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On the Existence, Uniqueness, and Stability of Solutions of the Equation $\rho_0 \mathfrak{X}_{tt} = E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} + \lambda \mathfrak{X}_{xxt}$

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1. INTRODUCTION

In a previous paper [1] the author established the existence of a unique solution of the initial-boundary value problem:

(E) $\quad \rho_0 \mathfrak{X}_{tt} = E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} + \lambda \mathfrak{X}_{xtx}; \quad (x, t) \in (0, 1) \times (0, \infty);$

(IC) $\quad \mathfrak{X}(x, 0) = x \quad$ and $\quad \mathfrak{X}_t(x, 0) = V^0(x), \quad 0 \leqslant x \leqslant 1,$

(BC) $\quad \mathfrak{X}(0, t) = 0 \quad$ and $\quad \mathfrak{X}(1, t) = 1, \quad t > 0;$

where $\rho_0 > 0$ is the constant mass density in the initial configuration and $\lambda > 0$. Equation (E) is a local statement of balance of momentum for materials for which the stress τ is related to strain $\mathfrak{X}_x$ and strain rate $\mathfrak{X}_{xt}$ through the constitutive equation

(CE) $$\tau = \sigma(\mathfrak{X}_x) + \lambda \mathfrak{X}_{xt},$$

where

$$\sigma(\mathfrak{X}_x) \overset{\text{def}}{=} \int_1^{\mathfrak{X}_x} E(\xi)\, d\xi.$$

The method of proof used to obtain a solution was nonconstructive; it relied on some nonstandard results for the linear heat equation (see [1, 3]).

Here we re-examine the same problem. We deduce via elementary arguments the existence and uniqueness of a generalized solution of (E), (IC), and (BC) (for a precise definition of a solution see Section 2). Equation (E) is satisfied in the sense in which balance of linear momentum is normally stated; that is

$$\int_{x_1}^{x_2} \rho_0[\mathfrak{X}_t(x, t_2) - \mathfrak{X}_t(x, t_1)]\, dx = \int_{t_1}^{t_2} [\tau(x_2, t) - \tau(x_1, t)]\, dt$$

$$= \int_{t_1}^{t_2} [\sigma(\mathfrak{X}_x(x_2, t)) - \sigma(\mathfrak{X}_x(x_1, t))]\, dt + \lambda[\mathfrak{X}_x(x_2, t_2) - \mathfrak{X}_x(x_1, t_2)]$$

$$- \lambda[\mathfrak{X}_x(x_2, t_1) - \mathfrak{X}_x(x_1, t_1)]. \tag{1.1}$$

We establish this result by showing that the solutions of certain finite difference approximations to (E) converge to the desired generalized solution.

2. Statement of Results

We are seeking a solution of the initial-boundary value problem (E), (IC), and (BC). Throughout, we shall assume that (a) the function $E(\cdot)$ is positive and C^2 on $(-\infty, \infty)$; and (b) the initial velocity field V^0 is $C^2[0, 1]$ and satisfies the compatibility conditions $V^0(0) = V^0(1) = 0$.

For any $T > 0$ we let S_T be the strip

$$S_T \stackrel{\text{def}}{=} \{(x, t) \mid 0 < x < 1, 0 < t < T\}.$$

The principal results of this investigation are contained in Theorems 1 and 2.

THEOREM 1. *For each $T > 0$ there exists a unique function $\mathfrak{X}$ defined on S_T with the following properties:*

(i) *$\mathfrak{X}$ is C^1 on $\bar{S}_T$ and assumes the data* (IC) *and* (BC);

(ii) *$\mathfrak{X}$ has strong $L_2(S_T)$ derivatives $\mathfrak{X}_{tt}$, $\mathfrak{X}_{tx}$, $\mathfrak{X}_{xx}$, $\mathfrak{X}_{txx}$ and $\mathfrak{X}_{ttx}$. Moreover, the functions $\mathfrak{X}_{tx}$ and $\mathfrak{X}_{xx}$ are in $L_\infty(S_T)$;*

(iii) *The mapping $t \to \mathfrak{X}_{tx}(\cdot, t) : [0, T] \to L_2(0, 1)$ is uniformly bounded and uniformly Hölder continuous with exponent $1/2$; i.e., there is a constant K, depending among other things on T, such that*

$$\| \mathfrak{X}_{tx}(\cdot, t_2) - \mathfrak{X}_{tx}(\cdot, t_1)\| = \left(\int_0^1 (\mathfrak{X}_{tx}(x, t_2) - \mathfrak{X}_{tx}(x, t_1))^2 \, dx\right)^{1/2}$$
$$\leqslant K \mid t_2 - t_1 \mid^{1/2}; \qquad \textit{and}$$

(iv) *$\mathfrak{X}$ satisfies* (E) *in the following generalized senses:*

(a) *For all x_1, x_2 in $[0, 1]$ and t_1, t_2 in $[0, T]$ Eq.* (1.1) *holds,*

(b)
$$\int_0^T \int_0^1 (\rho_0 \mathfrak{X}_{tt} - E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} - \lambda \mathfrak{X}_{xtx})\phi \, dx \, dt = 0$$

for all $C^\infty(S_T)$ functions ϕ which vanish in a neighborhood of $x = 0$, $x = 1$, and $t = T$.

REMARK (a) Assertion (iv-b) implies that (E) is satisfied almost everywhere in S_T.

For any function f we let

$$| f | = \max_{x \in [0,1]} | f(x) | \qquad \text{and} \qquad \| f \| = \left(\int_0^1 f^2(x) \, dx\right)^{1/2}.$$

THEOREM 2. *There exists a constant M, depending on $|V^0|$ and $|V_x^0|$ and tending to zero as $|V^0|$ and $|V_x^0|$ tends to zero, such that*

$$|\mathfrak{X} - x|(t) + |\mathfrak{X}_x - 1|(t) + |\mathfrak{X}_{xx}|(t) + |\mathfrak{X}_t|(t) + \|\mathfrak{X}_{tx}\|(t) \leqslant M, \quad t \geqslant 0.$$

Moreover,

$$\lim_{t\to\infty} |\mathfrak{X} - x|(t) + |\mathfrak{X}_x - 1|(t) + |\mathfrak{X}_{xx}|(t) + |\mathfrak{X}_t|(t) + \|\mathfrak{X}_{tx}\|(t) = 0.$$

We now outline this proof of Theorems 1 and 2. To establish Theorem 1 we finite difference the equation (E) in x. We let $0 < h < 1$ be some number such that $1/h$ is an integer, say N. We then consider the system of ordinary differential equations

$$(\mathrm{E})_h \qquad \rho_0 h \ddot{\mathfrak{X}}_k = \left[\sigma\left(\frac{\mathfrak{X}_{k+1} - \mathfrak{X}_k}{h}\right) - \sigma\left(\frac{\mathfrak{X}_k - \mathfrak{X}_{k-1}}{h}\right)\right] + \lambda\left[\left(\frac{\dot{\mathfrak{X}}_{k+1} - \dot{\mathfrak{X}}_k}{h}\right) - \left(\frac{\dot{\mathfrak{X}}_k - \dot{\mathfrak{X}}_{k-1}}{h}\right)\right]; \quad k = 1,\dots, N-1;$$

$$(IC)_h \qquad \mathfrak{X}_k(0) = kh \quad \text{and} \quad \dot{\mathfrak{X}}_k(0) = V^0(kh) \qquad k = 1,\dots, N-1$$

$$(\mathrm{BC})_h \qquad \mathfrak{X}_0(t) = 0 \quad \text{and} \quad \mathfrak{X}_N(t) = 1.$$

The function σ is defined by $\sigma(\gamma) = \int_1^\gamma E(\xi)\, d\xi$. We study the behavior of these solutions as the mesh size h tends to zero. Making use of a priori estimates (which are independent of h) we show that interpolates of the above solutions converge (as h tends to zero) to a function $\mathfrak{X}$ with the properties enunciated in the statement of Theorem 1.

