that the identity map $I: H_r \to H_c$ does not have completely bounded norm 1 (see Corollary 4.5 for a more precise result).

**Theorem 4.3.** Given an operator space $V$ and a Hilbert space $H$, we have complete isometries

(a) $V \otimes H_c \cong V \hat{\otimes} H_c$,
(b) \((H_c)^* \otimes V \cong (H_c)^* \otimes V\),
(c) \(H_c \otimes V \cong H_c \otimes V\),
(d) \(V \otimes (H_c)^* \cong V \otimes (H_c)^*\)

**Proof.** We have from [2, 5] that the projective operator norms for \(V \otimes H_c\) dominate the Haagerup norms. Conversely, let us suppose that \(u \in \mathcal{M}_n(V \otimes H)\) satisfies \(\|u\|_h < 1\). For some \(m \in \mathbb{N}\) we may let \(u = v \otimes \xi\), where \(v \in \mathcal{M}_{m,n}(V)\), and \(\xi \in \mathcal{M}_{m,n}(H_c)\) satisfy \(\|v\|, \|\xi\| < 1\). Letting \(\xi = [\xi_{ij}]\), we fix an orthonormal basis \(e_1, \ldots, e_p\) for the finite dimensional subspace of \(H\) spanned by the \(\xi_{ij}\). Letting \(\xi = \sum c_{ij} e_k\), we claim that \(\|\xi\| = \|c\|\), where \(c\) is the \(mp \times n\) matrix

\[
\begin{bmatrix}
c_{1j}^1 \\
c_{2j}^2 \\
v_j^2 \\
c_{pj}^p
\end{bmatrix}
\]

In fact since \(\xi \in \mathcal{M}_{m,n}(H_c) = \mathcal{B}(\mathbb{C}^n, H^m)\), we have that

\[
\|\xi\|^2 = \sup \{ \|\xi(\alpha)\|^2 : \alpha \in \mathbb{C}^n, \|\alpha\| \leq 1 \}
= \sup \left\{ \left\| \sum_j \xi_{ij} \alpha_j + \sum_j \xi_{2j} \alpha_j + \ldots + \sum_j \xi_{mj} \alpha_j \right\|^2 : \alpha \in \mathbb{C}^n, \|\alpha\| \leq 1 \right\}
= \sup \left\{ \sum_i \left\| \sum_j \xi_{ij} \alpha_j \right\|^2 : \alpha \in \mathbb{C}^n, \|\alpha\| \leq 1 \right\}
= \sup \left\{ \sum_i \left\| \sum_{i,k} c_{ij}^k e_k \alpha_j \right\|^2 : \alpha \in \mathbb{C}^n, \|\alpha\| \leq 1 \right\}
= \sup \{ \|c(\alpha)\|^2 : \alpha \in \mathbb{C}^n, \|\alpha\| \leq 1 \} = \|c\|^2.
\]

From (4.2), the matrix \(e = [e_1 \cdots e_p] \in \mathcal{M}_{1,p}(H_c) = \mathcal{B}(\mathbb{C}^p, H)\) satisfies \(\|e\| = 1\). Noting that

\[
v \otimes e = [v_{ij} \otimes e_1 \cdots v_{ij} \otimes e_p] \in \mathcal{M}_{n, mp}(V \otimes H),
\]

we have that

\[
u = v \otimes \xi = \sum_l v_{hl} \otimes \xi_{lj} = \sum_{l,k} v_{hl} \otimes e_k c_{lj}^k = l[v \otimes e]c,
\]
where $I$ is the identity matrix in $\mathbb{M}_n(\mathbb{C})$. It follows that by definition, $\|u\|_\wedge < 1$.

The proof for (b) follows by symmetry. We have from Corollary 3.3 and (2.21) that the maps

$$V^{**} \otimes (H_c)^* \subseteq (V^* \otimes H_c)^*$$

$$V^{**} \hat{\otimes} (H_c)^* \subseteq (V^* \hat{\otimes} H_c)^*$$

are completely isometric, and thus from (a), the Haagerup and spatial norms coincide on $V^{**} \otimes (H_c)^*$. On the other hand, the injection $V \otimes (H_c)^* \subseteq V^{**} \otimes (H_c)^*$ is completely isometric with respect to both of these norms, and we obtain (d). The proof for (c) is the same.

Given a Hilbert space $H$, we let $\mathcal{K}(H, K) \subseteq B(H, K)$ denote the operator space of compact operators.

**COROLLARY 4.4.** Given Hilbert spaces $H$ and $K$, we have complete isometries:

(a) $H \hat{\otimes} K \cong H \hat{\otimes} K \cong H \hat{\otimes} K \cong (H \otimes K)_c$

(b) $H \hat{\otimes} K \cong K \hat{\otimes} H \cong \mathcal{K}(H^*, K)$

(c) $H \hat{\otimes} K \cong H \hat{\otimes} K \cong B(K, H^*)_c$.

