On some self-dual codes and unimodular lattices in
dimension 48

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Received 18 March 2004; received in revised form 2 June 2004; accepted 3 June 2004
Available online 21 January 2005

Abstract

In this paper, binary extremal self-dual codes of length 48 and extremal unimodular lattices in
dimension 48 are studied through their shadows and neighbors. We relate an extremal singly even
self-dual [48, 24, 10] code whose shadow has minimum weight 4 to an extremal doubly even self-
dual [48, 24, 12] code. It is also shown that an extremal odd unimodular lattice in dimension 48
whose shadow has minimum norm 2 relates to an extremal even unimodular lattice. Extremal singly
even self-dual [48, 24, 10] codes with shadows of minimum weight 8 and extremal odd unimodular
lattice in dimension 48 with shadows of minimum norm 4 are investigated.
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MSC: 94B05; 11H71

1. Introduction

Self-dual codes and unimodular lattices are studied from several viewpoints (see [7]
for an extensive bibliography). Also many relationships between self-dual codes and
unimodular lattices are known and there are similar situations. In this paper, binary self-dual codes of length 48 and unimodular lattices in dimension 48 are studied through their shadows and neighbors. These are powerful tools in the study of self-dual codes and unimodular lattices.

Shadows for binary self-dual codes were introduced by Conway and Sloane [5], in order to derive new upper bounds for the minimum weight of singly even self-dual codes, and to provide restrictions on the weight enumerators of extremal singly even self-dual codes. There are two possible weight enumerators for extremal singly even self-dual [48, 24, 10] codes ($W_{48,1}$ and $W_{48,2}$ given in Section 2) and these weight enumerators are characterized by the minimum weights in shadows, namely $W_{48,1}$ and $W_{48,2}$ correspond to shadows of minimum weights 4 and 8, respectively. Shadows for odd unimodular lattices appeared in [6] and also in [7, p. 440] in order to derive new upper bounds for the minimum norm of odd unimodular lattices, and to provide restrictions on the theta series of extremal odd unimodular lattices. Similar to self-dual codes, there are two possible theta series of extremal odd unimodular lattices in dimensions 48 ($\theta_{L_{48},1}$ and $\theta_{L_{48},2}$ given in Section 2) and the two theta series are characterized by the minimum norms in shadows, namely $\theta_{L_{48},1}$ and $\theta_{L_{48},2}$ correspond to shadows of minimum norms 2 and 4, respectively.

It was shown in [3] that an extremal doubly even self-dual [48, 24, 12] code has an extremal singly even self-dual [48, 24, 10] code whose shadow has minimum weight 4 as a neighbor. In Section 3, we give the converse assertion of this, that is, we show that one of the two doubly even self-dual neighbors of an extremal singly even self-dual [48, 24, 10] code whose shadow has minimum weight 4 is an extremal doubly even self-dual [48, 24, 12] code. From the uniqueness of an extremal doubly even self-dual [48, 24, 12] code established by Houghton, Lam, Thiel and Parker [11], it follows that there are exactly ten inequivalent extremal singly even self-dual [48, 24, 10] codes whose shadow has minimum weight 4. For the other case where shadows have minimum weight 8, we demonstrate that there are exactly 64 inequivalent extremal singly even self-dual [48, 24, 10] neighbors of the extended quadratic residue code $QR_{48}$ of length 48. However, these are not all the extremal singly even self-dual [48, 24, 10] codes whose shadow has minimum weight 8. In fact, we construct an example of an extremal singly even self-dual [48, 24, 10] code, neither of whose doubly even self-dual neighbors is an extremal doubly even self-dual [48, 24, 12] code.

In Section 4, we show that a similar situation holds for unimodular lattices. It is shown that an extremal even unimodular lattice has an extremal odd unimodular neighbor whose shadow has minimum norm 2, and conversely, every extremal odd unimodular lattice in dimension 48 whose shadow has minimum norm 2 has an extremal even unimodular neighbor. Unlike the case of codes, the classification of extremal odd unimodular lattices whose shadows have minimum norm 2 is not feasible at present, since extremal even unimodular lattices in dimension 48 have not been classified yet. Also, even for the known three extremal even unimodular lattices $P_{48p}$, $P_{48q}$ and $P_{48n}$ (see [7, p. xli]), the classification of extremal odd unimodular neighbors appears to be a considerably difficult problem. For the other case where shadows have minimum norm 4, we construct an extremal odd unimodular lattice as a neighbor of $P_{48q}$. We also show that the odd unimodular neighbors of extremal even unimodular lattices do not exhaust all extremal odd
unimodular lattices, by constructing an extremal odd unimodular lattice whose shadow has minimum norm 4, neither of whose even unimodular neighbors is extremal.

2. Definitions and basic results

Let $C$ be a binary self-dual code, that is, $C = C^\perp$ where $C^\perp$ is the dual code of $C$. A self-dual code $C$ is called doubly even if all codewords have weight $\equiv 0 \pmod{4}$ and singly even if some codeword has weight $\equiv 2 \pmod{4}$. The minimum weight $d$ of a self-dual code $C$ of length $n$ is bounded by $d \leq 4[n/24] + 4$ unless $n \equiv 22 \pmod{24}$ when $d \leq 4[n/24] + 6$ [13] and [18]. In addition, a self-dual [24k, 12k, 4k + 4] code is doubly even [18]. Hence the minimum weight $d$ of a singly even self-dual code of length $n$ is bounded by $d \leq 4[n/24] + 2$ if $n \equiv 0 \pmod{24}$. We say that a self-dual code meeting the upper bound is extremal.