Theorem 2 follows from the estimates of Section 3.

We point out that for fixed h the above system has a rather interesting physical interpretation. The equations govern the motion of a system of $N + 1$ particles, each with mass $m = \rho_0 h$, moving on a straight line. A given particle is connected to its nearest neighbors by a nonlinear spring and linear viscous damper (see Figs. 1 and 2).

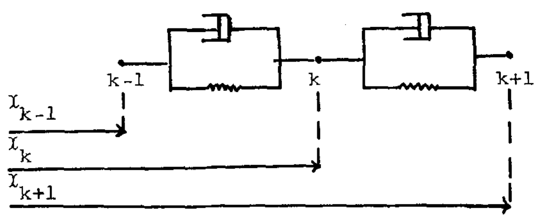

FIG. 1.

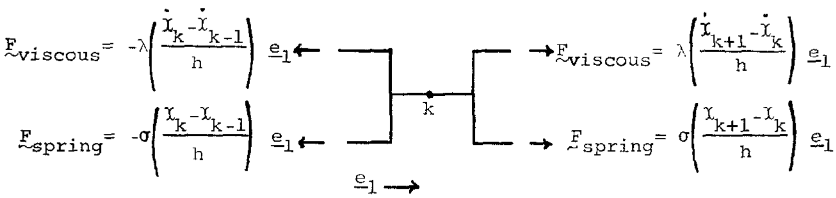

FIG. 2.

3. A PRIORI ESTIMATES

We here derive certain a priori estimates for smooth solutions of (E), (IC), and (BC). Analogues estimates hold for the discrete system $(E)_h$, $(IC)_h$, and $(BC)_h$. The estimates for the discrete system will be independent of the mesh size h.

We introduce the following notation. For functions f defined on $[0, 1] \times [0, \infty)$ we let

$$|f|(t) = \max_{x \in [0,1]} |f(x, t)|, \qquad |f|_T = \max_{t \in [0,T]} |f|(t),$$

$$\|f\|(t) = \left(\int_0^1 f^2(x, t)\, dx\right)^{1/2}, \qquad \|f\|_T = \max_{t \in [0,T]} \|f\|(t),$$

and

$$|||f|||_T = \left(\int_0^T \int_0^1 f^2(x, t)\, dx\, dt\right)^{1/2}. \tag{3.1}$$

For a given $0 < h < 1$ such that $1/h$ is an integer, say N, and a given function f defined on $\Sigma_h \times [0, \infty)$ where

$$\Sigma_h = \{x \in [0, 1] \mid x = kh, \qquad k = 0, 1, \ldots, N.\}, \tag{3.2}$$

we let

$$|f|_h(t) = \max_{x \in \Sigma_h} |f(x, t)|, \qquad |f|_{h,T} = \max_{t \in [0,T]} |f|_h(t),$$

$$\|f\|_h(t) = \left(\sum_{x \in \Sigma_h} hf^2(x, t)\right)^{1/2}, \qquad \|f\|_{h,T} = \max_{t \in [0,T]} \|f\|_h(t),$$

and

$$|||f|||_{h,T} = \left(\int_0^T \sum_{x \in \Sigma_h} hf^2(x, t)\, dt\right)^{1/2}. \tag{3.3}$$

The norms in (3.3) are the discrete analogues of the norms defined in (3.1).

We now record some facts which will be of use later.

REMARK 3.1. (a) Suppose that Φ is C^2 in x on $[0, 1] \times [0, \infty)$ and satisfies $\Phi(0, t) = 0$ and $\Phi(1, t) = 1$ for all $t \geqslant 0$. Then,

$$\begin{aligned} \| \Phi - x \|(t) &\leqslant | \Phi - x |(t) \leqslant \| \Phi_x - 1 \|(t) \leqslant | \Phi_x - 1 |(t) \\ &\leqslant \| \Phi_{xx} \|(t) \leqslant | \Phi_{xx} |(t); \end{aligned} \tag{3.4}$$

$$\begin{aligned} \| \Phi - x \|_h(t) &\leqslant | \Phi - x |_h(t) \leqslant \| \Delta_h\Phi - 1 \|_h(t) \leqslant | \Delta_h\Phi - 1 |_h(t) \\ &\leqslant \| \Delta_h{}^2\Phi \|_h(t) \leqslant | \Delta_h{}^2\Phi |(t). \end{aligned} \tag{3.4$'$}$$

Here

$$\Delta_h\Phi(x, t) = \frac{\Phi(x, t) - \Phi(x - h, t)}{h}, \ (x, t) \in \Sigma_h \times [0, \infty), \tag{3.5}$$

and

$$\Delta_h{}^2\Phi(x, t) = \frac{\Phi(x + h, t) - 2\Phi(x, t) + \Phi(x - h, t)}{h^2}, \ (x, t) \in \Sigma_h \times [0, \infty). \tag{3.6}$$

(b) Suppose that Ψ is uniformly continuous and integrable on $[0, \infty)$. Then

$$\lim_{t\to\infty} \Psi(t) = 0.$$

Now and in the sequel $\mathfrak{X}$ will denote a smooth solution of (E), (IC), and (BC). For a given $0 < h < 1$ we let $\vec{\mathfrak{X}}^h : [0, \infty) \to R^{N+1}$ be the unique solution of the discrete problem $(\mathrm{E})_h$, $(\mathrm{IC})_h$, and $(\mathrm{BC})_h$, and we define $\mathfrak{X}^h : \Sigma_h \times [0, \infty) \to R$ by

$$\mathfrak{X}^h(kh, t) = (\vec{\mathfrak{X}}^h)_k(t), \qquad k = 0, 1, \ldots, N = \frac{1}{h}. \tag{3.7}$$

Our basic estimates will follow from the following identities which must be satisfied by a solution of (E), (IC), and (BC):

$$\begin{aligned} \rho_0 \| \mathfrak{X}_t \|^2(t_2) &+ 2 \int_0^1 \int_1^{\mathfrak{X}_x(x,t_2)} \sigma(\mu)\, d\mu\, dx + 2\lambda \int_{t_1}^{t_2} \| \mathfrak{X}_{tx} \|^2(t)\, dt \\ &= \rho_0 \| \mathfrak{X}_t \|^2(t_1) + 2 \int_0^1 \int_1^{\mathfrak{X}_x(x,t_1)} \sigma(\mu)\, d\mu\, dx; \end{aligned} \tag{3.8}$$

$$\begin{aligned} \lambda \| \mathfrak{X}_{xx} \|^2(t_2) &+ 2 \int_{t_1}^{t_2} \int_0^1 E(\mathfrak{X}_x)\, \mathfrak{X}_{xx}^2\, dx\, d\tau \\ &= \lambda \| \mathfrak{X}_{xx} \|^2(t_1) + 2\rho_0 \int_{t_1}^{t_2} \int_0^1 \mathfrak{X}_{xx}\mathfrak{X}_{tt}\, dx\, d\tau \\ &= \lambda \| \mathfrak{X}_{xx} \|^2(t_1) + 2\rho_0 \int_0^1 (\mathfrak{X}_{xx}\mathfrak{X}_t(x, t_2) - \mathfrak{X}_{xx}\mathfrak{X}_t(x, t_1)\, dx \\ &\quad + 2\rho_0 \int_{t_1}^{t_2} \| \mathfrak{X}_{tx} \|^2(\tau)\, d\tau; \end{aligned} \tag{3.9}$$

and

$$2\rho_0 \int_{t_1}^{t_2} \| \mathfrak{X}_{tt} \|^2(\tau)\, d\tau + \lambda \| \mathfrak{X}_{tx} \|^2(t_2)$$

$$= 2 \int_{t_1}^{t_2} \int_0^1 E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} \mathfrak{X}_{tt}\, dx\, d\tau + \lambda \| \mathfrak{X}_{tx} \|^2(t_1). \tag{3.10}$$

