A symptotic Behavior of Nonlinear Difference Systems*

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In this paper we study a general variational stability, introduced mainly for weakly stable difference systems. Moreover, we obtain asymptotic formulae for these systems, which state new results about asymptotic behavior for perturbed systems under general hypotheses.

1. INTRODUCTION

The theory of stability in the sense of Lyapunov is well known and is used in concrete problems of real life. It is obvious that, in applications, asymptotic stability is more important than stability, because the desirable feature is to know the size of the region of asymptotic stability. On the other hand, the variation of parameter formulae for nonlinear summary difference equations, obtained by Agarwal [1] and Bainov and Simeonov [3], allows the study of nonlinear perturbations of systems with certain stability properties. However, in the study of asymptotical stability, it is difficult to work with non-exponential types of stability.

In this paper, we shall extend the study of exponential stability to a variety of reasonable systems called $h$-systems. Also, in this work we shall introduce the concept of $h$-stability. Thus, we shall obtain results about stability for weakly stable difference systems (at least, for systems with stabilities weaker than those given by exponential stability and uniform Lipschitz stability). See Medina [9, 10], Dannan and Elaydi [5], Brauer and Strauss [4], and Pinto [11].

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Consider the nonlinear difference system
\[ x(n+1) = f(n, x(n)), \]
along with its associated variational system
\[ z(n+1) = f_z(n, x(n, n_0, x_0))z(n); \]
and the perturbed system
\[ y(n+1) = f(n, y(n)) + g(n, y(n)), \]
where \( f, g : \mathbb{N}_0 \times D \to \mathbb{R}^m \) are two functions and \( D \subseteq \mathbb{R}^m \) is a domain with \( 0 \in D \), \( \mathbb{N}_0 = \{a, a + 1, \ldots, a + k, \ldots \} \) \((a) \) is a non-negative integer, \( \mathbb{N}_0 = \mathbb{N} \) the set of positive integers), and let \( f_z = \partial f / \partial x \) exist and be continuous and invertible on \( \mathbb{N}_0 \times D \), \( f(n, 0) = g(n, 0) = 0 \), \( x(n) = x(n, n_0, x_0) \) is the solution of (1) for \( n \geq n_0 \) with \( x(n_0, n_0, x_0) = x_0 \).

We shall study the stability of the perturbed system (3) knowing the stability of the variational system (2) with respect to the solution \( x(n) = x(n, n_0, x_0) \) of system (1), and assuming that the perturbation \( g = g(n, y) \) satisfies
\[ |g(n, y)| \leq \sum_{i=1}^{P} \lambda_i(n) \omega_i(|y|), \quad p \in \mathbb{N} \] (4)
with \( \lambda_i : \mathbb{N}_0 \to [0, \infty) \) \((1 \leq i \leq p) \) properly summable functions and \( \omega_i : [0, \infty) \to [0, \infty) \) \((1 \leq i \leq p) \) suitable non-decreasing and positive functions on \((0, \infty)\). Thus, the results in this paper extend many of the classical stabilities that have appeared in the literature, see [1, 4–8].

2. PRELIMINARIES

In order to establish our main results, we shall use the following:

**Lemma 1** [6, Lemma 4.6.2]. Assume that \( f : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}^m \) and \( f \) possesses partial derivatives on \( \mathbb{N}_0 \times \mathbb{R}^m \). Let the solution \( x(n) = x(n, n_0, x_0) \) of system (1) exist for \( n \geq n_0 \) and let
\[ H(n, n_0, x_0) = \frac{\partial f(n, x(n, n_0, x_0))}{\partial x}. \]
Then
\[ \Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}. \]
exists and is the solution of

\[ \Phi(n + 1, n_0, x_0) = H(n, n_0, x_0) \Phi(n, n_0, x_0), \]

\[ \Phi(n_0, n_0, x_0) = I. \]

If \( x(n) \) and \( y(n) \) are the solutions of Eqs. (1) and (3), respectively, and \( x(n_0) = y(n_0) \), then the following analogue of Alekseev's formula holds, Bainov and Simeonov [3],

\[ y(n) = x(n) + \sum_{l=n_0}^{n-1} \int_{l}^{n} \Phi(n, l + 1, u(y(l), \tau)) \times g(l, y(l)) \, d\tau, \quad (5) \]

where \( \Phi(n, n_0, x(n, n_0, x_0)) \) is the fundamental matrix of system (2), and

\[ u(y(n), \tau) = f(n, y(n)) + \tau \cdot g(n, y(n)), \quad \tau \in [0, 1]. \]

A generalization of (5) can be found in Agarwal [2].

**Definition 0** [9]. System (1) is called an h-system around the null solution, or more briefly an h-system, if there exist a positive function \( h : N_{\alpha} \rightarrow \mathbb{R} \) and a constant \( c \geq 1 \) such that

\[ |x(n, n_0, x_0)| \leq c|x_0|h(n)h(n_0)^{-1}, \quad n \geq n_0 \quad (6) \]

for \( |x_0| \) small enough (here \( h(n)^{-1} = 1/h(n) \)).

