Journal of Differential Equations **162**, 451–472 (2000)
doi:10.1006/jdeq.1999.3686, available online at http://www.idealibrary.com on IDEAL®

An Application of Conley Index Techniques to a Model of Bursting in Excitable Membranes

William M. Kinney

Mathematics Department, Bethel College, St. Paul, Minnesota 55112
E-mail: b-kinney@bethel.edu

Received November 19, 1998; revised May 4, 1999

Assumptions about a model of bursting activity in pancreatic β-cells are stated and a neighborhood of the attractor in this model is constructed. Conley index results and techniques are used to give a sufficient condition for a singular isolating neighborhood to isolate a nonempty attractor. Finally, this theorem is applied to the bursting model. © 2000 Academic Press

AMS subject classifications: primary 34D15; secondary 34C28.

Key Words: singular perturbation; isolating neighborhood; Conley index; Hodgkin–Huxley; fast-slow systems.

1. INTRODUCTION

The usual geometric approach to proving the existence of attractors for ordinary differential equations is to find a so-called *trapping region* for the flow. A trapping region is a compact neighborhood in the phase space whose boundary consists of points which immediately enter the neighborhood under the flow. Because of its global nature, this can be a very difficult problem, even in low dimensions. Even when it can be done, the construction of the trapping region is often "unnatural" in the sense that the region must be "shaved" in order to get the desired boundary behavior.

In this paper, we take a different approach in the case where the differential equation studied contains a small parameter, and the system is degenerate when the parameter is set to zero. We use an approach based on Conley index theory. In particular, we will use and modify some results and techniques found in [4], [5], and [10]. Our main theorem (Theorem 4.9) will give a sufficient condition for a certain kind of neighborhood (called an *isolating* neighborhood) to contain a nonempty attractor for small values of the parameter.

Our approach has the advantage of being more "natural" in the sense that for many applications, such as the one given here, the neighborhood that contains the attractor is constructed using only the qualitative features of the flow when the parameter is zero. There is no need for detailed

0022-0396/00 $35.00

estimates about the derivatives in various regions of the phase space to make sure that the boundary behavior is right.

Once our neighborhood in constructed, we will be able to prove the existence of an attractor by looking at a small class of points on the boundary, and finding a kind of Liapunov function near these points that satisfies certain conditions (and in this application, the Liapunov functions are very simple —in fact, *linear*).

We will then use this approach to prove the existence of an attractor for a system based on a model of electrical activity in pancreatic β-cells. This system was developed as a Hodgkin–Huxley type model by Chay and Keizer [2, 3] and studied from a qualitative mathematical viewpoint by Rinzel in [11] and Terman in [12, 13].

Experimentally, the membrane potential of β-cells is found to undergo a transition from steady state to sustained "bursting" oscillations as the glucose concentration is varied. These oscillations exhibit fast-slow behavior. They are close to steady-state for a period of time, then there is a fast transition into rapid oscillatory behavior for a period of time, then there is a fast transition to the near steady-state behavior. This behavior repeats itself in both periodic and quasiperiodic modes. In fact, in [12], Terman proved the existence of chaotic behavior in the number of "spikes per burst."

In the same paper, Terman also proved the existence of a periodic orbit using a fixed point argument for a Poincaré map of the system. In [6], the author constructed a Poincaré section and used the Conley index theory developed in [9] to prove the same result for this application. This construction will be shown in a future paper.

An outline of this paper is as follows. In Section 2, we restate the qualitative assumptions given in [12] for the pancreatic β-cell model. In Section 3, we construct a neighborhood of the attractor in the model. In Section 4, we present the relevant Conley index theory that will be used. In Section 5, we use the results of the preceding section for the bursting problem. And in Section 6, we prove the main theorem of this paper (Theorem 4.9).

Related approaches to singularly perturbed systems using Conley index theory can be found in [1], [5], [8], and [10].

2. THE BURSTING PROBLEM

For completeness, we fully restate the hypotheses that we need as stated by Terman in [12]. We consider the following system of differential equations.

$$\begin{aligned}\dot{v} &= f_1(v, w, y)\\ \dot{w} &= f_2(v, w, y)\\ \dot{y} &= \varepsilon g(v, w, y, k)\end{aligned} \tag{1}$$

In the model, v represents the potential difference between the inside and outside of the cell (v is the "bursting variable"), w represents the potassium channel state variable, y is related to the calcium concentration, and the parameter k is related to the glucose concentration. The parameter $\varepsilon \geqslant 0$ and is considered to be small.

The functions f_1, f_2, and g are assumed to be smooth. We will make qualitative assumptions on the nonlinearities in these functions below.

When $\varepsilon = 0$ in (1), $\dot{y} = 0$. Thus, when $\varepsilon = 0$, (1) can be understood by studying the following one-parameter family of equations.

$$\begin{aligned}\dot{v} &= f_1(v, w, y)\\ \dot{w} &= f_2(v, w, y)\end{aligned} \tag{2}$$

where $y \in \mathbf{R}$ is a parameter.

The family (2) is sometimes called the *fast-subsystem* of (1). Systems (1) and (2) are equivalent when $\varepsilon = 0$. We will use (2) to help us understand (1) for $0 < \varepsilon \ll 1$.

Let Φ^{ε} denote the flow generated by (1) at the parameter value ε, and let φ^{y} denote the flow generated by (2) at the parameter value y. The equivalence of (2) and (1) at $\varepsilon = 0$ can be made precise by the equation

$$\Phi^0((v, w, y), t) = (\varphi^y((v, w), t), y)$$

We make the following qualitative assumptions about (2).

Assumption 2.1. The restpoint set of (2) consists of a smooth S-shaped curve $\mathscr{S}$ in $((v, w), y)$-space (phase-parameter space). That is, there exists numbers $\lambda < \rho$ such that

(a) If $y < \lambda$, then (2) has precisely one restpoint, which we denote by l_y.

(b) If $y > \rho$, then (2) has precisely one restpoint, which we denote by u_y.

(c) If $\lambda < y < \rho$, then (2) has precisely three restpoints, which are denoted by l_y, m_y, and u_y.

(d) The restpoint at the "left knee," where $y = \lambda$, is denoted by K_λ, and the restpoint at the "right knee," where $y = \rho$, is denoted by K_ρ.

(e) The union of all the above restpoints forms a smooth curve denoted by $\mathscr{S}$.

Assumption 2.2. As solutions of (2), each of the restpoints l_y is an attractor and each of the restpoints m_y is a nondegenerate saddle. For each $y \in (\lambda, \rho)$, we denote the two trajectories in the unstable manifold $W^U(m_y)$ by $M_y^+(t)$ and $M_y^-(t)$ respectively.

Assumption 2.3. There exists an $h \in (\lambda, \rho)$ such that $M_h^+(t)$ is homoclinic to m_h. That is, $\lim_{t \to \pm\infty} M_h^+(t) = m_h$. Also, $\lim_{t \to +\infty} M_y^+(t) = l_y$ for all $y \in (\lambda, h)$.

Assumption 2.4. There exists $\eta_0 > 0$ such that if $h < y < \rho + \eta_0$, then there exists an asymptotically stable periodic solution $p_y(t)$ of (2). This periodic solution surrounds u_y, but not l_y or m_y. The union of these periodic solutions as y varies defines a continuous branch of solutions, which terminate at $M_h^+(t)$ as $y \to h$. Let $\mathscr{P}$ denote the union of all these periodic solutions.

