The connectivity of Z-transformation graphs of perfect matchings of polyominoes

Heping Zhang

Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China
Received 7 July 1993; revised 11 November 1994

Abstract

A polyomino, or any shaped chessboard, consists of finite cells of a plane square grid as its connected subgraph such that each interior face is surrounded by a cell. The Z-transformation graph $Z(G)$ of a polyomino $G$ is a graph in which the vertices are the perfect matchings of $G$ and two vertices are adjacent provided that the union of the corresponding two perfect matchings of $G$ contain exactly one cycle and the cycle consists of the four edges of a cell. This paper presents some properties of polyominoes with perfect matchings and mainly shows that the connectivity of $Z(G)$ reaches its minimum degree with only two exceptions.

1. Introduction

Polyominoes, or chessboards, which first appeared in games, have attracted some mathematicians' considerable attention, because many interesting combinatorial subjects can be produced from them, such as hypergraphs [1], domination problem [2,3], rook polynomial [5], etc. In addition, Motoyama and Hosoya studied perfect matchings of polyominoes and obtained some interesting results by introducing king and domino polynomials, which can be applied in physics and chemistry [4,7,8]. In Ref. [12] Zhang and the present author extended Z-transformation graphs of perfect matchings of hexagonal systems [10,11] to general plane bipartite graphs. It has been shown that most results are still valid. But the main result of Ref. [11], i.e. the connectivity of Z-transformation graph of a hexagonal system is equal to its minimum degree, does not hold. In this article, however, we show that the result holds for Z-transformation graphs of perfect matchings of polyominoes with only two exceptions.

2. Definitions and previous work

Throughout this paper a bipartite graph in question is always colored by black and white so that any two adjacent vertices receive different colors. Recall that a perfect
matching (written simply as PM) of a graph \( G = (V(G), E(G)) \) is a set of mutually disjoint edges of \( G \) saturating all vertices of \( G \). Let \( G \) be a bipartite graph with a PM \( M \). The edges of \( M \) are called \( M \)-double edges and the other edges of \( G \) \( M \)-single edges, which are usually designated by double lines and single lines in the figure. We call an edge of \( G \) a fixed edge if it belongs to either all PMs of \( G \) or no PMs of \( G \). The former is the so-called ‘fixed double edge’ and the latter is the so-called ‘fixed single edge’. It is clear that the edges of \( G \) adjacent to a fixed double edge must be fixed single edges. A connected bipartite graph is said to be elementary or normal if it has no fixed edges. It turns out that an elementary bipartite graph different from \( K_2 \) (i.e. a complete graph with two vertices) is 2-connected and can be produced by the Ear Construction [6].

Let \( M \) be a PM of a bipartite graph \( G \) and \( C \) a cycle and \( P \) a path of \( G \). If \( M \) and \( E(G) \setminus M \) saturate alternately the edges of \( C \) (resp. \( P \)), then \( C \) (resp. \( P \)) is called an \( M \)-alternating cycle (resp. path). We define the symmetric difference of two sets \( M_1 \) and \( M_2 \) as \( M_1 \oplus M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2) \). It is obvious that the symmetric difference of two distinct PMs of a graph consists of mutually disjoint \( M_1 \)- and \( M_2 \)-alternating cycles. Suppose that \( M \) is a PM of \( G \) and \( C \) an \( M \)-alternating cycle of \( G \). Then the symmetric difference of \( M \) and the edge-set \( E(C) \) of a cycle \( C \) is another PM of \( G \) instead of \( M \), which is simply denoted by \( M \oplus C \) and illustrated in Fig. 1. Let \( G_0 \) be a plane bipartite graph shown in Fig. 1 with a PM \( M \) and two \( M \)-alternating cycles \( C_1 \) and \( C_2 \). The other three PMs of \( G_0 \) are \( M \oplus C_1 \), \( M \oplus C_2 \) and \( M \oplus C_1 \oplus C_2 \). It is easily verified that the three edges \( e_1, e_2 \) and \( e_3 \) represented by heavy lines in Fig. 1 are fixed single edges.

**Lemma 1.** Let \( M \) be a PM of a graph \( G \) and \( C \) an \( M \)-alternating cycle. Then \( C \) is both \( M \)- and \( M \oplus C \)-alternating cycle and thus each edge of \( C \) is not fixed edge.