The function \( h \) as well as the constant \( c \) depends only on \( f \). If \( h \) is a bounded function, then an h-system allows the following types of stability:

**Definition 1.** The zero solution of system (1), or more briefly (1), is said to be an h-system in variation if the variational system (2) is an h-system, that is, there exist \( \delta_1 > 0 \) and \( c_1 \geq 1 \) such that for \( |z_0| < \delta_2 \), we have

\[ |z(n, n_0, z_0)| \leq c_1|z_0|h(n)h(n_0)^{-1}. \quad (7) \]

for \( n \geq n_0 \geq a. \)

The last condition is equivalent to

\[ |\Phi(n, n_0, x_0)| \leq c_1h(n)h(n_0)^{-1}, \quad (8) \]

for \( n \geq n_0 \geq a \), and hence, any variational h-system is an h-system because

\[ x(n, n_0, x_0) = \left[ \int_{0}^{1} \Phi(n, n_0, \tau x_0) \, d\tau \right] x_0. \]
**Definition 2** [9]. The zero solution of system (1), or more briefly (1), is said to be h-stable if a positive and bounded function \( h: N^a \to \mathbb{R} \) and a constant \( c \geq 1 \) exist, such that

\[
|\phi(n, n_0, x_0)| \leq c|x_0|h(n)h(n_0)^{-1},
\]

for \( n \geq n_0 \) and \( |x_0| \) is small enough.

**Definition 3** [9]. The zero solution of system (1), or more briefly (1), is said to be h-stable in variation if there is a positive and bounded function \( \Phi(n, n_0, x_0) \leq ch(n)h(n_0)^{-1} \), for \( n \geq n_0 \) and \( |x_0| \) small enough.

In effect, system (1) is h-stable in variation if the variational system (2) is h-stable; thus Definition 3 implies Definition 2. Moreover, Definitions 2 and 3 include several classical concepts of stability such as exponential stability and Lipschitz stability [4–6, 12].

We shall use the following theorem, which gives an explicit pointwise estimate, independent of \( u \), for a function \( u = u(n) \) which satisfies the inequality

\[
u(n) \leq c + \sum_{i=1}^{p} \left[ \sum_{j=n_0}^{n-1} \lambda_i(j) \omega_i(u(j)) \right], \quad p \in \mathbb{N}, \tag{10}\]

where

(i) The functions \( \omega_i: [0, \infty) \to [0, \infty), 1 \leq i \leq p \), are continuous and non-decreasing, \( \omega_i(u) > 0 \) for \( u > d \) and \( \omega_{i+1}/\omega_i \) \( (1 \leq i \leq p - 1) \) are non-decreasing on \( (d, \infty) \).

(ii) \( u: \mathbb{N} \to [d, \infty) \) and \( \lambda: \mathbb{N} \to [d, \infty) \) are functions, \( c \) is a constant such that \( c > d \).

We define the functions:

(i) \( W_i(u) = \int_{a_i}^{u} ds/\omega_i(s) \), \( u > 0 \) \( (1 \leq i \leq p) \) and \( W_i^{-1} \) is their inverse function.

(ii) \( \varphi_i(u) = u \) and

\[
\varphi_i(u) = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1, \quad 1 \leq i \leq p, \tag{11}\]

where \( \phi(u) = W_i^{-1}[W_i(u) + \alpha_i], \alpha_i \geq 0 \) is a constant. Thus, we can establish the following theorem;
THEOREM A [8]. Let $d \in \mathbb{R}$ and assume (1) and (11) hold. Let $m \in \mathbb{N}$ such that

$$
\alpha_i(m) =: \sum_{j=1}^{m} \lambda_i(j) \leq \frac{ds}{\omega_i(s)} \quad (1 \leq i \leq p),
$$

where the functions $\varphi_i$ ($0 \leq i \leq p - 1$) are given in (11) with $\alpha_i = \alpha_i(m)$.

If the function $u$ satisfies the inequality (10), then

$$
u(n) \leq W_p^{-1} \left[ W_p(\varphi_{p-1}(c)) + \sum_{j=1}^{n-1} \lambda_p(j) \right],$$

for any $n \leq m$.

Remark 1. The functions $\varphi_i$ are increasing and independent on $u_i$.

The condition

$$\int_{1}^{\infty} \frac{ds}{\omega_i(s)} = \infty \quad (i = 1, 2, \ldots, p) \quad (12)$$

implies that any function $\varphi_i$ is defined for all $n \in N_a$ and all $c \geq 0$.

The dual condition

$$\int_{0^+}^{1} \frac{ds}{\omega_i(s)} = \infty \quad (i = 1, 2, \ldots, p) \quad (13)$$

implies that any function $\varphi_i$ is defined for all $n \in N_a$ and $c \geq 0$ small enough. Moreover, condition (13) implies the stability property.

$$\varphi_i(0^+) = 0 \quad (i = 1, 2, \ldots, p). \quad (14)$$

See [8].

3. MAIN RESULTS

In this section we shall obtain some h-stability criteria for the perturbed system (3) satisfying (4).