We remark that we will use the same notation for the restpoint m_y, for instance, whether we consider it to be a restpoint of (2) or a restpoint of (1) at $\varepsilon = 0$. In the following assumption let $\omega(\gamma_0)$ denote the ω-limit set of a point $\gamma_0 \in \mathbf{R}^3$ with respect to the flow Φ^0.

Assumption 2.5. There exists a neighborhood $\mathscr{U}_\rho$ of K_ρ such that if the point $\gamma_0 = (v_0, w_0, y_0) \in \mathscr{U}_\rho$, then either $\omega(\gamma_0) = m_{y_0}$, $\omega(\gamma_0) = l_{y_0}$, or $\omega(\gamma_0) = p_{y_0}$. There exists a neighborhood $\mathscr{U}_\lambda$ of K_λ such that if $\gamma_0 \in \mathscr{U}_\lambda$, then either $\omega(\gamma_0) = m_{y_0}$, $\omega(\gamma_0) = l_{y_0}$, or $\omega(\gamma_0) = u_{y_0}$.

These assumptions are illustrated in Figs. 1 and 2.

The final assumptions are stated below and are illustrated in Fig. 3.

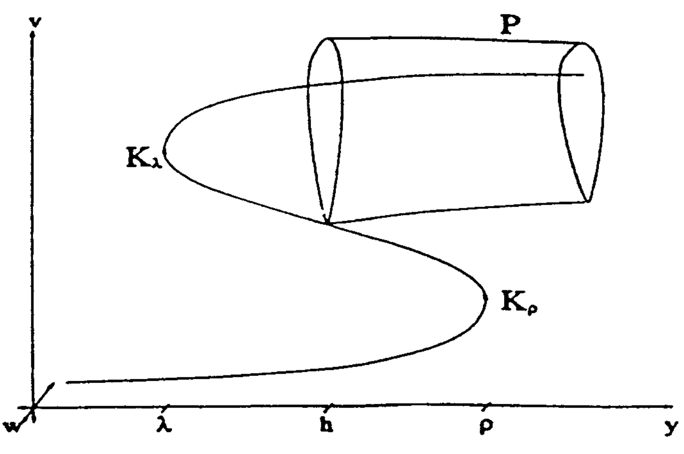

FIG. 1. The “slow manifold” for (1).

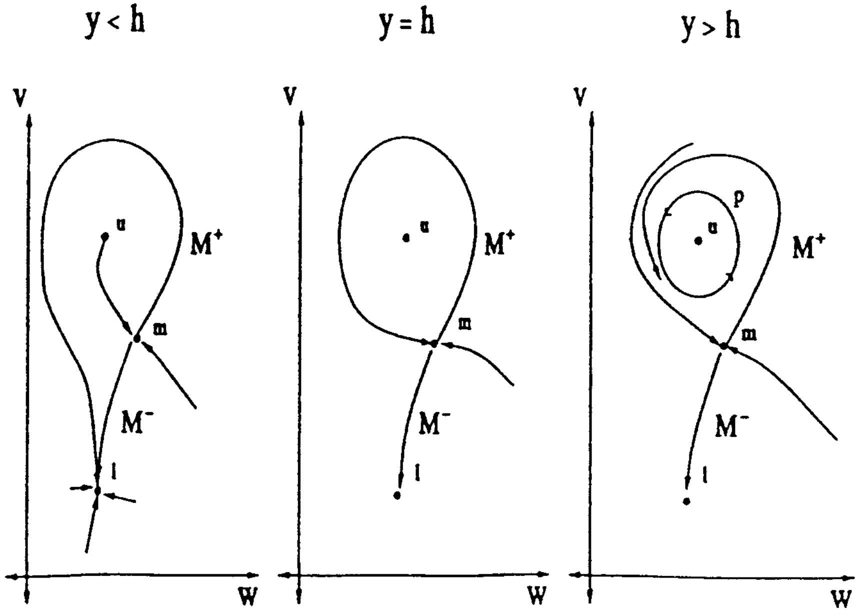

FIG. 2. The (w, v)-phase portrait for various values of y.

Assumption 2.6. There exists $k_\rho < k_\lambda$ such that if $k \in (k_\rho, k_\lambda)$, then there exists a smooth function $v = h(w, y, k)$ such that $g(v, w, y, k) = 0$ if and only if $v = h(w, y, k)$. Moreover, $g(v, w, y, k) < 0$ if and only if $v > h(w, y, k)$. If $\mathscr{M}_k = \{(v, w, y) \mid v = h(w, y, k)\}$, then $\mathscr{M}_k \cap \mathscr{S} = m_{y_k}$ for some $y_k \in (\lambda, \rho)$.

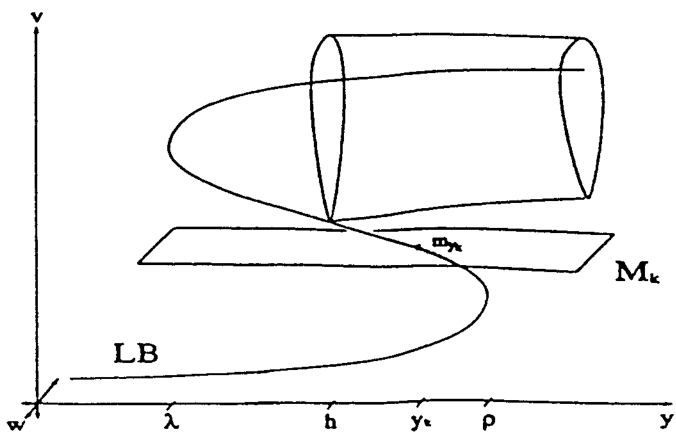

FIG. 3. The slow manifold along with the zero set $\mathscr{M}_k$.

Let $\mathcal{M}_k^+ := \{(v, w, y) \mid v > h(w, y, k)\}$ and $\mathcal{M}_k^- := \{(v, w, y) \mid v < h(w, y, k)\}$. Let LB denote the union of the restpoints l_y.

Assumption 2.7. If $k_\rho < k < k_\lambda$, then $LB \subset \mathcal{M}_k^-$. Moreover, there exists a unique $k_h \in (k_\rho, k_\lambda)$ such that $y_{k_h} = h$. If $k_\rho \leqslant k \leqslant k_h$, then $\mathcal{P} \subset \mathcal{M}_k^+$.

By using the assumptions above and looking at the figures, it should be intuitively clear that there exists an attractor whose solution curves exhibit the following kind of behavior.

Begin with an initial condition near the lower branch LB. If this point is close enough to LB, the slow dynamics will dominate and the solution will travel up LB toward the right knee K_ρ.

As the solution passes K_ρ, the fast dynamics become dominant and the solution rapidly moves toward the manifold $\mathcal{P}$ of periodic solutions. When it is close enough to $\mathcal{P}$, the slow dynamics become dominant and the solution slowly travels to the left along $\mathcal{P}$ as it oscillates rapidly around $\mathcal{P}$.

As the solution approaches the place where $\mathcal{P}$ limits to the homoclinic orbit of (2) at $y = h$ and then passes it, the behavior is much more sensitive. The solution could continue traveling up the middle branch for a while and then move down to the lower branch, or it could move down to the lower branch very quickly. In either case, the solution come back near the lower branch and the behavior repeats itself.

