Advances in Applied Mathematics **24,** 22–56 (2000)
doi:10.1006/aama.1999.0665, available online at http://www.idealibrary.com on IDEAL

Enumeration of m-Ary Cacti

Miklós Bóna,[1] Michel Bousquet, Gilbert Labelle,
and Pierre Leroux

Received January 22, 1999; accepted August 3, 1999

The purpose of this paper is to enumerate various classes of cyclically colored m-gonal plane cacti, called m-ary cacti. This combinatorial problem is motivated by the topological classification of complex polynomials having at most m critical values, studied by Zvonkin and others. We obtain explicit formulae for both labelled and unlabelled m-ary cacti, according to (i) the number of polygons, (ii) the vertex-color distribution, (iii) the vertex-degree distribution of each color. We also enumerate m-ary cacti according to the order of their automorphism group. Using a generalization of Otter's formula, we express the species of m-ary cacti in terms of rooted and of pointed cacti. A variant of the m-dimensional Lagrange inversion is then used to enumerate these structures. The method of Liskovets for the enumeration of unrooted planar maps can also be adapted to m-ary cacti. © 2000 Academic Press

1. INTRODUCTION

A *cactus* is a connected simple graph in which each edge lies in exactly one elementary cycle. It is equivalent to say that all blocks (two-connected components) of a cactus are edges or elementary cycles, i.e., polygons. An *m-gonal cactus* (m-cactus for short) is a cactus all of whose polygons are m-gons, for some fixed $m \geq 2$. By convention, a 2-cactus is simply a tree. These graphs were previously called "Husimi trees," and their definition was given by Harary and Uhlenbeck [12] following a paper by Husimi [13] on the cluster integrals in the theory of condensation in statistical mechanics. See also Riddell [18] and Uhlenbeck and Ford [22]. Their enumeration

[1] Present address: Dept. of Math., Univ. of Florida, Gainesville.
[2] With the partial support of FCAR (Québec) and CRSNG (Canada).

0196-8858/00 $35.00

according to the number of polygons was carried out in [12]. See also Harary and Palmer [11] and [13].

A *plane* m-cactus is an embedding of an m-cactus into the plane so that every edge is incident with the unbounded region. An *m-ary cactus* is a plane m-cactus whose vertices are cyclically m-colored $1, 2, \dots, m$ counterclockwise within each m-gon. For technical reasons, we also consider a single vertex colored in any one of the m colors to be an m-ary cactus. A quaternary ($m = 4$) cactus is shown in Fig. 1.

We define the *degree* of a vertex in a m-ary cactus to be the number of m-gons adjacent to that vertex. Note that it is twice the number of edges adjacent to the given vertex, for $m \geq 3$. Given an m-ary cactus κ, let n_{ij} denote the number of vertices of color i and degree j of κ and set $\mathbf{n}_i = (n_{i0}, n_{i1}, n_{i2}, \dots)$. The *vertex-degree distribution* of κ is given by the $m \times \infty$ matrix $N = (n_{ij})$, where $1 \leq i \leq m$ and $j \geq 0$. Note that $n_i = \sum_j n_{ij}$ is the number of vertices of color i and $n = \sum_i n_i$ is the total number of vertices of κ. The *vertex-color distribution* of κ is defined to be the vector $\mathbf{n} = (n_1, n_2, \dots, n_m)$. Also, let p denote the number of polygons in κ.

For the quaternary cactus of Fig. 1, the distributions are

$$\mathbf{n}_1 = (0,7,1,0,1,0,\dots) = 1^7 2^1 4^1, \qquad \mathbf{n}_2 = (0,7,3,0,0,0,\dots) = 1^7 2^3,$$

$$\mathbf{n}_3 = (0,8,1,1,0,0,\dots) = 1^8 2^1 3^1, \qquad \mathbf{n}_4 = (0,9,2,0,0,0,\dots) = 1^9 2^2,$$

$$n_1 = 9, \quad n_2 = 10, \quad n_3 = 10, \quad n_4 = 11, \quad n = 40, \quad \text{and } p = 13.$$

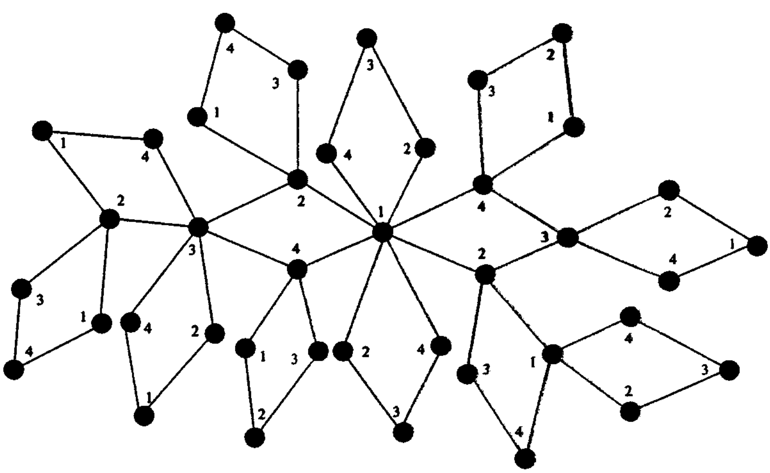

FIG. 1. A quaternary cactus.

Clearly, for any m-ary cactus with n vertices and p polygons we have

$$\sum_j jn_{ij} = p, \qquad \text{for all } i, \tag{1}$$

since each polygon contains exactly one vertex of color i, and also

$$n = (m-1)p + 1, \tag{2}$$

as one sees easily by induction on p.

The goal of this paper is to enumerate various classes of m-ary cacti according to the number n of vertices or p of polygons, to the vertex color distribution $\mathbf{n} = (n_1, n_2, \dots, n_m)$, and to the vertex-degree distribution $N = (n_{ij})_{1 \le i \le m,\ j \ge 0}$. The species we enumerate are the following:

(1) $\mathscr{K}$, the class of all m-ary cacti.

(2) $\mathscr{K}^{\bullet_i}$, the class of m-ary cacti *pointed* at vertex of color i (see Fig. 5).

(3) $\mathscr{K}^{\diamond}$, the class of *rooted* (i.e., pointed at a polygon) cacti (see Fig. 6).

(4) $\mathscr{A}_i$, the class of m-ary cacti, *planted* at a vertex v of color i, i.e., pointed at v with a pair of half edges attached to v contributing to its degree (see Fig. 4).

(5) $\bar{\mathscr{K}}$, the class of *asymmetric* m-ary cacti.

(6) $\mathscr{K}_{=s}$ and $\mathscr{K}_{\ge s}$, the classes of m-ary cacti whose automorphism group is of order s, and a multiple of s, respectively, where $s \ge 2$.

The motivation for the enumeration of m-ary cacti comes from the topological classification of polynomials having m critical values. More precisely, two complex polynomials $p_1(z)$ and $p_2(z)$ are said to be *topologically equivalent* if there exists two orientation-preserving homeomorphisms of the plane, h_1 and h_2, such that $h_1(p_1(z)) = p_2(h_2(z))$. Also, a complex number v is called a *critical value* of the polynomial $p(z)$ if the equation $p(z) = v$ has at least one multiple root; all the roots of the equation are then called *critical points*. Now if a polynomial $p(z)$ has m critical values $\{v_1, v_2, \dots, v_m\}$, we form a simple curvilinear m-gon joining these m critical values $\{v_1, \dots, v_m\}$. Then the preimage under p of this polygon yields an m-ary cactus whose vertex-degree distribution corresponds to the multiplicities of the critical points. For example, Fig. 2 shows the cactus corresponding to a degree 8 polynomial $p(z) = c_0 + c_1 z + \cdots + c_8 z^8$ having three critical values v_1, v_2, v_3, whose derivative is of the form $p'(z) = (z-b)(z-1)^3(z+\frac{1}{2})^2(z-i)$, where $b \in \mathbf{C}$ is chosen so that $p(b) = p(1) = v_1$, and $p(-\frac{1}{2}) = v_2$, $p(i) = v_3$.

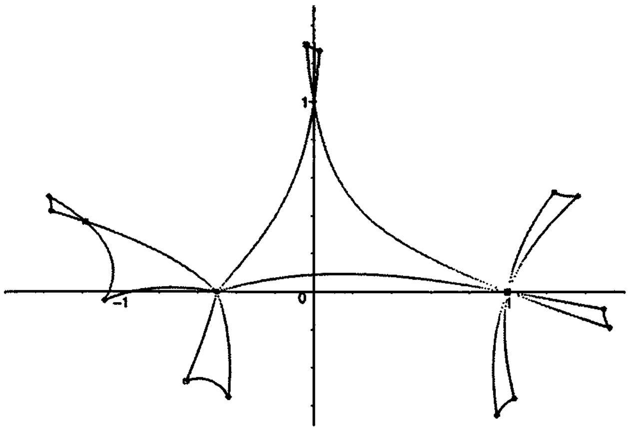

FIG. 2. Cactus associated to a polynomial of degree 8, having three critical values.

This is a crucial step in the topological classification but the equivalence classes of polynomials are in fact represented by the orbits of m-ary cacti under the action of the braid group. See [7] and [14] for more details. The enumeration of these orbits is an open problem.

This work extends to general $m \geq 2$, previous results of Labelle and Leroux [16] on bicolored plane trees. It also extends results of Goulden and Jackson [9] on the enumeration of rooted m-ary cacti. They show that rooted m-ary cacti with p polygons, having vertex-degree distribution $N = (n_{ij})$ are in one-to-one correspondence with decompositions of the circular permutation $(1, 2, \ldots, p)$ as the product $g_1 g_2 \cdots g_m$ of m permutations, where g_i has cyclic type $(1^{n_{i1}} 2^{n_{i2}} \cdots)$.

In Section 2, we state the main functional equations relating to the various species of m-ary cacti. We show that all these species can be expressed in terms of *planted* m-ary cacti which, themselves, satisfy functional equations opening the way to Lagrange inversion. Of particular importance is a dissymmetry theorem which relates (ordinary) m-ary cacti to pointed and rooted m-ary cacti. This theorem is closely related to the dissimilarity characteristic theorem for trees, due to Otter and extended to cacti by Harary and Norman [10]. The treelike structure of a cactus can be emphasized by using an equivalent representation, where a white (= color 0) vertex is placed within each polygon and joined to the vertices of the polygon, after which the edges of the polygons can be erased. This gives a bijection between m-ary cacti having p polygons and $(1 + m)$-colored trees having p vertices of color 0, all of degree m. The bijection is illustrated in Fig. 3 for a ternary ($m = 3$) cactus.

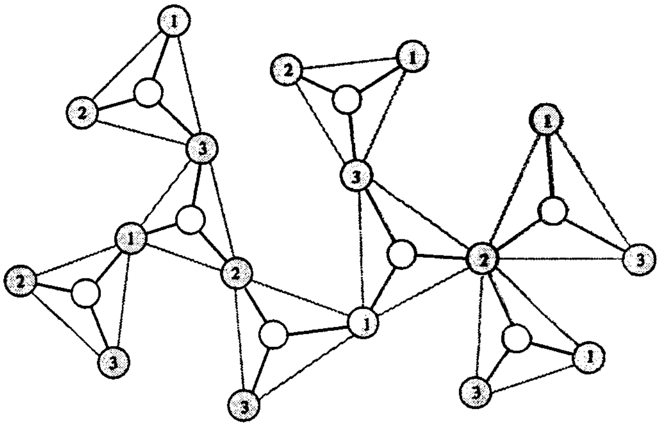

FIG. 3. Tree-like structure of a ternary cactus.

In Section 3 we establish a particular form of multidimensional Lagrange inversion, which is well adapted to the present situation. It extends the previously known two-dimensional case, in the spirit of Chottin's formulae [5, 6], and use the crucial observation due to Goulden and Jackson [9] that a certain Jacobian matrix reduces to a rank-1 matrix. We then use these results in Section 4 to enumerate both labelled and unlabelled m-ary cacti, including the special classes of planted, pointed, and rooted m-ary cacti, according to the number of vertices (or of polygons), to their vertex-color distribution and their vertex-degree distribution. We also enumerate m-ary cacti according to the order of their automorphism group, including the asymmetric ones.

An alternate method can be used for the enumeration of unlabelled m-ary cacti. It is based on a paper of Liskovets [17] on the enumeration of nonrooted planar maps which uses the concept of quotient of a labelled planar map under an automorphism. See Bousquet [2].