The function σ in (3.8) is defined by

$$\sigma(\mu) = \int_1^{\mu} E(\xi)\, d\xi. \tag{3.11}$$

Equations (3.8)–(3.10) are derived by multiplying (E) by $\mathfrak{X}_t$, $\mathfrak{X}_{xx}$, and $\mathfrak{X}_{tt}$, respectively, integrating the resulting expressions over $(0, 1) \times (t_1, t_2)$, and by making use of the fact that the boundary conditions imply that

$$\mathfrak{X}_t(0, t) = \mathfrak{X}_t(1, t) = 0. \tag{3.11}$$

For the discrete system $(\mathrm{E})_h$, $(\mathrm{IC})_h$, and $(\mathrm{BC})_h$ we obtain:

$$\rho_0 \| \mathfrak{X}_t^h \|_h^2(t_2) + 2 \sum_{x \in \Sigma_h} h \int_1^{\Delta_h \mathfrak{X}^h(x, t_2)} \sigma(\mu)\, d\mu + 2\lambda \int_{t_1}^{t_2} \| \Delta_h \mathfrak{X}_t^h \|_h^2(\tau)\, d\tau$$

$$= \rho_0 \| \mathfrak{X}_t^h \|_h^2(t_1) + 2 \sum_{x \in \Sigma_h} h \int_1^{\Delta_h \mathfrak{X}^h(x, t_1)} \sigma(\mu)\, d\mu; \tag{3.8$'$}$$

$$\lambda \| \Delta_h{}^2 \mathfrak{X}^h \|_h^2(t_2) + 2 \int_{t_1}^{t_2} \sum_{x \in \Sigma_h} h \left[\frac{\sigma(\Delta_h \mathfrak{X}^h(x, t)) - \sigma(\Delta_h \mathfrak{X}^h(x - h, t))}{h} \right] \Delta_h{}^2 \mathfrak{X}^h(x - h, t)$$

$$= \lambda \| \Delta_h{}^2 \mathfrak{X}^h \|_h^2(t_1) + 2\rho_0 \sum_{x \in \Sigma_h} h(\Delta_h{}^2 \mathfrak{X}^h \mathfrak{X}_t^h(x, t_2) - \Delta_h{}^2 \mathfrak{X}^h \mathfrak{X}_t^h(x, t_1))$$

$$+ 2\rho_0 \int_{t_1}^{t_2} \| \Delta_h \mathfrak{X}_t^h \|_h^2(\tau)\, d\tau; \tag{3.9$'$}$$

and

$$\rho_0 \int_{t_1}^{t_2} \| \mathfrak{X}_{tt}^h \|_h^2(\tau)\, d\tau + \lambda \| \Delta_h \mathfrak{X}_t^h \|_h^2(t_2)$$

$$= 2 \int_{t_1}^{t_2} \sum_{x \in \Sigma_h} h \left[\frac{\sigma(\Delta_h \mathfrak{X}^h(x, \tau)) - \sigma(\Delta_h \mathfrak{X}^h(x - h, \tau))}{h} \right] \mathfrak{X}_{tt}^h(x - h, \tau)\, d\tau$$

$$+ \lambda \| \Delta_h \mathfrak{X}_t^h \|_h^2(t_1). \tag{3.10$'$}$$

Equations (3.8)′–(3.10)′ are established in exactly the same way as equations (3.8)–(3.10) except now, integration over (0, 1) is replaced by summation over the lattice Σ_h.

Henceforth, M will denote a generic constant which depends on ρ_0, λ, $|V^0|$, and $|V_x^0|$ and tends to zero as $|V^0|$ and $|V_x^0|$ tend to zero.[1]

LEMMA 3.1. *There is a constant M such that*

$$\|\mathfrak{X}_t\|^2(t) + \int_0^t \|\mathfrak{X}_{tx}\|^2(\tau)\,d\tau + \|\mathfrak{X}_{xx}\|^2(t) + \int_0^t \|\mathfrak{X}_{xx}\|^2(\tau)\,d\tau$$
$$+ \int_0^t \|\mathfrak{X}_{tt}\|^2(\tau)\,d\tau + \|\mathfrak{X}_{tx}\|^2(t) + \int_0^t \|\mathfrak{X}_{txx}\|^2(\tau)\,d\tau \leqslant M. \quad (3.12)$$

PROOF. The hypothesis $E > 0$ implies that $\int_0^1 \int_1^{\mathfrak{X}_x(x,t)} \sigma(\mu)\,d\mu\,dx \geqslant 0$. Hence (3.8), with $t_1 = 0$, implies that

$$\rho_0 \|\mathfrak{X}_t\|^2(t) + 2\lambda \int_0^t \|\mathfrak{X}_{tx}\|^2(\tau)\,d\tau \leqslant \rho_0 \|V^0\|^2 \leqslant \rho_0 |V^0|^2. \quad (3.13)$$

Equation (3.13) establishes the boundedness of the first two terms in (3.12).

If we set $t_1 = 0$ in (3.9), apply Schwarz's inequality to $\int_0^1 \mathfrak{X}_{xx}\mathfrak{X}_t(x, t_2)\,dx$, and make use of (3.13) and the positivity of E we obtain the inequality:

$$\lambda \|\mathfrak{X}_{xx}\|^2(t) \leqslant 2\rho_0 \|V^0\| \, \|\mathfrak{X}_{xx}\|(t) + \frac{\rho_0^2}{\lambda} \|V^0\|^2$$
$$\leqslant 2\rho_0 |V^0| \, \|\mathfrak{X}_{xx}\|(t) + \frac{\rho_0^2}{\lambda} |V^0|^2. \quad (3.14)$$

The boundedness of $\|\mathfrak{X}_{xx}\|(t)$ independent of t now follows from (3.14). If we denote this bound by M^1, then Remark 3.1a with $\Phi = \mathfrak{X}$ implies

$$|\mathfrak{X} - x|(t) \leqslant |\mathfrak{X}_x - 1|(t) \leqslant M^1. \quad (3.15)$$

Since E is positive and smooth it follows that E has a positive maximum E_1 and a positive minimum E_0 on $|\xi - 1| \leqslant M^1$. The latter fact together with (3.9) establishes the boundedness of $\int_0^t \|\mathfrak{X}_{xx}\|^2(\tau)\,d\tau$ for all $t \geqslant 0$.

We now look at (3.10) with $t_1 = 0$. We obtain the inequality:

$$2\rho_0 \int_0^t \|\mathfrak{X}_{tt}\|^2(\tau)\,d\tau \leqslant 2E_1 M \left[\int_0^t \|\mathfrak{X}_{tt}\|^2(\tau)\,d\tau\right]^{1/2} + \lambda \|V_x^0\|^2; \quad (3.16)$$

[1] Recall that $\mathfrak{X}(x, 0) = x$ is fixed for all problems under consideration.

where M is bound for $(\int_0^t \| \mathfrak{X}_{xx} \|^2(\tau)\, d\tau)^{1/2}$. The boundedness of the last three terms in (3.12) now follows from (3.16) and the fact that by hypotheses

$$\lambda \mathfrak{X}_{xtx} = \rho_0 \mathfrak{X}_{tt} - E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} .$$

LEMMA 3.2. *There exists a constant M such that*

$$| \mathfrak{X}_t |(t) \leqslant M \qquad \text{and} \qquad | \mathfrak{X}_{xx} |(t) \leqslant M. \tag{3.17}$$

Moreover,

$$\lim_{t\to\infty} | \mathfrak{X}_t |(t) = 0 \tag{3.18}$$

$$\lim_{t\to\infty} | \mathfrak{X}_{xx} |(t) = 0. \tag{3.19}$$

PROOF. That $| \mathfrak{X}_t |(t) \leqslant M$ follows from the bound $\| \mathfrak{X}_{tx} \|(t) \leqslant M$, the boundary $\mathfrak{X}_t(0, t) = \mathfrak{X}_t(1, t) = 0$, and Remark $(3.1)_a$ with $\Phi \equiv \mathfrak{X}_t$.