**Proof.** We need only verify the last complete isometry in each of these relations.

Let us fix unit vectors $\xi_0 \in H^*$, $\eta_0 \in K^*$. The maps

$$H_c = B(\mathbb{C}, H) \subseteq B(H): \xi \mapsto \xi_0^* \otimes \xi$$

$$K_c = B(\mathbb{C}, K) \subseteq B(K): \eta \mapsto \eta_0^* \otimes \eta$$

are completely isometric. From the definition of the spatial tensor product, it follows that the map

$$H_c \hat{\otimes} K_c \rightarrow B(H \otimes K): \xi \otimes \eta \mapsto (\xi_0^* \otimes \eta_0^*) \otimes (\xi \otimes \eta)$$

is completely isometric. But the latter injection also determines the operator space structure on $(H \otimes K)_c$, and the first relation in (a) follows. The second relation is proved in the same manner, since we have, for example, that

$$H_c = B(H^*, \mathbb{C}) \subseteq B(H^*): \xi^* \mapsto \xi^* \otimes \xi_0$$

is completely isometric.
From (2.21), Theorem 4.1, and Corollary 4.2, the map
\[ H_r \hat{\otimes} K_c \subseteq \text{CB}((H_r)^*, K_c) = \text{B}(H^*, K) \]
is completely isometric. Since the algebraic tensor product is mapped onto the finite rank operators, (b) is immediate.

From (2.14), Theorem 4.1, and Corollary 4.2,
\[ \text{B}(K, H^*) = \text{CB}(K_c, (H^*)_c) = (H_r \hat{\otimes} K_c)^*. \]

The finite dimensional symmetric case of (c) in Corollary 4.4, i.e., the relation
\[ M_n^\ast \cong \text{M}_{1,n} \otimes \text{M}_{n,1}, \]
was first proved in [2] (see the remarks in Section 1). Letting \( e_{ij} \) be dual (via the trace) to the usual matrix units \( e_{ij} \), the complete isometry is given by \( e_{ij} \mapsto e_{ij} \otimes e_{ji} \). Using the injections \( H_r \subseteq \text{B}(H^*) \) and \( K_c \subseteq \text{B}(K) \) determined by fixed unit vectors \( \xi_0^* \in H, \eta_0^* \in K^* \) (see above), we obtain from [3] a complete isometry of \( \text{B}(H, K)_c \) into the free product \( \text{B}(H^*) \ast \text{B}(K) \).

In particular, this provides a "concrete" representation of the operator space of trace class operators \( \text{B}(H)_c \) as a linear space of operators.

Given Hilbert spaces \( H \) and \( K \), we have a natural contraction \( H^* \otimes K \to \text{B}(H, K) \), determined by \( (\xi^*, \eta) \mapsto \xi^* \otimes \eta \). An operator \( T \in \text{B}(H, K) \) is said to be Hilbert–Schmidt if it is in the range of this map, and then the Hilbert–Schmidt norm of \( T \) is the norm of its preimage (the latter is unique). The following is related to a result of Mathes [12].

**Corollary 4.5.** Given Hilbert spaces \( H \) and \( K \), and a linear map \( T: H_r \to K_c \), we have that \( \| T \|_{cb} \) coincides with the Hilbert–Schmidt norm of \( T \).

**Proof.** This is a consequence of the complete isometries
\[ \text{CB}(H_r, K_c) \cong (H_r \hat{\otimes} (K^*)_c)^* = ((H \otimes K^*)_c)^* \cong (H^* \otimes K)_c. \]

Another approach to this mapping space can be made by using results from Section 5. From the definition of the Hilbert factorization spaces, we have
\[ \text{CB}(H_r, K_c) = \Gamma_2(H_r, K_c). \]

Letting \( H \) and \( K \) be column Hilbert spaces, we obtain a result proved only at the level of normed spaces in [5] (as pointed out in [11], the isometric argument worked for general \( V \):
Corollary 4.6. Given a column Hilbert space $H$ and an operator space $V$, we have complete isometries
\[
(K_c)^* \otimes V \otimes H_c \cong V \overset{\hat{}}{\otimes} \mathcal{B}(H, K),
\]
\[
((K_c)^* \otimes V \otimes H_c)^* \cong \mathcal{CB}(V, \mathcal{B}(H, K)).
\]

Proof. The second relation follows from the first and (2.14). Using the commutativity and associativity of the projective operator tensor product, we have
\[
(K_c)^* \otimes V \otimes H_c \cong (K_c)^* \otimes (V \otimes H_c) \cong V \otimes ((K_c)^* \otimes H_c) \cong V \otimes \mathcal{B}(H, K).
\]

5. Hilbert Space Factorizations

Throughout this section we restrict our attention to operator spaces $V$ which are norm complete. It is easy to see that this implies that the spaces $\mathcal{M}_n(V)$ are all complete. If $H$ is a Hilbert space, we shall simply write $H$ for the column space $H_c$.