In the last section, we present some related enumerative results, concerning labelled *free* m-ary cacti and unlabelled plane m-gonal cacti having p polygons. We also state a closely related result due to Bousquet-Mélou and Schaeffer [4] on rooted m-ary constellations, having p polygons.

Three tables are given in the paper, containing numerical results which illustrate some of the formulas.

We have used the species formulation as a helpful unifying framework in this paper. A basic reference for the theory of species is found in [1]. However, the paper remains accessible to anyone with a knowledge of Pólya theory applied to graphical enumeration (see [11]).

2. FUNCTIONAL EQUATIONS FOR m-ARY CACTI

2.1. *Vertex-Color Distribution*

We consider the class $\mathscr{K}$ of m-ary cacti as an m-sort species. This means that an m-ary cactus is seen as a structure constructed on an m-tuple of sets $(U_1, U_2, \ldots, U_m)$, the elements of U_i being the (labels for) vertices of color i. Moreover, the relabelling bijections, and in particular, the automorphisms of m-ary cacti are required to preserve the sorts of elements, i.e., the colors. Although we are interested in the enumeration of unlabelled cacti, it is easier to establish the functional equations by giving bijections between labelled structures. If we ensure that these bijections are natural, that is, that they commute with any relabelling, thus defining isomorphisms of species, then the consequences for both the labelled (exponential) generating function

$$\mathscr{K}(x_1, x_2, \ldots, x_m) = \sum_{n_1, n_2, \ldots, n_m} |\mathscr{K}[n_1, n_2, \ldots, n_m]| \frac{x_1^{n_1} \cdots x_m^{n_m}}{n_1! \cdots n_m!} \tag{3}$$

and the unlabelled (ordinary) generating function

$$\tilde{\mathscr{K}}(x_1, x_2, \ldots, x_m) = \sum_{n_1, n_2, \ldots, n_m} \tilde{\mathscr{K}}(n_1, n_2, \ldots, n_m) x_1^{n_1} \cdots x_m^{n_m} \tag{4}$$

are automatic. Here $\mathscr{K}[n_1, n_2, \ldots, n_m]$ denotes the set of m-ary cacti over the multiset $[n_1] + [n_2] + \cdots + [n_m]$, with $[n] = \{1, 2, \ldots, n\}$, and $\tilde{\mathscr{K}}(n_1, n_2, \ldots, n_m)$ denotes the number of unlabelled m-ary cacti having n_i vertices of color i, for $i = 1, \ldots, m$.

Note that the plane embedding of an m-ary cactus κ is completely characterized by the specification, for each vertex v of κ, of a circular permutation on the polygons adjacent to v. We now present functional equations related to the m-sort species $\mathscr{A}_i$, of m-ary cacti, *planted* at a vertex of color i, $\mathscr{K}^{\bullet_i}$, of m-ary cacti, *pointed* at a vertex of color i, $\mathscr{K}^{\diamond}$, of *rooted* m-ary cacti.

The following notations are used: X_i denotes the species of singletons of sort (or color) i, $\mathscr{C}$ denotes the species of (nonempty) circular permutations, L denotes the species of lists (linear orders) and $\hat{\mathscr{A}}_i := \prod_{j \neq i} \mathscr{A}_j$ denotes the product of all $\mathscr{A}_j$ except $\mathscr{A}_i$.

PROPOSITION 1. *We have the following isomorphisms of species, for* $i = 1, \ldots, m$:

$$\mathscr{A}_i = X_i L\left(\hat{\mathscr{A}}_i\right), \tag{5}$$

$$\mathscr{K}^{\bullet_i} = X_i\left(1 + \mathscr{C}\left(\hat{\mathscr{A}}_i\right)\right), \tag{6}$$

$$\mathscr{K}^{\diamond} = \mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_m. \tag{7}$$

Proof. The plane embedding of a planted m-ary cactus determines a linear order on the neighboring polygons of the pointed vertex. If this vertex, say of color 1, is removed, each of these adjacent polygons can be simply decomposed into the product of $m-1$ planted m-ary cacti with roots of color $2, 3, \ldots, m$. Since this data completely specifies the planted cactus, we have Eq. (5). See Fig. 4 for an illustration of the equation $\mathscr{A}_1 = X_1 L(\mathscr{A}_2 \mathscr{A}_3)$ in the ternary case.

Equation (6) is similar to (5) except that for pointed cacti the polygons adjacent to the pointed vertex can freely rotate around it. Figure 5 illustrates the equation $\mathscr{K}^{\bullet_3} = X_3(1 + C(\mathscr{A}_1 \mathscr{A}_2))$. Equation (7) is immediate; see Fig. 6. ∎

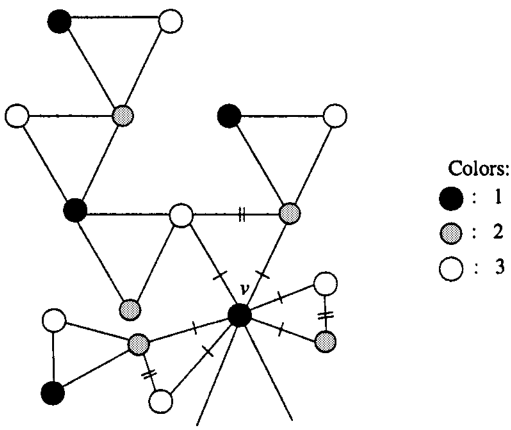

FIG. 4. A planted ternary cactus.

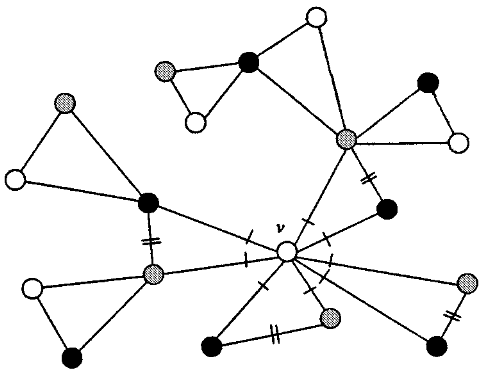

FIG. 5. A ternary cactus pointed at vertex v.

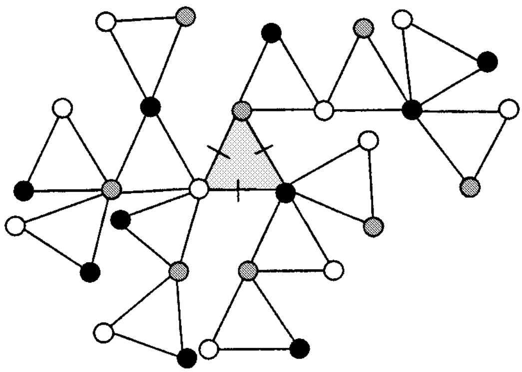

FIG. 6. A rooted ternary cactus.

Remark that Eqs. (5) and (7) are essentially due to Goulden and Jackson [9].

Recall that in a connected graph g, a vertex x belongs to the *center* of g if the maximal distance from x to any other vertex is minimal. In particular, if g is a cactus, then it is easy to see that the center of g is either a single vertex or a polygon. Now let κ be an m-ary cactus. In this case we define the *center* in a slightly different way: if the previous definition yields a vertex as the center of κ, then we leave this definition unchanged. If the previous definition yields a polygon p as the center of κ, then we take the color-1 vertex of p to be the center of κ. So now the center of an m-ary cactus is always a vertex.

THEOREM 2 Dissymmetry theorem for m-ary cacti. *There is an isomorphism of species*

$$\mathscr{K}^{\bullet_1} + \mathscr{K}^{\bullet_2} + \cdots + \mathscr{K}^{\bullet_m} = \mathscr{K} + (m-1)\mathscr{K}^{\diamond}. \tag{8}$$

Proof. For clarity, we prove the theorem for $m = 3$. That is, we establish an isomorphism $\mathscr{K}^{\bullet_1} + \mathscr{K}^{\bullet_2} + \mathscr{K}^{\bullet_3} = \mathscr{K} + 2\mathscr{K}^{\diamond}$, the proof for general m being analogous.

The left-hand side corresponds to cacti which have been pointed at a vertex, of color 1, 2, or 3. The first term on the right-hand side corresponds to cacti which have been pointed in a canonical way, at their center. So what remains to construct is a natural bijection from triangular cacti pointed *not* in their center onto two cases of $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3$ structures.

Suppose that a ternary cactus κ has been pointed at a vertex x of color 1 which is different from the center c of κ (see Fig. 7). Let the shortest path from x to c start with the edge $e = \{x, y\}$, and let t be the unique triangle containing e. Then we cut the three edges of t and thus separate

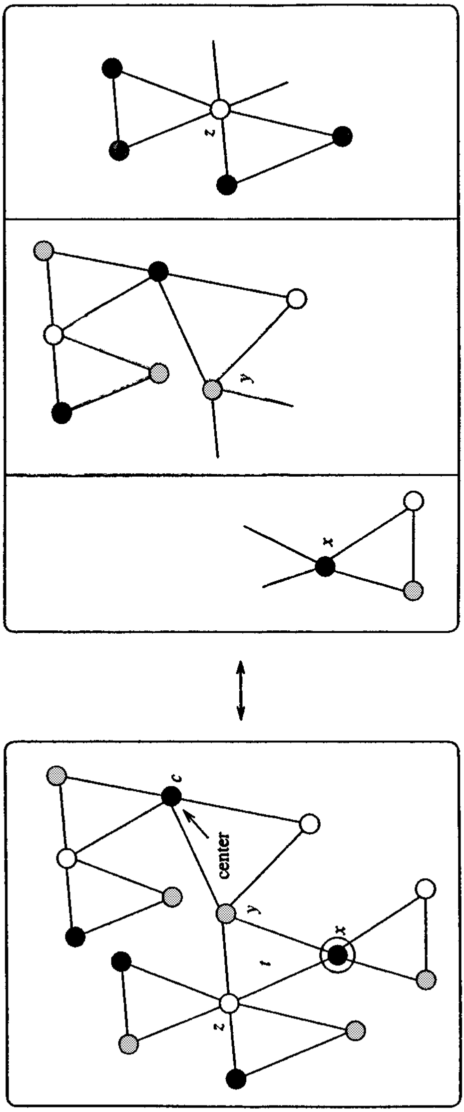

FIG. 7. $\mathscr{K}^{\bullet 1} + \mathscr{K}^{\bullet 2} + \mathscr{K}^{\bullet 3} = \mathscr{K} + 2\mathscr{K}^{\diamond}$ $(m = 3)$.

the cactus into three smaller cacti which are planted in a vertex of color 1, 2, and 3, respectively. We thus obtain an $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3$ structure. It is easy to see that we could have obtained this structure in another way. Indeed, if the vertices of t are x, y, and z, then pointing the cactus at z would have given the same decomposition. So this operation does define a map into $2\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3$.