Let $C$ be a singly even self-dual code and let $C_0$ denote the subcode of codewords having weight $\equiv 0 \pmod{4}$. Then $C_0$ is a subcode of codimension 1. The shadow $S$ of $C$ is defined to be $C_0^\perp \setminus C$. There are cosets $C_1, C_2, C_3$ of $C_0$ such that $C_0^\perp = C_0 \cup C_1 \cup C_2 \cup C_3$ where $C = C_0 \cup C_2$ and $S = C_1 \cup C_3$. Recall that two self-dual codes $C$ and $C'$ of length $n$ are said to be neighbors if $\dim C \cap C' = n/2 - 1$. If $C$ is a singly even self-dual code of length divisible by eight then $C$ has two doubly even self-dual neighbors, namely, $C_0 \cup C_1$ and $C_0 \cup C_3$.

The largest minimum weight of singly even self-dual codes of length 48 is 10 and there are two possible weight enumerators $W_{48,i}$ (resp. $S_{48,i}$) of these codes (resp. their shadows) as follows [5]:

$$
\begin{align*}
W_{48,1} &= 1 + 704y^{10} + 8976y^{12} + 56896y^{14} + \cdots, \\
S_{48,1} &= y^4 + 44y^8 + 17021y^{12} + \cdots, \\
W_{48,2} &= 1 + 768y^{10} + 8592y^{12} + 57600y^{14} + \cdots, \\
S_{48,2} &= 54y^8 + 16976y^{12} + \cdots.
\end{align*}
$$

Let $L$ be a unimodular lattice, that is, $L$ is a lattice with $L = L^*$ where $L^*$ is the dual lattice under the standard inner product $(x, y)$. A unimodular lattice $L$ is called even if all vectors have even norms and odd if some vector has an odd norm. Rains and Sloane [19] showed that the minimum norm $\min(L)$ of an $n$-dimensional unimodular lattice $L$ is bounded by $\min(L) \leq 2[n/24] + 2$ unless $n = 23$ when $\min(L) \leq 3$. Recently Gaulter [8] showed that any $24k$-dimensional unimodular lattice meeting the upper bound has to be even, which was conjectured by Rains and Sloane. Hence the minimum norm of an $n$-dimensional odd unimodular lattice $L$ is bounded by $\min(L) \leq 2[n/24] + 1$ if $n \equiv 0 \pmod{24}$. We say that a unimodular lattice meeting the upper bound is extremal.

Let $L$ be an odd unimodular lattice and let $L_0$ denote its sublattice of vectors of even norms. Then $L_0$ is a sublattice of $L$ of index 2 [6]. The shadow $S$ of $L$ is defined to be $L^*_0 \setminus L$. There are cosets $L_1, L_2, L_3$ of $L_0$ such that $L^*_0 = L_0 \cup L_1 \cup L_2 \cup L_3$ where $L = L_0 \cup L_2$ and $S = L_1 \cup L_3$. Recall that lattices $L$ and $L'$ are neighbors if both lattices contain a sublattice of index 2 in common. If $L$ is an odd unimodular lattice in dimension $8k$ then there are two even unimodular neighbors of $L$, that is, $L_0 \cup L_1$ and $L_0 \cup L_3$. 


The largest minimum norm of odd unimodular lattices in dimension 48 is 5. By [6, Section 1], one can determine the possible theta series $\theta_{L_{48}}$ (resp. $\theta_{S_{48}}$) of extremal odd unimodular lattices (resp. their shadows) in dimension 48 as follows:

\[
\begin{align*}
\theta_{L_{48,1}} &= 1 + 385024q^5 + 26398208q^6 + \cdots, \\
\theta_{S_{48,1}} &= 2q^2 + 2256q^4 + 52318616q^6 + \cdots, \\
\theta_{L_{48,2}} &= 1 + 393216q^5 + 26201600q^6 + \cdots, \\
\theta_{S_{48,2}} &= 2400q^4 + 52313600q^6 + \cdots.
\end{align*}
\]

In Sections 3 and 4, we use the following lemma (see e.g. [15] and [17] for the proof).

**Lemma 2.1.** Let $\Lambda$ be an even unimodular lattice in dimension 48. For any $\alpha \in \mathbb{R}^{48}$, the harmonic theta series

\[
\vartheta_{\Lambda}(q) = \sum_{x \in \Lambda} ((\alpha, x)^2 - \frac{1}{48}(\alpha, \alpha)(x, x)) q^{(x, x)/2}
\]

is a cusp form of weight 26, and in particular

\[
\vartheta_{\Lambda}(q) = c(q - 48q^2 + \cdots)
\]

for some constant $c$.

Let $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$ denote the ring of integers modulo $m$. In Section 4, it is necessary to deal with Type II codes over $\mathbb{Z}_4$ and $\mathbb{Z}_6$. A code $C$ of length $n$ over $\mathbb{Z}_{2k}$ (or a $\mathbb{Z}_{2k}$-code of length $n$) is a $\mathbb{Z}_{2k}$-submodule of $\mathbb{Z}_{2k}^n$. Two codes are equivalent if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. The dual code $C^\perp$ of $C$ is defined as $C^\perp = \{ x \in \mathbb{Z}_{2k}^n \mid x \cdot y = 0 \text{ for all } y \in C \}$ where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. A code $C$ is self-dual if $C = C^\perp$. The Euclidean weight of a codeword $x$ is $\sum_{i=1}^n \min(x_i^2, (2k-x_i)^2)$. The minimum Euclidean weight $d_E(C)$ of $C$ is the smallest Euclidean weight among all nonzero codewords of $C$. A self-dual code is called Type II if it has the property that every Euclidean weight is divisible by $4k$, and is called Type I otherwise [2]. A Type II $\mathbb{Z}_4$-code of length $n$ and minimum Euclidean weight $8[n/24] + 8$ is extremal [1].