Given complete operator spaces $V$ and $W$, we say that a linear map $\phi: V \to W$ factors through a column Hilbert space if there is a Hilbert space $H$ and a commutative diagram of completely bounded maps
\[
\begin{array}{ccc}
H & \overset{\sigma}{\longrightarrow} & \overset{\tau}{\longrightarrow} \\
V & \overset{\phi}{\longrightarrow} & W
\end{array}
\]

We define $\gamma_2(\phi) = \inf \{ \| \sigma \|_{cb} \| \tau \|_{cb} \}$, where the infimum runs over all possible factorizations, letting $\gamma_2(\phi) = \infty$ if no such factorization exists. We define $\Gamma_2(V, W)$ to be the linear space of linear maps $\phi: V \to W$ for which $\gamma_2(\phi) < \infty$. Although the fact that $\gamma_2$ is a norm follows from Theorem 5.3, it is more instructive to see a direct proof. What we need is

Lemma 5.1. Given operator spaces $V$ and $W$ and linear maps $\phi_j: V \to W$ ($j = 1, 2$), we have that
\[
\gamma_2(\phi_1 + \phi_2) \leq \gamma_2(\phi_1) + \gamma_2(\phi_2).
\]

Proof. Given diagrams
\[
\begin{array}{ccc}
H_j & \overset{\sigma_j}{\longrightarrow} & \overset{\tau_j}{\longrightarrow} \\
V & \overset{\phi_j}{\longrightarrow} & W
\end{array}
\]
(j = 1, 2), and letting $H = H_1 \oplus H_2$ (the usual Hilbert space direct sum, with the column operator structure) we obtain a third diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\sigma} & V \\
\downarrow{\tau} & & \phi \rightarrow W \\
V & \xrightarrow{\phi} & W
\end{array}
$$

where $\varphi = \varphi_1 + \varphi_2$, $\sigma(v) = (\sigma_1(v), \sigma_2(v))$, and $\tau(\xi_1, \xi_2) = \tau_1(\xi_1) + \tau_2(\xi_2)$. We note that it is better not to use matrices of maps between operator spaces which are not column (or row) Hilbert spaces since Theorem 4.1 is not applicable. We claim that

$$
\|\sigma\|_{cb} \leq (\|\sigma_1\|_{cb}^2 + \|\sigma_2\|_{cb}^2)^{1/2} \tag{5.2}
$$

To prove the first inequality, we take dilations for the maps $\sigma_j: V \rightarrow B(\mathbb{C}, H_j)$. We have

$$
\sigma_j(v) = S_j \pi(v_j) R_j,
$$

where $\pi_j$ are representations of $V$ on Hilbert spaces $K_j$, and

$$
\begin{array}{ccc}
C & \xrightarrow{R_j} & K_j \\
S_j \rightarrow & & H_j
\end{array}
$$

are diagrams with $\|S_j\| = 1$, and $\|R_j\| = \|\sigma_j\|_{cb}$. It follows that

$$
\sigma = S \pi(v) R,
$$

where $S = S_1 \oplus S_2$, $\pi = \pi_1 \oplus \pi_2$, and $R = \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix}$. Since $\|S\| = 1$ and $\|R\| = (\|R_1\|^2 + \|R_2\|^2)^{1/2}$ (we may identify $R$ with the vector $R(1)$), we obtain the first inequality in (5.2). Turning to the second inequality, consider the map

$$
\tau^*: W^* \rightarrow (H)^* = H_1^* \oplus H_2^*,
$$

where we place the row structures on the Hilbert spaces $H_1^*$, $H_2^*$ and $H_1^* \oplus H_2^*$. We have that

$$
\tau^*(w^*) = \begin{bmatrix} \tau_1^*(w^*) \\ \tau_2^*(w^*) \end{bmatrix};
$$

hence using dilations for the maps $\tau_j^*: W^* \rightarrow B(H_j, \mathbb{C})$, the preceding argument gives us the inequality for $\|\tau\|_{cb} = \|\tau^*\|_{cb}$ in (5.2). In this situation we must consider diagrams of the form

$$
\begin{array}{ccc}
H_j & \xrightarrow{R_j} & K_j \\
S_j \rightarrow & & \mathbb{C}
\end{array}
$$