To establish $(3.17)_2$ we observe that (E) may be regarded as an ordinary differential equation for $\mathfrak{X}_{xx}$. Solving for $\mathfrak{X}_{xx}$ we obtain:

$$\begin{aligned} \mathfrak{X}_{xx}(x, t_2) = {} & \frac{\rho_0}{\lambda} \mathfrak{X}_t(x, t_2) + \exp\left(-\int_{t_1}^{t_2} \frac{E(\mathfrak{X}_x(x, \eta))\, d\eta}{\lambda}\right) \\ & \times \left[\mathfrak{X}_{xx}(x, t_1) - \frac{\rho_0}{\lambda} \mathfrak{X}_t(x, t_1)\right] - \exp\left(-\int_{t_1}^{t_2} \frac{E(\mathfrak{X}_x)(x, \eta)\, d\eta}{\lambda}\right) \\ & \times \left\{\int_{t_1}^{t_2} \rho_0 \mathfrak{X}_t(x, \tau) \frac{E(\mathfrak{X}_x(x, \tau))}{\lambda^2} \exp\left(\int_{t_1}^{\tau} \frac{E(\mathfrak{X}_x(x, \eta))\, d\eta}{\lambda}\right) d\tau\right\}. \end{aligned} \tag{3.20}$$

An immediate consequence of (3.20) is the inequality:

$$\begin{aligned} | \mathfrak{X}_{xx} |(t_2) \leqslant {} & \frac{\rho_0}{\lambda} | \mathfrak{X}_t |(t_2) + e^{-E_0(t_2 - t_1)} \left[| \mathfrak{X}_{xx} |(t_1) + \frac{\rho_0}{\lambda} | \mathfrak{X}_t |(t_1)\right] \\ & + [\max_{t\in[t_1, t_2]} | \mathfrak{X}_t |(t)] \frac{\rho_0}{\lambda} [1 - e^{-E_1(t_2 - t_1)}], \end{aligned} \tag{3.21}$$

where $E_0 = \min_{|\xi-1|\leqslant M^1} E(\xi)$ and $E_1 = \max_{|\xi-1|\leqslant M^1} E(\xi)$ and M^1 is the upper bound for $\| \mathfrak{X}_{xx} \|(\cdot)$. Equations $(3.17)_1$ and (3.21) with $t_1 = 0$ then yield $(3.17)_2$.

To establish (3.18) it suffices to show that $\lim_{t\to\infty} \| \mathfrak{X}_{tx} \|(t) = 0$. We shall show that if $\mathfrak{X}$ is a solution, then the map $t \to \| \mathfrak{X}_{tx} \|^2(t)$ is uniformly continuous on $[0, \infty)$. The integrability of $\| \mathfrak{X}_{tx} \|^2(\cdot)$ on $[0, \infty)$ and Remark (3.1-b) will then imply the desired result.

To show that $\| \mathfrak{X}_{tx} \|^2(\cdot)$ is uniformly continuous we look at (3.10). The result is:

$$\lambda \mid \| \mathfrak{X}_{tx} \|^2(t_2) - \| \mathfrak{X}_{tx} \|^2(t_1) \mid$$
$$\leqslant 2\rho_0 \int_{t_1}^{t_2} \| \mathfrak{X}_{tt} \|^2(\tau)\, d\tau + 2E_1 \left(\int_{t_1}^{t_2} \| \mathfrak{X}_{xx} \|^2(\tau)\, d\tau \right)^{1/2} \left(\int_{t_1}^{t_2} \| \mathfrak{X}_{tt} \|^2(\tau)\, d\tau \right)^{1/2}.$$

The integrability of $\| \mathfrak{X}_{tt} \|^2(\cdot)$ and $\| \mathfrak{X}_{xx} \|^2(\cdot)$ imply that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for $| t_2 - t_1 | < \delta$

$$\int_{t_1}^{t_2} \| \mathfrak{X}_{xx} \|^2(\tau)\, d\tau \leqslant \frac{\epsilon}{2E_1^{\,2}} \lambda \rho_0 \qquad \text{and} \qquad \int_{t_1}^{t_2} \| \mathfrak{X}_{tt} \|^2 \leqslant \frac{\epsilon\lambda}{2\rho_0}.$$

The uniform continuity of $\| \mathfrak{X}_{tx} \|^2(\cdot)$ now follows.

To show that (3.19) is valid we return to the inequality (3.21). Since $| \mathfrak{X}_t |(\cdot)$ tends to zero at infinity, we may, given any $\epsilon > 0$, find a number $t_1(\epsilon)$ such that

$$\max_{t \geqslant t_1(\epsilon)} | \mathfrak{X}_t |(t) \leqslant \frac{\epsilon\lambda}{3\rho_0}.$$

We may now find a $t_2 \geqslant t_1$ such that the remaining term in (3.21) is bounded by $\epsilon/3$ thereby establishing (3.19).

REMARK 3.2. (a) Summarizing the results of Lemmas 3.1 and 3.2 we immediately obtain Theorem 2.

(b) The conclusions of Lemmas 3.1 and 3.2 are valid for the solutions $\mathfrak{X}^h$ of the discrete problem $(\mathrm{E})_h$, $(\mathrm{IC})_h$, and $(\mathrm{BC})_h$ provided we replace all spatial derivatives by the appropriate difference quotients and all norms by their discrete counterparts. We point out that all upper bounds are independent of the mesh size h.

We shall need an additional estimate for solutions of the discrete problem $(\mathrm{E})_h$, $(\mathrm{IC})_h$, and $(\mathrm{BC})_h$.

LEMMA 3.3. *Let $T > 0$ be given and let $S_T{}^h = \Sigma_h \times [0, T]$. Then, there exists a constant C, depending on ρ_0, λ, $| V^0 |$, $| V_x{}^0 |$, $| V_{xx}^0 |$, and T, and independent of h, such that*

$$\| \mathfrak{X}_{tt}^h \|_{h,T} \leqslant C \qquad \textit{and} \qquad \int_0^T \| \Delta_h \mathfrak{X}_{tt}^h \|_h^2(\tau)\, d\tau = \| \Delta_h \mathfrak{X}_{tt}^h \|_{h,T}^2 \leqslant C. \quad (3.22)$$

PROOF. If we differentiate $(\text{E})_h$ with respect to t we see that the velocity field $V^h \overset{\text{def}}{=} \mathfrak{X}_t^h$ satisfies the following system of equations:

$$\rho_0 h \dot{V}_k = E_{k+1,k}\left(\frac{V_{k+1} - V_k}{h}\right) - E_{k,k-1}\left(\frac{V_k - V_{k-1}}{h}\right)$$
$$+ \lambda \left\{\left(\frac{\dot{V}_{k+1} - \dot{V}_k}{h}\right) - \left(\frac{\dot{V}_k - \dot{V}_{k-1}}{h}\right)\right\}, \qquad k = 1, 2,..., N-1. \tag{3.23}$$

Here

$$E_{k+1,k}(t) = E\left(\frac{\mathfrak{X}_{k+1} - \mathfrak{X}_k}{h}\right),$$
$$\mathfrak{X}_k(t) = \mathfrak{X}^h(kh, t), \qquad \text{and}$$
$$V_k(t) = V^h(kh, t).$$