Let $C$ be a $\mathbb{Z}_k$-code and let $e_1, \ldots, e_n$ be an orthogonal basis of an $n$-dimensional Euclidean space satisfying $(e_i, e_j) = k\delta_{ij}$. Then we define the lattice $A_k(C)$ obtained from $C$ by Construction A as

\[
A_k(C) = \left\{ \frac{1}{k} \sum_{i=1}^n x_i e_i \mid x_i \in \mathbb{Z}, (x_i \text{ (mod } k)) \in C \right\}.
\]

In general, the set of vectors $f_1, \ldots, f_n$ in an $n$-dimensional lattice $L$ with $(f_i, f_j) = k\delta_{ij}$ is called a $k$-frame of $L$. $L$ has a $k$-frame if and only if $L$ is obtained by Construction A from some $\mathbb{Z}_k$-code. If $C$ is a Type II $\mathbb{Z}_{2k}$-code with minimum Euclidean weight $d_E$, then $A_{2k}(C)$ is an even unimodular lattice with minimum norm $\min(d_E/2k, 2k)$ [2].
3. Self-dual codes of length 48

3.1. Weight enumerator $W_{48,1}$

We relate extremal singly even self-dual codes with weight enumerator $W_{48,1}$ to extremal doubly even self-dual codes.

**Lemma 3.1.** Let $C$ be a doubly even self-dual $[48, 24]$ code with weight enumerator $W_C = \sum_{i=0}^{12} a_i y^i$. If $a_4 = 1$, then $a_8 \geq 44$ and $a_8 \equiv 2 \pmod{6}$.

**Proof.** Let $\Lambda \subset \mathbb{R}^{48}$ be the even unimodular lattice constructed from $C$ by Construction A. Then $\Lambda$ contains a 2-frame $\{e_i | 1 \leq i \leq 48\}$, $(e_i, e_j) = 2\delta_{ij}$. Let $\Lambda_k$ be the set of the vectors of norm $k$ in $\Lambda$. We have that $\Lambda_2 = D_4 \cup A^{44}_1$, where we may assume without loss of generality

$$D_4 = \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} \cup \left\{\pm \frac{1}{2} e_1 \pm \frac{1}{2} e_2 \pm \frac{1}{2} e_3 \pm \frac{1}{2} e_4\right\},$$

$$A^{44}_1 = \{\pm e_i | 5 \leq i \leq 48\}.$$

Let $C_8$ be the set of codewords of weight 8 in $C$. Then

$$A_4 = \{\pm e_i \pm e_j | 1 \leq i < j \leq 48\}$$

$$\cup \left\{\frac{1}{2} e_1 \pm \frac{1}{2} e_2 \pm \frac{1}{2} e_3 \pm \frac{1}{2} e_4 \pm e_i | 5 \leq i \leq 48\right\}$$

$$\cup \bigcup_{v \in C_8} \left\{\sum_{i \in \text{Supp}(v)} e_i e_i | e_i = \pm \frac{1}{2}\right\}$$

where $\text{Supp}(v)$ denotes the support of $v$. In particular, we have

$$|A_2| = 112,$$

$$|A_4| = 4 \cdot \binom{48}{2} + 32 \cdot 44 + 2^8 \cdot a_8. \quad (1)$$

In Lemma 2.1, we take $\alpha = e_j$ ($j = 5, \ldots, 48$). Then the constant $c$ is given by

$$c = \sum_{x \in A_2} \left((e_j, x)^2 - \frac{1}{48}(e_j, e_j)(x, x)\right)$$

$$= (e_j, e_j)^2 + (e_j, -e_j)^2 - \frac{1}{12} |A_2|$$

$$= -\frac{4}{3}.$$ 

Therefore the coefficient of $q^2$ in $\vartheta_A(q)$ is

$$\sum_{x \in A_4} \left((\alpha, x)^2 - \frac{1}{6}\right) = 64.$$
Note that, for \( j = 5, \ldots, 48 \), we have
\[
\sum_{x \in \Lambda_4} (e_j, x)^2 = 4 \cdot 47 \cdot 2^2 + 32 \cdot 2^2 + 2^8 a_8^{(j)},
\]
where
\[
a_8^{(j)} = \left| \{ v \in C_8 \mid v_j = 1 \} \right|.
\]
Thus
\[
2^4 \cdot 55 + 2^8 a_8^{(j)} = \frac{1}{6} |\Lambda_4| = 64.
\]
If we simplify this equality by using (1), we find
\[
6a_8^{(j)} = a_8 + 4.
\]
Hence
\[
\frac{44(a_8 + 4)}{6} = \sum_{j=5}^{48} a_8^{(j)} \leq \sum_{j=1}^{48} a_8^{(j)} = \sum_{v \in C_8} \left| \{ j \mid v_j = 1 \} \right| = 8a_8
\]
and we obtain \( a_8 \geq 44. \)

**Remark 3.2.** Let \( C \) be a doubly even self-dual \([48, 24, 4]\) code with \( a_4 = 1 \). From the above proof, we have the following fact. If the codeword \( v \) of weight 4 and each codeword of weight 8 in \( C \) are disjoint, then the codewords of weight 8 form a 1-design on 44 points by deleting the support of \( v \), and \( a_8 = 44 \). Conversely, if \( a_8 = 44 \) then the codeword of weight 4 and each codeword of weight 8 in \( C \) are disjoint.