To see that the algorithm is reversible, take any 3-tuple of cacti which are planted in vertices x, y, and z of color 1, 2, and 3, respectively. Join x, y, and z by a triangle to get a cactus, and look for its center c. If c comes from the component of x, then we can point either y or z in the cactus, if c comes from the component of y, then we can point either x or z and finally, if c comes from the component of z, then we can point either x or y. It is then a simple matter to number each of these cases in order to make the correspondence bijective, completing the proof. ∎

COROLLARY 3. *The species $\mathscr{K}$ of m-ary cacti can be written as*

$$\mathscr{K} = \sum_{i=1}^{m} \mathscr{K}^{\bullet_i} - (m-1)\mathscr{K}^{\diamond}$$
$$= \sum_{i=1}^{m} X_i\Big(1 + C\big(\hat{\mathscr{A}}_i\big)\Big) - (m-1)\prod_{i=1}^{m}\mathscr{A}_i. \tag{9}$$

The consequences for the labelled and unlabelled generating functions then follow from general principles. For $i = 1, \ldots, m$, we have, with $\mathbf{x} = (x_1, x_2, \ldots, x_m)$,

$$\mathscr{A}_i(\mathbf{x}) = x_i \frac{1}{1 - \hat{\mathscr{A}}_i(\mathbf{x})}, \tag{10}$$

$$\tilde{\mathscr{A}}_i(\mathbf{x}) = x_i \frac{1}{1 - \tilde{\hat{\mathscr{A}}}_i(\mathbf{x})}, \tag{11}$$

from which it follows that $\tilde{\mathscr{A}}_i(\mathbf{x}) = \mathscr{A}_i(\mathbf{x})$ since $\tilde{\hat{\mathscr{A}}}_i = \hat{\mathscr{A}}_i$. This expresses the fact that planted cacti are asymmetric structures. Moreover,

$$\mathscr{K}^{\bullet_i}(\mathbf{x}) = x_i\left(1 + \log\frac{1}{1 - \hat{\mathscr{A}}_i(\mathbf{x})}\right), \tag{12}$$

$$\tilde{\mathscr{K}}^{\bullet_i}(\mathbf{x}) = x_i\left(1 + \sum_{d\geq 1}\frac{\phi(d)}{d}\log\frac{1}{1 - \hat{\mathscr{A}}_i(\mathbf{x}^d)}\right), \tag{13}$$

where $\mathbf{x}^d := (x_1^d, x_2^d, \ldots, x_m^d)$ and ϕ is the Euler function. We also have

$$\mathscr{K}^{\diamond}(\mathbf{x}) = \tilde{\mathscr{K}}^{\diamond}(\mathbf{x}) = \mathscr{A}_1(\mathbf{x}) \cdots \mathscr{A}_m(\mathbf{x}) \tag{14}$$

and finally,

$$\tilde{\mathscr{K}}(\mathbf{x}) = \sum_{i=1}^{m} \tilde{\mathscr{K}}^{\bullet_i}(\mathbf{x}) - (m-1)\mathscr{K}^{\diamond}(\mathbf{x}). \tag{15}$$

2.2. *Vertex-Degree Distribution*

In order to enumerate m-ary cacti according to their degree distributions, we introduce weights in the form of monomials $w(\kappa) = \prod_{i,j} r_{ij}^{n_{ij}}$ with $i = 1, \ldots, m$, and $j \geq 0$, for a cactus κ having vertex-degree distribution $N = (n_{ij})$. In other words, the variable r_{ij} acts as a counter for (or marks) vertices of color i and degree j. We also use the notation $\mathbf{r}_i$ to denote the sequence $(r_{i0}, r_{i1}, \ldots)$. We denote by $\mathscr{K}_w, \mathscr{K}_w^{\diamond}$, and $\mathscr{K}_w^{\bullet_i}$ the corresponding species of m-ary cacti, weighted in this manner. We denote by $\mathscr{A}_{i,\mathbf{r}}$ the species of planted (at a vertex of color i) m-ary cacti similarly weighted by degree. The functional equations (5)–(9) can then be extended as follows:

$$\mathscr{A}_{i,\mathbf{r}} = X_i\left(r_{i,1} + r_{i,2}\hat{\mathscr{A}}_{i,\mathbf{r}}^2 + r_{i,3}\hat{\mathscr{A}}_{i,\mathbf{r}}^3 + \cdots\right) \tag{16}$$

where $\hat{\mathscr{A}}_{i,\mathbf{r}} = \prod_{j \neq i} \mathscr{A}_{i,\mathbf{r}}$,

$$\mathscr{K}_w^{\bullet_i} = X_i\left(r_{i,0} + r_{i,1}\mathscr{C}_1\left(\hat{\mathscr{A}}_{i,\mathbf{r}}\right) + r_{i,2}\mathscr{C}_2\left(\hat{\mathscr{A}}_{i,\mathbf{r}}\right) + \cdots\right), \tag{17}$$

where $\mathscr{C}_k$ denotes the species of circular permutations of length k,

$$\mathscr{K}_w^{\diamond} = \prod_{i=1}^{m} \mathscr{A}_{i,\mathbf{r}} \tag{18}$$

and

$$\mathscr{K}_w = \sum_{i=1}^{m} \mathscr{K}_w^{\bullet_i} - (m-1)\mathscr{K}_w^{\diamond}. \tag{19}$$

The important point here is that the weights behave multiplicatively, with respect to the operations of product and partitional composition. The consequences for the labelled and unlabelled generating functions are

$$\mathscr{A}_{i,\mathbf{r}}(\mathbf{x}) = x_i\Big(r_{i,1} + r_{i,2}\hat{\mathscr{A}}_{i,\mathbf{r}}(\mathbf{x}) + r_{i,3}\hat{\mathscr{A}}^2_{i,\mathbf{r}}(\mathbf{x}) + \cdots\Big), \tag{20}$$

$$\tilde{\mathscr{A}}_{i,\mathbf{r}}(\mathbf{x}) = \mathscr{A}_{i,\mathbf{r}}(\mathbf{x}), \tag{21}$$

$$\mathscr{K}_w^{\bullet_i}(\mathbf{x}) = x_i\left(r_{i,0} + \sum_{h\geq 1}\frac{r_{i,h}}{h}\hat{\mathscr{A}}^h_{i,\mathbf{r}}(\mathbf{x})\right), \tag{22}$$

$$\tilde{\mathscr{K}}_w^{\bullet_i}(\mathbf{x}) = x_i\left(r_{i,0} + \sum_{h\geq 1}\frac{r_{i,h}}{h}\sum_{d|h}\phi(d)\hat{A}^{h/d}_{i,\mathbf{r}^d}(\mathbf{x}^d)\right), \tag{23}$$

where $\mathbf{r}^d$ denotes the set of variables $\{r^d_{i,j}\}$, for $i = 1,\ldots,m$, $j \geq 0$. We also have

$$\mathscr{K}_w^{\diamond}(\mathbf{x}) = \tilde{\mathscr{K}}_w^{\diamond}(\mathbf{x}) = \prod_{i=1}^{m}\mathscr{A}_{i,\mathbf{r}}(\mathbf{x}) \tag{24}$$

and finally,

$$\tilde{\mathscr{K}}_w(\mathbf{x}) = \sum_{i=1}^{m}\tilde{\mathscr{K}}_w^{\bullet_i}(\mathbf{x}) - (m-1)\mathscr{K}_w^{\diamond}(\mathbf{x}). \tag{25}$$

2.3. *One-Sort m-Ary Cacti*

If neither the vertex-color nor the vertex-degree distribution are desired, but only the number of vertices or, equivalently, of polygons, then the enumeration is easier to carry out since one-dimensional Lagrange inversion will suffice. Indeed, we can consider the various species of m-ary cacti introduced earlier as one-sort species, as Fig. 1 suggests. This means that the underlying set (of vertex labels) is independent of the colors and that the relabellings can be arbitrary, although isomorphisms are still required to preserve colors. We use the same letters $\mathscr{K}$, $\mathscr{K}^{\bullet_i}$, $\mathscr{K}^{\diamond}$, $\mathscr{A}_i$ to denote these one-sort species. Equations (6)–(8) are still valid in this context, with the following simplifications: first, all singleton species X_i should be replaced by X; second, the addition of 1 modulo m to the colors induces isomorphisms of species $\mathscr{A}_1 \cong \mathscr{A}_2 \cong \cdots \cong \mathscr{A}_m$, and we write $\mathscr{A}$ for this common species, and also $\mathscr{K}^{\bullet_1} \cong \mathscr{K}^{\bullet_2} \cong \cdots \cong \mathscr{K}^{\bullet_m}$.

Equation (5) then simplifies to

$$\mathscr{A} = XL(\mathscr{A}^{m-1}) = \frac{X}{1-\mathscr{A}^{m-1}}, \tag{26}$$

which implies $\mathscr{A} = X + \mathscr{A}^m$. Moreover, Eq. (7) reduces to

$$\mathscr{K}^{\diamond} = \mathscr{A}^m = \mathscr{A} - X, \tag{27}$$

while (6) reduces to

$$\mathscr{K}^{\bullet_i} = X\big(1 + C(\mathscr{A}^{m-1})\big). \tag{28}$$

Finally, the dissymmetry theorem for m-ary cacti takes the form

$$\begin{aligned} \mathscr{K} &= \mathscr{K}^{\bullet} - (m-1)\mathscr{K}^{\diamond} \\ &= mX\big(1 + C(\mathscr{A}^{m-1})\big) - (m-1)(\mathscr{A} - X), \end{aligned} \tag{29}$$

where $\mathscr{K}^{\bullet}$ denotes the one-sort species of pointed at (any color) m-ary cacti.

3. MULTIDIMENSIONAL LAGRANGE INVERSION TECHNIQUES

In this section we establish a special form of multidimensional Lagrange inversion, which can be directly applied to m-ary cacti. First recall the standard form, due to Good (see Theorem 1.2.9, 1 of [8] or the equivalent formula (28b) of [1]).

THEOREM 4 Good's Lagrange Inversion Formula. *Let $A_1, A_2, \dots, A_m$ be formal power series in the variables $x_1, x_2, \dots, x_m$ such that the relations $A_i = x_i R_i(A_1, A_2, \dots, A_m)$ are satisfied for all $i = 1, \dots, m$. Then for any formal power series $F(t_1, t_2, \dots, t_m)$ we have*

$$\begin{aligned} &\big[x_1^{n_1} \cdots x_m^{n_m}\big] F\big(A_1(x), \dots, A_m(x)\big) \\ &\quad = \big[t_1^{n_1} \cdots t_m^{n_m}\big] F(\mathbf{t}) |K(\mathbf{t})| R_1^{n_1}(\mathbf{t}) \cdots R_m^{n_m}(\mathbf{t}), \end{aligned} \tag{30}$$

where $\mathbf{t} = (t_1, t_2, \dots, t_m)$ and $K(\mathbf{t})$ is the $m \times m$ matrix whose (i, j)th entry is

$$K(\mathbf{t})_{ij} = \delta_{ij} - \frac{t_j}{R_i(\mathbf{t})} \cdot \frac{\partial R_i(\mathbf{t})}{\partial t_j}. \tag{31}$$

There is a particularly simple two-dimensional case of this formula, the alternating case, which we call the *Chottin formula*. In the papers [5, 6], Chottin worked extensively on the two-dimensional Lagrange inversion and its combinatorial proof.

THEOREM 5 Chottin Formula. *Let $A(x, y)$ and $B(x, y)$ be two formal power series satisfying the relations $A = x\Phi(B)$ and $B = y\Psi(A)$, where $\Phi(t)$ and $\Psi(s)$ are given formal power series. Then, for any non-negative integers α and β we have*

$$[x^n y^m] A^\alpha B^\beta = \left(1 - \frac{(n-\alpha)(m-\beta)}{nm}\right)[s^{n-\alpha} t^{m-\beta}]\Phi^n(t)\Psi^m(s),$$
$$n \geq 1, \quad m \geq 1. \tag{32}$$

We extend this result into m dimensions.

THEOREM 6 Generalized Chottin Formula. *Let $A_1, A_2, \ldots, A_m$ be formal power series in the variables $x_1, x_2, \ldots, x_m$ such that for $i = 1, \ldots, m$, the relations $A_i = x_i \Phi_i(\hat{A}_i)$ are satisfied, where the Φ_i are given formal power series of one variable, and $\hat{A}_i = \prod_{j \neq i} A_j$. Also let $n_1, \ldots, n_m$ be integers ≥ 1 and let $\alpha_1, \ldots, \alpha_m$ be non-negative integers. Set $n = \sum_{i=1}^m n_i$ and $\alpha = \sum_{i=1}^m \alpha_i$. Suppose that the following coherence conditions are satisfied*

$$n_i \geq \alpha_i, \qquad \frac{n-\alpha}{m-1} = \beta \text{ is an integer},$$

and set $\beta_i = \beta - n_i + \alpha_i$. Then

$$\left[x_1^{n_1} \cdots x_m^{n_m}\right] A_1^{\alpha_1} \cdots A_m^{\alpha_m} = D \cdot \left[s_1^{\beta_1} \cdots s_m^{\beta_m}\right] \Phi_1^{n_1}(s_1) \cdots \Phi_m^{n_m}(s_m), \tag{33}$$

where

$$D = \prod_{i=1}^m \left(1 + \frac{\beta_i}{n_i}\right) - \sum_{j=1}^m \frac{\beta_j}{n_j} \prod_{i \neq j} \left(1 + \frac{\beta_i}{n_i}\right). \tag{34}$$

Proof. We use Theorem 4 with $R_i(t_1, \ldots, t_m) = \Phi_i(\widehat{t_i})$, where $\widehat{t_i} = \prod_{j \neq i} t_j$. We take advantage of some useful observations made by Goulden and Jackson in [9] to compute the determinant $|K(\mathbf{t})|$. Indeed, for $i = 1, 2, \ldots, m$, we have

$$t_i \frac{\partial R_i}{\partial t_i} = 0,$$

as $R_i(\mathbf{t})$ does not depend on t_i, and for $j \neq i$,

$$t_j \frac{\partial R_i}{\partial t_j} = \widehat{t_i} \Phi_i'\left(\widehat{t_i}\right)$$

which is independent of j. We set $\Psi_i(\widehat{t_i}) = \widehat{t_i}\Phi_i'(\widehat{t_i})$ and write $\Psi_i = \Psi_i(\widehat{t_i})$, $\Phi_i = \Phi_i(\widehat{t_i})$.