(5.4)
Noting that
\[ \tau \circ \sigma(v) = \tau_1(\sigma_1(v)) + \tau_2(\sigma_2(v)), \]
the remainder of the argument is analogous to that used in showing the Haagerup norm is subadditive [6]. Thus we use the fact that for real
\[ A, B > 0, \]
\[ AB = \inf \left\{ \frac{1}{2} (tA^2 + t^{-1}B^2) : t > 0 \right\}. \]
Given \( \varepsilon > 0 \), we choose \( \sigma_j, \tau_j, \) and reals \( t_j > 0 \) such that
\[ \frac{1}{2} \left( t_j \| \sigma_j \|^2_{cb} + t_j^{-1} \| \tau_j \|^2_{cb} \right)^{1/2} < \gamma_2(\varphi_j) + \varepsilon. \]
We then have that
\[ \varphi = \tau \circ \sigma = \tau' \circ \sigma', \]
where
\[ \sigma'(v) = (t_1^{1/2} \sigma_1(v), t_2^{1/2} \sigma_2(v)) \]
\[ \tau' (\xi_1, \xi_2) = t_1^{-1/2} \tau_1 (\xi_1) + t_2^{-1/2} \tau_2 (\xi_2). \]
We thus have that
\[ \gamma_2(\varphi) \leq \| \sigma' \|_{cb} \| \tau' \|_{cb} \]
\[ \leq (t_1 \| \sigma_1 \|^2_{cb} + t_2 \| \sigma_2 \|^2_{cb})^{1/2} (t_1^{-1} \| \tau_1 \|^2_{cb} + t_2^{-1} \| \sigma_2 \|^2_{cb})^{1/2} \]
\[ \leq \frac{1}{2} \left( t_1 \| \sigma_1 \|^2_{cb} + t_2 \| \sigma_2 \|^2_{cb} + t_1^{-1} \| \tau_1 \|^2_{cb} + t_2^{-1} \| \sigma_2 \|^2_{cb} \right) \]
\[ \leq \gamma_2(\varphi_1) + \gamma_2(\varphi_2) + 2\varepsilon, \]
and we are done.

It is immediate that \( \| \varphi \|_{cb} \leq \gamma_2(\varphi) \), and that if \( V \) or \( W \) is a column Hilbert space, then \( \| \varphi \|_{cb} = \gamma_2(\varphi) \); i.e., we have \( \Gamma_2(V, W) = CB(V, W) \).

Given linear maps \( \psi : V_1 \to V \) and \( \theta : W \to W_1 \) it is clear that \( \theta \phi \psi : V_1 \to W_1 \) satisfies
\[ \gamma_2(\theta \phi \psi) \leq \| \theta \|_{cb} \gamma_2(\varphi) \| \psi \|_{cb}. \]

As before, a matrix \( \varphi = [\varphi_{ij}] \in \mathbb{M}_n \Gamma_2(V, W) \) determines a map \( \varphi : V \to \mathbb{M}_n(W) \) by \( \varphi(v) = [\varphi_{ij}(v)] \). We claim that any such \( \varphi \) has a factorization of the form

\[ \mathbb{M}_n(H) \quad \sigma \quad \tau_1 \quad \tau_{1,n} \]
\[ V \xrightarrow{\varphi} \mathbb{M}_n(W) \]

(5.5)
where \( \sigma: V \to \mathbb{M}_{1,n}(H) \) and \( \tau: H \to \mathbb{M}_{1,n}(W) \) are completely bounded. To see this, we may by definition assume that we have commutative diagrams

\[
\begin{array}{c}
\sigma_{ij} \\
\alpha \downarrow \\
V \\
\phi_{ij} \\
\tau \uparrow \\
W
\end{array}
\]

where \( \sigma_{ij} \) and \( \tau_{ij} \) are completely bounded, and we let \( H = \bigoplus H_{ij} \) be the Hilbert space direct sum with the column structure. Writing \( \varepsilon_{ij} \) (resp., \( \varepsilon_{ij}^* \)) for the usual injection \( H_{ij} \to H \) (resp., projection \( H \to H_{ij} \)), each linear map \( \alpha: V \to H \) is given by a column matrix \( \alpha = (\alpha_{11}, ..., \alpha_{mn}) \), where \( \alpha_{ij} = \varepsilon_{ij} \circ \alpha \), and each linear map \( \beta: H \to W \) is given by a row matrix \( \beta = [\beta_{11}, ..., \beta_{mn}] \), with \( \beta_{ij} = \beta \circ \varepsilon_{ij} \). Composition is then given by

\[
\beta \circ \alpha = \sum_{ij} \beta_{ij} \circ \alpha_{ij}.
\]