Introduce the following conditions:

(U) $u(y(n), \tau) = f(n, y(n)) + \tau \cdot g(n, y(n))$ $\in \mathcal{D}$ for $n \in N_a$, $y \in D$, and $\tau \in [0, 1]$.

(M) $|\Phi(n, n_0, x_o)| \leq ch(n)h(n_o)^{-1}$, for $a \leq n_0 \leq n < \infty$, $x_o \in \mathcal{D}$; where $c \geq 1$ is a constant and $h : N_a \rightarrow \mathbb{R}$ is a positive function.
The perturbation $g = g(n, y)$ satisfies (4) with $(\lambda_i, \omega_i)(1 \leq i \leq p)$ satisfying (I)–(II).

For any $n_0 \in N_a$,

$$\sum_{l=n_0}^{\infty} \lambda_i(l)r_i\left(h(l)h(n_0)^{-1}\right)h(l+1)^{-1} < \infty \quad (1 \leq i \leq p).$$

The functions $\omega_i(1 \leq i \leq p)$ satisfy condition (I) and for any $i, 1 \leq i \leq p$, there is a function $r_i$ defined on $(0, \infty)$ such that

$$\omega_i(\alpha u) \leq r_i(\alpha)\omega_i(u), \quad \text{for } \alpha > 0. \tag{15}$$

**Theorem 1.** Assume that:

- (H$_1$) Conditions (U), (M), (P), and (S) hold.
- (H$_2$) There exists $\delta > 0$ such that

$$K_i(\delta) := \sup\{\alpha_i(n_0, y_0)/n_0 \geq a, 0 < |y_0| < \delta\} \leq c^{-1} \int_{\varphi_i(c)}^\infty \frac{ds}{\omega_i(s)} \quad (1 \leq i \leq p), \tag{16}$$

where $c$ is the constant satisfying (M), and

$$\alpha_i(n_0, y_0) = \frac{h(n_0)}{|y_0|} \sum_{j=n_0}^{\infty} \lambda_i(j)\left|h(j)h(n_0)^{-1}\right| \cdot r_i(|y_0|h(j)h(n_0)^{-1}) \quad (1 \leq i \leq p),$$

and

$$\varphi_i = \psi_i \circ \psi_{i-1} \circ \cdots \circ \psi_1, \quad \psi_i(u) = W_i^{-1}[W_i(u) + cK_i(\delta)]. \tag{17}$$

Then, for all $n_0 \geq a$ and $|y_0|$ small enough any solution $y(n) = y(n, n_0, y_0)$ of (3) satisfies

$$|y(n, n_0, y_0)| \leq \varphi_p(c)|y_0|h(n)h(n_0)^{-1}, \quad n \geq n_0.$$  

**Proof.** We note that from condition (M) and the formula

$$x(n, n_0, x_0) = \left[\int_0^1 \Phi(n, n_0, \tau x_0) d\tau\right]x_0,$$
it follows that the solution $x(n, n_0, x_0)$ of system (1) satisfies the estimate

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h(n_0)^{-1},$$

for $a \leq n_0 \leq n < \infty$ and $x_0 \in D$.

Thus, using (U), (M), and (P) we have

$$|y(n)| \leq c|y_0|h(n)h(n_0)^{-1} + \sum_{j=n_0}^{n-1} ch(n)h(j+1)^{-1} \cdot \sum_{i=1}^{p} \lambda_i(j) \omega_i(|y(j)|),$$

(18)

or denoting $l(n, n_0, y_0) = |y_0|h(n)h(n_0)^{-1}$ for $0 < |y_0| < \delta$, then by (S) we have that $u(n) = |y(n)|/l(n, n_0, y_0)$ satisfies for $|y_0| \neq 0$

$$u(n) \leq c + c \cdot \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{\lambda_i(j)r_i(l(j, n_0, y_0))}{l(j+1, n_0, y_0)} \omega_i(u(j)).$$

So, by Theorem A it follows that

$$u(n) \leq W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + c \cdot \sum_{j=n_0}^{n-1} \frac{\lambda_p(j)r_p(l(j, n_0, y_0))}{l(j+1, n_0, y_0)}\right].$$

for $n \geq n_0 \geq a$, where $\varphi_i$ are the functions defined in (17). The inequalities (16) show that this estimation is valid for every $n \geq n_0 \geq a$ (by taking $c \cdot \lambda_i(n)h(n+1)^{-1} \cdot r_i(|y_0|h(n)h(n_0)^{-1})$ instead $\lambda_i; \quad i = 1, 2, \ldots, p$, and $m = \infty$ in Theorem A) and that the function in the right member is bounded by $\varphi_p(c)$, that is

$$W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + c \cdot \sum_{j=n_0}^{\infty} \frac{\lambda_p(j)r_p(l(j, n_0, y_0))}{l(j+1, n_0, y_0)}\right] \leq \varphi_p(c).$$

Thus, it follows that $u(n) \leq \varphi_p(c)$, for $n \geq n_0$, that is, for $|y_0|$ small enough,

$$|y(n, n_0, y_0)| \leq \varphi_p(c)|y_0|h(n)h(n_0)^{-1}, \quad n \geq n_0.$$

Therefore, the perturbed system (3) is h-stable because, from Corollary 2 of [8], we have $\varphi_p(c) \geq c > 1$.  ■
Remark 2. The existence of \( d \), hypothesis \( H_2 \) of Theorem 1, will be discussed in Remark 5.