In addition V^h satisfies:

$$\left.\begin{aligned} V_k(0) &= V^0(kh) \\ \dot{V}_k(0) &= \Delta_h{}^2 V^0(kh), \qquad \text{and}^2 \\ V_0(t) &= V_N(t) = 0, \qquad t \geqslant 0. \end{aligned}\right\} \tag{3.24}$$

If we now multiply (3.23) by $\dot{V}_k$, sum the resulting expression over the indicies $0, 1,..., N$, and make use of the boundary condition $(3.24)_3$ we obtain the identity:

$$\frac{1}{2}\frac{d}{dt}\left\{\rho_0 \| V_t{}^h \|_h^2 + \sum_{k=1}^{N} hE_{k+1,k}[\Delta_h V^h((k+1)h, t)]^2\right\} + \lambda \| \Delta_h V_t{}^h \|_h^2$$
$$= \sum_{k=0}^{N} hE'_{k+1,k}[\Delta_h V^h((k+1)h, t)]^3, \tag{3.25}$$

where $E'(\xi) \equiv d/(d\xi)\, E(\xi)$. Integrating (3.25) over $(0, t)$ with $t \leqslant T$ yields

$$\rho_0 \| V_t{}^h \|_h^2(t) + \sum_{k=0}^{N} hE_{k+1,k}[\Delta_h V^h((k+1)h, t)]^2 + 2\lambda \int_0^t \| \Delta_h V_t{}^h \|_h^2(\tau)\, d\tau$$
$$= \rho_0 \| V_t{}^h \|_h^2(0) + \sum_{k=0}^{N} hE_{k+1,k}[\Delta_h V^h((k+1)h, 0)]^2$$
$$+ 2\int_0^t \sum_{k=0}^{N} hE'_{k+1,k}[\Delta_h V^h((k+1)h, \tau)]^3\, d\tau$$
$$\leqslant \rho_0 \mid V_{xx}^0 \mid^2 + E(1) \mid V_{xx}^0 \mid^2 + 2E' \int_0^t \sum_{k=0}^{N} h \mid \Delta_h V^h((k+1)h, \tau)\mid^3 d\tau, \tag{3.26}$$

[2] Here we are making use of the fact that $\sigma(1) = 0$ and that $\Delta_h \mathfrak{X}^h(x, 0) = 1$ for $x \in \Sigma_h$.

where $E_1' = \max_{|\xi-1|\leqslant M^1} |E'(\xi)|$ and M^1 is the common upper bound for $|\mathfrak{X}_x - 1|(\cdot)$ and $|\Delta_h\mathfrak{X}^h - 1|(\cdot)$.

We now observe that the last term in (3.26) is dominated by

$$2E' \int_0^t \left[\sum_{k=0}^{N} h \mid \Delta_h V^h((k+1)h, \tau)|^4\right]^{3/4} d\tau,$$

which in turn is dominated by

$$2E' T^{1/4} \left(\int_0^t \| \Delta_h^2 V^h \|^2 \| \Delta_h V^h \|^2 \, d\tau\right)^{3/4}.$$

The inequalities

$$\| \Delta_h V^h \|^2(t) \leqslant M, \qquad t \geqslant 0,$$

$$\int_0^t \| \Delta_h^2 V^h \|^2(\tau) \, d\tau \leqslant M, \qquad t \geqslant 0$$

imply that the right side of (3.26) is bounded by

$$(\rho_0 + E(1)) \mid V_{xx}^0 \mid^2 + 2E' M^{3/2} T^{1/4},$$

which establishes (3.19).

Corollary. *A direct consequence of the preceding lemma and the uniform bound for* $\| \Delta_h^2 \mathfrak{X}^h \|$ *is the existence of a constant* C, *depending on* ρ_0, $|V^0|$, $|V_x^0|$, $|V_{xx}^0|$, *and* T, *and independent of* h *such that*

$$\| \Delta_h^2 \mathfrak{X}_t^h \|_{h,T} \leqslant C \qquad and \qquad | \Delta_h \mathfrak{X}_t^h |_{h,T} \leqslant C. \tag{3.27}$$

We are now in a position to prove: *There exists at most one function* $\mathfrak{X}$ *with the properties stated in Theorem* 1.

Proof. Since any function $\mathfrak{X}$ which satisfies (E) in the sense indicated by (iv-b) will also satisfy (E) in the sense indicated by (1.1), it suffices to show that there exists at most one function $\mathfrak{X}$ satisfying (E) in the sense of (iv-b).

We assume there exists two, say $\mathfrak{X}^2$ and $\mathfrak{X}^1$. Straight-forward manipulation shows that their difference $W = \mathfrak{X}^2 - \mathfrak{X}^1$ must satisfy

$$\int_0^T \int_0^1 (\rho_0 W_{tt} - E(\mathfrak{X}_x^2) W_{xx} - \lambda W_{xtx})\phi \, dx \, dt$$

$$= \int_0^T \int_0^1 (E(\mathfrak{X}_x^2) - E(\mathfrak{X}_x^1)) \mathfrak{X}_{xx}^1 \phi \, dx \, dt$$

for all ϕ in $L_2(S_T)$ which vanish on $x = 0$, $x = 1$, and $t = T$. W also satisfies homogeneous initial and boundary conditions. If we now take

$$\phi = \begin{cases} W_t & \text{on} \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant t^1 \quad \text{with} \quad 0 < t^1 < T \\ 0 & \text{on} \quad 0 \leqslant x \leqslant 1, \quad t^1 < t \leqslant T, \end{cases}$$

then we obtain the identity

$$\int_0^1 (\rho_0 W_t^2 + E(\mathfrak{X}_x^2)\, W_x^2)(x, t)\, dx + 2\lambda \int_0^t \int_0^1 W_{tx}^2\, dx\, dt$$
$$= 2 \int_0^t \int_0^1 \Big\{(E(\mathfrak{X}_x^2) - E(\mathfrak{X}_x^1))\, \mathfrak{X}_{xx}^1 W_t + E'(\mathfrak{X}_x^2)\, \mathfrak{X}_{tx}^2 W_t W_x$$
$$- \frac{E'(\mathfrak{X}_x^2)\, \mathfrak{X}_{tx}^2 W_x^2)}{2} \Big\}\, dx\, dt.$$

We now bound the terms on the right-hand side of the above equation:

$$\left| 2 \int_0^t \int_0^1 (E(\mathfrak{X}_x^2) - E(\mathfrak{X}_x^1))\, \mathfrak{X}_{xx}^1 W_t\, dx\, dt \right|$$
$$\leqslant |\, E' \,|\, |\, \mathfrak{X}_{xx}^1 \,|_T \int_0^t \int_0^1 (W_x^2 + W_t^2)\, dx\, dt;$$

$$\left| 2 \int_0^t \int_0^1 E(\mathfrak{X}_x^2)\, \mathfrak{X}_{tx}^2 W_t W_x \right| \leqslant |\, E' \,|\, |\, \mathfrak{X}_{tx}^2 \,|_T \int_0^t \int_0^1 (W_x^2 + W_t^2)\, dx\, dt;$$

and

$$\left| \int_0^t \int_0^1 E'(\mathfrak{X}_x^2) \mathfrak{X}_{tx}^2 W_x^2\, dx\, dt \right| \leqslant |\, E' \,|\, |\, \mathfrak{X}_{tx}^2 \,|_T \int_0^t \int_0^1 W_x^2\, dx\, dt,$$

where

$$|\, E' \,| = \max_{|\xi| \leqslant M^1} |\, E'(\xi)|$$

and M^1 is the common upper bound for $\mathfrak{X}_x^1$ and $\mathfrak{X}_x^2$.