We define

\[
\sigma \in \mathbb{M}_{1,n} \text{CB}(V, H) = \text{CB}(V, \mathbb{M}_{1,n}(H))
\]

by \( \sigma = [\sigma_1 \cdots \sigma_n] \), where the \( n^2 \times 1 \) matrices \( \sigma_i \in \text{CB}(V, H) \) are defined by

\[
\sigma_1 = ((\sigma_{11}, ..., \sigma_{n1}), (0, ..., 0), ..., (0, ..., 0))
\]

\[
\sigma_2 = ((0, ..., 0), (\sigma_{12}, ..., \sigma_{n2}), ..., (0, ..., 0))
\]

\[
\vdots
\]

and we define

\[
\tau \in \mathbb{M}_{m,n} \text{CB}(H, W) = \text{CB}(H, \mathbb{M}_{m,n}(W))
\]

by \( \tau = (\tau_1, ..., \tau_n) \), where the \( \tau_j \in \text{CB}(H, W) \) are defined by

\[
\tau_1 = [[\tau_{11} 0_2 \cdots 0_n] [\tau_{12} 0_2 \cdots 0_n] \cdots [\tau_{1n} 0_2 \cdots 0_n]]
\]

\[
\tau_2 = [[0 \tau_{21} 0_3 \cdots 0_n] [0 \tau_{22} 0_3 \cdots 0_n] \cdots [0 \tau_{2n} 0_3 \cdots 0_n]]
\]

\[
\vdots
\]

Using (5.6), it is evident that \( \tau_i \circ \sigma_j = \tau_j \circ \sigma_{ij} = \varphi_{ij} \), and thus

\[
(\tau)_{1,n}(\sigma(v)) = \left[ \tau(\sigma_1(v)) \tau(\sigma_2(v)) \cdots \tau(\sigma_n(v)) \right]
\]

\[
\times \left[ \begin{array}{cccc}
\tau_1(\sigma_1(v)) & \tau_1(\sigma_2(v)) & \cdots & \tau_1(\sigma_n(v)) \\
\tau_2(\sigma_1(v)) & \tau_2(\sigma_2(v)) & \cdots & \tau_2(\sigma_n(v)) \\
\vdots & \vdots & \ddots & \vdots \\
\end{array} \right]
\]

\[
= [\varphi_{ij}(v)].
\]
We define the $\gamma_2$ norm of a matrix $\varphi = [\varphi_{ij}] \in \mathbb{M}_n \Gamma_2(V, W)$ by

$$\gamma_2(\varphi) = \inf \{ \| \sigma \|_{cb} \| \tau \|_{cb} \},$$

where the infimum extends over all factorizations (5.5). Once again one can use dilations to show that $\gamma_2$ is subadditive, and thus a norm. In this more general context one replaces $\mathbb{C}$ by $\mathbb{C}^n$ in the diagrams (5.3) and (5.4), and one uses the inequality $\| R \| \leq (\| R_1 \|^2 + \| R_2 \|^2)^{1/2}$. Rather than going into the details, we note that in Corollary 5.4 we prove that the $\gamma_2$'s determine an operator space structure on $\Gamma_2(V, W)$.

**Proposition 5.2.** Given complete operator spaces $V$ and $W \subseteq W_1$, the corresponding inclusion

$$\Gamma_2(V, W) \subseteq \Gamma_2(V, W_1)$$

is completely isometric.

**Proof.** Suppose that we have a diagram

$$\begin{array}{ccc}
\mathbb{M}_{1,n}(H) & & \mathbb{M}_n(W) \\
\sigma \downarrow & & \downarrow \tau_{1,n} \\
V \rightarrow & & \mathbb{M}_n(W_1) \subseteq \mathbb{M}_n(W)
\end{array}$$

where $\sigma \in \text{CB}(V, \mathbb{M}_{1,n}(H))$, $\tau \in \text{CB}(H, \mathbb{M}_{n,1}(W_1))$, and $\| \sigma \|_{cb}$, $\| \tau \|_{cb} < 1$. Then letting $\sigma$ be the row matrix $[\sigma_1 \cdots \sigma_n]$ and $\tau$ be the column matrix $(\tau_1, \ldots, \tau_n)$, we have that $\varphi_{ij}(v) = \tau_i \sigma_j(v)$. Letting $H_0 \subseteq H$ be the closed linear span of the subspaces $\sigma_j(V)$, we have that $\sigma(V) \subseteq \mathbb{M}_{1,n}(H_0)$. On the other hand, since $\tau_i \sigma_j(v) = \varphi_{ij}(v) \in W$, and $W$ is complete and thus closed in $W_1$, it follows that $\tau_i(x) \in W$ for all $x \in H_0$. Thus we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{M}_{1,n}(H_0) & & \mathbb{M}_n(W) \\
\sigma \downarrow & & \downarrow \tau_{1,n} \\
V \rightarrow & & \mathbb{M}_n(W_1)
\end{array}$$

and we are done.
Given operator spaces $V$ and $W$, each linear map $\varphi: V \to W^*$ determines a linear function $B: W \otimes V \to C$ by