Remark 3. An important class of admissible functions \( \omega_i \), is any polynomial system

\[
\omega_i(u) = u^{\gamma_i}, \quad \gamma_i \geq 1 \quad (1 \leq i \leq p)
\]

for which, if \( \lambda_i(n) \cdot h^{\gamma_i}(n)h(n + 1)^{-1} \in l_1(N_\omega), i = 1, 2, \ldots, p \), then Eq. (3) is also \( h \)-stable.

(b) If we replace the condition \( (H_2) \) of Theorem 1 by a hypothesis less restrictive, namely,

\[
K_i =: \sup_i \{ \alpha_i(n_0, y_0) | n_0 \geq a, 0 < |y_0| < \infty \}
\]

\[
< c^{-1} \cdot \int_0^\infty \frac{ds}{\omega_i(s)} \quad (1 \leq i \leq p), \quad \quad \quad (H_2)'
\]

where \( c \) is the constant satisfying \( (M) \), and \( \alpha_i(n_0, y_0) \) is the same as in Theorem 1, then we obtain that the zero solution of Eq. (3) is globally \( h \)-stable.

By considering \( f \equiv 0 \) and \( g = f \) in (3), we obtain a useful criteria for the Lipschitz stability.

Corollary 1. Suppose that for \( (n, x) \in N_\omega \times D \),

\[
|f(n, x)| \leq \sum_{i=1}^p \lambda_i(n) \omega_i(|x|),
\]

where \( \omega_i (1 \leq i \leq p) \) satisfy \( (1) \), and \( \lambda_i (1 \leq i \leq p) \) are non-negatives and \( \lambda_i \in l_1(N_\omega) \). Further, assume that for \( 0 < |x_0| < \delta \) (respectively, every \( 0 \neq x_0 \in D \)),

\[
q_i(\delta) =: \sup \left\{ \frac{r_i(|x_0|)}{|x_0|} \Bigg| 0 < |x_0| < \delta \right\}
\]

is finite and verifies

\[
K_i(\delta) = q_i(\delta) \cdot \sum_{j=0}^{\infty} \lambda_i(j) < \int_{\psi_i(c)}^{\infty} \frac{ds}{\omega_i(s)} \quad (1 \leq i \leq p), \quad (19)
\]

where \( \varphi_i \) are the same as in (17), Theorem 1.
Then, for all \( n_0 \geq a \) and \(|x_0|\) small enough, any solution \( x(n) = x(n, n_0, x_0) \) of system
\[
x(n + 1) = f(n, x(n))
\]
is Lipschitz stable, that is, there exists a constant \( c \geq 1 \) such that
\[
|x(n, n_0, x_0)| \leq c|x_0|, \quad n \geq n_0,
\]
for \(|x_0|\) small enough (respectively globally Lipschitz stable if \(|x_0| < \infty\)).

In fact, it suffices to take \( h = 1 \) in Theorem 1.


**Corollary 2.** Assume that (1) is exponentially asymptotically stable in variation and that (1) holds, where the functions \( \omega_i \) (1 \( \leq i \leq p \)) satisfy (S) such that
\[
r_i(\alpha u) \leq \alpha r_i(u) \leq Mu \quad (1 \leq i \leq p)
\]
for \( 0 < \alpha < 1, 0 < u < \delta \), where \( M > 0 \) is a constant; and the functions \( \lambda_i \), \( i = 1, 2, \ldots, p \) are nonnegatives and \( \lambda_i \in l_1(N_0) \). Further, suppose that for some \( \delta > 0 \), \( K_i(\delta) \) (defined in (19)) satisfies
\[
cK_i(\delta) < \int_{\varphi_i(\delta)}^{\infty} \frac{ds}{\omega_i(s)}, \quad i = 1, 2, \ldots, p,
\]
where \( \varphi_i \) is show in (17) of Theorem 1, and \( c \) is the constant in (M ) for \( h(n) = e^{-\alpha n} \).

Then, the perturbed system (3) is exponentially asymptotically stable.

The stabilities considered in Corollaries 1 and 2 are Lipschitz stabilities, that is, for which (2) holds.

In the next corollary we shall study \( h \)-stability and Lipschitz stability simultaneously, assuming that the perturbations are “polynomials.”

**Corollary 3.** Assume that (1) is uniformly \( h \)-stable in variation, that is, \( h \)-stable in variation such that \( h(n)h(n_0)^{-1} \leq M \) for \( n \geq n_0 \geq a \) and \( M \geq 1 \) a constant. Furthermore, the perturbation \( g(n, y) \) satisfies on \( N_0 \times D \) condition (4) for \( \omega_i(u) = u^{\gamma_i}, \gamma_i \geq 1, i = 1, 2, \ldots, p \), where \( \lambda_i(n)/h(n + 1) \in l_1(N_0) \). Then, the statements of Theorem 1 remain valid.