The definition of $K(\mathbf{t})$ then yields, after routine transformations,

$$|K(\mathbf{t})| = \frac{\prod_{i=1}^{m}(\Phi_i + \Psi_i)}{\prod_{i=1}^{m}\Phi_i} \cdot \left|\delta_{ij} - \frac{\Psi_i}{\Psi_i + \Phi_i}\right|.$$

Let $M_{ij} = -\Psi_i/(\Psi_i + \Phi_i)$ and note that the rank of M is 1 since all its columns are equal. So, by the Sherman–Morrison formula [20] we have $|I + M| = 1 + \operatorname{trace}(M)$. Therefore, the previous equation yields

$$|K(\mathbf{t})| = \frac{\prod_{i=1}^{m}(\Phi_i + \Psi_i)}{\prod_{i=1}^{m}\Phi_i} \cdot \left(1 - \sum_{i=1}^{m} \frac{\Psi_i}{\Psi_i + \Phi_i}\right). \tag{35}$$

It follows from the Lagrange inversion formula (30) that

$$\begin{aligned}
&[x_1^{n_1} \cdots x_m^{n_m}] A_1^{\alpha_1} \cdots A_m^{\alpha_m} \\
&\quad = [t_1^{n_1} \cdots t_m^{n_m}] t_1^{\alpha_1} \cdots t_m^{\alpha_m} \cdot |K(\mathbf{t})| \prod_{i=1}^{m} \Phi_i^{n_i} \\
&\quad = [t_1^{n_1-\alpha_1} \cdots t_m^{n_m-a_m}] \left(\prod_{i=1}^{m} \Phi_i^{n_i-1}(\Phi_i + \Psi_i)\right)\left(1 - \sum_{i=1}^{m} \frac{\Psi_i}{\Psi_i + \Phi_i}\right).
\end{aligned} \tag{36}$$

Now let us define the coefficients c_{i,β_i} by

$$\Phi_i^{n_i}\left(\widehat{t_i}\right) = \sum_{\beta_i \geq 0} c_{i,\beta_i} \hat{t}_i^{\beta_i}, \tag{37}$$

which implies, by the definition of Ψ_i that

$$\Phi_i^{n_i-1}\left(\widehat{t_i}\right)\Psi_i\left(\widehat{t_i}\right) = \sum_{\beta_i \geq 0} \frac{\beta_i}{n_i} c_{i,\beta_i} \hat{t}_i^{\beta_i}. \tag{38}$$

Recall that $n = \sum_{i=1}^{m} n_i$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Then $t_1^{n_1-\alpha_1} \cdots t_m^{n_m-\alpha_m} = \hat{t}_1^{\beta_1} \cdots \hat{t}_m^{\beta_m}$ if and only if $\beta - \beta_i = n_i - \alpha_i$ for all i, where $\beta = \sum_{i=1}^{m} \beta_i$. Summing these equations for $i = 1, \ldots, m$ yields $(m-1)\beta = n - \alpha$ and also $\beta_i =$

$\beta - n_i + \alpha_i$. We then conclude that (36) equals

$$\left[\hat{t}_1^{\beta_1} \cdots \hat{t}_m^{\beta_m}\right]\left(\prod_{i=1}^{m} \Phi_i^{n_i-1}(\Phi_i + \Psi_i)\right) \cdot \left(1 - \sum_{i=1}^{m} \frac{\Psi_i}{\Psi_i + \Phi_i}\right)$$
$$= D \cdot \left[s_1^{\beta_1} \cdots s_m^{\beta_m}\right] \Phi_1^{n_1}(s_1) \cdots \Phi_m^{n_m}(s_m),$$

where D is given by (34), completing the proof. ∎

The following special cases are particularly useful.

(1) $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$, with the condition that $(n-1)/(m-1) = p$ is a positive integer. Then we find that $\beta = p - 1$, $\beta_i = p - n_i$ and $D = p^{m-1}/\prod_{i=1}^m n_i$, and we have

$$\left[x_1^{n_1} \cdots x_m^{n_m}\right] A_1 \cdots A_m$$
$$= \frac{p^{m-1}}{\prod_{i=1}^m n_i} \cdot \left[s_1^{p-n_1} \cdots s_m^{p-n_m}\right] \Phi_1^{n_1}(s_1) \cdots \Phi_m^{n_m}(s_m). \tag{39}$$

(2) $\alpha_1 = 0$, $\alpha_2 = \cdots = \alpha_m = k \geq 1$, with the condition that $(\sum_i a_i)/(m-1) = q$ is an integer. Then we find that $\beta = q - k$, $\beta_1 = q - a_1 - k$, $\beta_i = q - a_i$, for $i = 2, \ldots, m$, and that $D = q^{m-2}k/\prod_{i \neq 1} a_i$, and we have, writing $\hat{A}_1 = A_2 A_3 \cdots A_m$,

$$\left[x_1^{a_1} \cdots x_m^{a_m}\right] \hat{A}_1^k(\mathbf{x})$$
$$= \frac{q^{m-2}k}{\prod_{i \neq 1} a_i} \cdot \left[s_1^{q-a_1-k} s_2^{q-a_2} \cdots s_m^{q-a_m}\right] \Phi_1^{a_1}(s_1) \cdots \Phi_m^{a_m}(s_m). \tag{40}$$

(3) Under the condition that $(\sum_i a_i)/(m-1) = q$ is an integer, it follows from (40) that for any formal power series $F(s)$ we have

$$\left[x_1^{a_1} \cdots x_m^{a_m}\right] F\left(\hat{A}_1\right) = \frac{q^{m-2}}{\prod_{i \neq 1} a_i} \cdot \left[s_1^{q-a_1-1} s_2^{q-a_2} \cdots s_m^{q-a_m}\right]$$
$$\times F'(s_1) \Phi_1^{a_1}(s_1) \cdots \Phi_m^{a_m}(s_m). \tag{41}$$

4. ENUMERATION OF m-ARY CACTI

4.1. *Coherence Conditions*

As observed in the introduction, there are some coherence conditions on the statistics of an m-ary cactus. We now state necessary and sufficient conditions for the existence of an m-ary cactus. The first one concerns the relationship between the number of vertices and the number of polygons. It is easily proved by induction on p.

LEMMA 7. *There exists an m-ary cactus having n vertices and p polygons if and only if*

$$n = p(m-1) + 1.$$

LEMMA 8. *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of non-negative integers and set $n = \sum_i n_i$. There exists an m-ary cactus having n vertices, p polygons, and vertex-color distribution $\vec{n}$ if and only if*

(1) $p = (n-1)/(m-1)$ *is an integer,*

(2) $p \geq 1 \Rightarrow n_i \leq p$ *for* $i = 1, \ldots, m$.

Proof. The conditions are clearly necessary. Sufficiency is proved by induction on p. If $p = 0$, then $n = 1$, and we have a 1-vertex cactus. If $p \geq 1$, then all components of $\vec{n}$ are strictly positive since otherwise, supposing, for example, that $n_1 = 0$, we find

$$n = \sum_{i=2}^{m} n_i \leq (m-1)p = n - 1,$$

a contradiction. Hence we have $n_i \geq 1$, for all i. If $p = 1$, then $n_i = 1$ for all i, and we have a cactus with a single polygon. If $p > 1$, we must have $n_i < p$ for some i, since otherwise $n = mp$ and $n = p(m-1) + 1$ leads to a contradiction. Assume, say, $n_m < p$ and define a new vector $\vec{n}'$ by $n'_m = n_m$ and $n'_i = n_i - 1$ for $i = 1, \ldots, m-1$. This vector $\vec{n}'$ satisfies the conditions 1 and 2 with $(n'-1)/(m-1) = p-1$ and we can apply the induction hypothesis to construct a cactus with vertex distribution $\vec{n}'$. It suffices then to add a new polygon to this cactus, attached to any existing vertex of color m to obtain a cactus with vertex-color distribution equal to $\vec{n}$. ∎

Observe that when conditions (1) and (2) are satisfied, $p \geq 1 \Rightarrow n_i \geq 1$ for all i.

LEMMA 9. *Let $N = (n_{ij})_{1 \leq i \leq m, j \geq 0}$ be an $m \times \infty$ matrix of non-negative integers, and set $n = \sum_{ij} n_{ij}$. There exists an m-ary cactus having n vertices and p polygons and whose vertex-degree distribution is given by the matrix N if and only if*

(1) $p = (n-1)/(m-1)$ *is an integer,*

(2) $\sum_j j n_{ij} = p$ *for all* i,

(3) $p \geq 1 \Rightarrow n_{i0} = 0$ *for all* i.

Proof. These conditions are clearly necessary. Sufficiency is again proved by induction on p. If $p = 0$, then $n = 1$ and we have a one vertex cactus. If $p \geq 1$, then we can prove that for all i, except possibly one, $n_{i1} \geq 1$. Indeed conditions (2) and (3) imply that $n_i = \sum_j n_{ij} \leq p$. Then, if $n_{i1} = 0$ for some i, we have $n_i \leq p/2$. If this occurs for two or more values of i, then $n = \sum_i n_i \leq (m-1)p = n - 1$, a contradiction. If $p = 1$, then $n_{i1} = 1$ for all i and we have a one polygon cactus. If $p > 1$, then either one $n_{i1} = 0$, say $n_{m1} = 0$, or all n_{i1} are ≥ 1. In the first case there must be some $j \geq 2$ with $n_{mj} \geq 1$; in the second case, there must exist some i, say $i = m$, and some $j \geq 2$, with $n_{mj} \geq 1$ since otherwise $n_i = n_{i1} = p$ for all i and $n = \sum n_i = mp = (m-1)p + p$, a contradiction. In either case we set $n'_{i1} = n_{i1} - 1$ for $i \neq m$, $n'_{mj} = n_{mj} - 1$, $n'_{m,j-1} = n_{m,j-1} + 1$, and $n'_{ij} = n_{ij}$ for other i, j. Then the matrix $N' = (n'_{ij})$ satisfies the conditions of the lemma with $p' = p - 1$ and we can apply the induction hypothesis to construct a cactus with vertex-degree distribution N'. It remains then to add a new polygon to this cactus, attached to any existing vertex of color m and degree $j - 1$ to obtain a cactus with vertex-degree distribution N. ▮

4.2. *Rooted or Labelled m-Ary Cacti*

As observed earlier, the species $\mathscr{K}^{\diamond}$ of rooted m-ary cacti is asymmetric. It follows that labelled m-ary cacti and rooted m-ary cacti are closely related. For example, in the one-sort case, we have

$$p\mathscr{K}_n = \mathscr{K}_n^{\diamond} = n!\tilde{\mathscr{K}}_n^{\diamond}, \tag{42}$$

where $\mathscr{K}_n$ and $\mathscr{K}_n^{\diamond}$ denote the number of m-ary cacti and rooted m-ary cacti, respectively, having n labelled vertices, and $\tilde{\mathscr{K}}_n^{\diamond}$ denotes the number of unlabelled m-ary cacti with n vertices, and where p is the number of polygons.