$$\langle B(\varphi), w \otimes v \rangle = \langle \varphi(v), w \rangle.$$  

**Theorem 5.3.** If $V$ and $W$ are operator spaces, then $\varphi \mapsto B(\varphi)$ determines a complete isometry

$$\Gamma_2(V, W^*) \cong (W \otimes V)^*.$$  

**Proof.** Suppose that $F: W \otimes^h V \to M_n$ satisfies $\|F\|_\cb < 1$. Dilating $F$, we have

$$F(w \otimes v) = T\pi_2(w)S\pi_1(v)R,$$

where $\pi_1: V \to B(K)$ and $\pi_2: W \to B(H)$ are representations, and we have a diagram

$$\begin{array}{ccc}
C^n & \overset{R}{\to} & K \\
& \underset{S}{\searrow} & \underset{T}{\to} & C^n
\end{array}$$

with $\|R\|, \|S\|, \|T\| < 1$. Letting $\sigma(v) = S\pi_1(v)R$ and $\bar{\tau}(w) = T\pi_2(w)$, we have that

$$\sigma \in CB(V, B(C^n, H)) = CB(V, M_{1,n}(H)),$$

and

$$\bar{\tau} \in CB(W, B(H, C^n)),$$

where $\|\sigma\|_\cb, \|\bar{\tau}\|_\cb < 1$. Under the identifications

$$CB(W, CB(H, C^n)) \cong CB(W \hat{\otimes} H, C^n) \cong CB(H, CB(W, C^n))$$

$\bar{\tau}$ corresponds to a map $\tau: H \to CB(W, C^n)$ satisfying $\|\tau\|_\cb = \|\bar{\tau}\|_\cb$, which is determined by

$$\bar{\tau}(w)(\xi) = \tau(\xi)(w).$$

More generally, given $\xi = [\xi_1, \ldots, \xi_n] \in M_{1,n}(H)$,

$$\bar{\tau}_{1,m}(w)\xi = [\bar{\tau}(w)\xi_1 \cdots \bar{\tau}(w)\xi_n] = [\tau(\xi_1)(w) \cdots \tau(\xi_n)(w)]$$

$$= [\tau(\xi_1) \cdots \tau(\xi_n)](w) - \tau_{1,n}(\xi)(w).$$

Letting $\xi = \sigma(v)$, we conclude that

$$F(w \otimes v) = \bar{\tau}(w)\sigma(v) = \tau_{1,n}(\sigma(v))(w).$$
and thus letting $\varphi = \tau_{1,n} \circ \sigma$, we have that
\[
F(w \otimes v) = [F_{y}(w \otimes v)] = [B(\tau_{1} \circ \sigma)(w \otimes v)] = B_{n}(\varphi)(w \otimes v),
\]
i.e., $F = B_{n}(\varphi)$, where $\gamma_{2}(\varphi) < 1$.

Conversely let us suppose that $\varphi: V \to \mathcal{M}_{n}(W^{*})$ satisfies $\gamma_{2}(\varphi) < 1$. Then we have a diagram
\[
\begin{array}{c}
\mathcal{M}_{1,n}(H) \\
\sigma \downarrow \\
V \xrightarrow{\varphi} \mathcal{M}_{n}(W^{*}),
\end{array}
\]
with $\|\sigma\|_{cb}, \|\tau\|_{cb} < 1$. We let
\[
\sigma(v) = S_{1} \pi_{1}(v) R,
\]
where $\pi_{1}$ is a representation of $V$ on a Hilbert space $K$, and $\mathcal{C}^{n} \xrightarrow{R} K \xrightarrow{S_{1}} H$ is a diagram with $\|R\|$ and $\|S_{1}\| < 1$. Reversing the argument above, we have that $\tau: H \to \text{CB}(W, \mathbb{C}^{n})$ determines a map $\tilde{\tau}: W \to \text{CB}(H, \mathbb{C}^{n})$ with the same completely bounded norm. Letting
\[
\tilde{\tau}(w) = T\pi_{2}(w)S_{2}
\]
with $\pi_{2}$ a representation of $W$ on $L$, and $S_{2}, T$ maps of norm less than one in the diagram $H \xrightarrow{S_{2}} L \xrightarrow{T} \mathbb{C}^{n}$, we have that $F = B_{n}(\varphi)$, where
\[
F(w \otimes v) = \tau_{1,n}(\sigma(v))(w) = \tilde{\tau}(w)\sigma(v) = T\pi_{2}(w)S_{2}\pi_{1}(v)R
\]
satisfies $\|F\|_{cb} < 1$.