Proof. In fact, for \( \omega_i(u) = u^{\gamma_i}, \omega_{i+1}/\omega_i \) is non-decreasing, if and only if, \( \gamma_i \leq \gamma_{i+1} \) and \( r_i = \omega_i \). Thus, if \( \gamma_i \geq 1 \), (16) follows provided that
\[
K_i(\delta) = (\delta M)^{\gamma_i-1} < c^{-1} \int_{\varphi_i(\delta)}^{\infty} s^{-\gamma_i} ds,
\]

(•)
where

\[ \alpha_i = \sum_{j=n_0}^{\infty} \lambda_i(j) \quad (1 \leq i \leq p). \]

The existence of such a \( \delta \) will follow from the fact that for \( \gamma_i > 1 \), \( K_i(0^+) = 0 \).

Since

\[
\int_{\tilde{\varphi}_{i-1}(c)}^{\infty} s^{\gamma_i} ds = \begin{cases} 
\infty, & \text{if } \gamma_i = 1 \\
 c^{-1}(\gamma_i - 1)^{-1}(\tilde{\varphi}_{i-1}(c))^{1-\gamma_i}, & \text{if } \gamma_i > 1,
\end{cases}
\]

we must study only the case, \( \gamma_i > 1 \). So, let \( \delta_1 > 0 \) satisfying (*) for \( i = 1 \):

\[ K_1(\delta_1) = (\delta_1 M)^{\gamma_1-1} \alpha_1 < c^{-1}/(\gamma_1 - 1), \quad \alpha_1 = \sum_{j=n_0}^{\infty} \lambda_i(j). \]

Moreover, also there exists \( \delta_2 \leq \delta_1 \), satisfying (*) for \( i = 2 \),

\[
K_2(\delta_2) = (\delta_2 M)^{\gamma_2-1} \alpha_2 < c^{-1}(\tilde{\varphi}_1(c))^{1-\gamma_2}/(\gamma_2 - 1) \\
= c^{-1}(\gamma_2 - 1)^{-1}(W_1^{-1}[W_1(c) + cK_1(\delta_1)])^{1-\gamma_2},
\]

because \( K_2(0^+) = 0 \). Thus, we find \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_p \) such that

\[ K_i(\delta_i) < c^{-1}(\gamma_i - 1)^{-1}(\tilde{\varphi}_{i-1}(c))^{1-\gamma_i}, \]

where

\[ \tilde{\varphi}_i = \tilde{\psi}_i \circ \tilde{\psi}_1, \quad \tilde{\psi}_i(u) = W_i^{-1}[W_i(u) + cK_i(\delta_i)]. \]

Since \( K_i \) \( (1 \leq i \leq p) \) are non-decreasing functions, for \( \delta = \delta_p \) it verifies

\[
K_p(\delta) \leq K_i(\delta_i) < c^{-1}(\gamma_i - 1)^{-1}(\tilde{\varphi}_{i-1}(c))^{1-\gamma_i} \\
\leq c^{-1}(\gamma_i - 1)^{-1}(\varphi_{i-1}(c))^{1-\gamma_i},
\]

because \( \varphi_i(c) \leq \tilde{\varphi}_i(c) \). Thus, \( \delta = \delta_p \) satisfies (*).

Remark 5. The method proposed in Corollary 3 to compute \( \delta \), which satisfies (16) is not only exclusive of \( \omega_i(u) = u^{\gamma_i}, \gamma_i \geq 1 \), it is rather proper of the situation: the \( K_i (1 \leq i \leq p) \) are non-decreasing functions such that \( K_i(0^+) = 0 \). For \( \omega_i(u) = u^{\gamma_i}, \gamma_i < 1 \) this last assertion does not satisfy the conclusion of Theorem 1.
The (asymptotic) stability in Theorem 1 need not be uniform. However, we have

**Corollary 4.** Assume that (1) is Lipschitz stable in variation and that the perturbation \( g \) verifies (4). In addition, suppose that \( \lambda_i, i = 1, 2, \ldots, p, \) are non-negative and \( \lambda_i \in l(N) \). Then, the perturbed system (3) is uniformly stable.

**Proof.** System (1) is Lipschitz stable in variation if and only if (1) is \( \h \)-stable with \( \h = 1 \), that is, if and only if (M) becomes

\[
|\Phi(n, n_0, x_0)| \leq c, \quad n \geq n_0 \geq a
\]

for \( x_0 \in D \). Thus, Theorem A is applied on (18) with \( \h = 1 \), obtaining

\[
|y(n, n_0, y_0)| \leq W_p^{-1}\left[W_p\left(\varphi_p^{-1}(c|y_0|)\right) + c \cdot \sum_{i=n_0}^{n-1} \lambda_p(i) \right],
\]

where \( \varphi \) is given by (11) with \( c\lambda \) instead of \( \lambda \) and \( m = \infty \). Hence, proceeding as in the proof of Theorem 1, we obtain that

\[
|y(n, n_0, y_0)| \leq \varphi_p(c|y_0|)
\]

is valid for \( |y_0| \) small enough, for all \( n \geq n_0 \). Thus, using (14), the statement of the corollary follows from the last inequality.