Intuitively, the invariant set consisting of all these kinds of solutions should be attracting because of the attracting behavior of the set of bounded solutions of (2) for each particular value of y.

3. THE ISOLATING NEIGHBORHOOD

In this section, we use the properties of the flow Φ^0 to qualitatively construct a neighborhood N that will isolate the desired attractor for $0 < \varepsilon \ll 1$. To make the description less cumbersome, we will give a pictorial description of N. A more precise mathematical description of N can be found in [6].

We construct the set N by constructing various pieces of it and then putting them all together at the end. Some of these pieces are the same as Terman constructs in [12].

From now on, we assume that $k \in (k_\lambda, k_h)$ has been fixed and exclude all reference to it. In what follows, the words "tube" and "cube" will refer to compact sets homeomorphic to the unit cube in $\mathbf{R}^3$ via a homeomorphism that preserves the orientation of the y-coordinate and that sends slices perpendicular to the y-axis to slices perpendicular to the y-axis.

Unless otherwise stated, we will hold to the following conventions. The words "left" and "right" will refer to the orientation of an object in $\mathbf{R}^3$

when the positive y-axis is pointing horizontally to the right. The words "front" and "back" will refer to the orientation of an object in $\mathbf{R}^3$ when the positive w-axis is pointing toward the reader. Finally, the words "top" and "bottom" will refer to the orientation of an object in $\mathbf{R}^3$ when the positive v-axis is pointing upward.

Given a set $A \subset \mathbf{R}^3$ and $y \in \mathbf{R}$, we will let A^y denote the set $A \cap (\mathbf{R}^2 \times \{y\})$.

We begin by noting that since the restpoint l_y is an attractor of (2) with $l_y \in \mathscr{M}^-$ for $y < \rho$, there exists a tube N_L and a number $\delta > 0$ chosen sufficiently small so that the following properties hold:

(1) $l_y \in N_L$ for $\lambda - \delta \leqslant y \leqslant \rho - \delta$.

(2) $N_L \subset \mathscr{M}^-$.

(3) The y-coordinates of the left and right sides of N_L are $\lambda - \delta$ and $\rho - \delta$ respectively.

(4) Points in $\partial N_L \cap \{(v, w, y) \mid \lambda - \delta < y < \rho - \delta\}$ immediately enter $int(N_L)$ under Φ^0.

Note that δ may be taken smaller if necessary. In what follows, we will do so.

Since the set $\mathscr{P}$ consists of attracting periodic solutions and since $\mathscr{P} \subset \mathscr{M}^+$ for $h < y < \rho + \eta_0$, we can construct a tube N_P and take δ small enough so that the following properties hold:

(1) All points in p_y and all points interior to p_y relative to $\mathbf{R}^2 \times \{y\}$ are in N_P for $h + \delta \leqslant y \leqslant \rho + \delta < \rho + \eta_0$. Also, $h + \delta < \rho - \delta$.

(2) $N_P \subset \mathscr{M}^+$.

(3) The y-coordinates of the left and right sides of N_P are $h + \delta$ and $\rho + \delta$ respectively.

(4) Points in $\partial N_P \cap \{(v, w, y) \mid h + \delta < y < \rho + \delta\}$ immediately enter $int(N_P)$ under Φ^0.

The sets N_L and N_P are shown in Fig. 4. We remark that we can take these sets as small as we like with respect to the condition that they still contain the prescribed orbits of Φ^0.

Since a (possibly degenerate) saddle-node bifurcation takes place at the right knee K_ρ, we can construct a cube N_ρ so that the following properties hold.

(1) $K_\rho \in int(N_\rho)$.

(2) $N_\rho \subset \mathscr{M}^-$.

(3) The y-coordinates of the left and right sides of N_ρ are $\rho - \delta$ and $\rho + \delta$ respectively.

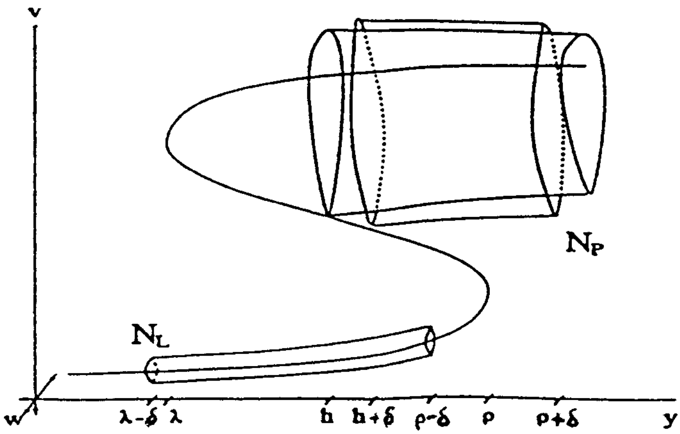

FIG. 4. The sets N_L and N_P.

(4) There exists a set $T_\rho \subset \partial N_\rho$ homeomorphic to the the closed unit square in $\mathbf{R}^2$ so that if $\xi \in T_\rho$, then ξ immediately exits N_ρ and has $\omega(\xi) \subset \mathscr{P}$ (T_ρ is the "top" of N_ρ).

(5) There exists a set $B_\rho \subset \partial N_\rho$ homeomorphic to the closed unit square in $\mathbf{R}^2$ so that if $\xi \in B_\rho$, then ξ immediately enters N_ρ (B_ρ includes the "bottom," "front," and "back" of N_ρ).

(6) All other points of ∂N_ρ stay in ∂N_ρ for a certain time interval containing zero (points in the open "left" and "right" sides of N_ρ).

Let PF_ρ denote the set homeomorphic to a cube in $\mathbf{R}^3$ obtained by using the flow Φ^0 to push the set T_ρ forward in time until the top of the resulting set is completely contained in N_P. This can be done uniformly by the compactness of T_ρ. Now let $N'_\rho := N_\rho \cup PF_\rho$. N'_ρ has the following properties.

(1) $K_\rho \in int(N'_\rho)$

(2) The y-coordinates of the left and right sides of N'_ρ are $\rho - \delta$ and $\rho + \delta$ respectively.

(3) All points on the front, back, bottom, and right faces of N'_ρ exit in backward time.

(4) All points on the top face of N'_ρ exit in forward time and immediately enter N_P.

(5) The flow Φ^0 on the left face of N'_ρ can be described as follows. There are two restpoints $l_{\rho-\delta}$ and $m_{\rho-\delta}$. One branch of the unstable manifold $W^U(m_{\rho-\delta})$ connects the two restpoints. The stable manifold $W^S(m_{\rho-\delta})$ enters the left face through its intersection with the front and back faces, and it separates left face into two components. The points in the same component as $l_{\rho-\delta}$ approach $l_{\rho-\delta}$ as $t \to +\infty$ and exit N'_ρ through

the intersection of the left face with the union of the front, bottom, and back faces in backward time. Points in the other component exit N'_ρ through the intersection of the left face with the top in forward time and exit though the intersection of the left face with the union of the front and back faces in backward time.

We can make an analogous construction of sets N_λ, PF_λ, and N'_λ near the left knee K_λ so that N'_λ has the following properties.

