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On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces

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ABSTRACT

Recently, the concept of statistical convergence of double sequences has been studied in intuitionistic fuzzy normed spaces by Mursaleen and Mohiuddine [Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos, Solitons Fractals, 41 (2009) pp. 2414–2421]. We know that ideal convergence is more general than statistical convergence for single or double sequences. This has motivated us to study the ideal convergence of double sequences in a more general setting. That is, in this paper, we study the concept of ideal convergence and ideal Cauchy for double sequences in intuitionistic fuzzy normed spaces.

1. Introduction and preliminaries

After the pioneering work of Zadeh [1], a huge number of research papers have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It has a wide range of applications in various fields: population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space [6] and of intuitionistic fuzzy 2-normed space [7] are the latest developments in fuzzy topology.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. The notion of I-convergence, which is a generalization of statistical convergence [8], was introduced by Kastyrko, Salat and Wilczynski [9] by using the ideal I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in [10]. Recently, the notion of statistical convergence of double sequences $x = (x_{jk})$ has been defined and studied by Mursaleen and Edely [11]; and for fuzzy numbers by Savaş and Mursaleen [12]. Quite recently, Das et al. [13] studied the notion of I- and I^*-convergence of double sequences in $\mathbb{R}$. In this paper we propose to study the I- and I^*-convergence of double sequences in intuitionistic fuzzy normed space.

We recall some notations and basic definitions used in this paper.

Definition 1.1 (*[6]*). The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed space* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

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(a) $\mu(x,t)+\nu(x,t)\le 1$,
(b) $\mu(x,t)>0$,
(c) $\mu(x,t)=1$ if and only if $x=0$,
(d) $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,
(e) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,
(f) $\mu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
(g) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
(h) $\nu(x,t)<1$,
(i) $\nu(x,t)=0$ if and only if $x=0$,
(j) $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,
(k) $\nu(x,t)\diamond\nu(y,s)\ge\nu(x+y,t+s)$,
(l) $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
(m) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm*.

Definition 1.2 ([6]). Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, a sequence $x=(x_k)$ is said to be *convergent* to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-L,t)>1-\epsilon$ and $\nu(x_k-L,t)<\epsilon$ for all $k\ge k_0$. In this case we write $(\mu,\nu)\text{-}\lim x=L$ or $x_k\xrightarrow{(\mu,\nu)}L$ as $k\to\infty$.

Definition 1.3 ([6]). Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, $x=(x_k)$ is said to be a *Cauchy sequence* with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-x_\ell,t)>1-\epsilon$ and $\nu(x_k-x_\ell,t)<\epsilon$ for all $k,\ell\ge k_0$.

Definition 1.4 ([8]). Let K be a subset of $\mathbb{N}$. Then the *asymptotic density of* K, denoted by $\delta(K)$, is defined as

$$\delta(K)=\lim_n\frac{1}{n}|\{k\le n:k\in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x=(x_k)$ is said to be *statistically convergent* to the number ℓ if, for each $\epsilon>0$, the set $K(\epsilon)=\{k\le n:|x_k-\ell|>\epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n\frac{1}{n}|\{k\le n:|x_k-\ell|>\epsilon\}|=0.$$

In this case we write $st\text{-}\lim x=\ell$.

Definition 1.5 ([8]). A number sequence $x=(x_k)$ is said to be a *statistically Cauchy* sequence if, for every $\epsilon>0$, there exists a number $N=N(\epsilon)$ such that

$$\lim_n\frac{1}{n}|\{j\le n:|x_j-x_N|\ge\epsilon\}|=0.$$

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine [14].

Definition 1.6 ([14]). Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, a double sequence $x=(x_{jk})$ is said to be statistically convergent to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) provided that, for every $\epsilon>0$ and $t>0$,

$$\delta(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L,t)\le 1-\epsilon\text{ or }\nu(x_{jk}-L,t)\ge\epsilon\})=0,$$

or equivalently

$$\lim_{mn}\frac{1}{mn}|\{j\le n,k\le m:\mu(x_{jk}-L,t)\le 1-\epsilon\text{ or }\nu(x_{jk}-L,t)\ge\epsilon\}|=0.$$

In this case we write $st^2_{(\mu,\nu)}\text{-}\lim x=L$.