It was shown in [3] that an extremal singly even self-dual \([48, 24, 10]\) code with weight enumerator \( W_{48,1} \) can be constructed from an extremal doubly even self-dual \([48, 24, 12]\) code as a neighbor. We prove its converse.

**Theorem 3.3.** Let \( C \) be an extremal singly even self-dual \([48, 24, 10]\) code with weight enumerator \( W_{48,1} \). Then either \( C_0 \cup C_1 \) or \( C_0 \cup C_3 \) is an extremal doubly even self-dual \([48, 24, 12]\) code.

**Proof.** We may assume that \( C_1 \) contains the unique vector of weight 4 in the shadow. In view of the shadow weight enumerator \( S_{48,1} \), there are at most 44 vectors of weight 8 in \( C_1 \). Then by **Lemma 3.1**, a doubly even self-dual neighbor \( C_0 \cup C_1 \) must contain all 44 vectors of weight 8 in the shadow. It follows that the other doubly even self-dual neighbor \( C_0 \cup C_3 \) has minimum weight 12. \( \square \)

**Remark 3.4.** Alternatively, one could prove **Theorem 3.3** by an argument similar to **Remark 4.3**. In fact, by considering the harmonic theta series of the lattice \( A_2(C_0 \cup C_3) \), one can show that \( C_0 \cup C_3 \) has no codeword of weight 8.

Let \( W_i \) be the weight enumerator of \( C_i (i = 1, 3) \). By [5, Theorem 5], one can easily see that \( W_1 - W_3 \) is of the form
\[
a_1y^4 + (38a_1 + a_2)y^8 + (429a_1 - 8a_2)y^{12} + (712a_1 + 28a_2)y^{16} + \cdots,
\]
for some integers $a_1$ and $a_2$. If $C_1$ has vectors of weights 4 and 8, then we have that $a_1 = 1$ and $a_2 = 6$. Thus

$$W_1 - W_3 = y^4 + 44y^8 + 381y^{12} + 880y^{16} + \cdots.$$ 

Therefore we have the following decomposition of $S_{48,1}$:

$$W_1 = y^4 + 44y^8 + 8701y^{12} + 268400y^{16} + \cdots$$

$$W_3 = 8320y^{12} + 267520y^{16} + \cdots.$$ 

### 3.2. Singly even self-dual neighbors of $QR_{48}$

Using the method in [16], we have found all extremal singly even self-dual [48, 24, 10] neighbors of $QR_{48}$. We have verified by MAGMA that all these codes are pairwise inequivalent. Hence we have the following:

**Proposition 3.5.** The extended quadratic residue code of length 48 has exactly 74 inequivalent extremal singly even self-dual [48, 24, 10] neighbors.

Since every singly even self-dual neighbor of the code $C = QR_{48}$ can be generated by $C \cap \langle v \rangle$ and $v$ for some $v \not\in C$, it is sufficient to give the vectors $v$ in order to present the

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<td>$C_{48,75}$</td>
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</tr>
</tbody>
</table>
74 codes given in Proposition 3.5 instead of listing generator matrices. In our search, the code $QR_{48}$ is defined as the extended cyclic code with generator polynomial

$$x^{23} + x^{19} + x^{18} + x^{14} + x^{13} + x^{12} + x^{10}$$
$$+ x^9 + x^7 + x^6 + x^5 + x^3 + x^2 + x + 1,$$

where a polynomial $a_0 + a_1 x + \cdots + a_{46} x^{46}$ is regarded as a codeword $(a_0, a_1, \ldots, a_{46})$, noting that the extended coordinate is chosen to be the 48th coordinate (see [12, pp. 190–191] for the form of a generator matrix of the cyclic code with a given generator polynomial).

In Table 1, we list the vectors $v$ written in octal using $0 = (000), 1 = (001), \ldots, 6 = (110)$ and $7 = (111)$. The ten codes $C_{48,i}$ ($i = 1, \ldots, 10$) have weight enumerator $W_{48,1}$ and the other codes have weight enumerator $W_{48,2}$. The orders of the automorphism groups are calculated by MAGMA and the results are listed in Table 2.

Let $w$ be a vector $\in \mathbb{F}_2^{48}$ of weight 4. Then

$$N = \{ u + w \mid u \in QR_{48} \setminus \langle w \rangle^\perp \} \cup (QR_{48} \cap \langle w \rangle^\perp)$$

is a singly even self-dual [48, 24, 10] neighbor of $QR_{48}$ whose shadow has minimum weight 4 [3]. Hence each of the codes $C_{48,i}$ ($i = 1, \ldots, 10$) is also defined by a vector $w$ of weight 4 in the shadow. The ten codes have the following supports of $w$:

$$\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 6\},$$
$$\{1, 2, 3, 7\}, \{1, 2, 4, 7\}, \{1, 3, 4, 7\}, \{1, 3, 6, 7\}, \{1, 4, 6, 7\},$$

respectively.

Recently, Houghten, Lam, Thiel and Parker [11] announced that their computer search for extremal doubly even self-dual codes of length 48 had been completed.
Theorem 3.6 (Houghten et al. [11]). The extended quadratic residue code of length 48 is the unique extremal doubly even self-dual [48, 24, 12] code up to equivalence.

Combined with Theorem 3.3, we have the following:

Proposition 3.7. There are exactly ten inequivalent extremal singly even self-dual [48, 24, 10] codes with weight enumerator $W_{48,1}$.

3.3. Weight enumerator $W_{48,2}$

We now consider extremal singly even self-dual [48, 24, 10] codes with weight enumerator $W_{48,2}$. Some examples of such codes were found as neighbors of $QR_{48}$.