(1) $K_\lambda \in int(N'_\lambda)$

(2) The y-coordinates of the left and right sides of N'_λ are $\lambda - \delta$ and $\lambda + \delta$ respectively.

(3) All points on the front, back, "top," and left faces of N'_λ exit in backward time. (The "top" of N'_λ is the same as the bottom of N_λ (see Figure 5).)

(4) All points on the bottom face of N'_λ exit in forward time and enter N_L.

The flow on the right face of N'_λ behaves in a similar way to the flow on the left face of N'_ρ, but this behavior will ultimately not matter (as the reader will soon see) because of our final construction. We now construct a set N'_M which adjoins N'_λ, N_P, and N_L.

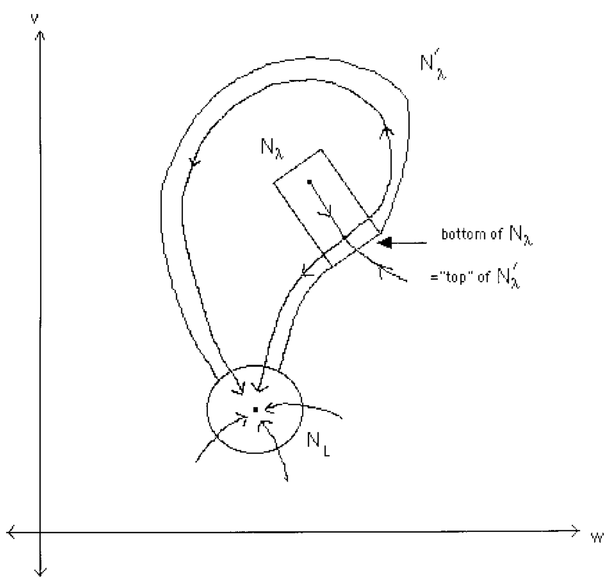

FIG. 5. How the "top" of N'_λ is the same set as the bottom of N_λ.

Since m_y is a hyperbolic saddle point for $y \in (\lambda, \rho)$, we can construct an "elongated" cube N_M near the middle branch that has the following properties.

(1) $N_M \subset \mathcal{M}^+$.

(2) The y-coordinates of the left and right sides of N_M are $\lambda + \delta$ and $h + \delta$ respectively.

(3) $m_y \in int(N_M)$ for all $y \in (\lambda + \delta, h + \delta)$.

(4) The left face of N_M is contained in the right face of N'_λ.

(5) All points on the front and back faces of N_M exit N_M in forward time. And all points on the top and bottom faces of N_M exit N_M in backward time.

(6) If necessary, choose $\delta > 0$ smaller than previously to insure that the branch of $W^S(m_y)$ which has the homoclinic orbit as its limit as $y \to h^+$ intersects $int_{\mathbf{R}^2 \times \{y\}}(N^y_M)$ on both sides of $W^U_{loc}(m_y)$ (see Fig. 6). (This can be done by continuity of the flow and the existence of the homoclinic orbit).

Now take the points on the bottom-front and the bottom-back edges of N_M and push them forward in time under Φ^0 until their orbits all meet with N_L. For each $y \in [\lambda + \delta, h + \delta]$, form a Jordan curve C_y in the following way: adjoin the forward orbits of these edge points at the given value of y with the bottom edge of N^y_M and the arc of $(\partial N_L)^y$ that connects the

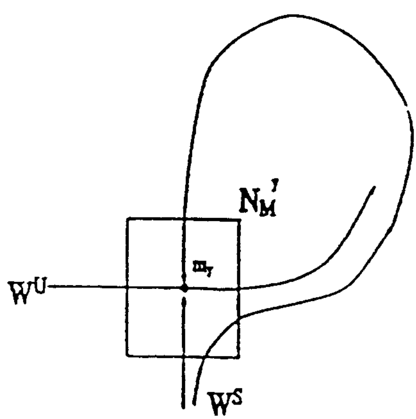

FIG. 6. Choosing δ small enough so that the specified branch of $W^S(m_y)$ intersects $int(N^y_M)$ on both sides of $W^U_{loc}(m_y)$.

forward orbits of the edge points of N_M and intersects the orbit M_y^-. Let D_y be the bounded component of $(\mathbf{R}^2 \times \{y\}) \backslash C_y$. Finally, let

$$N'_M := \bigcup_{\lambda+\delta \leqslant y \leqslant h+\delta} (D_y \cup C_y).$$

N'_M has the following properties:

(1) The homoclinic orbit M_h^+ lies in $int(N'_M)$.

(2) The y-coordinates of the left and right faces of N'_M are $\lambda+\delta$ and $h+\delta$ respectively.

(3) The left face of N'_M is contained in the right face of N'_λ.

(4) The bottom face of N'_M is contained in the boundary of N_L.

(5) If ξ is in the right face of N'_λ but not in the left face of N'_M, then ξ leaves N'_λ in backward time and does not enter N'_M.

(6) If $\xi \in \partial N'_M \cap \{(v, w, y) \mid \lambda+\delta < y < h+\delta\}$, then ξ leaves N'_M in backward time through the "top" of N'_M. (which is really the bottom of N_M.)

(7) The flow on the right face of N'_M can be described as follows. There are two restpoints, $u_{h+\delta}$ and $m_{h+\delta}$, and the stable periodic orbit $p_{h+\delta}$. Both branches of $W^S(m_{h+\delta})$ enter through the "top" of N'_M. The entire stable manifold separates all the points in the right face except those in $W^U(m_{h+\delta})$, on $p_{h+\delta}$, and interior to $p_{h+\delta}$ into two types of limiting behavior. On the component containing $p_{h+\delta}$, points approach $p_{h+\delta}$ in forward time and leave through the "top" of N'_M in backward time. On the other component, points leave the right face in forward time through the

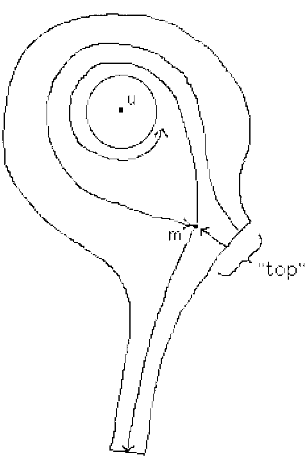

FIG. 7. The flow on the right side of N'_M.

bottom of N'_M. All points that are not restpoints, not in $W^U(m_{h+\delta})$, and not in p or interior to p leave the right face in backward time through the "top" of N'_M. (see Fig. 7).

As a final touch-up, make the tube N_P "skinnier" if necessary so that the left face of N_P is contained in the interior of the right face of N'_M with respect to $\mathbf{R}^2 \times \{h+\delta\}$.

Finally, we define

$$N := N_L \cup N_P \cup N'_\rho \cup N'_\lambda \cup N'_M.$$

The set N is pictured in Fig. 8.

Let $S := Inv(N, \Phi^0) := \{\xi \mid \Phi^0(\xi, \mathbf{R}) \subset N\}$ and let $S_\partial := S \cap \partial N$. The following proposition is an easy consequence of the properties on the preceding pages.