Observing that $E(\cdot)$ is bounded from below on $|\, \xi \,| \leqslant M^1$ by some number $E_0 > 0$ we see that $T(t) \overset{\text{def}}{=} \int_0^1 (W_t^2 + W_x^2)(x, t)\, dx$ satisfies the inequality:

$$0 \leqslant T(t) \leqslant L_T \int_0^t T(\tau)\, d\tau, \qquad 0 \leqslant t \leqslant T,$$

where

$$L_T = \frac{|\, E' \,|\, [|\, \mathfrak{X}_{xx}^1 \,|^T + 2\, |\, \mathfrak{X}_{tx}^2 \,|_T]}{\min(\rho_0\,,\, E_0)}.$$

The uniqueness of the solution now follows.

4. Proof of Theorem 1

We point out that for each $0 < h < 1$ (with $1/h$ and integer) there is a *unique solution* of the discrete problem $(E)_h$, $(IC)_h$, and $(BC)_h$ (and therefore a well-defined function $\mathfrak{X}^h : \Sigma_h \times [0, \infty) \to R$).

We now let $0 < h_1 < 1$ (with $1/h_1 \overset{\text{def}}{=} N_1$ an integer) be fixed, $h_n \overset{\text{def}}{=} (h_1)^n$ and $\mathfrak{X}^{(n)} \overset{\text{def}}{=} \mathfrak{X}^{(h_n)}$. $\bar{\mathfrak{X}}^{(n)}$ will be an arbitrary extension of $\mathfrak{X}^{(n)}$ to $[0, 1] \times [0, \infty)$ with the following properties:

(i) $\bar{\mathfrak{X}}^{(n)}$ in C^1 and has continuous derivatives $\bar{\mathfrak{X}}^{(n)}_{tt}$, $\bar{\mathfrak{X}}^{(n)}_{tx} = \bar{\mathfrak{X}}^{(n)}_{xt}$, and $\bar{\mathfrak{X}}^{(n)}_{ttx} = \bar{\mathfrak{X}}^{(n)}_{txt} = \bar{\mathfrak{X}}^{(n)}_{xtt}$ on $[0, 1] \times [0, \infty)$. Moreover, for $x \neq kh_n$, $k = 0, 1, -(N_1)^n$, the derivative $\bar{\mathfrak{X}}^{(n)}_{xx}$ and $\bar{\mathfrak{X}}^{(n)}_{xxt} = \bar{\mathfrak{X}}^{(n)}_{xtx} = \bar{\mathfrak{X}}^{(n)}_{txx}$ exist. These functions are continuous on $[kh_n, (k + 1) h_n] \times [0, \infty)$ and are, for each t, piecewise continuous on $[0, 1]$.

(ii) On $\Sigma_{h_n} \times [0, \infty)$ $\bar{\mathfrak{X}}^{(n)} = \mathfrak{X}^{(n)}$, $\bar{\mathfrak{X}}^{(n)}_x = \Delta_{h_n}\mathfrak{X}^{(n)}$, $\bar{\mathfrak{X}}^{(n)}_t = \mathfrak{X}^{(n)}_t$, $\bar{\mathfrak{X}}^{(n)}_{tx} = \Delta_{h_n}\mathfrak{X}^{(n)}_t$, and $\bar{\mathfrak{X}}^{(n)}_{tt} = \mathfrak{X}^{(n)}_{tt}$.

(iii) There exists a number $M^{\#}$, depending only on the upper bound M of (3.12) and independent of n, such that (3.12) holds with M replaced by $M^{\#}$ and $\mathfrak{X}$ replaced by $\bar{\mathfrak{X}}^{(n)}$.

(iv) For each $T > 0$ there exists a constant $C^{\#}$, depending only on the constant C in (3.22) and (3.27) and independent of n, such that $\| \bar{\mathfrak{X}}^{(n)}_{tt} \|^T \leqslant C^{\#}$, $\int_0^T \| \bar{\mathfrak{X}}^{(n)}_{ttx} \|^2(\tau)\, d\tau = ||| \bar{\mathfrak{X}}^{(n)}_{ttx} |||^2_T \leqslant C^{\#}$, and $\| \bar{\mathfrak{X}}^{(n)}_{txx} \|_T \leqslant C^{\#}$.

To show that at least one such extension exists one need only consider the function $\bar{\mathfrak{X}}^{(n)}$ defined by

$$\bar{\mathfrak{X}}^{(n)}(x, t) = \begin{cases} [\mathfrak{X}^{(n)}(h_n, t)] \dfrac{x}{h_n}, \quad 0 \leqslant x \leqslant h_n, \quad t \geqslant 0; \\ -[\Delta^2_{h_n}\mathfrak{X}^{(n)}(kh_n, t)] \dfrac{(x - kh_n)^3}{h_n} \\ + 2[\Delta^2_{h_n}\mathfrak{X}^{(n)}(kh_n, t)](x - kh_n)^2 \\ + [\Delta_{h_n}\mathfrak{X}^{(n)}(kh_n, t)](x - kh_n) \\ + \mathfrak{X}^{(n)}(kh_n, t), \quad kh_n \leqslant x \leqslant (k + 1)h_n, \quad t \geqslant 0 \quad \text{and} \\ k =, 1\ 2, \ldots (N_1^n - 1). \end{cases}$$

We shall now demonstrate that the extensions converge to a limit function $\mathfrak{X}$ having the properties enunciated in Theorem 2. Since the problem has at most one solution, the limit function $\mathfrak{X}$ will be independent of the particular procedure used to extend the functions $\mathfrak{X}^{(n)}$ and independent of the particular h_1 used to generate the sequence of functions $\{\mathfrak{X}^{(n)}\}$.

Let $T > 0$ be an arbitrary but fixed number and S_T be the strip

$$S_T \overset{\text{def}}{=} \{(x\ \ t) \mid 0 < x < 1, 0 < t < T\}.$$

K will denote some constant depending on the function V^0, the parameters λ and ρ_0 and the upper limit T. The a priori bounds of the previous section guarantee the existence of a constant K such that all elements of the sequence $\{\mathfrak{X}^{(n)}\}$ satisfy the following inequality:

$$|||\mathfrak{X}^{(n)}||| \overset{\text{def}}{=} \max \begin{Bmatrix} |\mathfrak{X}^{(n)}|_T, & |\mathfrak{X}_x^{(n)}|_T, & |\mathfrak{X}_t^{(n)}|_T, & |\mathfrak{X}_{xx}^{(n)}|_T, \\ |\mathfrak{X}_{tx}^{(n)}|_T, & \|\mathfrak{X}_{tt}^{(n)}|_T, & \|\mathfrak{X}_{txx}^{(n)}\|_T, & \\ |||\mathfrak{X}_{ttx}^{(n)}|||_T & & & \end{Bmatrix} \leqslant K. \quad (4.1)$$

LEMMA 4.1. *We are guaranteed the existence of a subsequence of the* $\mathfrak{X}^{(n)}$*'s (which we again denote by* $\{\mathfrak{X}^{(n)}\}$*) and a function* $\mathfrak{X}$ *with the following properties:*

(i) $\mathfrak{X}$ *is continuous on* $\bar{S}_T$ *and* $\lim_{n\to\infty} |\mathfrak{X}^{(n)} - \mathfrak{X}|_T = 0$;

(ii) $\mathfrak{X}$ *is* C^1 *in* x *on* $\bar{S}_T$ *and* $\lim_{n\to\infty} |\mathfrak{X}_x^{(n)} - \mathfrak{X}_x|_T = 0$;

(iii) $\mathfrak{X}$ *has strong* $L_2(S_T)$ *derivatives* $\mathfrak{X}_t$, $\mathfrak{X}_{tx}$, $\mathfrak{X}_{tt}$, $\mathfrak{X}_{xx}$, $\mathfrak{X}_{txx}$, *and* $\mathfrak{X}_{ttx}$. *Moreover, the functions* $\mathfrak{X}_x^{(n)}$ *and* $\mathfrak{X}_{tx}^{(n)}$ *converge strongly in* $L_2(S_T)$ *to* $\mathfrak{X}_t$ *and* $\mathfrak{X}_{tx}$, *while the functions* $\mathfrak{X}_{tt}^{(n)}$, $\mathfrak{X}_{xx}^{(n)}$, $\mathfrak{X}_{txx}^{(n)}$, *and* $\mathfrak{X}_{ttx}^{(n)}$ *converge weakly in* $L_2(S_T)$ *to* $\mathfrak{X}_{tt}$, $\mathfrak{X}_{xx}$, $\mathfrak{X}_{txx}$, *and* $\mathfrak{X}_{ttx}$.