If (13) fails, then the stability cannot be endured. However, we can obtain:

**Corollary 5.** If in Theorem 1 condition (12) holds and

\[
\left\{y \in \mathbb{R}^m : |y| \leq \delta \cdot \varphi_p(c) h(n) h(n_0)^{-1}\right\} \subset D \quad \text{for } n \geq n_0 \geq a,
\]

then all solutions of (3) are bounded on \( N_a \). Moreover, they tend to zero if \( h \) does so as \( n \to \infty \).

In fact, condition (12) implies that the estimate in Theorem 1 is valid for any \( n_0 \geq a \), \( y_0 \) and \( n \in N_a \). Then, the statement of Corollary 5 follows at once.

**Remark 6.** If in Corollary 5 conditions (12) and (13) hold simultaneously, then the global (asymptotic) \( \h \)-stability in variation of (1) implies the global (asymptotic) stability of the perturbed system (3).

This fact extends several classic results about stability for finite difference systems, but it is not valid if (13) does not hold.
4. ASYMPTOTIC FORMULAE

Our objective is to obtain asymptotic formulae for systems resulting from the perturbation of an \( h \)-system, thus establishing new results about asymptotic behavior for weakly stable difference systems.

**Theorem 2.** Assume that conditions (U), (M), (P), (S), and (R) hold.

(i) If \( D = \mathbb{R}^m \) and the functions \( \omega_i \) (defined in (1)) satisfy (12) for \( i = 1, 2, \ldots, p \), then for all \( n_0 \in \mathbb{N} \) and \( y_0 \in \mathbb{R}^m \) the solution \( y(n) = y(n, n_0, y_0) \) of system (3) satisfies an estimate of the form \(|y(n, n_0, y_0)| \leq Kh(n)h(n_0)^{-1}\), for \( n \geq n_0 \), where \( K = K(n_0, y_0) \) is a positive constant. Moreover,

\[
y(n, n_0, y_0) = x(n, n_0, y_0) + h(n) \cdot \tilde{o}(1) \quad \text{as } n \to \infty, \tag{21}
\]

where \( x(n, n_0, y_0) \) is the solution of system (1) with \( x(n, n_0, y_0) = y_0 \), and \( \tilde{o}(1) \) is a convergent function as \( n \to \infty \). If \( h(n)^{-1} \cdot \int_0^1 \Phi(n, l + 1, u(y(l), \tau)) \, d\tau \to 0 \) as \( n \to \infty \), for \( n \geq l \geq n_0 \), then

\[
y(n) = x(n) + h(n) \cdot o(1) \quad \text{as } n \to \infty. \tag{22}
\]

(ii) If \( D \) is a proper subset of \( \mathbb{R}^m \), and the functions \( \omega_i \) satisfy (13) for \( i = 1, 2, \ldots, p \), then the preceding statements are valid for \(|y_0| \leq \delta^* \) small enough, namely

\[
|y(n, n_0, y_0)| \leq K(n_0, y_0)h(n)h(n_0)^{-1},
\]

where \( K(n_0, y_0) \to 0 \) if \( |y_0| \to 0 \).

**Proof.** Let \( y(n) = y(n, n_0, y_0) \) be a solution of the perturbed system (3), for \( n \geq n_0 \geq a \). By the discrete Alekseev's formula (5), the solution \( y(n) \) satisfies Eq. (5). Thus, using (U), (M), and (P) it follows that

\[
|y(n)| \leq c|y_0|h(n)h(n_0)^{-1} + \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} ch(n)h(j + 1)^{-1} \lambda_i(j) \omega_i(|y(j)|).
\]

Thus,

\[
\frac{|y(n)|}{h(n)h(n_0)^{-1}} = c|y_0| + \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{c \lambda_i(j) \omega_i(|y(j)|)}{h(j + 1)h(n_0)^{-1}}.
\]
By (5) it follows that
\[
\frac{|y(n)|}{\beta(n,n_0)} \leq c|y_0| + \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{c\lambda_i(j)r_i(\beta(j,n_0))}{\beta(j+1,n_0)} \omega_i \left( \frac{|y(j)|}{\beta(j,n_0)} \right),
\]
where \(\beta(n,n_0) = h(n)h(n_0)^{-1}\), for \(n \geq n_0\).

Now, we apply Theorem A to \(\nu(n) = |y(n)|/\beta(n,n_0)\) thus establishing that
\[
|y(n)| \leq h(n)h(n_0)^{-1}W_p^{-1}\left[ W_p(\varphi_{p+1}(c|y_0|) + \alpha_p(n)) \right], \quad (23)
\]
where
\[
\alpha_i(n_0) = c \cdot \sum_{j=n_0}^{\infty} \frac{\lambda_i(j)r_i(\beta(j+1)n_0^{-1})}{h(j+1)h(n_0)^{-1}}, \quad i = 1, 2, \ldots, p.
\]

Thus, the estimation (23) is valid for every \(n \geq n_0 \geq a\) and the function in the right member is bounded by \(\varphi_p(c|y_0|)\).