Definition 1.7 ([14]). Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, a sequence $x=(x_{jk})$ is said to be *statistically Cauchy* with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon>0$ and $t>0$, there exist $N=N(\epsilon)$ and $M=M(\epsilon)$ such that, for all $j,p\ge N$ and $k,q\ge M$,

$$\delta(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_{pq},t)\le 1-\epsilon\text{ or }\nu(x_{jk}-x_{pq},t)\ge\epsilon\})=0.$$

Definition 1.8 ([9]). If X is a non-empty set then a family I of subsets of X is called an *ideal* in X if and only if

(a) $\emptyset \in I$,
(b) $A, B \in I$ implies $A \cup B \in I$,
(c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

where $P(X)$ is the power set of X. I is called *nontrivial ideal* if $X \notin I$.

Definition 1.9 ([9]). Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a *filter* on X if and only if

(a) $\emptyset \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) For each $A \in F$ and $B \supset A$ we have $B \in F$.

Definition 1.10 ([9]). A nontrivial ideal I in X is called an *admissible ideal* if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{x\} \in I$ for each $x \in X$.

Let $I \subset P(X)$ be a nontrivial ideal. Then a class $F(I) = \{M \subset X : M = X \setminus A, \text{ for some } A \in I\}$ is a filter on X, called the *filter associated with the ideal* I.

Definition 1.11 ([9]). An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the *condition* (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\bigcup_{n\in} B_n \in I$.

Definition 1.12 ([9]). Let $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then, a sequence $x = (x_k)$ is said to be I-convergent to L if, for every $\epsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I.$$

In this case we write $I\text{-}\lim x = L$.

Definition 1.13 ([9]). Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be I-Cauchy if, for each $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that

$$\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\} \in I.$$

2. I_2-Convergence in an IFNS

In this section, we study the concept of ideal convergence of double sequences in an intuitionistic fuzzy normed space. Throughout the paper we take I_2 as a nontrivial ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.1. Let I_2 be a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A double sequence $x = (x_{jk})$ of elements of X is said to be I_2-*convergent* to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for each $\epsilon > 0$ and $t > 0$,

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - L, t) \geq \epsilon\} \in I_2. \tag{1}$$

In this case we write $I_2^{(\mu,\nu)}\text{-}\lim x = L$.

Theorem 2.1. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, for every $\epsilon > 0$ and $t > 0$, the following statements are equivalent:*

(i) $I_2^{(\mu,\nu)}\text{-}\lim x = L$.
(ii) $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \epsilon\} \in I_2$ *and* $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L, t) \geq \epsilon\} \in I_2$.
(iii) $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) > 1 - \epsilon$ *and* $\nu(x_{jk} - L, t) < \epsilon\} \in F(I_2)$.
(iv) $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) > 1 - \epsilon\} \in F(I_2)$ *and* $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L, t) < \epsilon\} \in F(I_2)$.
(v) $I_2\text{-}\lim \mu(x_{jk} - L, t) = 1$ *and* $I_2\text{-}\lim \nu(x_{jk} - L, t) = 0$.

Theorem 2.2. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. If a double sequence $x = (x_{jk})$ is I_2- convergent with respect to the intuitionistic fuzzy norms (μ, ν), then $I_2^{(\mu,\nu)}$-limit is unique.*

Proof. Suppose that $I_2^{(\mu,\nu)}$- $\lim x = L_1$ and $I_2^{(\mu,\nu)}$- $\lim x = L_2$. Given $\epsilon > 0$, choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$ and $r\Diamond r < \epsilon$. Then, for any $t > 0$, define the following sets as:

$$K_{\mu,1}(r,t) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L_1, t/2) \le 1-r\},$$
$$K_{\mu,2}(r,t) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L_2, t/2) \le 1-r\},$$
$$K_{\nu,1}(r,t) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu(x_{jk} - L_1, t/2) \ge r\},$$
$$K_{\nu,2}(r,t) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu(x_{jk} - L_2, t/2) \ge r\}.$$