(iv) $\mathfrak{X}_t$, $\mathfrak{X}_{tx}$, *and* $\mathfrak{X}_{xx}$ *are in* $L_\infty(S_T)$ *and* $\max(|\mathfrak{X}_t|_T, |\mathfrak{X}_{tx}|_T, |\mathfrak{X}_{xx}|_T) \leqslant K$.

PROOF. Assertion (i) follows from Ascoli's Theorem and the observation that the $\mathfrak{X}^{(n)}$'s are C^1 and satisfy

$$\max(|\mathfrak{X}^{(n)}|_T, |\mathfrak{X}_x^{(n)}|_T, |\mathfrak{X}_t^{(n)}|_T) \leqslant K.$$

Noting that the sequence of derivatives $\{\mathfrak{X}_x^{(n)}\}$ are continuous and satisfy

$$|\mathfrak{X}_x^{(n)}|_T \leqslant K,$$

and

$$|\mathfrak{X}_x^{(n)}(x_2, t_2) - \mathfrak{X}_x^{(n)}(x_1, t_1)|_T \leqslant K(|x_2 - x_1| + |t_2 - t_1|)$$

we see that the argument used above yields assertion (ii).

To establish (iii) we observe that (4.1) implies that the derivatives $\mathfrak{X}_t^{(n)}$, $\mathfrak{X}_{tx}^{(n)}$, $\mathfrak{X}_{tt}^{(n)}$, $\mathfrak{X}_{xx}^{(n)}$, $\mathfrak{X}_{ttx}^{(n)}$ are bounded in $L_2(S_T)$ independent of n. The weak compactness of bounded closed sets in $L_2(S_T)$ guarantees the existence of a subsequence of the $\mathfrak{X}^{(n)}$'s (which we again denote by $\{\mathfrak{X}^{(n)}\}$) and functions $a_1 - a_6$ in $L_2(S_T)$ such that

$$\mathfrak{X}_t^{(n)} \rightharpoonup a_1, \qquad \mathfrak{X}_{tx}^{(n)} \rightharpoonup a_2, \qquad \text{etc.}^3$$

That $a_1 - a_6$ are the appropriate strong derivatives of $\mathfrak{X}$ is immediate.

[3] Here $\rightharpoonup$ denotes weak convergence in $L_2(S_T)$.

We now observe that the weak convergence of the sequences $\{\mathfrak{X}_{tx}^{(n)}\}$ and $\{\mathfrak{X}_{tt}^{(n)}\}$ to a_2 and a_3 respectively implies that the sequence $\{\mathfrak{X}_t^{(n)}\}$ is converging strongly to a_1. The same argument applied to the sequences $\{\mathfrak{X}_{ttx}^{(n)}\}$ and $\{\mathfrak{X}_{txx}^{(n)}\}$ establishes $\lim_{n\to\infty} ||| \mathfrak{X}_{tx}^{(n)} - a_2 |||_T = 0$.

Assertion (iv) follows from the a priori bounds for the functions $\mathfrak{X}_t^{(n)}$, $\mathfrak{X}_{tx}^{(n)}$, and $\mathfrak{X}_{txx}^{(n)}$.

Actually certain sharper results may be obtained.

LEMMA 4.2. *The subsequence $\{\mathfrak{X}^{(n)}\}$ of the preceding lemma and the function $\mathfrak{X}$ may be chosen such that*

$$\lim_{n\to\infty} | \mathfrak{X}_t^{(n)} - \mathfrak{X}_t |_T = 0 \tag{4.2}$$

and

$$\lim_{n\to\infty} \| \mathfrak{X}_{tx}^{(n)} - \mathfrak{X}_{tx} \|_T = 0. \tag{4.3}$$

Equations (4.2) *and* (4.3) *together with the continuity of the $\mathfrak{X}^{(n)}$'s and the fact that the mappings $t \to \mathfrak{X}_{tx}^{(n)}(\cdot, t)$ from* $[0, T]$ *into* $L_2(0, 1)$ *are uniformly Hölder continuous with exponent $\frac{1}{2}$, i.e., that*

$$\| \mathfrak{X}_{tx}^{(n)}(t_2) - \mathfrak{X}_{tx}^{(n)}(t_1)\| \leqslant K \, | t_2 - t_1 |^{1/2} \tag{4.4}$$

imply that $\mathfrak{X}_t$ is continuous on $\bar{S}_T$ and that $t \to \mathfrak{X}_{tx}(\cdot, t)$ is a uniformly Hölder continuous map of $[0, T]$ *into* $L_2(0, 1)$ *which satisfies* (4.4).

PROOF. To establish the lemma we look at the sequence of functions $\{\mathfrak{X}_{tx}^{(n)}\}$. The vanishing of $\mathfrak{X}_t^{(n)}$ at zero and one guarantees, for each n and t, the existence of an $x^*(n, t)$ such that $\mathfrak{X}_{tx}^{(n)}(x^*, t) = 0$. We then obtain $| \mathfrak{X}_{tx}^{(n)} |(t) \leqslant \| \mathfrak{X}_{txx}^{(n)} \|(t)$. The bound, $\| \mathfrak{X}_{txx}^{(n)} \|_T \leqslant K$, then implies that on any dense set of t points $\mathfrak{J} = \{0 = t_1 < t_2 < \cdots\}$ in $[0, T]$ we may find a subsequence of the $\mathfrak{X}_{tx}^{(n)}$'s (which we again denote by $\{\mathfrak{X}_{tx}^{(n)}\}$) and functions $\{f(t_i)\}$ in $L_2(0, 1)$ such that

$$\lim_{n\to\infty} \| \mathfrak{X}_{tx}^{(n)}(\cdot, t_i) - f(t_i)\| \to 0, \qquad t_i \in \mathfrak{J}.$$

Noting that $||| \mathfrak{X}_{ttx} |||_T \leqslant K$ implies

$$\| \mathfrak{X}_{tx}^{(n)}(t) - \mathfrak{X}_{tx}^{(n)}(\tau)\| \leqslant K \, | t - \tau |^{1/2},$$

we may conclude

(a) $$\lim_{n\to\infty} \sup_{t\in\mathfrak{J}} \| \mathfrak{X}_{tx}^{(n)}(\cdot, t) - f(t)\| = 0,$$

(b) $f : \mathfrak{I} \to L_2(0, 1)$ satisfies (4.4) and is therefore uniquely extendable to a continuous map $F : [0, T] \to L_2(0, 1)$ satisfying (4.4). It then follows that $F = \mathfrak{X}_{tx}$ and $\lim_{n\to\infty} \| \bar{\mathfrak{X}}_{tx}^{(n)} - \mathfrak{X}_{tx} \|_T = 0$. That $\lim_{n\to\infty} | \bar{\mathfrak{X}}_t^{(n)} - \mathfrak{X}_t |_T = 0$ is a consequence of the inequality $| \bar{\mathfrak{X}}_t^{(n)} - \mathfrak{X}_t |_T \leqslant \| \bar{\mathfrak{X}}_{tx}^{(n)} - \mathfrak{X}_{tx} \|_T$.