Hence,
\[
|y(n)| \leq h(n)h(n_0)^{-1} \varphi_p(c|y_0|), \quad n \geq n_0 \geq a.
\]

It follows that for \(y_0\) small enough, there is a positive constant \(K = K(n_0,y_0)\) such that
\[
|y(n,n_0,y_0)| \leq Kh(n)h(n_0)^{-1}, \quad n \geq n_0.
\]

Now, for these solutions \(y(n)\) of system (3), we have
\[
\left| \sum_{j=n_0}^{n-1} \int_0^1 \Phi(n,j+1,u(y(j),\tau)) \cdot g(j,y(j)) \, d\tau \right|
\leq \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} c \cdot h(n)h(j+1)^{-1} \cdot \lambda_i(j) \omega_i(|y(j)|)
\leq c \cdot \beta(n,n_0) \cdot \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{c\lambda_i(j)r_i(\beta(j,n_0))}{\beta(j+1,n_0)} \omega_i \left( \frac{|y(j)|}{\beta(j,n_0)} \right)
\leq c \cdot L \cdot \beta(n,n_0) \sum_{i=1}^{p} \sum_{j=n_0}^{n-1} \frac{\lambda_i(j)r_i(\beta(j,n_0))}{\beta(j+1,n_0)} ,
\]
where \(L = \sum_{i=1}^{p} \omega_i(K)\).
Then, for every solution \( y(n) \) of system (3) the solution \( x(n) \) of system (1), given by

\[
x(n) = y(n) - \sum_{j=n_0}^{n-1} \int_0^1 \Phi(n, j + 1, u(y(j), \tau)) \cdot g(j, y(j)) \, d\tau,
\]

has the property

\[
y(n) = x(n) + h(n) \cdot \tilde{\sigma}(1) \quad \text{as } n \to \infty.
\]

If (12) holds and \( h(n)^{-1} \cdot \int_0^1 \Phi(n, l + 1, u(y(l), \tau)) \, d\tau \to 0 \), as \( n \to \infty \), then for \( n^* \leq n \)

\[
h(n)^{-1} \sum_{j=n_0}^{n^*-1} \int_0^1 \Phi(n, j + 1, u(y(j), \tau)) g(j, y(j)) \, d\tau
\]

\[+ h(n)^{-1} \sum_{j=n^*}^{n-1} \int_0^1 \Phi(n, j + 1, u(y(j), \tau)) g(j, y(j)) \, d\tau
\]

tends to zero as \( n, n^* \to \infty \). Thus, (22) follows.

The proof of (ii) is similar. \( \blacksquare \)

Remark 7. If \( \alpha(n_0) \to 0 \) as \( n_0 \to \infty \), then \( K = K(n_0, y_0) \to c|y_0| \) as \( n_0 \to \infty \).

Remark 8. We remark that under the conditions of Theorem 2, the error given by the asymptotic formula (21) is always dominated by \( h \). However, if the function \( h \) is not a good majorant then all the information can be added to the error. Moreover, a difficulty like this can be solved assuming that the function \( h \) satisfies the condition \( \lim_{n \to \infty} h(n) \) exists, and the proof of Theorem 2 remains valid without modifications.

The next corollary of Theorem 2 considers the particular case when \( h = 1 \).

Corollary 6. Assume that:

(a) Condition (U) holds.

(b) \( |\Phi(n, n_0, x_0)| \leq c \) for \( a \leq n_0 \leq n \) and \( x_0 \in D \).

(c) The following inequality is valid, \( |g(n, y)| \leq \sum_{l=n_0}^{\infty} \lambda_i(y) |\omega_i(y)| \) for \( n \in \mathbb{N} \) and \( y \in D \), where the functions \( \lambda_i \) satisfy

\[
\sum_{l=n_0}^{\infty} \lambda_i(l) < \infty
\]

for \( i = 1, 2, \ldots, p \) and \( \omega_i (i = 1, 2, \ldots, p) \) satisfy condition (1).
Then

(i) If \( D = \mathbb{R}^m \) and the functions \( \omega_i \) satisfy (12) for \( i = 1, 2, \ldots, p \), then for all \( n_0 \in \mathbb{N} \) and \( y_0 \in \mathbb{R}^m \), the solution \( y(n) = y(n, n_0, y_0) \) for (3) satisfies an estimate of the form \( |y(n, n_0, y_0)| \leq K, n \geq n_0 \), where \( K > 0 \) is a constant independent of \( n_0 \). Moreover, each solution satisfies the relation

\[
y(n, n_0, y_0) = x(n, n_0, y_0) + o(1) \quad \text{as} \quad n \to \infty.
\]

In addition, if \( \int_0^1 \Phi(n, l + 1, u(y(l), \tau)) \, d\tau \to 0 \) as \( n \to \infty \) for \( n \geq l \geq n_0 \), then

\[
y(n) = x(n) + o(1) \quad \text{as} \quad n \to \infty.
\]

(ii) If \( D \) is a proper subset of \( \mathbb{R}^m \) and the functions \( \omega_i \) satisfy (13) for \( i = 1, 2, \ldots, p \), then the preceding statements are valid for \( |y_0| \leq \delta^* \) small enough, namely \( |y(n, n_0, y_0)| \leq K(y_0) \), where \( K(y_0) \to 0 \) if \( |y_0| \to 0 \).