Since $I_2^{(\mu,\nu)}$- $\lim x = L_1$, we have

$$K_{\mu,1}(r,t) \quad \text{and} \quad K_{\nu,1}(r,t) \in I_2.$$

Furthermore, using $I_2^{(\mu,\nu)}$- $\lim x = L_2$, we get

$$K_{\mu,2}(r,t) \quad \text{and} \quad K_{\nu,2}(r,t) \in I_2.$$

Now let $K_{\mu,\nu}(r,t) = (K_{\mu,1}(r,t) \cup K_{\mu,2}(r,t)) \cap (K_{\nu,1}(r,t) \cup K_{\nu,2}(r,t)) \in I_2$. Then we see that $K_{\mu,\nu}(r,t) \in I_2$. This implies that its complement $K^C_{\mu,\nu}(r,t)$ is a non-empty set in $F(I_2)$. If $(j,k) \in K^C_{\mu,\nu}(r,t)$, then we have two possible cases. That is, $(j,k) \in K^C_{\mu,1}(r,t) \cap K_{\mu,2}(r,t)$ or $(j,k) \in K^C_{\nu,1}(r,t) \cap K_{\nu,2}(r,t)$. We first consider that $(j,k) \in K^C_{\mu,1}(r,t) \cap K^C_{\mu,2}(r,t)$. Then we have

$$\mu(L_1 - L_2, t) \ge \mu\left(x_{jk} - L_1, \frac{t}{2}\right) * \mu\left(x_{jk} - L_2, \frac{t}{2}\right) > (1-r)*(1-r) > 1-\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get $\mu(L_1 - L_2, t) = 1$ for all $t > 0$, which yields $L_1 = L_2$. On the other hand, if $(j,k) \in K^C_{\nu,1}(r,t) \cap K^C_{\nu,2}(r,t)$, then we may write that

$$\nu(L_1 - L_2, t) \le \nu\left(x_{jk} - L_1, \frac{t}{2}\right) \Diamond \nu\left(x_{jk} - L_2, \frac{t}{2}\right) < r\Diamond r < \epsilon.$$

Therefore, we have $\nu(L_1 - L_2, t) = 0$, for all $t > 0$, which implies that $L_1 = L_2$. Therefore, in all cases, we conclude that the $I_2^{(\mu,\nu)}$-limit is unique.

This completes the proof of the theorem. □

Theorem 2.3. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS and I_2 be an admissible ideal. Then*

(i) *If (μ,ν)- $\lim x_{jk} = L$ then $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L$.*
(ii) *If $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L_1$ and $I_2^{(\mu,\nu)}$- $\lim y_{jk} = L_2$ then $I_2^{(\mu,\nu)}$- $\lim(x_{jk} + y_{jk}) = (L_1 + L_2)$.*
(iii) *If $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L$ then $I_2^{(\mu,\nu)}$- $\lim \alpha x_{jk} = \alpha L$.*

Proof. (i) Suppose that (μ,ν)- $\lim x_{jk} = L$. Then for each $\epsilon > 0$ and $t > 0$ there exists a positive integer N such that

$$\mu(x_{jk} - L, t) > 1-\epsilon \quad \text{and} \quad \nu(x_{jk} - L, t) < \epsilon$$

for each $j, k > N$. Since the set

$$A(\epsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L, t) \le 1-\epsilon \text{ or } \nu(x_{jk} - L, t) \ge \epsilon\}$$

is contained in $\{1, 2, 3, \ldots, N-1\}$ and the ideal I_2 is admissible, $A(\epsilon) \in I_2$. Hence $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L$.

(ii) Let $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L_1$ and $I_2^{(\mu,\nu)}$- $\lim y_{jk} = L_2$. For a given $\epsilon > 0$, choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$ and $r\Diamond r < \epsilon$. Then, for any $t > 0$, we define the following sets: $K_{\mu,1}(r,t)$, $K_{\mu,2}(r,t)$, $K_{\nu,1}(r,t)$ and $K_{\nu,2}(r,t)$ as above. Since $I_2^{(\mu,\nu)}$- $\lim x_{jk} = L_1$, we have