**Corollary 5.4.** For any operator spaces $V$ and $W$, $\Gamma_{2}(V, W)$ is an operator space.

**Proof.** From Proposition 5.2 and Theorem 5.3, we have the completely isometric injection
\[
\Gamma_{2}(V, W) \hookrightarrow \Gamma_{2}(V, W^{**}) \cong (W^{*} \otimes_{h} V)^{*}.
\]

Given vector spaces $V, W,$ and $X$, and a function $F: W \otimes V \to X$, we define a function $\hat{F}: V \to \text{Lin}(W, X)$ by
\[
\hat{F}(v)(w) = F(w \otimes v).
\]

**Corollary 5.5.** Suppose that $V, W,$ and $X$ are operator spaces. Then the map $F \to \hat{F}$ determines a complete isometry
\[
\Gamma_{2}(W \otimes V, X) \cong \Gamma_{2}(V, \Gamma_{2}(W, X)).
\]
Proof. First let us say that \( X \) is a dual operator space—say that \( X = (X_*)^* \). Then from Theorem 5.3 we have the natural complete isometries

\[
\Gamma^h_2(W \otimes V, X) \cong (X_* \otimes W \otimes V)^* \cong \Gamma^h_2(V, (X_* \otimes W)^*) \cong \Gamma^h_2(V, \Gamma^h_2(W, X)).
\]

In general we have a commutative diagram

\[
\begin{array}{ccc}
\Gamma^h_2(W \otimes V, X) & \xrightarrow{\iota_1} & \Gamma^h_2(V, \Gamma^h_2(W, X)) \\
\downarrow & & \downarrow \\
\Gamma^h_2(W \otimes V, X^{**}) & \cong & \Gamma^h_2(V, \Gamma^h_2(W, X^{**}))
\end{array}
\]

where the bottom map is completely isometric by the previous argument, and the vertical maps \( \iota_1, \iota_2 \) are completely isometric injections by Proposition 5.2.

Since the algebraic tensor product \( W \otimes V \) is dense in \( W \otimes^h V \), we have that \( \varphi \in \Gamma^h_2(W \otimes^h V, X^{**}) \) lies in the image of \( \iota_1 \) if and only if \( \varphi(w \otimes v) \in X \) for all \( v \in V \) and \( w \in W \). But that in turn will happen if and only if \( \hat{\varphi}(v)(w) \) lies in \( X \) for all \( v \) and \( w \), i.e., if and only if \( \hat{\varphi}(v) \in \Gamma^h_2(W, X) \), or equivalently, \( \varphi \) is in the image of \( \iota_2 \).

The above arguments apply equally well to factorizations through row Hilbert spaces. Letting \( \tilde{\Gamma}^h_2(V, W) \) be the corresponding space, Proposition 5.2 remains valid, and we obtain

\[
\begin{align*}
\tilde{\Gamma}^h_2(V, W^*) & \cong (V \otimes^h W)^* \\
\tilde{\Gamma}^h_2(V \otimes W, X) & \cong \tilde{\Gamma}^h_2(V, \tilde{\Gamma}^h_2(W, X)).
\end{align*}
\]

The following generalizes Theorem 3.11 in [2] (which considered only the isometric case).

**Corollary 5.6.** The natural map \( V \otimes^h W \hookrightarrow \tilde{\Gamma}^h_2(V^*, W) \) is a complete isometry.

**Proof.** This is apparent from the row version of Theorem 5.3 and the diagram

\[
V \otimes^h W \hookrightarrow V^{**} \otimes W^{**} \hookrightarrow (V^* \otimes W^*)^* = \tilde{\Gamma}^h_2(V^*, W^{**}),
\]

where we have used Corollary 3.3 for the second isometry.
We recall from Banach space theory that if $V$ and $W$ are Banach spaces, and $\varphi: V \to W$ is a linear map, then the 2-summing norm $\pi_2(\varphi)$ is defined to be the least constant $C$ such that for any $v_1, \ldots, v_n \in V$, we have that

$$\left( \sum \| \varphi(v_i) \|^2 \right)^{1/2} \leq C \sup \left\{ \sum (| \langle v_i, f \rangle |^2)^{1/2} : \| f \| \leq 1, f \in V^* \right\}.$$ 

We say that $\varphi$ is 2-summing if $\pi_2(\varphi) < \infty$.

We say that an operator space $V$ is minimal if it has the form $V = \text{Min } V_0$ for some normed space $V_0$. Equivalently, letting $K$ the closed unit ball of $V^*$, we have isometric inclusions

$$\mathbb{M}_n(V) \subset \mathbb{M}_n(C(K)) = C(K, \mathbb{M}_n).$$

In particular, given $v \in \mathbb{M}_n(V)$, we have that

$$\| v \| = \sup \{ \| \langle v, f \rangle \| : f \in K \}.$$ (5.9)

For any operator space $V$, compact set $L$, and linear map $\varphi: V \to C(L)$, we have that $\| \varphi \|_{cb} = \| \varphi \|$ (see [1]). It follows that if $W$ is minimal, i.e., we have a complete injection $W \subset C(L)$, then for any linear map $\varphi: V \to W$,

$$\| \varphi \|_{cb} = \| \varphi \|.$$

**Theorem 5.7.** Suppose that $V$ and $W$ are complete operator spaces and that $\varphi: V \to W$ is a linear map. If $V$ is minimal, then $\pi_2(\varphi) \leq \gamma_2(\varphi)$. If $W$ is minimal, then $\pi_2(\varphi) \geq \gamma_2(\varphi)$.