5. EXAMPLES AND APPLICATIONS

Example 1. Examples of h-systems are those systems which are exponentially stable, Lipschitz stable, uniformly stable, etc.

Example 2. Consider the Emden–Fowler difference equation

\[
\Delta^2 y(n) = p(n) y^{\gamma_1}(n) + q(n) y^{\gamma_2}(n),
\]

(24)

where \( \{p(n)\}^\gamma_1 \) and \( \{q(n)\}^\gamma_2 \) are sequences of real numbers; \( \gamma_1, \gamma_2 (\neq 0, 1) \) are real numbers and \( \Delta \) is the forward difference operator with unit spacing; i.e., \( \Delta u(n) = u(n - 1) - u(n) \), and \( \Delta^2 u(n) = \Delta(\Delta u(n)) \).

If we define \( u_i(n) = y(n + i - 1) \), \( 1 \leq i \leq 2 \), then Eq. (24) can be written as

\[
u(n + 1) = Au(n) + g(n, u(n)),
\]

(25)

where

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad g(n, u(n)) = p(n) u_1(n) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + q(n) u_2(n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

and

\[
u(n) = (u_1(n), u_2(n)) = (y(n), y(n + 1)).
\]

A fundamental matrix solution of equation

\[
u(n + 1) = A\nu(n),
\]

(26)
is given by

$$\Phi(n, 1) = \begin{bmatrix} n & 1 \\ n + 1 & 1 \end{bmatrix}.$$  

Moreover, we have

$$|g(n, u(n))| \leq |p(n)| |u_1(n)|^{\gamma_1} + |q(n)| |u_2(n)|^{\gamma_2}$$

$$\leq \lambda_1(n) \omega_1(|u(n)|) + \lambda_2(n) \omega_2(|u(n)|),$$

where \(\lambda_1(n) = |p(n)|\) and \(\lambda_2(n) = |q(n)|\), \(\omega_1(\nu) = \nu^{\gamma_1}\) and \(\omega_2(\nu) = \nu^{\gamma_2}\).

On the other hand, Eq. (26) is an h-system, since

$$|\Phi(n, 1)| \leq c h(n) h(1)^{-1},$$

where \(h(n) = 2n + 3\) and \(c \geq 5\), for \(n \geq n_0 = 1\).

Suppose that

$$\sum_{l=1}^{\infty} \frac{(2l + 3)^{\gamma_i}}{5^{\gamma_i} \cdot (2l + 5)} \cdot \lambda_i(l) < \infty \quad \text{for } i = 1, 2.$$

Under the above assumptions by Theorem 2 we have:

(A) If \(D = \mathbb{R}^2\) and \(\gamma_i \leq 1\; i = 1, 2\), then for all \(n_0 \geq 1\) and \(z_0 \in \mathbb{R}^2\),

the solution \(u(n) = u(n, n_0, z_0)\) of system (25) satisfies

$$u(n, n_0, z_0) = \Phi(n, n_0) z_0 + h(n) \cdot \delta(1) \quad \text{as } n \to \infty.$$  

Thus, it follows that

$$y(n) = nz_0^1 + z_0^2 + n \cdot \delta(1) \quad \text{as } n \to \infty, \quad (27)$$

$$z_0 = (z_0^1, z_0^2).$$

(B) If \(D\) is a proper subset of \(\mathbb{R}^2\) and \(\gamma_i > 1\; (1 = 1, 2)\), then the asymptotic formula (27) is valid for \(|z_0|\) small enough.

Now, we will show two applications of Theorem 2 to the linear case,

$$y(n + 1) = (A(n) + B(n)) y(n), \quad (28)$$

where \(A(n)\) and \(B(n)\) are \(m \times m\) discrete matrices.
Application 1. Assume that
\[ x(n) = A(n)x(n), \quad n \geq n_0 \]  \hspace{1cm} (29)
is an h-system. Then for every fundamental matrix \( \Psi \) of system (28), with
\[ \sum_{n=n_0}^{\infty} (h(n)/h(n+1)) \cdot |B(n)| < \infty, \] there exists a constant and invertible matrix \( C \) such that
\[ \Psi(n) = \Phi(n,n_0)C + h(n) \cdot o(1) \quad \text{as} \quad n \to \infty, \]
where \( \Phi(n,n_0) = \prod_{n=n_0}^{n-1} A(n) \) is the fundamental matrix of system (29).

Application 2. Assume that for the fundamental matrix \( \Phi(n,n_0) \) of (29) we have
\[ \sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)} \cdot |\Phi^{-1}(n+1,n_0)B(n)\Phi(n,n_0)| < \infty. \]

Then for each fundamental matrix \( \Psi \) of (28) there is an invertible and constant matrix \( C \) such that
\[ \Psi(n) = \Phi(n,n_0)[C + o(1)] \quad \text{as} \quad n \to \infty. \]

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REFERENCES