$$K_{\mu,1}(r,t) \quad \text{and} \quad K_{\nu,1}(r,t) \in I_2.$$

Furthermore, using $I_2^{(\mu,\nu)}$- $\lim x = L_2$, we get

$$K_{\mu,2}(r,t) \quad \text{and} \quad K_{\nu,2}(r,t) \in I_2.$$

Now let $K_{\mu,\nu}(r,t) = (K_{\mu,1}(r,t) \cup K_{\mu,2}(r,t)) \cap (K_{\nu,1}(r,t) \cup K_{\nu,2}(r,t))$. Then $K_{\mu,\nu}(r,t) \in I_2$, which implies that $K^C_{\mu,\nu}(r,t)$ is a non-empty set in $F(I_2)$. Now we have to show that $K^C_{\mu,\nu}(r,t) \subset \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu((x_{jk}+y_{jk}) - (L_1+L_2), t) > 1-\epsilon$ and $\nu((x_{jk}+y_{jk}) - (L_1+L_2), t) < \epsilon\}$. If $(j,k) \in K^C_{\mu,\nu}(r,t)$, then we have $\mu(x_{jk} - L_1, \frac{t}{2}) > 1-r$, $\mu(y_{jk} - L_2, \frac{t}{2}) > 1-r$, $\nu(x_{jk} - L_1, \frac{t}{2}) < r$, and $\nu(y_{jk} - L_2, \frac{t}{2}) < r$. Therefore

$$\mu((x_{jk}+y_{jk}) - (L_1+L_2), t) \ge \mu\left(x_{jk} - L_1, \frac{t}{2}\right) * \mu\left(y_{jk} - L_2, \frac{t}{2}\right) > (1-r)*(1-r) > 1-\epsilon.$$

and

$$\nu((x_{jk}+y_{jk})-(L_1+L_2),t)\leq \nu\left(x_{jk}-L_1,\frac{t}{2}\right)\diamondsuit\mu\left(y_{jk}-L_2,\frac{t}{2}\right)< r\diamondsuit r<\epsilon.$$

This shows that $K^C_{\mu,\nu}(r,t)\subset\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu((x_{jk}+y_{jk})-(L_1+L_2),t)>1-\epsilon$ and $\nu((x_{jk}+y_{jk})-(L_1+L_2),t)<\epsilon\}$. Since $K^C_{\mu,\nu}(r,t)\in F(I_2)$, $I_2^{(\mu,\nu)}\text{-}\lim(x_{jk}+y_{jk})=(L_1+L_2)$.

(iii) This is obvious for $\alpha=0$. Now let $\alpha\neq 0$. Then for a given $\epsilon>0$ and $t>0$,

$$B(\epsilon)=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L,t)>1-\epsilon \text{ and } \nu(x_{jk}-L,t)<\epsilon\}\in F(I_2). \tag{2}$$

It is sufficient to prove that, for each $\epsilon>0$ and $t>0$,

$$B(\epsilon)\subset\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(\alpha x_{jk}-\alpha L,t)>1-\epsilon \text{ and } \nu(\alpha x_{jk}-\alpha L,t)<\epsilon\}.$$

Let $(j,k)\in B(\epsilon)$. Then we have

$$\mu(x_{jk}-L,t)>1-\epsilon \quad\text{and}\quad \nu(x_{jk}-L,t)<\epsilon.$$

So we have

$$\begin{aligned}\mu(\alpha x_{jk}-\alpha L,t)&=\mu\left(x_{jk}-L,\frac{t}{|\alpha|}\right)\geq\mu\left(x_{jk}-L,t\right)*\mu\left(0,\frac{t}{|\alpha|}-t\right)=\mu(x_{jk}-L,t)*1\\&=\mu(x_{jk}-L,t)>1-\epsilon.\end{aligned}$$

Furthermore,

$$\begin{aligned}\nu(\alpha x_{jk}-\alpha L,t)&=\nu\left(x_{jk}-L,\frac{t}{|\alpha|}\right)\leq\nu\left(x_{jk}-L,t\right)\diamondsuit\nu\left(0,\frac{t}{|\alpha|}-t\right)=\mu(x_{jk}-L,t)\diamondsuit 0\\&=\mu(x_{jk}-L,t)<\epsilon.\end{aligned}$$

Hence we have

$$B(\epsilon)\subset\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(\alpha x_{jk}-\alpha L,t)>1-\epsilon \text{ and } \nu(\alpha x_{jk}-\alpha L,t)<\epsilon\},$$

and from (2) we conclude that $I_2^{(\mu,\nu)}\text{-}\lim\alpha x_{jk}=\alpha L$.

This completes the proof of the theorem. □

3. I_2^*-convergence in an IFNS

In this section, we introduce the concept of I_2^*-convergence of double sequences in an intuitionistic fuzzy normed space.