Let $N_{48}$ be the singly even self-dual code generated by $QR_{48} \cap \langle x, y \rangle \perp$ and $\langle x, y \rangle$ where

$$x = (00000100000101010000011110001000000000010000),$$

$$y = (00000000001111000101000001010000010000001100000010100110).$$

Then the code $N_{48}$ is an extremal singly even self-dual [48, 24, 10] code with weight enumerator $W_{48,2}$. The two doubly even self-dual neighbors $D_{48,1}$ and $D_{48,2}$ of $N_{48}$ are defined as the codes generated by $N_{48} \cap \langle v \rangle \perp$ and $v$ where

$$v = (0001000101001110000100000000000000000001000000000000),$$

$$v = (00000000001001000000000000000000000000000000001000000001001100).$$

respectively. The two neighbors $D_{48,1}$ and $D_{48,2}$ have the following weight enumerators

$$1 + 24y^8 + 17104y^{12} + 535767y^{16} + 3994032y^{20} + 7683360y^{24} + \cdots,$$

$$1 + 30y^8 + 17056y^{12} + 535935y^{16} + 3993696y^{20} + 7683780y^{24} + \cdots,$$

respectively. Therefore we have the following:

Proposition 3.8. There exists an extremal singly even self-dual [48, 24, 10] code with weight enumerator $W_{48,2}$, neither of whose doubly even self-dual neighbors is an extremal doubly even self-dual [48, 24, 12] code.

4. Unimodular lattices in dimension 48

4.1. Theta series $\theta_{L_{48,1}}$

We show that an extremal even unimodular lattice in dimension 48 has an extremal odd unimodular neighbor whose shadow has minimum norm 2, and conversely, every extremal odd unimodular lattice whose shadow has minimum norm 2 has an extremal even unimodular neighbor.

Proposition 4.1. Let $\Lambda$ be an extremal even unimodular lattice in dimension 48. Then $\Lambda$ has an extremal odd unimodular neighbor $\Gamma$ with theta series $\theta_{L_{48,1}}$.

Proof. Since $\Lambda$ is extremal, there exists a vector $x \in \Lambda$ with $(x,x) = 8$ by [7, p. 52]. Put $A_+ = \{ v \in \Lambda \mid (x,v) \equiv 0 \pmod{2} \}$. 

$$A_+ = \{ v \in \Lambda \mid (x,v) \equiv 0 \pmod{2} \}.$$
Note that there exists a vector \( y \in A \) such that \( (x, y) \) is odd. Indeed, otherwise we would have \( \frac{1}{2}x \in \Lambda^* = \Lambda \). \((\frac{1}{2}x, \frac{1}{2}x) = 2 < \min(\Lambda)\), a contradiction. Fix such a vector \( y \). We claim that the lattice \( \Gamma \) generated by \( \Lambda_x \) and \( \frac{1}{2}x + y \) is an odd unimodular lattice with \( \min(\Gamma) = 5 \).

Since \( \min \Lambda_x \geq 6 \), it suffices to show that \( (u, u) \geq 5 \) for all \( u \in \frac{1}{2}x + y + \Lambda_x \). Since \( (u, \frac{1}{2}x) \) is a non-zero half integer, we may assume without loss of generality \( (u, \frac{1}{2}x) \leq -\frac{1}{2} \). Then \((u + \frac{1}{2}x, u + \frac{1}{2}x) \leq (u, u) + 1 \). Since \( u + \frac{1}{2}x \) is a nonzero vector in \( \Lambda \), we obtain \( 6 \leq (u, u) + 1 \).

Finally note that \( \frac{1}{2}x \) is a vector of the shadow of \( \Gamma \). Hence \( \Gamma \) has theta series \( \theta_{L_{48,1}} \). \( \square \)

Now we prove the converse of the above result.

**Theorem 4.2.** Let \( \Gamma \) be an odd unimodular lattice in dimension 48. If \( \min(\Gamma) = 5 \) and the shadow of \( \Gamma \) has minimum norm 2, then \( \Gamma \) has an extremal even unimodular neighbor.

**Proof.** Let \( \alpha \) be a vector of norm 2 in the shadow \( S \) of \( \Gamma \). Then \( S \) contains 2256 vectors of norm 4. Let \( \Lambda \) be the lattice generated by the even sublattice \( \Gamma_x \) of \( \Gamma \) and \( \alpha \). Then \( \Lambda \) is an even unimodular lattice. Since \( \pm \alpha \) are the only vectors of norm 2 in \( S \), if we can show \( |A_4| = 2256 \) where \( A_i \) is the set of the vectors of norm \( i \) in \( \Lambda \), then the common even neighbor \( \Lambda' \) of \( \Gamma \) and \( \Lambda \) is extremal.

By Lemma 2.1, we have
\[
\sum_{\lambda \in A_4} \left( (\alpha, \lambda)^2 - \frac{1}{6} \right) = -48 \sum_{\lambda \in A_2} \left( (\alpha, \lambda)^2 - \frac{1}{12} \right) = -\frac{47.48}{6}.
\]
(2)

We claim \( (\alpha, \lambda) = 0 \) for all \( \lambda \in A_4 \). Indeed, since \( \alpha \notin \Gamma_x \) and \( \lambda \notin \Gamma_x \), we have \( \alpha + \lambda \in \Gamma_x \), hence
\[
6 = \min(\Gamma) \leq \min(\Gamma_x) \leq (\alpha + \lambda, \alpha + \lambda) = 2 + 4 + 2(\alpha, \lambda)
\]
hence \( (\alpha, \lambda) \geq 0 \). Replacing \( \lambda \) by \(-\lambda \), we obtain \( (\alpha, \lambda) \leq 0 \). Therefore \( (\alpha, \lambda) = 0 \). Now (2) gives \( |A_4| = 47.48 = 2256 \). \( \square \)

**Remark 4.3.** Alternatively, one could consider the harmonic theta series of \( \Lambda' \) with respect to \( \alpha \). Then
\[
\sum_{x \in \Lambda'_4} \left( (\alpha, x)^2 - \frac{1}{6} \right) = 0,
\]
but \( (\alpha, x) = \pm \frac{1}{2} \) for all \( x \in \Lambda'_4 \). This forces \( \Lambda'_4 = \emptyset \).