**Proof.** Let us suppose that $V$ is minimal. Given $\varphi \in \Gamma_2(V, W)$, with $\gamma_2(\varphi) < 1$, we have that $\varphi = \tau \circ \sigma$, where the maps

$$V \overset{\sigma}{\longrightarrow} H \overset{\tau}{\longrightarrow} W$$

satisfy $\| \sigma \|_{cb}, \| \tau \|_{cb} < 1$. Since the column matrix $(\sigma(v_1), \ldots, \sigma(v_n))$ is a vector in $H^n$, and $V$ is minimal, we have from (5.9),

$$\left( \sum \| \varphi(v_i) \|^2 \right)^{1/2} \leq \| \tau \| \left( \sum \| \sigma(v_i) \|^2 \right)^{1/2}$$

$$= \| \tau \| \| (\sigma(v_1), \ldots, \sigma(v_n)) \|$$

$$\leq \| \tau \|_{cb} \| \sigma \|_{cb} \| (v_1, \ldots, v_n) \|$$

$$\leq \sup \{ \| (f(v_1), \ldots, f(v_n)) \| : f \in K \}.$$

$$= \sup \left\{ \left( \sum | \langle v_i, f \rangle |^2 \right)^{1/2} : \| f \| \leq 1, f \in V^*_n \right\},$$

and thus $\pi_2(\varphi) < 1$. 

Conversely let us suppose that $W$ is minimal, and that as a mapping of the underlying Banach spaces, we have that $\gamma_2(\varphi) \leq 1$. From the Pietsch Factorization Theorem (see [15, Corollary 1.8]) there exist a probability measure on the dual ball $K$ of $V$, a probability measure $\lambda$ on $K$, and a factorization of $\varphi$ given by the diagram

$$V \xrightarrow{i} C(K) \xrightarrow{j} L^2(K, \lambda) \xrightarrow{\sigma} W,$$

(5.10)

where $i$ and $j$ are the inclusion maps, and $\|\theta\| \leq 1$. Since $C(K)$ is minimal, $\|i\|_{cb} = \|i\| = 1$. Letting $\pi$ be the multiplication representation of $C(K)$ on $L^2(K, \lambda)$, we have that $j(f) = \pi(f)\zeta$, where $\zeta$ is the unit vector determined by the function $1$ in $H = L^2(K, \lambda)$. Thus letting $H$ have the column operator space structure, $\|j\|_{cb} \leq 1$. Finally, since $W$ is minimal, we have that $\|\theta\|_{cb} = \|\theta\| \leq 1$. We thus have a complete contractive factorization $V \to H_c \to W$, and $\gamma_2(\varphi) \leq 1$.

Given Banach spaces $V$ and $W$, any linear map $\varphi: V \to W$ may also be regarded as a linear map from $\text{Min } V$ to $\text{Min } W$, i.e., we may regard $\text{Min}$ as a functor by letting $\text{Min } \varphi = \varphi$.

**Corollary 5.8.** Given Banach spaces $V$ and $W$ and a linear map $\varphi: V \to W$, we have that

$$\pi_2(\varphi) = \gamma_2(\text{Min } \varphi).$$

It would be tempting to seek an operator space analogue for the $\pi_2$ norm. In fact it is fairly evident that this is already provided by the $\gamma_2$ norm. Restricting our attention to single maps rather than matrices of maps, let us suppose that $\varphi: V \to W$ satisfies $\gamma_2(\varphi) < 1$. Then we have a diagram (5.1) with $\sigma$, $\tau$ complete contractions. Fixing a $C^*$-algebra $A$ containing $V$, we may assume that $\sigma(v) = S\pi(v)\zeta$, where $\pi$ is a representation of $A$ on a Hilbert space $H$, and $\zeta \in H$ is a unit vector, which we identify with a map $A \to H$. We may replace $H$ by the subspace $[\pi(A)\zeta]$.

Letting $p(a) = \pi(a)\zeta$, $\zeta$, we may identify $[\pi(A)\zeta]$ with the Hilbert space $H_\rho = L^2(A, p)$ that one associates with the state $p$ by the GNS construction. Letting $j(a) = a\zeta$, and $\theta(\eta) = \tau(S\eta)$, we obtain the diagram of complete contractions

$$V \xrightarrow{i} A \xrightarrow{j} (H_\rho)_c \xrightarrow{\theta} W.$$

Conversely, if $\varphi$ has such a decomposition, it follows from the above proof that $\pi_2(\varphi) < 1$. In light of Pietsch's characterization of $\pi_2$ (see (5.10)) this would certainly seem to be the correct condition for $\pi_2(\varphi) < 1$.
REFERENCES