Definition 3.1. Let $(X,\mu,\nu,*,\diamondsuit)$ be an intuitionistic fuzzy normed space. We say that a double sequence $x=(x_{jk})$ of elements in X is said to be I_2^*-*convergent* to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if there exists a subset $K=\{(j_m,k_m):j_1<j_2<\cdots;k_1<k_2<\cdots\}$ of $\mathbb{N}\times\mathbb{N}$ such that $K\in F(I_2)$ (i.e. $\mathbb{N}\times\mathbb{N}\setminus K\in I_2$) and $(\mu,\nu)\text{-}\lim_m x_{j_mk_m}=L$.

In this case we write $I_2^{*,(\mu,\nu)}\text{-}\lim x=L$, and L is called the I_2^*-limit of the sequence $x=(x_{jk})$ with respect to the intuitionistic fuzzy norm (μ,ν).

Theorem 3.1. *Let $(X,\mu,\nu,*,\diamondsuit)$ be an IFNS and I_2 be an admissible ideal. If $I_2^{*,(\mu\nu)}\text{-}\lim x=L$ then $I_2^{(\mu\nu)}\text{-}\lim x=L$.*

Proof. Suppose that $I_2^{*,(\mu,\nu)}\text{-}\lim x=L$. Then $K=\{(j_m,k_m):j_1<j_2<\cdots;k_1<k_2<\cdots\}\in F(I_2)$ (i.e. $\mathbb{N}\times\mathbb{N}\setminus K=H(\text{say})\in I_2$) such that $(\mu,\nu)\text{-}\lim_m x_{j_mk_m}=L$. But then for each $\epsilon>0$ and $t>0$ there exists a positive integer N such that $\mu(x_{j_mk_m}-L,t)>1-\epsilon$ and $\nu(x_{j_mk_m}-L,t)<\epsilon$ for all $m>N$. Since $\{(j_m,k_m)\in K:\mu(x_{j_mk_m}-L,t)\leq 1-\epsilon$ or $\nu(x_{j_mk_m}-L,t)\geq\epsilon\}$ is contained in $\{j_1<j_2<\cdots<j_{N-1};k_1<k_2<\cdots<k_{N-1}\}$ and the ideal I_2 is admissible, we have

$$\{(j_m,k_m)\in K:\mu(x_{j_mk_m}-L,t)\leq 1-\epsilon \text{ or } \nu(x_{j_mk_m}-L,t)\geq\epsilon\}\in I_2.$$

Hence

$$\begin{aligned}&\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L,t)\leq 1-\epsilon \text{ or } \nu(x_{jk}-L,t)\geq\epsilon\}\\&\quad\subseteq H\cup\{j_1<j_2<\cdots<j_{N-1};k_1<k_2<\cdots<k_{N-1}\}\in I_2\end{aligned}$$

for all $\epsilon>0$ and $t>0$. Therefore, we conclude that $I_2^{(\mu,\nu)}\text{-}\lim x=L$.

Remark 3.1. The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.1. Let $(\mathbb{R}, |.|)$ denote the space of all real numbers with the usual norm, and let $a*b = ab$ and $a\diamond b = \min\{a+b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider

$$\mu(x, t) := \frac{t}{t + |x|} \quad \text{and} \quad \nu(x, t) := \frac{|x|}{t + |x|}.$$

Then $(\mathbb{R}, \mu, \nu, *, \diamond)$ is an IFNS.

Let $\mathbb{N} \times \mathbb{N} = \bigcup_{i,j} \Delta_{ij}$ be a decomposition of $\mathbb{N} \times \mathbb{N}$ such that, for any $(m, n) \in \mathbb{N} \times \mathbb{N}$, each Δ_{ij} contains infinitely many $(i, j)'$s, where $i \geq m, j \geq n$ and $\Delta_{ij} \cap \Delta_{mn} = \emptyset$ for $(i, j) \neq (m, n)$. Now we define a sequence $x_{mn} = \frac{1}{ij}$ if $(m, n) \in \Delta_{ij}$. Then

$$\mu(x_{mn}, t) = \frac{t}{t + |x_{mn}|} \longrightarrow 1$$

and

$$\nu(x_{mn}, t) = \frac{|x_{mn}|}{t + |x_{mn}|} \longrightarrow 0$$

as $m, n \to \infty$. Hence $I_2^{(\mu\nu)}\text{-}\lim_{m,n} x_{mn} = 0$.