THEOREM 10. *Let p be a positive integer and set $n = p(m-1) + 1$. Then the numbers $\tilde{\mathscr{K}}_n^{\diamond}$, of rooted* (*unlabelled*) *m-ary cacti, and $\mathscr{K}_n$, of labelled m-ary cacti, having n vertices* (*and p polygons*), *are given by*

$$\tilde{\mathscr{K}}_n^{\diamond} = \frac{1}{n}\binom{mp}{p} \tag{43}$$

and

$$\mathscr{K}_n = \frac{(n-1)!}{p}\binom{mp}{p}. \tag{44}$$

Proof. It follows from (26) and (27) that the one-sort species $\mathscr{A}$ and $\mathscr{K}^{\diamond}$ of planted and rooted m-ary cacti, respectively, satisfy $\mathscr{A}(x) = x/(1 - \mathscr{A}^{m-1}(x))$ and $\tilde{\mathscr{K}}^{\diamond}(x) = \mathscr{K}^{\diamond}(x) = \mathscr{A}(x) - x$. The result follows easily from Lagrange inversion since

$$\begin{aligned}
\tilde{\mathscr{K}}_n^{\diamond} &= [x^n](\mathscr{A}(x) - x) \\
&= \frac{1}{n}[t^{n-1}](1 - t^{m-1})^{-n} \\
&= \frac{1}{n}[t^{(m-1)p}](1 - t^{m-1})^{-((m-1)p+1)} \\
&= \frac{1}{n}\binom{mp}{p}.
\end{aligned}$$

The second result then follows from (42). ∎

Remark 11. Formula (43) also represents the number of (unlabelled) m-ary ordered rooted trees having p internal vertices and n leaves. A direct bijection can be given between rooted m-ary cacti and m-ary ordered rooted trees, which also explains the functional equation $\mathscr{A} = X + \mathscr{A}^m$. See [2] and [3].

Suppose now that a vector $\vec{n} = (n_1, n_2, \ldots, n_m)$ satisfies the conditions of Lemma 8. Let $\mathscr{K}_{\vec{n}}$ denote the number of m-ary cacti over the multiset of vertices $([n_1], [n_2], \ldots, [n_m])$, that is, of labelled cacti with vertex-color distribution $\vec{n}$. Similarly let $\mathscr{K}_{\vec{n}}^{\diamond}$ denote the number of labelled rooted m-ary cacti with vertex distribution $\vec{n}$. Then we have

$$p \cdot \mathscr{K}_{\vec{n}} = \mathscr{K}_{\vec{n}}^{\diamond} = \left(\prod_{i=1}^{m} n_i!\right)\tilde{\mathscr{K}}_{\vec{n}}^{\diamond}, \tag{45}$$

where $\tilde{\mathscr{K}}_{\vec{n}}^{\diamond}$ is the number of unlabelled rooted cacti with vertex-color distribution $\vec{n}$.

THEOREM 12. *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of nonnegative integers satisfying the coherence conditions of Lemma* 8, *with $p \geq 1$. Then the number of unlabelled rooted m-ary cacti having vertex distribution $\vec{n}$ is given by*

$$\tilde{\mathscr{K}}_{\vec{n}}^{\diamond} = \frac{1}{p}\prod_{i=1}^{m}\binom{p}{n_i}. \tag{46}$$

Proof. Recall that $\tilde{\mathscr{K}}^{\diamond}(\mathbf{x}) = \mathscr{K}^{\diamond}(\mathbf{x}) = \mathscr{A}_1(\mathbf{x})\mathscr{A}_2(\mathbf{x}) \cdots \mathscr{A}_m(\mathbf{x})$ and that the $\mathscr{A}_i(\mathbf{x})$ satisfy functional equation (10). Hence we can use the special case 1 of the generalized Chottin formula, that is, formula (39), with $\Phi_i(s) = L(s) = 1/(1-s)$ for all i. Hence we find that

$$\begin{aligned}\tilde{\mathscr{K}}^{\diamond}_{\vec{n}} &= [x_1^{n_1} \cdots x_m^{n_m}]\mathscr{A}_1(\mathbf{x}) \cdots \mathscr{A}_m(\mathbf{x}) \\ &= \frac{p^{m-1}}{\prod_{i=1}^m n_i} \prod_{i=1}^m [s_i^{p-n_i}](1-s_i)^{-n_i} \\ &= \frac{p^{m-1}}{\prod_{i=1}^m n_i} \prod_{i=1}^m \binom{p-1}{n_i-1},\end{aligned}$$

which implies (46). ∎

Putting together equations (45)–(46) yields the following.

COROLLARY 13. *If the conditions of Lemma* 8 *are satisfied, the number of labelled m-ary cacti with vertex-color distribution* $\vec{n} = (n_1, n_2, \ldots, n_m)$ *is given by*

$$\mathscr{K}_{\vec{n}} = p^{m-2} \prod_{i=1}^m (p - n_i + 1)^{\langle n_i - 1 \rangle}, \tag{47}$$

where $x^{\langle k \rangle}$ *denotes the rising factorial* $x(x+1)\cdots(x+k-1)$.

Remark 14. This extends to general $m \geq 2$ the formula $n_1^{\langle n_2-1 \rangle} n_2^{\langle n_1 - 1 \rangle}$ for the number of labelled plane bicolored trees with vertex-color distribution (n_1, n_2) (see formula (2.7) of [16]).

To find the number $\mathscr{K}_N$ of labelled m-ary cacti having vertex-*degree* distribution $N = (n_{ij})$, where $i = 1, \ldots, m$ and $j \geq 0$, a similar approach can be followed. As for the vertex-color distribution, we have

$$p \cdot \mathscr{K}_N = n_1! \cdots n_m! \tilde{\mathscr{K}}^{\diamond}_N, \tag{48}$$

where $n_i = \sum_j n_{ij}$ and $\tilde{\mathscr{K}}^{\diamond}_N$ denotes the number of (unlabelled) rooted m-ary cacti having vertex-degree distribution N. Recall that $\mathbf{n}_i = (n_{i0}, n_{i1}, n_{i2}, \ldots)$ is the degree distribution for vertices of color i. The following result, due to Goulden and Jackson [9], expresses the number $\tilde{\mathscr{K}}^{\diamond}_{\mathrm{N}}$ in terms of the multinomial coefficients $\binom{n_i}{\mathbf{n}_i}$.

THEOREM 15 [9]. *Let* $N = (n_{ij})_{1 \leq i \leq m, j \geq 0}$ *be an* $m \times \infty$ *matrix of nonnegative integers satisfying the coherence conditions of Lemma* 9, *with* $n =$

$\sum_{ij} n_{ij}$ *and* $p = (n-1)/(m-1) \geq 1$. *Then the number of rooted* m*-ary cacti having* n_{ij} *vertices of color* i *and degree* j, *is given by*

$$\tilde{\mathscr{K}}_N^{\diamond} = \frac{p^{m-1}}{\prod_{i=1}^{m} n_i} \prod_{i=1}^{m} \binom{n_i}{\mathbf{n}_i}. \tag{49}$$

Proof. Recall that $\mathscr{K}_w^{\diamond}(\mathbf{x}) = \tilde{\mathscr{K}}_w^{\diamond}(\mathbf{x}) = \prod_{i=1}^{m} \mathscr{A}_{i,\mathbf{r}}(\mathbf{x})$ and also recall Eqs. (20). Again, we use the generalized Chottin formula (39), with

$$\Phi_i(s) = \Psi_{\mathbf{r}_i}(s) := r_{i1} + r_{i2}s + r_{i3}s^2 + \cdots. \tag{50}$$

Then we have

$$\tilde{\mathscr{K}}_N^{\diamond} = \left[\prod_{i,j} r_{ij}^{n_{ij}}\right]\left[\prod_i x_i^{n_i}\right]\mathscr{K}_w^{\diamond}(\mathbf{x}) = \frac{p^{m-1}}{\prod_i n_i}\left[\prod_{i,j} r_{ij}^{n_{ij}}\right]\left[\prod_i s_i^{p-n_i}\right]\prod_{i=1}^{m} \Phi_i^{n_i}(s_i), \tag{51}$$

which implies (49). ∎

COROLLARY 16. *The number* $\mathscr{K}_N$ *of labelled* m*-ary cacti having vertex-degree distribution* N, *assuming that the conditions of Lemma* 9 *are satisfied, with* $p \geq 1$, *is given by*

$$\mathscr{K}_N = p^{m-2} \prod_{i=1}^{m} (n_i - 1)! \binom{n_i}{\mathbf{n}_i}. \tag{52}$$

Remark 17. It is well known that the number of ways to label an unlabelled structure κ over an underlying multiset $[n_1, n_2, \ldots, n_m]$ is $n_1! n_2! \cdots n_m! / |\mathrm{Aut}(\kappa)|$, where $\mathrm{Aut}(\kappa)$ denotes the (color-preserving) automorphism group of κ. It follows that

$$\mathscr{K}_N = \sum_{\kappa} \frac{n_1! n_2! \cdots n_m!}{|\mathrm{Aut}(\kappa)|}, \tag{53}$$

where the sum is taken over all unlabelled m-ary cacti κ with vertex-degree distribution N. It also follows that

$$\sum_{\kappa} \frac{1}{|\mathrm{Aut}(\kappa)|} = \frac{1}{p}\tilde{\mathscr{K}}_N^{\diamond}. \tag{54}$$

This formula can be used, as in [7], to check that all unlabelled cacti with a given degree distribution have been found.

4.3. *Pointed m-ary Cacti (Unlabelled)*

Recall that $\mathscr{K}^{\bullet} = \mathscr{K}^{\bullet}(X)$ denotes the one-sort species of m-ary cacti which are pointed at a vertex of any color. We have

$$\tilde{\mathscr{K}}^{\bullet}(x) = \tilde{\mathscr{K}}^{\bullet_1}(x) + \cdots + \tilde{\mathscr{K}}^{\bullet_m}(x) = m\tilde{\mathscr{K}}^{\bullet_1}(x).$$

THEOREM 18. *Let p be a positive integer and set $n = p(m-1)+1$. Then the number $\tilde{\mathscr{K}}_n^{\bullet}$ of pointed m-ary cacti having n vertices (and p polygons), is given by*

$$\tilde{\mathscr{K}}_n^{\bullet} = \frac{1}{p}\sum_{d|p}\phi(d)\binom{pm/d}{p/d}, \tag{55}$$

where ϕ is the Euler function.

Proof. We have $\tilde{\mathscr{K}}_n^{\bullet} = m\tilde{\mathscr{K}}_n^{\bullet_1}$ and $\mathscr{K}^{\bullet_1} = X(1 + C(\mathscr{A}^{m-1}))$. By Lagrange inversion, we find for $p \geq 1, n \geq m$,

$$\begin{aligned}
\tilde{\mathscr{K}}_n^{\bullet_1} &= [x^n]\big(\tilde{\mathscr{K}}_n^{\bullet_1}(x) - x\big)\\
&= [x^n]x\sum_{d\geq 1}\frac{\phi(d)}{d}\log\frac{1}{1-\mathscr{A}^{m-1}(x^d)}\\
&= \sum_{d|n-1}\frac{\phi(d)}{d}[x^{(n-1)/d}]\log\frac{1}{1-\mathscr{A}^{m-1}(x)}\\
&= \sum_{d|n-1}\frac{\phi(d)}{n-1}(m-1)[t^{(n-d-1)/d}]t^{m-2}(1-t^{m-1})^{-(n+d-1)/d}\\
&= \sum_{d|p}\frac{\phi(d)}{p}[t^{(p-d)/d}](1-t)^{-(n+d-1)/d}\\
&= \sum_{d|p}\frac{\phi(d)}{p}\binom{pm/d-1}{p/d-1}\\
&= \frac{1}{mp}\sum_{d|p}\phi(d)\binom{pm/d}{p/d},
\end{aligned}$$

which completes the proof. ∎

We now wish to compute the numbers $\tilde{\mathscr{K}}_{\vec{n}}^{\bullet_i}$ and $\tilde{\mathscr{K}}_N^{\bullet_i}$ of (unlabelled) m-ary cacti pointed at a vertex of color i, with vertex-color distribution $\vec{n}$

and vertex-degree distribution N, respectively. For symmetry reasons, it is sufficient to consider the case $i = 1$ since we have

$$\tilde{\mathscr{K}}^{\bullet i}_{\vec{n}} = \tilde{\mathscr{K}}^{\bullet 1}_{\sigma^{i-1}\vec{n}} \qquad \text{and} \qquad \tilde{\mathscr{K}}^{\bullet i}_{N} = \tilde{\mathscr{K}}^{\bullet 1}_{\sigma^{i-1}N}, \tag{56}$$

where σ denotes a cyclic shift of the components of $\vec{n}$ or of the rows of N, i.e.,

$$(\sigma\vec{n})_i = n_{i+1} \qquad \text{and} \qquad (\sigma N)_{ij} = n_{i+1,j}, \tag{57}$$

the sum $i+1$ being taken modulo m. We introduce the following notations:

$$\vec{e}_k = (\delta_{ki}),\, i = 1,\ldots,m, \qquad \mathbf{e}_h = \left(\delta_{jh}\right)_{j\geq 0}, \qquad E_{r,s} = \left(\delta_{ir}\cdot\delta_{js}\right)_{1\leq i\leq m,\, j\geq 0}. \tag{58}$$

THEOREM 19. *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of non-negative integers satisfying the coherence conditions of Lemma 8, with $n = \sum_i n_i$ and $p = (n-1)/(m-1) \geq 1$. Then the number of m-ary cacti pointed at a vertex of color 1 and having vertex-color distribution $\vec{n}$ is given by*

$$\tilde{\mathscr{K}}^{\bullet 1}_{\vec{n}} = \frac{p - n_1 + 1}{p^2}\sum_d \phi(d)\binom{p/d}{(n_1-1)/d}\prod_{i\neq 1}\binom{p/d}{n_i/d}, \tag{59}$$

where the sum is taken over all d such that d divides p and all components of $\vec{n} - \vec{e}_1$.