Completion of the Proof of Theorem 1. We shall now show that the limit function $\mathfrak{X}$ of the preceding two lemma's is the desired *generalized solution.* That $\mathfrak{X}$ and $\mathfrak{X}_t$ satisfy the initial and boundary conditions (IC) and (BC) follows from the uniform convergence of the functions $\bar{\mathfrak{X}}^{(n)}$ and $\bar{\mathfrak{X}}_t^{(n)}$ and from the properties of the $\bar{\mathfrak{X}}^{(n)}$'s (see the opening remarks at the beginning of the section). That $\mathfrak{X}$ has the desired smoothness properties is a consequence of Lemmas 4.1 and 4.2.

We shall now show that $\mathfrak{X}$ satisfies (1.1). Let $\Sigma_\infty = \bigcup_{n\geqslant 0} \Sigma_{h_n}$ and let x_2 and x_1 be in Σ_∞. It then follows that there is an $N(x_2, x_1)$ such that for all $n \geqslant N(x_2, x_1)$ the solutions $\mathfrak{X}^{(n)}$ of the discrete problem $(\mathrm{E})_{h_n}$, $(\mathrm{IC})_{h_n}$, and $(\mathrm{BC})_{h_n}$ are defined at x_1 and x_2. We then have

$$\begin{aligned}
&\int_{x_1}^{x_2} \rho_0[\mathfrak{X}_t(x, t_2) - \mathfrak{X}_t(x, t_1)]\, dx \\
&\quad = \int_{x_1}^{x_2} \rho_0[\mathfrak{X}_t(x, t_2) - \bar{\mathfrak{X}}_t^{(n)}(x, t_2)]\, dx - \int_{x_1}^{x_2} \rho_0[\mathfrak{X}_t(x\ t_1) - \bar{\mathfrak{X}}_t^{(n)}(x, t_1)]\, dx \\
&\qquad + \int_{x_1}^{x_2} \rho_0\bar{\mathfrak{X}}_t^{(n)}(x, t_2)\, dx - \sum_{\{x\in\Sigma_{h_n} | x_1 \leqslant x \leqslant x_2\}} \rho_0 h_n \mathfrak{X}_t^{(n)}(x, t_2) \\
&\qquad - \int_{x_1}^{x_2} \rho_0\bar{\mathfrak{X}}_t^{(n)}(x, t_1)\, dx + \sum_{\{x\in\Sigma_{h_n} | x_1 \leqslant x \leqslant x_2\}} \rho_0 h_n \mathfrak{X}_t^{(n)}(x, t_1) \\
&\qquad + \sum_{\{x\in\Sigma_{h_n} | x_1 \leqslant x \leqslant x_2\}} \rho_0 h_n[\mathfrak{X}_t^{(n)}(x, t_2) - \mathfrak{X}_t^{(n)}(x, t_1)] \\
&\quad \overset{\text{def}}{=} I_1^{(n)} - I_2^{(n)} + I_3^{(n)} - I_4^{(n)} + \sum_{\{x\in\Sigma_{h_n} | x_1 \leqslant x \leqslant x_2\}} \rho_0 h_n[\mathfrak{X}_t^{(n)}(x, t_2) - \mathfrak{X}_t^{(n)}(x, t_1)].
\end{aligned}$$

Since $\mathfrak{X}^{(n)}$ satisfies the discrete problem and since $\Delta_{h_n}\mathfrak{X}^{(n)}(x, t) = \bar{\mathfrak{X}}_x^{(n)}(x, t)$ for $x \in \Sigma_{h_n}$, it follows that

$$\begin{aligned}
0 &\leqslant \left| \int_{x_1}^{x_2} \rho_0[\mathfrak{X}_t(x, t_2) - \mathfrak{X}_t(x, t_1)]\, dx - \int_{t_1}^{t_2} [\sigma(\mathfrak{X}_x(x_2, t)) - \sigma(\mathfrak{X}_x(x_1, t))]\, dt \right. \\
&\qquad \left. - \lambda[\mathfrak{X}_x(x_2, t_2) - \mathfrak{X}_x(x_1, t_2)] + \lambda[\mathfrak{X}_x(x_2, t_1) - \mathfrak{X}_x(x_1, t_1)] \right| \\
&\leqslant \sum_{i=1}^{4} | I_i^{(n)} | + \sum_{i=1}^{3} | E_i^{(n)} |, \qquad n \geqslant N(x_1, x_2),
\end{aligned}$$

where

$$E_1^{(n)} = \int_{t_1}^{t_2} [\sigma(\mathfrak{X}_x(x_2, t)) - \sigma(\bar{\mathfrak{X}}_x^{(n)}(x_2 + h_n, t))]\, dt,$$

$$E_2^{(n)} = \int_{t_1}^{t_2} [\sigma(\mathfrak{X}_x(x_1, t)) - \sigma(\bar{\mathfrak{X}}_x^{(n)}(x_1 + h_n, t))]\, dt, \qquad \text{and}$$

$$\begin{aligned} E_3^{(n)} &= \lambda \mid \mathfrak{X}_x(x_2, t_2) - \bar{\mathfrak{X}}_x^{(n)}(x_2 + h_n, t_2)| + \lambda \mid \mathfrak{X}_x(x_1, t_2) - \bar{\mathfrak{X}}_x^{(n)}(x_1 + h_n, t_2)| \\ &\quad + \lambda \mid \mathfrak{X}_x(x_2, t_1) - \bar{\mathfrak{X}}_x^{n)}(x_2 + h_n, t_1)| \\ &\quad + \lambda \mid \mathfrak{X}_x(x_1, t_1) - \bar{\mathfrak{X}}_x^{(n)}(x_1 + h_n, t_1)|. \end{aligned}$$

One now lets $n \to \infty$ and uses the uniform convergence of $\bar{\mathfrak{X}}_t^{(n)}$ and $\bar{\mathfrak{X}}_x^{(n)}$ to $\mathfrak{X}_t$ and $\mathfrak{X}_x$ to obtain (1.1) wherever x_1 and x_2 are points of Σ_∞. Since Σ_∞ is dense in [0, 1], and since all quantities appearing in (1.1) are continuous on $\bar{S}_T$ one may infer that (1.1) holds for all x_1 and x_2.

A similar argument may be used to establish that for all ϕ in $C^\infty(\bar{S}_T)$ which vanish in a neighborhood of $x = 0$, $x = 1$, and $t = T$, the following identity holds

$$\int_0^T \int_0^1 \{-\rho_0 \mathfrak{X}_t \phi_t + \sigma(\mathfrak{X}_x)\, \phi_x - \lambda \mathfrak{X}_x \phi_{xt}\}\, dx\, dt = \int_0^1 V^0(x)\, \phi(x)\, dx + \lambda \int_0^1 V_x{}^0 \phi_x\, dx.$$

Using the smoothness of $\mathfrak{X}$ (see Lemma 4.1), we then have

$$\int_0^T \int_0^1 (\mathfrak{X}_{tt} - E(\mathfrak{X}_x)\, \mathfrak{X}_{xx} - \lambda \mathfrak{X}_{txx}) \phi\, dx\, dt = 0$$

for all ϕ of the above type thereby completing the proof of the theorem.

REMARK 4.1. The uniqueness theorem now implies that the selection procedure used to obtain the solution was unnecessary and that the full sequence $\{\bar{\mathfrak{X}}^{(n)}\}$ converges to $\mathfrak{X}$.[4]

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[4] Of course convergence is in the senses indicated by Lemmas 4.1 and 4.2.