PROPOSITION 3.1. *The set S_∂ consists of the following points.*

(1) *The restpoint $l_{\lambda-\delta}$.*

(2) *The restpoint $m_{\rho-\delta}$ and the points on the unstable manifold $W^U(m_{\rho-\delta})$ which do not intersect the open right face of N_L or int(N_P).*

(3) *Points on the periodic orbit $p_{\rho+\delta}$, the restpoint $u_{\rho+\delta}$, and points in the unstable manifold $W^U(u_{\rho+\delta})$.*

(4) *The restpoint $m_{h+\delta}$ and the points in the unstable manifold $W^U(m_{h+\delta})$ which do not intersect the open left face of N_P or int(N_L).*

The main goal of this paper is to show that, in order to prove the existence of a nonempty attractor of Φ^ε ($\varepsilon \ll 1$) inside N, it will suffice to verify

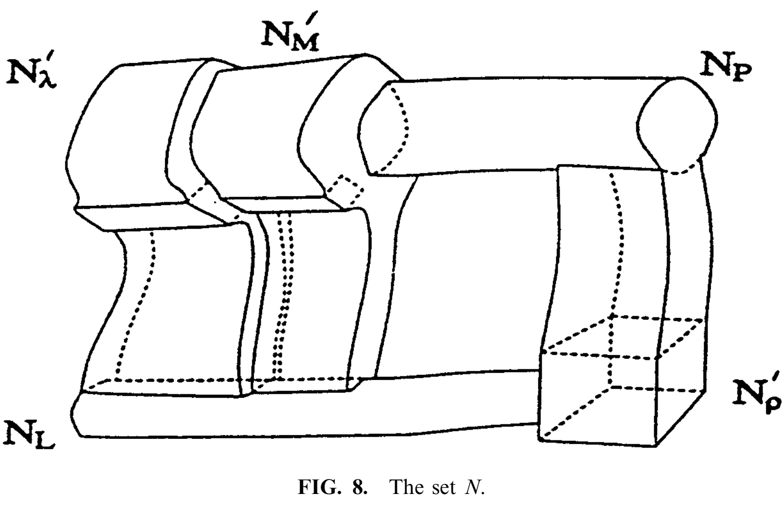

FIG. 8. The set N.

that the points in the preceding proposition satisfy a certain condition. Once the theory is understood, this condition is relatively easy to verify for this particular application.

4. THE CONLEY INDEX

4.1. *Basic Results*

Let X be a locally compact metric space and let $\varphi: X \times \mathbf{R} \to X$ be a continuous flow on X. We will often suppress X and φ in our notation.

Given $N \subset X$, let $Inv(N) = \{x \mid \varphi(x, t) \in N \text{ for all } t \in \mathbf{R}\}$. $Inv(N)$ is the maximal invariant subset of N.

Let $S \subset X$ be a compact invariant set. A subset $A \subset S$ is called an *attractor in S* if there exists a neighborhood U of A such that $\omega(U \cap S) = A$.

Two interesting theorems of Conley [4] give us ways of finding attractors.

THEOREM 4.1. *Suppose* $U \subset S$ *and, for some* $t_0 > 0$, $\varphi(cl(U), t_0) \subset int(U)$. *Then* $\omega(U)$ *is an attractor contained in* $int(U)$.

THEOREM 4.2. *Suppose N is a compact subset of S with the property that each point of* ∂N *is carried out of N in backward time. Then* $Inv(N)$ *is an attractor.* (*However,* $Inv(N)$ *could be empty.*)

An *isolating neighborhood* is a compact set N such that $Inv(N) \subset int(N)$. Equivalently, N is an isolating neighborhood if every point in ∂N eventually leaves N in either forward or backward time. An *isolated invariant set* is a set S for which there exists an isolating neighborhood N such that $S = Inv(N)$.

A set $L \subset N$ is called *positively invariant in N* if $x \in L, t > 0$, and $\varphi(x, [0, t]) \subset N$ imply that $\varphi(x, [0, t]) \subset L$. A set $L \subset N$ is called an *exit set for N* if $x \in N, t_1 > 0$, and $\varphi(x, t_1) \notin N$ imply that there exists a $t_0 \in [0, t_1]$ such that $\varphi(x, [0, t_0]) \subset N$ and $\varphi(x, t_0) \in L$.

DEFINITION 4.1. Let S be an isolated invariant set. A compact pair (N, L) is called an *index pair* for S if the following conditions hold.

(a) $S = Inv(cl(N \backslash L))$ and $N \backslash L$ is a neighborhood of S.

(b) L is positively invariant in N.

(c) L is an exit set for N.

The next two theorems allow us to define the Conley index of an isolated invariant set.

THEOREM 4.3. *Let S be an isolated invariant set. Then there exists an index pair (N, L) for S. Furthermore, given any isolating neighborhood N of S, there exists $L' \subset N' \subset N$ such that (N', L') is an index pair for S.*

THEOREM 4.4. *Let (N, L) and (N', L') be index pairs for an isolated invariant set S, then the pointed spaces $(N/L, [L])$ and $(N'/L', [L'])$ are homotopy equivalent.*

DEFINITION 4.2. Let S be an isolated invariant set and let (N, L) be an index pair for S. The *Conley index of S*, denoted by $h(S)$, is the homotopy type $[N/L]$ of the pointed space $(N/L, [L])$.

Here is a basic result we will need.

PROPOSITION 4.1. *If $h(S)$ is not the homotopy type of a pointed one-point space, then S is nonempty.*

Once the Conley index has been defined as above, it is its *continuation* properties which give it power in applications.

Let Λ be a compact, locally contractible, connected metric space. Given a family of continuous flows $\{\varphi^\lambda\}_{\lambda \in \Lambda}$ on X, we can define a flow Φ on $X \times \Lambda$ by the equation

$$\Phi((x, \lambda), t) := (\varphi^\lambda(x, t), \lambda).$$

Φ is called the *parameter flow* associated with the family $\{\varphi^\lambda\}_{\lambda \in \Lambda}$ and this family is said to be *continuously parameterized* if Φ is continuous.

The following proposition on the stability of isolating neighborhoods under perturbation is an easy consequence of the definition of an isolating neighborhood.

PROPOSITION 4.2. *Let N be an isolating neighborhood for the flow φ^{λ_0} for some $\lambda_0 \in \Lambda$. Then there is an $\varepsilon > 0$ such that N is an isolating neighborhood for φ^λ if $d(\lambda, \lambda_0) < \varepsilon$.*

Index *pairs* do not behave so nicely under perturbation however. Because of this, the following theorems are nontrivial.

THEOREM 4.5. *Let N be an isolating neighborhood for φ^{λ_0}. Choose $\varepsilon > 0$ such that if $d(\lambda, \lambda_0) < \varepsilon$, then N is an isolating neighborhood for φ^λ. Then $h(Inv(N, \varphi^\lambda)) = h(Inv(N, \varphi^{\lambda_0}))$.*

To extend the preceding perturbation theorem to a global continuation theorem, we must use the parameter flow. First, we give some notation. Given $N \subset X \times \Lambda$ and $\lambda \in \Lambda$, define the slice N^λ by

$$N^\lambda := N \cap (X \times \{\lambda\}).$$

DEFINITION 4.3. Let S^{λ_0} and S^{λ_1} be isolated invariant sets for φ^{λ_0} and φ^{λ_1} respectively. S^{λ_0} and S^{λ_1} are said to be related by continuation if there exists an isolating neighborhood $N \subset X \times \Lambda$ for the parameter flow Φ such that $Inv(N^{\lambda_0}, \varphi^{\lambda_0}) = S^{\lambda_0}$ and $Inv(N^{\lambda_1}, \varphi^{\lambda_1}) = S^{\lambda_1}$.