Now suppose that $I_2^{*,(\mu,\nu)}\text{-}\lim_{m,n} x_{mn} = 0$. Then there exists a subset $K = \{(m_1 < m_2) < \cdots ; n_1 < n_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \in F(I_2)$ and $(\mu, \nu)\text{-}\lim_j x_{m_j n_j} = 0$. Since $K \in F(I_2)$, there is a set $H \in F(I_2)$ such that $K = \mathbb{N} \times \mathbb{N} \setminus H$. Now, from the definition of I_2, there exists, say, $p \in \mathbb{N}$ such that

$$H \subset \left(\bigcup_{m=1}^{p}\left(\bigcup_{n=1}^{\infty} \Delta_{mn}\right)\right) \bigcup \left(\bigcup_{n=1}^{p}\left(\bigcup_{m=1}^{\infty} \Delta_{mn}\right)\right).$$

But then $\Delta_{p+1,q+1} \subset K$, and therefore

$$x_{m_j n_j} = \frac{1}{(p+1)^2} > 0$$

for infinitely many $(m_j, n_j)'$s from K. This contradicts that $(\mu, \nu)\text{-}\lim_j x_{m_j n_j} = 0$. Therefore the assumption $I_2^{*,(\mu,\nu)}\text{-}\lim_{m,n} x_{mn} = 0$ is wrong. Hence the converse of the theorem need not be true.

This completes the proof of the theorem. $\square$

Remark 3.2. From the above example we have seen that I_2^*-convergence implies I_2-convergence but not conversely. Now the question arises under what condition the converse may hold. For this we define the condition (AP) and see that under this condition the converse holds.

Definition 3.2. An admissible ideal $I_2 \subset P(\mathbb{N} \times \mathbb{N})$ is said to satisfy the *condition* (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from I_2 there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\bigcup_{n\in\mathbb{N}} B_n \in I_2$.

Theorem 3.2. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and the ideal I_2 satisfy the condition* (AP). *If $x = (x_{jk})$ is a sequence in X such that $I_2^{(\mu,\nu)}\text{-}\lim x = L$, then $I_2^{*,(\mu,\nu)}\text{-}\lim x = L$.*

Proof. Suppose I_2 satisfies the condition (AP) and $I_2^{(\mu,\nu)}\text{-}\lim x = L$. Then for each $\epsilon > 0$ and $t > 0$,

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - L, t) \geq \epsilon\} \in I_2.$$

We define the set A_p for $p \in \mathbb{N}$ and $t > 0$ as

$$A_p = \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : 1 - \frac{1}{p} \leq \mu(x_{jk} - L, t) < 1 - \frac{1}{p+1} \text{ or } \frac{1}{p+1} < \nu(x_{jk} - L, t) \leq \frac{1}{p}\right\}.$$

Obviously $\{A_1, A_2, \ldots\}$ is countable and belongs to I_2, and $A_i \cap A_j = \emptyset$ for $i \neq j$. By the condition (AP), there is countable family of sets $\{B_1, B_2, \ldots\} \in I_2$ such that $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in I_2$. From the definition of the associate filter $F(I_2)$ there is a set $K \in F(I_2)$ such that $K = \mathbb{N} \times \mathbb{N} \setminus B$. To prove the theorem we have to show that the subsequence $(x_{jk})_{(j,k)\in K}$ is convergent to L with respect to the intuitionistic fuzzy norm (μ, ν). Let $\eta > 0$ and $t > 0$. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \eta$. Then

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \eta \text{ or } \nu(x_{jk} - L, t) \geq \eta\}$$
$$\subset \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \frac{1}{q} \text{ or } \nu(x_{jk} - L, t) \geq \frac{1}{q}\right\} \subset \bigcup_{i=1}^{q+1} A_i.$$

Since $A_i \triangle B_i$, $i = 1, 2, \ldots, q+1$ are finite, there exists $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{q+1} B_i\right) \bigcap \{(j,k) : j \geq j_0 \text{ and } k \geq k_0\} = \left(\bigcup_{i=1}^{q+1} A_i\right) \bigcap \{(j,k) : j \geq j_0 \text{ and } k \geq k_0\}. \tag{3}$$

If $j \geq j_0$, $k \geq k_0$ and $(j,k) \in K$ then $(j,k) \notin \cup_{i=1}^{q+1} B_i$. Therefore, by (3), we have $(j,k) \notin \cup_{i=1}^{q+1} A_i$. Hence, for every $j \geq j_0$, $k \geq k_0$ and $(j,k) \in K$, we have

$$\mu(x_{jk} - L, t) > 1 - \eta \quad \text{and} \quad \nu(x_{jk} - L, t) < \eta.$$

Since $\eta > 0$ is arbitrary, we have $I_2^{*,(\mu,\nu)}$- $\lim x = L$.