### 4.2. Theta series \( \theta_{L_{48,2}} \)

We now consider extremal odd unimodular lattices with theta series \( \theta_{L_{48,2}} \). First we construct such a lattice as a neighbor of \( P_{48q} \). Next we construct another such lattice, neither of whose even unimodular neighbors is extremal. A new extremal Type II \( \mathbb{Z}_4 \)-code of length 48 is also found.

Recently it was shown in [9] that \( P_{48p} \) and \( P_{48q} \) are constructed from some Type II \( \mathbb{Z}_6 \)-codes by Construction A. Let \( C_{48X}^{(6)} \) be the Type II \( \mathbb{Z}_6 \)-code with \( A_6(C_{48X}^{(6)}) = P_{48X} \) given...
in [9] for $X = p, q$. Note that any Type II (resp. Type I) $\mathbb{Z}_6$-code can be regarded as a pair $(B, T)$ of a binary doubly even (resp. singly even) self-dual code $B$ and a ternary self-dual code $T$. Suppose that $C_{48}^{(6)}$ is the pair of a binary doubly even self-dual code $B_{48}$ and a ternary self-dual code $T_{48}$. Note that $B_{48}$ and $B_{48}$ are equivalent to the doubly even self-dual code $d_{48}^+$ (see e.g. [20, Section 11] for a definition of $d_{48}^+$).

Let $B'_{48}$ be a singly even self-dual neighbor of $B_{48}$. Then $(B'_{48}, T_{48})$ is a Type I $\mathbb{Z}_6$-code. Since $(B_{48}, T_{48})$ and $(B'_{48}, T_{48})$ have a subcode of index 2 in common, $A_6((B'_{48}, T_{48}))$ is a neighbor of $A_6((B_{48}, T_{48}))$. Among the neighbors of $B_{48}$, we have found a singly even self-dual code $B'_{48}$ such that $A_6(C_{48}^{(6)})$ is an extremal odd unimodular lattice with kissing number 393216, where $C_{48}^{(6)} = (B'_{48}, T_{48})$.

**Proposition 4.4.** There exists an extremal odd unimodular lattice with theta series $\theta_{L_{48}}$, which is a neighbor of the extremal even unimodular lattice $P_{48}$.

A generator matrix of $C_{48}^{(6)}$ is given by $(I, M)$ where in order to save space, $M$ is given using the form $m_1, m_2, \ldots, m_{24}$ where $m_j$ is the $j$-th row:

$$
\begin{align*}
3512140002111111315555501, & \quad 4205151510400442420431, \quad 3513115115113331131330, \\
33345202213333135553323, & \quad 5152000221531551535323, \quad 35304052025513331335503, \\
311220022535355315101, & \quad 1352000543533553353303, \quad 5334040203555313511125, \\
244004022124442040031, & \quad 2400220400542420003042451, \quad 4204200242300042442451, \\
2044220222032244222211, & \quad 00002042004425042424253, \quad 20024202242240120402251, \\
42000204204204420442435, & \quad 4424402044244040342453, \quad 4404042420002044100411, \\
402404022400422404450235, & \quad 442444420200404242045451, \quad 244444042404242020511, \\
4324444420222204424222231, & \quad 0034222402202202233, \quad 500240042444240422240453.
\end{align*}
$$

There is no reason to believe that the lattice $A_6(C_{48}^{(6)})$ is a unique extremal odd unimodular lattice with theta series $\theta_{L_{48}}$, which is a neighbor of an extremal even unimodular lattice. However, we do not try to search for more, since this will not lead to a complete classification of extremal odd unimodular lattice with theta series $\theta_{L_{48}}$. Indeed, there are odd unimodular lattices with theta series $\theta_{L_{48}}$, which are not neighbors of an extremal even unimodular lattice. The rest of the paper is devoted to a construction of such lattices.

**Lemma 4.5.** Let $C$ be an extremal Type II $\mathbb{Z}_4$-code of length 48 with the property that $C$ contains the all-one-vector. Let $\Lambda = A_4(C)$ and $\{v_1, \cdots, v_{48}\}$ be the 4-frame of $\Lambda$. For $k = 3, 4, 5, 6$, define $A_k = \langle \Lambda \cap (v_k)^* \rangle$ where

$$
\begin{align*}
v_3 &= \frac{1}{8} \sum_{i=1}^{48} f_i = \frac{1}{4}(1, 1, \ldots, 1), \\
v_4 &= v_3 - \frac{1}{2} f_1 - \frac{1}{2} f_2 = \frac{1}{4}(-3, -3, 1, \ldots, 1), \\
v_5 &= v_4 + f_1 = v_3 + \frac{1}{2} f_1 - \frac{1}{2} f_2 = \frac{1}{4}(5, -3, 1, \ldots, 1), \\
v_6 &= v_3 - f_1 = \frac{1}{4}(-7, 1, \ldots, 1).
\end{align*}
$$
Then we have the following.