THEOREM 4.6. *If* S^{λ_0} *and* S^{λ_1} *are related by continuation, then* $h(S^{\lambda_0}, \varphi^{\lambda_0}) = h(S^{\lambda_1}, \varphi^{\lambda_1})$.

We also have need to recall the definition of the *chain recurrent set* of a flow φ on X.

DEFINITION 4.4. Given $\varepsilon, T > 0$ and $x, y \in X$, an (ε, T)-*chain* from x to y is a finite sequence

$$\{(x_i, t_i)\} \subset X \times [0, \infty),\ i = 1, ..., n$$

such that $x = x_1$, $t_i \geqslant T$, and $d(\varphi(x_i, t_i), x_{i+1}) \leqslant \varepsilon$ for each $i = 1, ..., n-1$ and $d(\varphi(x_n, t_n), y) \leqslant \varepsilon$. If there exists an (ε, T)-chain from x to y, then we write $x \succcurlyeq_{(\varepsilon, T)} y$. If $x \succcurlyeq_{(\varepsilon, T)} y$ for all $\varepsilon, T > 0$, then we write $x \succcurlyeq y$.

DEFINITION 4.5. The *chain recurrent set of X under the flow* φ is defined by

$$\mathscr{R}(X) = \mathscr{R}(X, \varphi) := \{x \in X \mid x \succcurlyeq x\}$$

4.2. *Singular Isolating Neighborhoods*

Now consider a family of differential equations on $\mathbf{R}^n$ of the form:

$$\dot{x} = f(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) \tag{3}$$

where f_0 and f_1 are smooth and $\varepsilon \geqslant 0$. Let φ^ε: $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ denote the flow generated by (3) at the parameter value $\varepsilon \geqslant 0$.

DEFINITION 4.6. A compact set $N \subset \mathbf{R}^n$ is called a *singular isolating neighborhood* for the family of flows $\{\varphi^\varepsilon\}_{\varepsilon \geqslant 0}$ if N is *not* an isolating neighborhood for φ^0, but there is an $\bar{\varepsilon} > 0$ such that N is an isolating neighborhood for all $\varepsilon \in (0, \bar{\varepsilon}]$.

Given a compact set $N \subset \mathbf{R}^n$, let $S = Inv(N, \varphi^0)$ and let $S_\partial = S \cap \partial N$.

DEFINITION 4.7. Let $g: S \to \mathbf{R}$. The *average of* g *on* S, which we will denote by the symbol $Ave(g, S)$, is the limit as $T \to \infty$ of the set of numbers $\{1/T \int_0^T g(\varphi^0(x, s))\, ds \mid x \in S\}$. If $Ave(g, S) \subset (0, \infty)$, then we say that g has strictly positive averages.

The following definition is a modification of a definition Conley gave in [5] that will apply more directly to our setting.

DEFINITION 4.8. A point $x \in S_\partial$ is called a *simple C-slow exit point* if there exists a compact set $K_x \subset S$ which is invariant under φ^0, a neighborhood U_x of the chain recurrent set $\mathscr{R}(K_x)$, an $\bar{\varepsilon} > 0$, and a differentiable function ℓ: $cl(U_x) \times [0, \bar{\varepsilon}] \to \mathbf{R}$ such that the following conditions are satisfied.

(a) $\omega(x, \varphi^0) \subset K_x$

(b) ℓ is of the form

$$\ell(z, \varepsilon) = \ell_0(z) + \varepsilon \ell_1(z)$$

(c1) There exists a neighborhood W_x of K_x such that $\ell_0|_{cl(U_x \cap (S \setminus W_x))} < -2\delta$ for some $\delta > 0$.

(c2) There exists a neighborhood V_x of $\mathscr{R}(K_x)$ such that $\ell_0|_{V_x \cap U_x} > -\delta$.

(c3) $\ell_0|_{S \cap cl(U_x)} \leqslant 0$.

(d) Let

$$g_0(z) = \nabla \ell_0(z) \cdot f_0(z) \text{ and } g_1(z) = \nabla \ell_0(z) \cdot f_1(z) + \nabla \ell_1(z) \cdot f_0(z)$$

Then $g_0 \equiv 0$ and g_1 has strictly positive averages on $\mathscr{R}(K_x)$.

We will be using the dual concept of C-slow *entrance* points.

DEFINITION 4.9. $x \in S_\partial$ is called a *simple C-slow entrance point* if it is a C-slow exit point under time reversal. Equivalently, the same conditions in the preceding definition hold with the modifications that $\alpha(x, \varphi^0) \subset K_x$ and that g_1 has strictly *negative* averages on $\mathscr{R}(K_x)$.

From now on, let S_∂^- and S_∂^+ denote the set of simple C-slow exit and simple C-slow entrance points respectively. The following theorems are modifications of theorems of Conley in [5, Lemma 3.2.B, Theorem 3.2.C]. Their proofs are modifications of Conley's proofs and are given in [7].

THEOREM 4.7. *If* $x \in S_\partial^-$ $(x \in S_\partial^+)$, *then there exists a neighborhood* W_S *of* S, *a neighborhood* Ω_x *of* x, *and an* $\bar{\varepsilon} > 0$ *such that for* $\varepsilon \in (0, \bar{\varepsilon}]$ *and* $y \in \Omega_x$, *we have* $\varphi^\varepsilon(y, [0, \infty)) \not\subset W_S$ $(\varphi^\varepsilon(y, (-\infty, 0]) \not\subset W_S)$.

THEOREM 4.8. *If* $S_\partial \subset S_\partial^+ \cup S_\partial^-$, *then* N *is a singular isolating neighborhood.*

The following condition will be needed. It is easy to verify in the bursting problem.

DEFINITION 4.10. A simple C-slow entrance point x is called a *strict simple C-slow entrance point* if there exists a neighborhood Θ_x of x and an $\bar{\varepsilon} > 0$ such that if $y \in \Theta_x \cap N$ and $\varepsilon \in (0, \bar{\varepsilon}]$, then there exists $t_y(\varepsilon) > 0$ for which

$$\varphi^\varepsilon(y, [0, t_y(\varepsilon)]) \subset N.$$

Such points will be easily detected in the context of our application because of the geometry of N.

We can now state our main theorem. We prove this theorem in Section 6.

THEOREM 4.9. *Let* S_∂^+ *be the set of simple C-slow entrance points in* N *and suppose that*

(a) $S_\partial = S_\partial^+$.

(b) *No points in* ∂N *leave* N *in forward time under* φ^0.

Then $Inv(N)$ *is an attractor for sufficiently small* $\varepsilon > 0$.

Furthermore, if

(c) $\bigcup_{x \in S_\partial^+} \mathscr{R}(K_x)$ *consists of strict simple C-slow entrance points.*

Then $h(Inv(N)) = [N/\varnothing]$ *for sufficiently small* $\varepsilon > 0$.

Remark 4.1. This theorem remains true if the word "simple" is deleted and Conley's original definition for C-slow entrance points holds (see [5]).

In the next section, we apply this theorem to the bursting problem.