Proof. Recall equation (13), with $i = 1$. In what follows we use the special case 3 of the generalized Chottin formula, i.e. (41), with $F(s) = \log[1/(1-s)]$ and $a_1 = (n_1-1)/d$, $a_2 = n_2/d, \ldots, a_m = n_m/d$, so that $q = p/d$. We find

$$\begin{aligned}
\tilde{\mathscr{K}}^{\bullet 1}_{\vec{n}} &= \left[x_1^{n_1}\cdots x_m^{n_m}\right]\left(\tilde{\mathscr{K}}^{\bullet i}(\mathbf{x}) - x_1\right)\\
&= [\mathbf{x}^{\vec{n}}]x_1\sum_{d\geq 1}\frac{\phi(d)}{d}\log\frac{1}{1-\hat{A}_1(\mathbf{x}^d)}\\
&= [\mathbf{x}^{\vec{n}-\vec{e}_1}]\sum_{d\geq 1}\frac{\phi(d)}{d}\log\frac{1}{1-\hat{A}_1(\mathbf{x}^d)}\\
&= \sum_{d\mid\vec{n}-\vec{e}_1}\frac{\phi(d)}{d}[\mathbf{x}^{(\vec{n}-\vec{e}_1)/d}]\log\frac{1}{1-\hat{A}_1(\mathbf{x})}
\end{aligned}$$

$$= \sum_{d|(p,\vec{n}-\vec{e}_1)} \phi(d) \frac{p^{m-2}}{\prod_{i=2}^m n_i} \cdot \left[s_1^{(p-n_1-d+1)/d} \right] \left(\frac{1}{1-s_1} \right)^{(n_1+d-1)/d}$$

$$\times \prod_{i=2}^m \left[s_i^{(q-n_i)/d} \right] \left(\frac{1}{1-s_i} \right)^{n_i/d}$$

$$= \sum_{d|(p,\vec{n}-\vec{e}_1)} \phi(d) \frac{p^{m-2}}{\prod_{i=2}^m n_i} \binom{(p-d)/d}{(n_1-1)/d} \prod_{i=2}^m \binom{(p-d)/d}{(n_i-d)/d},$$

which is equivalent to (59). ∎

THEOREM 20. *Let $N = (n_{ij})$ be an $m \times \infty$ matrix of non-negative integers satisfying the coherence conditions of Lemma* 9, *with $n = \sum_{ij} n_{ij}$ and $p = (n-1)(m-1) \geq 1$. Then the number of m-ary cacti pointed at a vertex of color* 1 *and having n vertices of color i and degree j, is given by*

$$\tilde{\mathscr{K}}_N^{\bullet_1} = \frac{p^{m-2}}{\prod_{i\neq 1} n_i} \sum_{h,d} \phi(d) \binom{(n_1-1)/d}{(\mathbf{n}_1-\mathbf{e}_h)/d} \prod_{i\neq 1} \binom{n_i/d}{\mathbf{n}/d}, \tag{60}$$

where the sum is taken over all ordered pairs (h,d) such that $n_{1h} \neq 0$ and d divides h, p and all components of $\mathbf{n}_1 - \mathbf{e}_h$ and of $\mathbf{n}_i$ with $i \geq 2$.

Proof. Recall that $n_i = \sum_j n_{ij}$ and $\mathbf{n}_i = (n_{ij})_{j\geq 0}$. We will use the special case 2 of the generalized Chottin formula, i.e. (40), with $\Phi_i(s) = \Psi_{r_i^d}(s)$ (see (50)), $k = h/d$, $a_1 = (n_1-1)/d$, $a_2 = n_2/d, \ldots, a_m = n_m/d$, for which $q = p/d$, and $A_i(\mathbf{x}) = \mathscr{A}_{i,\mathbf{r}^d}(\mathbf{x})$. Since $p \geq 1$, we have, using formula (23) with $i = 1$,

$$\tilde{\mathscr{K}}_N^{\bullet_1} = \left[\prod_{ij} r_{ij}^{n_{ij}} \right] \left[\prod_i x_i^{n_i} \right] \left(\tilde{\mathscr{K}}_w^{\bullet_1}(\mathbf{x}) - r_{1,0} x_1 \right)$$

$$= [\mathbf{r}^N][\mathbf{x}^{\vec{n}}] \sum_h \frac{x_1 r_{1,h}}{h} \sum_{d|h} \phi(d) \hat{\mathscr{A}}_{1,\mathbf{r}^d}^{h/d}(\mathbf{x}^d)$$

$$= \sum_h \frac{1}{h} [\mathbf{r}^{N-E_{1,h}}] \sum_{d|(h,\vec{n}-\vec{e}_1)} \phi(d) \left[x_1^{(n_1-1)/d} x_2^{n_2/d} \cdots x_m^{n_m/d} \right] \hat{\mathscr{A}}_{1,\mathbf{r}^d}^{h/d}(\mathbf{x})$$

$$= \frac{p^{m-2}}{\prod_{i\neq 1} n_i} \sum_{h,d} \phi(d) [\mathbf{r}^{N-E_{1,h}}] \left[s_1^{(p-n_1-h+1)/d} \right] \Psi^{(n_1-1)/d}(s_1)$$

$$\times \prod_{i\neq 1} \left[s_i^{(p-n_i)/d} \right] \Psi_{\mathbf{r}_i^d}^{n_j/d}(s_i)$$

$$= \frac{p^{m-2}}{\prod_{i\neq 1} n_i} \sum_{h,d} \phi(d) \left[\mathbf{r}_1^{(\mathbf{n}_1-\mathbf{e}_h)/d} \right] \left[s_1^{(p-n_1-h+1)/d} \right] \Psi_{\mathbf{r}_1^d}^{(n_1-1)/d}(s_1)$$

$$\times \prod_{i\neq 1} \left[\mathbf{r}_i^{n_i/d}\right]\left[s_i^{(p-n_i)/d}\right]\Psi_{\mathbf{r}_i}^{n_i/d}(s_i)$$

$$= \frac{p^{m-2}}{\prod_{i\neq 1} n_i} \sum_{d,h} \phi(d) \binom{(n_1-1)/d}{(\mathbf{n}_1-\mathbf{e}_h)/d} \prod_{i=2}^{m} \binom{n_i/d}{\mathbf{n}_i/d}, \tag{61}$$

where the summation is taken over all ordered pairs (h, d) such that $n_{1,h} \neq 0$ and d divides h, p, $\mathbf{n}_1 - \mathbf{e}_h$, and all $\mathbf{n}_i$ with $i \geq 2$. ∎

4.4. *m-Ary Cacti (Unlabelled)*

In order to enumerate unlabelled and unrooted m-ary cacti, two methods can be used. The first one uses the dissymmetry theorem for cacti (see Theorem 2) which expresses the species of m-ary cacti in terms of pointed and of rooted cacti; see below. The second is Liskovets' method for the enumeration of unlabelled planar maps [17]. It uses the Cauchy–Frobenius theorem (alias Burnside's Lemma) and the concept of quotient of a planar map under an automorphism; see [2] and [3] for the application of Liskovets' method to the enumeration of m-ary cacti.

THEOREM 21. *Let p be a positive integer and set $n = p(m-1) + 1$. Then the number $\tilde{\mathscr{K}}_n$ of (unlabelled) m-ary cacti having n vertices (and p polygons), is given by*

$$\tilde{\mathscr{K}}_n = \frac{1}{p}\left(\frac{1}{n}\binom{mp}{p} + \sum_{\substack{d\mid p\\ d<p}} \phi(p/d)\binom{md}{d}\right), \tag{62}$$

where ϕ is the Euler function.

Proof. Using the dissymmetry formula (29) for one-sort m-ary cacti, we find

$$\widetilde{\mathscr{K}_n} = \widetilde{\mathscr{K}_n^{\bullet}} - (m-1)\tilde{\mathscr{K}}_n^{\diamond} \tag{63}$$

and the result follows from (43) and (55). ∎

See Table I for some numerical values of $\tilde{\mathscr{K}}_n$.

THEOREM 22. *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of nonnegative integers satisfying the coherence conditions of Lemma* 8, *with $n = \sum_i n_i$ and*

TABLE I

Number of Unlabelled m-ary and m-gonal Cacti Having p Polygons and n Vertices

	$m = 2$				$m = 3$			
p	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$
0	1	1	1	1	1	1	1	1
1	2	1	1	1	3	1	1	1
2	3	2	0	1	5	3	0	1
3	4	3	1	2	7	6	3	2
4	5	6	2	3	9	19	10	7
5	6	10	8	6	11	57	54	19
6	7	28	18	14	13	258	222	86
7	8	63	61	34	15	1110	1107	372
8	9	190	170	95	17	5475	5346	1825
9	10	546	538	280	19	27,429	27,399	9143
10	11	1708	1654	854	21	143,379	142,770	47,801
11	12	5346	5344	2694	23	764,970	764,967	254,990
12	13	17,428	17,252	8714	25	4,173,906	4,170,672	1,391,302

	$m = 4$				$m = 5$			
p	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$
0	1	1	1	1	1	1	1	1
1	4	1	1	1	5	1	1	1
2	7	4	0	1	9	5	0	1
3	10	10	6	3	13	15	10	3
4	13	44	28	11	17	85	60	17
5	16	197	193	52	21	510	505	102
6	19	1228	1140	307	25	4051	3876	811
7	22	7692	7688	1936	29	33,130	33,125	6626
8	25	52,828	52,364	13,207	33	291,925	290,700	58,385
9	28	373,636	373,560	93,496	37	2,661,255	2,661,100	532,251
10	31	2,735,952	2,732,836	683,988	41	25,059,670	25,049,020	5,011,934
11	34	20,506,258	20,506,254	5,127,163	45	241,724,380	241,724,375	48,344,880
12	37	156,922,676	156,899,748	39,230,669	49	2,379,912,355	2,379,812,100	475,982,471

	$m = 6$				$m = 7$			
p	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$	n	$\tilde{\mathcal{K}}_n$	$\bar{\mathcal{K}}_n$	$\tilde{\mathcal{H}}_n$
0	1	1	1	1	1	1	1	1
1	6	1	1	1	7	1	1	1
2	11	6	0	1	13	7	0	1
3	16	21	15	4	19	28	21	4
4	21	146	110	25	25	231	182	33
5	26	1101	1095	187	31	2100	2093	300
6	31	10,632	10,326	1772	37	23,884	23,394	3412
7	36	107,062	107,056	17,880	43	285,390	285,383	40,770
8	41	1,151,802	1,149,126	191,967	49	3,626,295	3,621,150	518,043
9	46	12,845,442	12,845,166	2,141,232	55	47,813,815	47,813,367	6,830,545
10	51	147,845,706	147,817,170	24,640,989	61	650,367,788	650,302,814	92,909,684

$p = (n-1)/(m-1) \geq 1$. Then the number $\tilde{\mathscr{K}}_{\vec{n}}$ of (unlabelled) m-ary cacti having vertex-color distribution $\vec{n}$ is given by

$$\tilde{\mathscr{K}}_{\vec{n}} = \frac{1}{p^2}\left(\prod_{i=1}^{m}\binom{p}{n_i} + \sum_{i,d}\phi(d)(p-n_i+1)\binom{p/d}{(n_i-1)/d}\prod_{j\neq i}\binom{p/d}{n_j/d}\right), \tag{64}$$

where the sum is taken over all pairs (i, d) such that $1 \leq i \leq m$, $d > 1$, d divides p and all components of $\vec{n} - \vec{e}_i$.