This completes the proof of the theorem. □

Theorem 3.3. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then the following conditions are equivalent:*

(i) $I_2^{*,(\mu,\nu)}$- $\lim x = L$.
(ii) *There exist two sequences $y = (y_{jk})$ and $z = (z_{jk})$ in X such that $x = y + z$, (μ,ν)- $\lim y = L$ and the set $\{(j,k) : z_{jk} \neq \theta\} \in I_2$, where θ denotes the zero element of X.*

Proof. Suppose that the condition (i) holds. Then there exists a set $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that

$$K \in F(I_2) \quad \text{and} \quad (\mu,\nu)\text{- }\lim_m x_{j_m,k_m} = L. \tag{4}$$

We define the sequences $y = (y_{jk})$ and $z = (z_{jk})$ as follows:

$$y_{jk} = \begin{cases} x_{jk}; & \text{if } (j,k) \in K \\ L; & \text{if } (j,k) \in K^C; \end{cases}$$

and $z_{jk} = x_{jk} - y_{jk}$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. For given $\epsilon > 0$, $t > 0$ and $(j,k) \in K^C$, we have

$$\mu(x_{jk} - L, t) = 1 > 1 - \epsilon \quad \text{and} \quad \nu(x_{jk} - L, t) = 0 < \epsilon.$$

Using (4), we have (μ,ν)- $\lim y = L$. Since $\{(j,k) : z_{jk} \neq \theta\} \subset K^C$, we have $\{(j,k) : z_{jk} \neq \theta\} \in I_2$.

Let the condition (ii) hold and $K = \{(j,k) : z_{jk} = \theta\}$. Obviously $K \in F(I_2)$ is an infinite set. Let $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$. Since $x_{j_m k_m} = y_{j_m k_m}$ and (μ,ν)- $\lim_m y_{j_m k_m} = L$, (μ,ν)- $\lim_m x_{j_m k_m} = L$. Hence $I_2^{*,(\mu\nu)} - \lim_{jk} x_{jk} = L$.

This completes the proof of the theorem. □

4. I_2- and I_2^*-Cauchy sequences on IFNS

In this section we define I_2- and I_2^*-Cauchy double sequences in intuitionistic fuzzy normed spaces and prove that I_2-convergence and I_2-Cauchy are equivalent in IFNS.

Definition 4.1. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{jk})$ is said to be I_2-*Cauchy* with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon > 0$ and $t > 0$, there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that, for all $j, p \geq N$, $k, q \geq M$,

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{pq}, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - x_{pq}, t) \geq \epsilon\} \in I_2.$$

Definition 4.2. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. We say that a double sequence $x = (x_{jk})$ is I_2^*-*Cauchy* with respect to the intuitionistic fuzzy norm (μ,ν) if there exists a subset $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \in F(I_2)$ and the subsequence $(x_{j_m k_m})$ is an ordinary Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν).

The following results are analogues to our Theorems 3.1 and 3.2, respectively, and can be proved on similar lines.

Theorem 4.1. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. If a double sequence $x = (x_{jk})$ is I_2^*-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν) then it is I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν).*

Theorem 4.2. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and ideal I_2 satisfy the condition* (AP). *If a double sequence $x = (x_{jk})$ is I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν) then it is also I_2^*-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν).*

Now, we prove the following characterization.