5. APPLICATION TO THE BURSTING PROBLEM

We now consider the bursting problem again, in particular, we show that N is a singular isolating neighborhood for system (1) and that the hypotheses of Theorem 4.9 are satisfied. This will imply the existence of a nonempty attractor.

The following proposition is an extension of Proposition 3.1.

PROPOSITION 5.1. *The points in* S_∂ *are simple C-slow entrance points. So* $S_\partial = S_\partial^+$.

Proof. We will prove that the restpoint $m_{h+\delta}$ and the points in the unstable manifold $W^U(m_{h+\delta})$ which do not intersect the open left face of N_P or $int(N_L)$ are simple C-slow entrance points. The demonstrations for the other points are similar.

Let x be one of these points. Then $\alpha(x)=\{m_{h+\delta}\}$. Let $K_x=\{m_{h+\delta}\}=\{(v_{h+\delta}, w_{h+\delta}, y_{h+\delta})\}$, so $\alpha(x)\subset K_x$. Under Φ^0, we have $\mathscr{R}(K_x)=K_x$. Let U_x be a neighborhood of $\mathscr{R}(K_x)$ chosen sufficiently small so that $U_x\cap\partial N=U_x\cap\partial N'_M$, $U_x\subset\mathscr{M}^+$, and $U_x\cap N_P=\varnothing$.

Let

$$\ell((v, w, y), \varepsilon)=\ell_0(v, w, y):=y-(h+\delta).$$

Now $\nabla\ell_0=(0, 0, 1)$ so $g_0(v, w, y)\equiv 0$ and $g_1(v, w, y)=g(v, w, y)$ (where g is the function in system (1)). So on the set $\mathscr{R}(K_x)=\{m_{h+\delta}\}\subset\mathscr{M}^+$, we have $g_1=g<0$. This implies that g_1 has strictly negative averages on $\mathscr{R}(K_x)$.

It is easy to see that there exists neighborhoods W_x and V_x of K_x and $\mathscr{R}(K_x)$, respectively, that satisfy conditions (c1), (c2), and (c3) of Definition 4.8. Choose W_x and V_x as in Fig. 9. The set W_x is chosen to be sticking out of the set U_x in the direction of $W^U(m_{h+\delta})$.

Finally, note that $\ell_0|_{S\cap cl(U_x)}\leqslant 0$.

Therefore, x is a simple C-slow entrance point. ∎

THEOREM 5.1. *N is a singular isolating neighborhood of* (1).

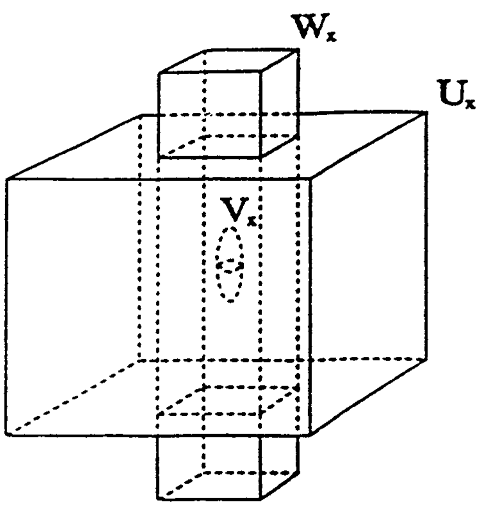

FIG. 9. Choosing the sets W_x and V_x near $m_{h+\delta}$ once U_x is chosen.

Proof. This follows immediately from the preceding proposition and Theorem 4.8. ▮

We finally come to our desired theorem for the bursting problem.

THEOREM 5.2. *For $\varepsilon > 0$ sufficiently small, $Inv(N)$ is a nonempty attractor of* (1). *Furthermore, $h(Inv(N))$ is the homotopy type of the disjoint union or a circle and a distinguished point.*

Proof. We have already shown that $S_\partial = S_\partial^+$. It is clear that no points in ∂N leave N in forward time under Φ^0 by our construction of N.

Thus, it is left to show that the set $R = \bigcup_{x \in S_\partial^+} \mathscr{R}(K_x)$ consists of strict simple C-slow entrance points. We have already proven that they are simple C-slow entrance points, thus it is left to show that Definition 4.10 holds. But this follows because R consists of points which are bounded away from $\mathscr{M}$, the zero set of g, and which enter N immediately in forward time when $\varepsilon > 0$ because of their location on N with respect to $\mathscr{M}$.

Thus, Theorem 4.9 implies that $Inv(N)$ is an attractor and that we have $h(Inv(N), \Phi^\varepsilon) = [N/\varnothing]$ for $\varepsilon > 0$ sufficiently small. Since N is homotopic to a circle, this implies that $[N/\varnothing]$ is the homotopy type of the disjoint union of a circle and a distinguished point.

Thus, by Proposition 4.1, $Inv(N) \neq \varnothing$. ▮

6. PROOF OF THEOREM 4.9

In order to prove Theorem 4.9, we need a few more definitions and a proposition.

Given $x \in \mathbf{R}^n$ and $\varepsilon > 0$, let $B_\eta(x)$ be the open ball of radius η centered at x. Given $Y \subset \mathbf{R}^n$, let $B_\eta(Y) := \bigcup_{y \in Y} B_\eta(y)$.

Let $Q_\eta := B_\eta(\bigcup_{x \in S_\partial^+ \cup S_\partial^-} \mathscr{R}(K_x))$.

Consider a one-parameter family of smooth bump functions μ_η: $\mathbf{R}^n \to [0, 1]$ satisfying:

(1) *supp* $\mu_\eta \subset Q_\eta$

(2) $Q_{\eta/2} \subset \mu_\eta^{-1}(1)$

Let ψ_η^ε denote the flow of the two parameter family

$$\dot{x} = f_0(x) + \varepsilon \mu_\eta(x) f_1(x) \tag{4}$$

Note that when η is sufficiently large, $\psi_\eta^\varepsilon = \varphi^\varepsilon$ in a neighborhood of N.

The following proposition is proven in [10] using a compactness argument in conjunction with parameterized versions of Definition 4.8 and Theorem 4.8.

PROPOSITION 6.1. *Assume that* $S_\partial \subset S_\partial^+ \cup S_\partial^-$, *and let* $r = diam(N)$. *Then there is a continuous function* $\tilde{\varepsilon}$: $(0, r] \to (0, \infty)$ *with the property that* N *is an isolating neighborhood for* ψ_η^ε *for all* (ε, η) *such that* $0 < \eta \leqslant r$ *and* $0 < \varepsilon \leqslant \tilde{\varepsilon}(\eta)$.

The proof of Theorem 4.9 is a conglomeration and modification of proofs found in [5] and [10].

Proof (of Theorem 4.9). First note that N is a singular isolating neighborhood by Theorem 4.8 and our first hypothesis.

To see that $Inv(N)$ is an attractor for small $\varepsilon > 0$, we use Theorem 4.2. That is, we must verify that all points of ∂N leave N in backward time for small $\varepsilon > 0$.

Given $\xi \in S_\partial = S_\partial^+$, Theorem 4.7 implies the existence of neighborhoods W_ξ of S, Ω_ξ of ξ, and an $\bar{\varepsilon}_\xi > 0$ such that $\varepsilon \in (0, \bar{\varepsilon}_\xi]$ and $\zeta \in \Omega_\xi$ imply that $\varphi^\varepsilon(\zeta, (-\infty, 0]) \not\subset W_\xi$.