Proof. Using the dissymmetry formula (15), we have

$$\tilde{\mathscr{K}}_{\vec{n}} = \sum_{i=1}^{m}\tilde{\mathscr{K}}_{\vec{n}}^{\bullet_i} - (m-1)\tilde{\mathscr{K}}_{\vec{n}}^{\diamond}. \tag{65}$$

The result follows from (46), (56), and (59). ∎

See Table II for some numerical values of $\tilde{\mathscr{K}}_{\vec{n}}$.

THEOREM 23. *Let $N = (n_{ij})$ be an $m \times \infty$ matrix of non-negative integers satisfying the coherence conditions of Lemma 9, with $n = \sum_{ij} n_{ij}$ and $p = (n-1)/(m-1) \geq 1$. Then the number $\tilde{\mathscr{K}}_N$ of (unlabelled) m-ary cacti*

TABLE II
The Number of Unlabelled m-ary Cacti (Rooted, Plain, Asymmetric) According to Their Vertex−Color Distribution

$\vec{n}$	$\tilde{\mathscr{K}}_{\vec{n}}^{\diamond}$	$\tilde{\mathscr{K}}_{\vec{n}}$	$\bar{\mathscr{K}}_{\vec{n}}$	$\vec{n}$	$\tilde{\mathscr{K}}_{\vec{n}}^{\diamond}$	$\tilde{\mathscr{K}}_{\vec{n}}$	$\bar{\mathscr{K}}_{\vec{n}}$
(7, 7)	226,512	17,424	17,424	(1, 3, 3)	1	1	0
(5, 6)	5292	536	523	(2, 2, 3)	3	1	1
(6, 6, 7)	28,224	3138	3135	(1, 4, 4)	1	1	0
(4, 4, 5)	225	39	36	(2, 3, 4)	6	2	1
(5, 6, 8)	10,584	1176	1176	(3, 3, 3)	16	4	4
(5, 5, 5)	1323	189	189	(3, 3, 5)	20	4	4
(4, 6, 7)	1960	248	242	(1, 3, 3, 3)	1	1	0
(5, 6, 6)	5488	692	680	(2, 2, 3, 3)	3	1	1
(3, 4, 4, 5)	50	10	10	(2, 3, 4, 4)	6	2	1
(6, 6, 6, 7)	21,952	2752	2736	(4, 4, 4, 4)	125	25	25

having n_{ij} vertices of color i and degree j, is given by

$$\tilde{\mathscr{K}}_N = p^{m-2}\left(\prod_{i=1}^{m}\frac{1}{n_i}\binom{n_i}{\mathbf{n}_i} + \sum_{i,h,d}\frac{\phi(d)}{\prod_{l\neq i} n_l}\binom{(n_i-1)/d}{(\mathbf{n}_i-\mathbf{e}_h)/d}\prod_{l\neq i}\binom{n_l/d}{\mathbf{n}_l/d}\right), \tag{66}$$

where the sum is taken over all triplets (i, h, d) such that $n_{ih} \neq 0$, $d > 1$, and d divides h, p, and all entries of the matrix $N - E_{ih}$.

Proof. The dissymmetry formula (19) gives

$$\tilde{\mathscr{K}}_N = \sum_{i=1}^{m}\tilde{\mathscr{K}}_N^{\bullet_i} - (m-1)\tilde{\mathscr{K}}_N^{\diamond}. \tag{67}$$

The result follows from (49), (56), and (60). ∎

See Table III for some numerical values of $\tilde{\mathscr{K}}_N$.

4.5. *Unlabelled m-Ary Cacti According to their Automorphisms*

We first consider asymmetric m-ary cacti, that is, cacti whose automorphism group is reduced to the identity. Let $\overline{K}$ denote the species of asymmetric m-ary cacti. We have already observed that the species $\mathscr{K}^{\diamond}$ of

TABLE III
The Number of Unlabelled m-ary Cacti (Rooted, Plain, Asymmetric) According to Their Vertex–Degree Distributions

m	N	$\tilde{\mathscr{K}}_N^{\bullet_i}$, $i = 1, \ldots, m$	$\tilde{\mathscr{K}}_N^{\diamond}$	$\tilde{\mathscr{K}}_N$	$\overline{\mathscr{K}}_N$
2	$(1^5 3^2, 2^7)$	(8, 7)	14	1	1
2	$(1^2 2^2 4^1, 1^2 2^4)$	(76, 90)	150	16	14
3	$(1^3 2^3, 1^3 2^3, 1^6 3^1)$	(600, 600, 702)	900	102	99
3	$(1^2 2^1, 1^2 2^1, 1^2 2^1)$	(12, 12, 12)	16	4	4
3	$(4, 1^4, 1^4)$	(1, 1, 1)	1	1	0
3	$(2^2, 1^2 2, 1^4)$	(1, 2, 2)	2	1	0
3	$(1 \cdot 3, 1^2 2, 1^4)$	(2, 3, 4)	4	1	1
3	$(1^2 2^2, 1^2 2^2, 1^4 2^1)$	(54, 54, 69)	81	15	12
3	$(1^3 2^1 4^1, 1^3 2^3, 1^7 2^1)$	(600, 720, 960)	1080	120	120
3	$(1^3 2^2, 1^3 2^2, 1^3 2^2)$	(280, 280, 280)	392	56	56
3	$(1^2 3^2, 1^4 2^2, 1^6 2^1)$	(120, 180, 212)	240	32	28
3	$(2^4, 1^4 2^2, 1^6 2^1)$	(20, 30, 36)	40	6	4
3	$(1^4 4^1, 1^4 2^2, 1^4 2^2)$	(252, 300, 300)	400	52	48
3	$(1^2 2^3, 1^4 2^2, 1^4 2^2)$	(504, 600, 600)	800	104	96
4	$(1^4 2^2, 1^4 2^2, 1^4 2^2, 1^6 2^1)$	(6000, 6000, 6000, 7008)	8000	1008	992

rooted m-ary cacti is asymmetric, i.e., that $\bar{\mathscr{K}}^{\diamond} = \mathscr{K}^{\diamond}$. The dissymmetry formulas (29) and (8), yields, in the one-sort case,

$$\bar{\mathscr{K}} = \bar{\mathscr{K}}^{\bullet} - (m-1)\mathscr{K}^{\diamond}, \tag{68}$$

and in the m-sort case,

$$\bar{\mathscr{K}} = \sum_{i=1}^{m} \bar{\mathscr{K}}^{\bullet_i} - (m-1)\mathscr{K}^{\diamond}. \tag{69}$$

Since $\mathscr{K}^{\bullet_i} = X(1 + C(\hat{\mathscr{A}}_i))$, the enumeration of (unlabelled) $\bar{\mathscr{K}}^{\bullet_i}$-structures uses the *asymmetry index series* Γ_C of the species C of circular permutations, instead of the cycle index series Z_C for the enumeration of unlabelled cacti (see [15, 1]), where

$$\Gamma_C(x_1, x_2, \dots) = \sum_{d \geq 1} \frac{\mu(d)}{d} \log \frac{1}{1 - x_d}, \tag{70}$$

compared to the cycle index series

$$Z_C(x_1, x_2, \dots) = \sum_{d \geq 1} \frac{\phi(d)}{d} \log \frac{1}{1 - x_d}, \tag{71}$$

where μ is the Möbius function. It follows that the enumeration formulas for asymmetric m-ary cacti will be very similar to those of unlabelled cacti. In fact it suffices to replace ϕ by μ in the formulas of the previous section. Hence we have the following theorem. (See Tables I–III for sample values.)

THEOREM 24. *Assume that the coherence conditions of Lemmas* 7, 8, *and* 9 *are satisfied, with* $p \geq 1$. *Then the corresponding enumerative formulas for* (*unlabelled*) *asymmetric m-ary cacti are as follows*:

$$\bar{\mathscr{K}}_n = \frac{1}{p}\left(\frac{1}{n}\binom{mp}{p} + \sum_{\substack{d \mid p \\ d < p}} \mu(p/d)\binom{md}{d}\right), \tag{72}$$

$$\bar{\mathscr{K}}_{\vec{n}} = \frac{1}{p^2}\left(\prod_{i=1}^{m}\binom{p}{n_i} + \sum_{i,d} \mu(d)(p - n_i + 1)\binom{p/d}{(n_i - 1)/d}\prod_{j \neq i}\binom{p/d}{n_j/d}\right), \tag{73}$$

$$\overline{\mathscr{K}}_N = p^{m-2}\left(\prod_{i=1}^{m}\frac{1}{n_i}\binom{n_i}{\mathbf{n}_i} + \sum_{i,h,d}\frac{\mu(d)}{\prod_{l\neq i} n_l}\binom{(n_i-1)/d}{(\mathbf{n}_i-\mathbf{e}_h)/d}\prod_{l\neq i}\binom{n_l/d}{\mathbf{n}_l/d}\right), \tag{74}$$

where the summation ranges of (73) *and* (74) *are the same as for* (64) *and* (66).

We now consider m-ary cacti admitting at least one nontrivial automorphism. Since automorphisms are required to preserve colors, the only possibilities are rotations around a central vertex. See Fig. 8. Observe that the order of such an automorphism must divide the number p of polygons. Let $s \geq 2$ be an integer. Let $\mathscr{K}_{=s}$, and $\mathscr{K}_{\geq s}$, denote the species of m-ary cacti whose automorphism groups (necessarily cyclic) are of order s, and a multiple of s, respectively. Then, following the notations of [16], Section 3, we have

$$\mathscr{K}_{=s} = \sum_{i=1}^{m} X_i C_{=s}\left(\hat{\mathscr{A}}_i\right), \tag{75}$$

$$\mathscr{K}_{\geq s} = \sum_{i=1}^{m} X_i C_{\geq s}\left(\hat{\mathscr{A}}_i\right). \tag{76}$$

We can determine the unlabelled generating series $\tilde{\mathscr{K}}_{\geq s}(\mathbf{x})$ and $\tilde{\mathscr{K}}_{=s}(\mathbf{x})$ by formulas (3.2) and (3.3) of [16], essentially due to Stockmeyer. See [1], Exercise 4.4.16, and [21]. Extracting coefficients in these series is similar to the computations of Section 4.3. We find the following.

THEOREM 25. *Let $s \geq 2$ be an integer and assume that the coherence conditions of Lemmas* 7, 8, *and* 9 *are satisfied, with p a multiple of s. The*

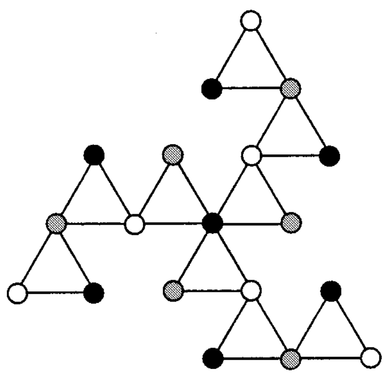

FIG. 8. A ternary cactus with a symmetry of order 3.

corresponding enumerative formulas for (unlabelled) m-ary cacti whose automorphism groups are of order s, and a multiple of s, respectively, are as follows:

$$\tilde{\mathscr{K}}_{=s,n} = \frac{s}{p} \sum_{d|\frac{p}{s}} \mu(d) \binom{pm/sd}{p/sd} \tag{77}$$

and

$$\tilde{\mathscr{K}}_{\geq s,n} = \frac{s}{p} \sum_{d|\frac{p}{s}} \phi(d) \binom{pm/sd}{p/sd}, \tag{78}$$

$$\tilde{\mathscr{K}}_{=s,\vec{n}} = \sum_{i=1}^{m} \frac{s(p-n_i+1)}{p^2} \sum_{d} \mu(d/s) \binom{p/d}{(n_i-1)/d} \prod_{j\neq i} \binom{p/d}{n_j/d}, \tag{79}$$

and

$$\tilde{\mathscr{K}}_{\geq s,\vec{n}} = \sum_{i=1}^{m} \frac{s(p-n_i+1)}{p^2} \sum_{d} \phi(d/s) \binom{p/d}{(n_i-1)/d} \prod_{j\neq i} \binom{p/d}{n_j/d}, \tag{80}$$

the second summations being taken over all integers d such that $s|d$ and d divides p and all components of $\vec{n} - \vec{e}_i$,

$$\tilde{\mathscr{K}}_{=s,N} = \sum_{i=1}^{m} \frac{p^{m-2}}{\prod_{j\neq i} n_j} \sum_{h,d} \mu(d/s) \binom{(n_i-1)/d}{(\mathbf{n}_i - \mathbf{e}_h)/d} \prod_{j\neq i} \binom{n_j/d}{\mathbf{n}_j/d} \tag{81}$$

and

$$\tilde{\mathscr{K}}_{\geq s,N} = \sum_{i=1}^{m} \frac{p^{m-2}s}{\prod_{j\neq i} n_j} \sum_{h,d} \phi(d/s) \binom{(n_1-1)/d}{(\mathbf{n}_i - \mathbf{e}_h)/d} \prod_{j\neq i} \binom{n_j/d}{\mathbf{n}_j/d}, \tag{82}$$

the second summations being taken over all pairs of integers $h, d \geq 1$ such that $n_{ih} \neq 0$, $s|d$, and d divides h and all entries in $N - E_{ih}$.