Theorem 4.3. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a double sequence $x = (x_{jk})$ is I_2-convergent with respect to the intuitionistic fuzzy norm (μ,ν) if and only if it is I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν).*

Proof. *Necessity*. Let $x = (x_{jk})$ be I_2-convergent to L with respect to the intuitionistic fuzzy norm (μ, ν), i.e., $I_2^{(\mu,\nu)}\text{-}\lim x = L$. Choose $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$ and $r \diamond r < \epsilon$. Then, for all $t > 0$, we have

$$A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - r \text{ or } \nu(x_{jk} - L, t) \geq r\} \in I_2. \tag{5}$$

This implies that

$$\emptyset \neq A^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) > 1 - r \text{ and } \nu(x_{jk} - L, t) < r\} \in F(I_2).$$

Let $(p, q) \in A^C$. Then we have

$$\mu(x_{pq} - L, t) > 1 - r \quad \text{and} \quad \nu(x_{pq} - L, t) < r.$$

Now let

$$B = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{pq}, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - x_{pq}, t) \geq \epsilon\}.$$

We need to show that $B \subset A$. Let $(j, k) \in B$. Then we have

$$\mu\left(x_{jk} - x_{pq}, \frac{t}{2}\right) \leq 1 - \epsilon \quad \text{or} \quad \nu\left(x_{jk} - x_{pq}, \frac{t}{2}\right) \geq \epsilon.$$

We have two possible cases. We first consider that $\mu(x_{jk} - x_{pq}, t) \leq 1 - \epsilon$. Then we have $\mu(x_{jk} - L, \frac{t}{2}) \leq 1 - r$; therefore $(j, k) \in A$. Otherwise, if $\mu(x_{jk} - L, \frac{t}{2}) > 1 - r$ then

$$1 - \epsilon \geq \mu(x_{jk} - x_{pq}, t) \geq \mu\left(x_{jk} - L, \frac{t}{2}\right) * \mu\left(x_{pq} - L, \frac{t}{2}\right) > (1-r) * (1-r) > 1 - \epsilon,$$

which is not possible. Hence $B \subset A$.

Similarly, consider that $\nu(x_{jk} - x_{pq}, t) \geq \epsilon$. Then we have $\nu(x_{jk} - L, \frac{t}{2}) \geq r$; therefore $(j, k) \in A$. Otherwise, if $\nu(x_{jk} - L, \frac{t}{2}) < r$, then

$$\epsilon \leq \nu(x_{jk} - x_{pq}, t) \leq \nu\left(x_{jk} - L, \frac{t}{2}\right) \diamond \nu\left(x_{pq} - L, \frac{t}{2}\right) < r \diamond r < \epsilon,$$

which is not possible. Hence $B \subset A$.

Thus in both cases we conclude that $B \subset A$. By (5), we have $B \in I_2$. Hence x is I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).

Sufficiency. Let $x = (x_{jk})$ be I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) but not I_2-convergent with respect to the intuitionistic fuzzy norm (μ, ν). Then there exist M and N such that

$$A(\epsilon, t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{MN}, t) \leq 1 - \epsilon \text{ or } \nu(x_{jk} - x_{MN}, t) \geq \epsilon\} \in I_2$$

and

$$B(\epsilon, t) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{jk} - L, \frac{t}{2}\right) > 1 - \epsilon \text{ and } \nu\left(x_{jk} - L, \frac{t}{2}\right) < \epsilon\right\} \in I_2$$

equivalently, $B^C(\epsilon, t) \in F(I_2)$. Since

$$\mu(x_{jk} - x_{MN}, t) \geq 2\mu\left(x_{jk} - L, \frac{t}{2}\right) > 1 - \epsilon,$$

and

$$\nu(x_{jk} - x_{MN}, t) \leq 2\nu\left(x_{jk} - L, \frac{t}{2}\right) < \epsilon,$$

if $\mu(x_{jk} - L, \frac{t}{2}) > (1-\epsilon)/2$ and $\nu(x_{jk} - L, \frac{t}{2}) < \epsilon/2$, respectively, we have $A^C(\epsilon, t) \in I_2$, and so $A(\epsilon, t) \in F(I_2)$, which is a contradiction, as x was I_2-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν). Hence x must be I_2-convergent with respect to the intuitionistic fuzzy norm (μ, ν).

This completes the proof of the theorem. □

Similarly, we can prove the following:

Theorem 4.4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then every double sequence $x = (x_{jk})$ is I_2^*-convergent with respect to the intuitionistic fuzzy norm (μ, ν) if and only if it is I_2^*-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).*

5. Conclusion

In the present paper we have studied a more general type of convergence for double sequences, that is, I_2-convergence as well as I_2-Cauchy in a more general setting, i.e. in an intuitionistic fuzzy normed space. These definitions and results provide new tools to deal with the convergence problems of double sequences occurring in many branches of science and engineering.

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