Since S_∂ is compact, we can cover it with finitely many such neighborhoods, say $\Omega_{\xi_1}, \Omega_{\xi_2}, ..., \Omega_{\xi_n}$. Let $\bar{\varepsilon}_1 := \min_i \bar{\varepsilon}_{\xi_i}$, $\Omega := \bigcup_i \Omega_{\xi_i}$, and $W_1 := \bigcap_i W_{\xi_i}$.

Since W_1 is a neighborhood of S and Ω is a neighborhood of S_∂, $cl[(N \backslash W_1) \cup (\partial N \backslash \Omega)] \cap S = \emptyset$. Because of this fact and assumption (b) of the theorem, points in $cl[(N \backslash W_1) \cup (\partial N \backslash \Omega)]$ must leave N in backward time under φ^0. Hence, given $\xi \in cl[(N \backslash W_1) \cup (\partial N \backslash \Omega)]$, there exists an $\bar{\varepsilon}_\xi > 0$ and a neighborhood Θ_ξ of ξ such that $\varepsilon \in (0, \bar{\varepsilon}_\xi]$ and $\zeta \in \Theta_\xi$ imply that $\varphi^\varepsilon(\zeta, (-\infty, 0]) \not\subset N$.

Since $cl[(N \backslash W_1) \cup (\partial N \backslash \Omega)]$ is compact, we can cover it with finitely many neighborhoods, say $\Theta_{\xi_1}, \Theta_{\xi_2}, ..., \Theta_{\xi_m}$. Let $\bar{\varepsilon}_2 := \min_i \bar{\varepsilon}_{\xi_i}$ and $W_2 := \bigcup_i \Theta_{\xi_i}$.

Let $W := \Omega \cup W_2$ and $\bar{\varepsilon} := \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$. Note that W is a neighborhood of ∂N.

Suppose $\zeta \in \partial N$ and $\varepsilon \in (0, \bar{\varepsilon}]$. If $\zeta \in W_2$, then $\varphi^\varepsilon(\zeta, (-\infty, 0]) \not\subset N$ and we are done. If $\zeta \in \Omega$, then $\varphi^\varepsilon(\zeta, (-\infty, 0]) \not\subset W_1$. Thus, we can choose $T > 0$ so that $\zeta_{-T} := \varphi^\varepsilon(\zeta, -T) \notin W_1$. If $\zeta_{-T} \notin N$, we are done. Otherwise, $\zeta_{-T} \in N \backslash W_1 \subset W_2$ so that $\varphi^\varepsilon(\zeta_{-T}, (-\infty, 0]) \not\subset N$. Therefore, $\varphi^\varepsilon(\zeta, (-\infty, 0]) \not\subset N$ and we are done. All points of ∂N leave N in backward time for $\varepsilon \in (0, \bar{\varepsilon}]$. Thus, $Inv(N)$ is an attractor for $\varepsilon \in (0, \bar{\varepsilon}]$.

Next, we show that $h(Inv(N)) = [N/\emptyset]$ for small $\varepsilon > 0$ if we make the additional assumption (c). Let $Q_\eta^+ := B_\eta(\bigcup_{x \in S_\partial^+} \mathscr{R}(K_x))$ and let μ_η and ψ_η^ε be defined as above (with Q_η^+ in place of Q_η).

By Proposition 6.1, we can find a continuous function $\tilde{\varepsilon}$: $(0, r] \to (0, \infty)$ with the property that N is an isolating neighborhood of ψ_η^ε for all (ε, η) such that $0 < \eta \leqslant r$ and $0 < \varepsilon \leqslant \tilde{\varepsilon}(\eta)$.

$(N, \emptyset)$ is an index pair for sufficiently small $\eta > 0$ and $0 < \varepsilon \leqslant \tilde{\varepsilon}(\eta)$ because it trivially satisfies the first two conditions for an index pair and because hypothesis (c) of Theorem 4.9 guarantees that it satisfies the third condition for an index pair when $\eta > 0$ and $\varepsilon > 0$ are sufficiently small.

Letting η increase to r while keeping $\varepsilon < \tilde{\varepsilon}(\eta)$, we see that the continuation property of the Conley index gives us our desired result. ▌

7. CONCLUSION

Theorem 4.9 has rather wide applicability to systems whose qualitative features are well understood. And though it is not as general as the theorem given in [10], we feel that it has two advantages. One advantage is its simpler statement. The other is that it applies more directly to certain kinds of problems, such as the bursting application given here.

In a future paper, we will construct a Poincaré section for this bursting model and use a theorem developed in [9] to prove the existence of a periodic solution. Another reference where many related ideas can be found is [8].

ACKNOWLEDGMENTS

Most of the work for this paper was done as a graduate student at the University of Minnesota under Rick Moeckel. I would like to thank him for his help and guidance. I would also like to thank Jim Reineck of SUNY Buffalo and Alan Parks of Lawrence University for their helpful comments. Finally, I would like to thank Dave Terman of Ohio State University for his suggestion of looking at the bursting problem as an interesting application of Conley index ideas.

REFERENCES

1. G. Carpenter, A Geometric approach to singular perturbation problems with applications to nerve impulse equations, *J. Differential Equations* **23** (1977), 335–367.
2. T. Chay, Chaos in a three-variable model of an excitable cell, *Phys. D.* **16** (1985), 233–242.
3. T. Chay and Keizer, Minimal model for membrane oscillations in the pancreatic β-cell, *Biophys. J.* **42** (1983), 181–190.
4. C. Conley, Isolated invariant sets and the morse index, *in* "CBMS Lecture Notes," Vol. 38, Amer. Math. Soc., Providence, RI, 1978.
5. C. Conley, A qualitative singular perturbation theorem, *in* "Global Theory of Dynamical Systems" (Z. Nitecki and C. Robinson, Eds.), Lect. Notes in Math., Vol. 819, Springer-Verlag, 1980, 65–89.
6. W. Kinney, "Oscillations in Singularly Perturbed Systems and the Conley Index," thesis, University of Minnesota, 1996.
7. W. Kinney, A new perspective on some results of Conley for singularly perturbed differential equations, in preparation.
8. K. Mischaikow, Conley index theory: Some recent developments, CDSNS94-181, Georgia Institute of Technology, 1994.

9. C. McCord, K. Mischaikow, and M. Mrozek, Zeta functions, periodic trajectories, and the Conley index, *J. Differential Equations* **121** (1995), 258–292.
10. K. Mischaikow, M. Mrozek, and J. Reineck, Singular Index Pairs, *J. Dynam. Differential Equations* **11** (1999), 399–425.
11. J. Rinzel, Bursting oscillations in an excitable membrane model, *in* "Ordinary and Partial Differential Equations" (B. D. Sleeman and R. J. Jarvis, Eds.), Springer-Verlag, New York, 1985, pp. 304–316.
12. D. Terman, Chaotic spikes arising from a model of bursting in excitable membranes, *SIAM J. Appl. Math.* **51** (1991), 1418–1450.
13. D. Terman, The transition from bursting to continuous spiking in excitable membrane models, *J. Nonlin. Sci.* **2** (1992), 135–182.