(1) The even neighbors of $A_3$ are $A$ and $A_6$.
(2) The even neighbors of $A_5$ are $A$ and $A_4$.
(3) min($A_3$) = 3 and min($A_4$) = 4.
(4) If min($A_6$) = 6, then min($A_5$) = 5.

**Proof.** Let $k = 3$ or $k = 5$. By the assumption, we have $2u_k \in A \setminus 2A$ and thus $[A : A \cap (u_k)^*] = 2$. Since $f_1 \notin (u_k)^*$, the dual lattice of $A \cap (u_k)^*$ is spanned by $v_k, f_1$ and $A \cap (u_k)^*$. Hence we have proved (1) and (2).

(3) Since $d_E(C) = 24$, we have min($A$) = 4 and the set of the vectors of norm 4 in $A$ is $\{ \pm f_1, \ldots, \pm f_{48} \}$. Hence we have min $A \cap (u_k)^* \geq d_E(C)/4 = 6$. Let $x \in A_k \setminus A \cap (u_k)^*$.
Then $x$ can be written as $\frac{1}{2}(a_1, \ldots, a_{48})$ where $a_i \in 2Z + 1$. Hence $(x, x) \geq (1/16) \times 48 = 3,$ and thus min($A_3$) = ($v_3, v_3$) = 3. Since $A_4$ is even, we have min($A_4$) = ($v_4, v_4$) = 4.

(4) Suppose that min($A_5$) < 5. Then min($A_5$) = 3 and we may assume $A_5$ contains a vector $v'' = \frac{1}{2}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{48})$ where $\varepsilon_i = \pm 1$ for each $i$. Set $v'' = v' - \frac{\varepsilon_1}{2}f_1 - \frac{\varepsilon_2}{2}f_2$. Then we have $(v', v'') = (v', v'') + 1 = 4$. Moreover we have

$$v'' - v_6 = (v', v_3) + \frac{1}{4}(5\varepsilon_1 - \varepsilon_2) - 6.$$  

Since $v'$ and $v_3$ are contained in $A_5$, $(v', v_3) = (v', v_3) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)$ is an integer and thus $(v'' - v_6, v_6)$ is also an integer. Since $v' - v_5 \in A$, we have

$$v'' - v_6 = (v' - v_5) - \frac{\varepsilon_1 - 3}{2}f_1 - \frac{\varepsilon_2 + 1}{2}f_2 \in A \cap (v_6)^*.$$  

Hence $v''$ is contained in $A_6$ and min $A_6 < 6$ as required. \square

**Remark 4.6.** Note that any Type II $\mathbb{Z}_q$-code contains a vector consisting of 1 or 3 [10]. Hence any extremal Type II code of length 48 satisfies the condition of the above lemma after taking suitable sign changes if necessary.

Two extremal ternary self-dual [48, 24, 15] codes are known, namely, the ternary extended quadratic residue code and the Pless symmetry code. We denote these codes by $C_{48q}^{(3)}$ and $C_{48p}^{(3)}$, respectively. By [7, pp. 148–150], the extremal even unimodular lattice $P_{48q}$ (resp. $P_{48p}$) is an even neighbor of $\Gamma_q = A_3(C_{48q}^{(3)})$ (resp. $\Gamma_p = A_3(C_{48p}^{(3)})$). We denote by $A_q$ (resp. $A_p$) the other even neighbor of $\Gamma_q$ (resp. $\Gamma_p$).

Now let $X = p$ or $q$. The odd unimodular lattice $\Gamma_X$ and its shadow have the following theta series:

$$\begin{align*}
1 + 96q^3 + 415104q^5 + 26398208q^6 + \cdots \\
96q^4 + 520390976q^6 + \cdots,
\end{align*}$$

respectively. Hence the lattice $AX$ has exactly 96 vectors of norm 4, since min($P_{48X}$) = 6. Such vectors of norm 4 can be written as $(1/2\sqrt{3})(\pm 1, \ldots, \pm 1)$ and are obtained from the 96 codewords of weight 48 of $C_{48X}^{(3)}$. These 96 codewords constitute the rows and their negatives of a Hadamard matrix of order 48 by [14]. Thus, the 96 vectors of norm 4 in $AX$ form a unique 4-frame in $AX$, and hence $AX$ can be written as $AX = A_4(C_{48X}^{(4)})$ for some
Proposition 4.8. Since \( \Lambda_X \) contains no vector of norm 4 except those of the 4-frame, we have \( d_E(C_{48\times}^{(4)}) = 24 \).

Let \( x \) be a vector of norm 3 in \( \Gamma_X \) and \( \{ \pm f_1, \ldots, \pm f_{48} \} \) be the 4-frame of \( \Lambda_X \). Since \( P_{48\times} \) and \( \Lambda_X \) are the even neighbors of \( \Gamma_X \), we have

\[
\Lambda_X \cap (x)^* = P_{48\times} \cap (x)^*.
\]

Since \( f_i \notin P_{48\times} \), the inner product \( (x, f_i) \) is not an integer for each \( i \). Hence \( x \) can be written as \( x = (1/4)(\pm 1, \ldots, \pm 1) \) with respect to the orthonormal basis \( \{ f_1/2, \ldots, f_{48}/2 \} \). We may assume \( x = (1/4)(1, \ldots, 1) \) by taking suitable sign changes if necessary. This means that the extremal Type II \( \mathbb{Z}_4 \)-code \( C_{48\times}^{(4)} \) satisfies the assumption of Lemma 4.5, and \( \Gamma_X \) and \( P_{48\times} \) can be regarded as \( A_3 \) and \( A_6 \) in Lemma 4.5. Define \( L_{48\times} \) to be the lattice \( \Lambda_X \) as in Lemma 4.5. From the extremality of \( P_{48\times} \) we obtain \( \min(\Lambda_3) = 5 \) by Lemma 4.5(4). This conclusion, together with Lemma 4.5(2) implies the following:

**Theorem 4.7.** There exist at least two extremal odd unimodular lattices with theta series \( \theta_{L_{48\times}} \), which are not neighbors of an extremal even unimodular lattice.