5. RELATED ENUMERATIVE RESULTS

5.1. *Plane m-Gonal Cacti*

Let $\mathscr{H}$ denote the one-sort species of plane m-gonal cacti (not m-colored). The case of an isolated vertex is included. If $\mathscr{H}^{\bullet}$ and $\mathscr{A}$ denote the species of pointed and of planted plane m-gonal cacti, respectively, then $\mathscr{A}$

coincides with the species introduced in Section 2.3, characterized by the functional equation $\mathscr{A} = XL(\mathscr{A}^{m-1})$, and $\mathscr{H}^{\bullet}$ is isomorphic to the species $\mathscr{K}^{\bullet_i}$, for any i, that is, satisfies

$$\mathscr{H}^{\bullet} = X\big(1 + C(\mathscr{A}^{m-1})\big). \tag{83}$$

See (26) and (28). However, the species $\mathscr{H}^{\diamond}$ of rooted (at a polygon) plane m-gonal cacti is no longer asymmetric. In fact, we have

$$\mathscr{H}^{\diamond} = C_m(\mathscr{A}), \tag{84}$$

where C_m denotes the species of circular permutations of length m. Another important difference resides in the form of the dissymmetry theorem which is more closely related to that of free (nonplane) m-gonal cacti. Indeed, we have (see [10] and [1], (4.2.16) and Fig. 4.2.5)

$$\mathscr{H}^{\bullet} + \mathscr{H}^{\diamond} = \mathscr{H} + \mathscr{A}\cdot\mathscr{A}^{m-1} \tag{85}$$

from which we deduce, since $\mathscr{A}^m = \mathscr{A} - X$, that

$$\begin{aligned} \mathscr{H} &= \mathscr{H} + \mathscr{H}^{\diamond} - \mathscr{A} + X \\ &= X\big(1 + C(\mathscr{A}^{m-1})\big) + C_m(\mathscr{A}) - \mathscr{A} + X. \end{aligned} \tag{86}$$

THEOREM 26. *Let p be a positive integer and set $n = p(m-1) + 1$. Then the numbers $\mathscr{H}_n$ and $\tilde{\mathscr{H}}_n$ of labelled and unlabelled m-gonal cacti, respectively, having n vertices (and p polygons) are given by*

$$\mathscr{H}_n = \frac{(n-1)!}{mp}\binom{mp}{p} \tag{87}$$

and

$$\tilde{\mathscr{H}}_n = \alpha_n + \beta_n - \gamma_n, \tag{88}$$

where

$$\alpha_n = \tilde{\mathscr{H}}_n^{\bullet} = \frac{1}{mp}\sum_{d\mid p}\phi\left(\frac{p}{d}\right)\binom{dm}{d}, \tag{89}$$

$$\beta_n = \tilde{\mathscr{H}}_N^{\diamond} = \frac{1}{mp}\sum_{d\mid(m,\,p-1)}\phi(d)\binom{pm/d}{(p-1)/d}, \tag{90}$$

and

$$\gamma_n = \tilde{\mathscr{A}}_n = \frac{1}{n}\binom{mp}{p}. \tag{91}$$

In the case where $m = 2$, we recover formulas of Walkup [23] for the number of plane trees. See also Labelle and Leroux ([16], (1.18)–(1.21)). It is also possible to derive similar formulas for the number of m-gonal plane cacti according to the vertex-degree distribution. See [16], (1.23)–(1.26) where the computations have been carried out in the case $m = 2$. Table I contains numerical values of $\tilde{\mathscr{H}}_n$, for $n = (m-1)p + 1$, and $m = 2, \ldots, 7$.

5.2. *Free (Labelled) m-Ary Cacti*

A *free m-ary cactus* can be informally defined as an m-ary cactus without the plane embedding. In other words, the m-gons attached to a vertex are free to take any position with respect to each other. Denoting by $\mathscr{F}$ the species of free m-ary cacti, we have the functional equations

$$\mathscr{F}^{\diamond} = \mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_m \tag{92}$$

and, for $i = 1, 2, \ldots, m$,

$$\mathscr{A}_i = X_i E\big(\hat{\mathscr{A}}_i\big), \tag{93}$$

where E denotes the species of sets, for which

$$E(x) = e^x, \qquad \tilde{E}(x) = (1-x)^{-1}$$

and

$$Z_E(x_1, x_2, \ldots) = \exp\left(\sum_{i \geq 1} \frac{x_i}{i} \right). \tag{94}$$

The computations of Section 4.1 for *labelled* m-ary cacti according to vertex-color distribution can be easily adapted to free m-ary cacti. In particular, we find the following result.

PROPOSITION 27. *Let $\vec{n} = (n_1, \ldots, n_m)$ be a vector of positive integers satisfying the coherence conditions of Lemma* 8 *with $p = (n-1)/(m-1) \geq 1$. Then the number $\mathscr{F}(\vec{n})$ of labelled free m-ary cacti having vertex-color distribution $\vec{n}$ is given by*

$$\mathscr{F}(\vec{n}) = p^{m-2} \prod_{i=1}^{m} \frac{(n_i - 1)!\, n_i^{p-n_i}}{(p-n_i)!}. \tag{95}$$

This extends Scoins [19] formula $n_1^{n_2-1} n_2^{n_1-1}$ for the number of labelled bicolored free trees with vertex-color distribution (n_1, n_2) to general $m \geq 2$.

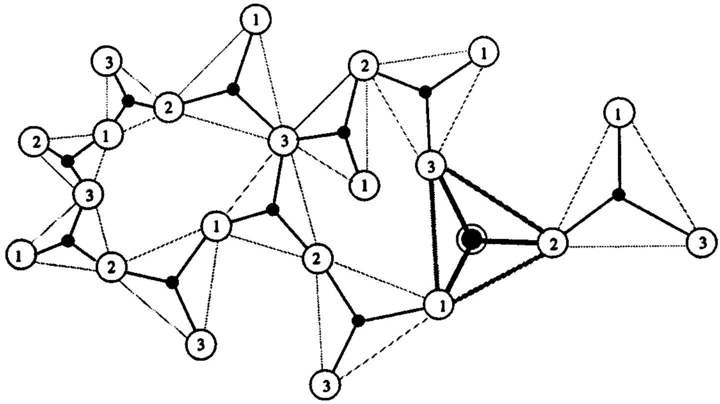

FIG. 9. A rooted ternary constellation.

5.3. *Constellations*

Another combinatorial object closely related to m-ary cacti is an m-ary constellation which is defined in a similar way as an m-ary cactus except that cycles of polygons are now allowed. Figure 9 shows a typical ternary constellation which is rooted, that is, has a distinguished polygon. Bousquet-Mélou and Schaeffer [4] have found that the number $\tilde{C}^{\diamond}(p)$ of unlabelled rooted m-ary constellations having p polygons is given by

$$\tilde{C}^{\diamond}(p) = \frac{(m+1)m^{p-1}}{((m-1)p+2)((m-1)p+1)}\binom{mp}{p}. \tag{96}$$

ACKNOWLEDGMENTS

We would like to thank Sacha Zvonkin, for introducing us to the problem of cactus enumeration, and Robert Cori and Gilles Schaeffer for useful discussions.

Note added in proof. For the vertex color distribution $\vec{n} = (m, m, \ldots, m)$, numerical computations show that the number $\tilde{\mathscr{K}}_{\vec{n}}$ of unlabelled m-ary cacti is $(m+1)^{m-2}$. The only vertex degree distribution in this case is $(1^{m-1}2, 1^{m-1}2, \ldots, 1^{m-1}2)$. In terms of topological classification of polynomials, this corresponds to the generic case of a polynomial of degree $m+1$ whose derivative has m distinct roots. The formula is easily established since there is an obvious bijection between these m-ary cacti and (free) trees with m labelled edges, whose number is $(m+1)^{m-2}$; indeed, pointing an arbitrary vertex leads to a tree with $m+1$ labelled vertices and Cayley's formula $(m+1)^{m-1}$ can be invoked. This fact is also mentioned in V. I. Arnold's "Topological Classification of Trigonometric Polynomials and Combinatorics of Graphs under an Equal Number of Vertices and Edges," *Funct. Anal. Appl.* **30** (1996), 1–14.

REFERENCES

1. F. Bergeron, G. Labelle, and P. Leroux, "Combinatorial Species and Tree-Like Structures," Encyclopedia of Mathematics and Its Applications, Vol. 67, Cambridge University Press, Cambridge, 1998.
2. M. Bousquet, "Espèces de structures et applications au dénombrement de cartes et de cactus planaires," Thèse de doctorat, UQAM, 1998; Publications du LaCIM, Vol. 24, LACIM, Montreal, 1999.
3. M. Bousquet, Quelques résultats sur les cactus m-aires, *Ann. Sci. Math. Québec*, in press.
4. M. Bousquet-Melou and G. Schaeffer, Enumeration of planar constellations, *Adv. Appl. Math.*, in press doi 10.0006/aama.1999.0673.
5. L. Chottin, Une démonstration combinatorie de la formule de Lagrange à deux variables, *Discrete Math.* **13** (1975), 215–224.
6. L. Chottin, Énumération d'arbres et formules d'inversion de séries formelles, *J. Combin. Theory Ser. B* **31** (1981), 23–45.
7. M. El Marraki, N. Hanusse, J. Zipperer, and A. Zvonkin, "Cacti, Braids and Complex Polynomials," Séminaire Lotharingien de Combinatoire (1997), Vol. 37, *available at* http://cartan.u-strasbg.fr/~slc
8. I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration," John Wiley and Sons, New York, 1983.
9. I. P. Goulden and D. M. Jackson, The Combinatorial Relationship Between Trees, Cacti and Certain Connection Coefficients for the Symmetric Group, *European J. Combin.* **13** (1992), 357–365.
10. F. Harary and R. Z. Norman, Dissimilarity characteristic of Husimi trees, *Ann. of Math.* **58** (1953), 134–141.
11. F. Harary and G. Palmer, "Graphical Enumeration," Academic, New York, 1973.
12. F. Harary and G. E. Uhlenbeck, On the number of Husimi trees, *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 315–322.
13. K. Husimi, Note on Mayer's theory of cluster integrals, *J. Chem. Phys.* **18** (1950), 682–684.
14. A. G. Khovanskii and S. Zdravkovka, Branched covers of S^2 and braid groups, *J. Knot Theory Ramifications* **5** (1996), 55–75.
15. G. Labelle, On asymmetric structures, *Discrete Math.* **99** (1992), 141–164.
16. G. Labelle and P. Leroux, Enumeration of (uni- or bicolored) plane trees according to their degree distribution, *Discrete Math.* **157** (1996), 227–240.
17. V. A. Liskovets, A census of non-isomorphic planar maps, *Colloq. Math. Soc. János Bolyai* **25** (1981), 479–494.
18. R. J. Riddell, "Contributions to the theory of condensation," Dissertation, University of Michigan, Ann Arbor, 1951.
19. H. I. Scoins, The number of trees with nodes of alternate parity, Proc. Cambridge Philos. Soc. **58** (1962), 12–16.
20. J. Sherman and W. J. Morrison, Adjustments of an inverse matrix corresponding to changes in the elements of a given row or a given column of the original matrix, *Ann. Math. Stat.* **20** (1949), 621.
21. P. K. Stockmeyer, "Enumeration of Graphs with Prescribed Automorphism Group," University of Michigan, Ann Arbor, 1971.
22. G. E. Uhlenbeck and G. W. Ford, "Lectures in Statistical Mechanics," American Mathematical Society, Providence, Rhode Island, 1963.
23. D. W. Walkup, The number of plane trees, *Mathematica* **19** (1972), 200–204.