**Proof.** We only need to show that the two lattices \( L_{48q} \) and \( L_{48p} \) are non-isometric. The lattice \( A_4(C_{48q}^{(4)}) \) (resp. \( A_4(C_{48p}^{(4)}) \)) is the unique neighbor of \( L_{48q} \) (resp. \( L_{48p} \)) which contains exactly 96 vectors of norm 4. If \( L_{48q} \) and \( L_{48p} \) were isometric, then \( A_4(C_{48q}^{(4)}) \) and \( A_4(C_{48p}^{(4)}) \) would be isometric. This would imply that \( C_{48q}^{(4)} \) and \( C_{48p}^{(4)} \) are equivalent, since \( A_4(C_{48q}^{(4)}) \) and \( A_4(C_{48p}^{(4)}) \) have a unique 4-frame.

The binary code \( \overline{C} = \{ x \ (\mod 2) \mid x \in C \} \) is called a residual code of a \( \mathbb{Z}_4 \)-code \( C \). We have verified that \( C_{48p}^{(4)} \) is equivalent to the direct sum of two copies of the extended Golay code. The residual code of \( C_{48q}^{(4)} \) is the extended quadratic residue code. Hence \( C_{48q}^{(4)} \) and \( C_{48p}^{(4)} \) are inequivalent. Therefore \( L_{48q} \) and \( L_{48p} \) are non-isometric. \( \square \)

The above proof also shows that there are at least two extremal Type II \( \mathbb{Z}_4 \)-codes of length 48. The only previously known extremal Type II \( \mathbb{Z}_4 \)-code of length 48 is the extended quadratic residue \( \mathbb{Z}_4 \)-code \( QR_{48}^{(4)} \) (cf. [1]). Note that the residual code of \( QR_{48}^{(4)} \) is the extended quadratic residue code.

**Proposition 4.8.** \( C_{48p}^{(4)} \) is a new extremal Type II \( \mathbb{Z}_4 \)-code of length 48.

We give a generator matrix of \( C_{48p}^{(4)} \). In order to save space, we list the matrix \( M \) of a generator matrix \( (I, \ M) \) in standard form using the form \( m_1, m_2, \ldots, m_{24} \) where \( m_j \) is the \( j \)-th row:

\[
\begin{align*}
3132010323223202220120200, & \quad 002200223201201003031003, \quad 0222000033221130232101200, \\
0200200001313323002201, & \quad 2200220312313321021201, \quad 2202222302111121212123, \\
000222202322322012121121, & \quad 2002222303202322322222, \quad 02022020320303111230000, \\
22002000220312313210022000022230, & \quad 210213310222202021202112, \\
220222201032021132121200, & \quad 220220221222033031001223, \quad 112213022022000222332, \\
000200201123103220001, & \quad 2130210023200020300102, \quad 133220312222020202132,
\end{align*}
\]
Finally we remark that $C^{(4)}_{48q}$ is equivalent to $QR^{(4)}_{48}$. In order to verify this, it suffices to show that the lattice $A_4(\mathcal{Q}R^{(4)}_{48})$ is a neighbor of $A_3(C^{(3)}_{48q})$. This fact can be seen from the isometry from the extremal even unimodular neighbor $P_{48q}$ of $A_3(C^{(3)}_{48q})$ to an even unimodular neighbor of $A_4(\mathcal{Q}R^{(4)}_{48})$ as established in [4]. Here we describe how to verify this by MAGMA. Let $\Lambda = A_4(\mathcal{Q}R^{(4)}_{48})$, where $\mathcal{Q}R^{(4)}_{48}$ is defined as the extended code of the cyclic $\mathbb{Z}_4$-code with generator polynomial

$$3x^{23} + 2x^{21} + 3x^{19} + 3x^{18} + 2x^{16} + 3x^{14} + x^{13} + x^{12} + 2x^{11} + x^{10} + x^9 + 3x^7 + x^6 + x^5 + 2x^4 + 3x^3 + 3x^2 + x + 1$$

by appending a 3 to the generators. Note that the extended coordinate is chosen to be the 48th coordinate and $\mathcal{Q}R^{(4)}_{48}$ contains the all-one vector. Define $A_3$ as in Lemma 4.5. Let $\{\pm v_1, \pm v_2, \ldots, \pm v_{48}\}$ be the vectors of norm 3 in $A_3$. Note that $v_i$ is a vector consisting only of $\pm 1/4$. Let $M$ be the matrix whose $i$th row is $v_i$. Then $4M$ is a Hadamard matrix of order 48. Hence the unimodular lattice $\Lambda'_3 = A_3 \cdot M^T / \sqrt{3}$ has the 3-frame $\{\pm(\sqrt{3}, 0, \ldots, 0), \pm(0, \sqrt{3}, 0, \ldots, 0), \ldots, \pm(0, \ldots, 0, \sqrt{3})\}$, and it can be written as $A_3(C)$ for some ternary code $C$. It is verified by MAGMA that $C$ is equivalent to the ternary extended quadratic residue code $C^{(3)}_{48q}$. Hence $A_3(C^{(3)}_{48q})$ is obtained as a neighbor of $A_4(\mathcal{Q}R^{(4)}_{48})$, as required.

References