**Proof.** It is obvious. \( \Box \)

The following important results were obtained in [12].
Lemma 2 (Zhang and Zhang [12]). Let $G$ be a connected plane bipartite graph with perfect matchings. Suppose that a cycle $C$ of $G$ lies on the boundary of one face. If $\frac{1}{2}|E(C)|$ independent edges of $C$ are not fixed single edges, then there exists a perfect matching $M$ of $G$ such that $C$ is an $M$-alternating cycle.

Theorem 3 (Zhang and Zhang [12]). Let $G$ be a plane bipartite graph. Then $G$ is an elementary bipartite graph if and only if the boundary of each face (including the infinite face) is an alternating cycle with respect to a PM of $G$.

We now define the $Z$-transformation graphs of perfect matchings of plane bipartite graphs as follows.

Definition 4. Let $G$ be a plane bipartite graph with PMs. $Z$-transformation graph of PMs of $G$, denoted by $Z(G)$, is defined as a graph in which the vertices are the PMs of $G$ and two vertices are adjacent if and only if the symmetric difference of the corresponding two PMs consists exactly of the boundary of one finite face of $G$.

$Z$-transformation graph of a plane bipartite graph $G_0$ as shown in Fig. 1 is the union of two $K_2$’s, which is referred to Fig. 2, where $M$ and $M \oplus C_2$ are not adjacent in $Z(G_0)$ because the interior region of $C_2$ is not a face of $G$. This example shows that $Z$-transformation graph of a plane bipartite graph with fixed edges is not necessarily connected. For plane elementary bipartite graphs, however, we have the following results.

Theorem 5 (Zhang and Zhang [12]). Let $G$ be a plane elementary bipartite graph different from $K_2$, then

(a) $Z(G)$ is a connected bipartite graph,

(b) $Z(G)$ has at most two vertices of degree one,

(c) $Z(G)$ is either a path or a graph of girth 4 different from cycles.

An appropriate modification of the definition for $Z$-transformation graph has been made so that Theorem 5 can be extended to general plane bipartite graphs [12]. It was shown that the connectivity of $Z$-transformation graph of a hexagonal system is equal to its minimum degree [11]. Many examples show that for a general plane elementary
bipartite graph the above result does not hold. Let us consider further examples. Let $T$ be a crossed polyomino shown in Fig. 3(a) and let $M$ be a PM and $a, b, c, d, e$ be the five cells of $T$. It is easy to construct $Z$-transformation graph $Z(T)$, which is referred to Fig. 3(b).

Let $T'$ be another polyomino as shown in Fig. 4 with two fixed single edges designated by heavy lines. Obviously, $Z(T') = Z(T) \times K_2$ (see Fig. 4), where ‘$\times$’ denotes an operation ‘Cartesian product’ between graphs. We can find that the connectivity of $Z(T')$ and $Z(T)$ is less than their minimum degree. In fact $Z(T)$ and $Z(T')$ are only two exceptions of our main result in this paper.

3. Preliminary results

A unit square in the infinite plane square grid is called a cell. A polyomino is a finite connected subgraph in the infinite plane square grid in which every interior face is surrounded by a cell and every edge belongs to at least one cell. $T$ and $T'$ denoted in Figs. 3 and 4 are two examples of polyominoes. Let $G$ be a polyomino. The periphery of the infinite face of $G$ is called the periphery of $G$, each edge of which is called a peripheral edge of $G$. It is well known that a polyomino is a plane bipartite graph. For convenience, we always place a polyomino considered on a plane so that one of the two edge directions is horizontal and the other is vertical. In what follows, we always restrict our attention to polyominoes with PMs.
We now describe an important configuration, called a zigzag chain. Let $c_1, c_2, \ldots, c_t$ be a series of cells of a polyomino $G$. If $c_{i-1}$ and $c_i$ share an edge $e_i$ ($1 \leq i \leq t$) and $e_2e_3\ldots e_t$ is a path, then the configuration consisting of $c_1, c_2, \ldots, c_t$ is called a zigzag chain with a central path $e_1e_2\ldots e_t$, where $e_1$ and $e_{t+1}$ are the edges of $c_1$ and $c_t$, respectively, as shown in Fig. 5.

**Lemma 6.** Let $c$ be a cell of a polyomino $G$ with PMs. If an edge $e$ of $c$ is a fixed single edge of $G$, then one of the two edges of $c$ adjacent to $e$ must be a fixed single edge.

**Proof.** Let $e_1$ and $e_2$ be two edges of $c$ adjacent to $e$. Suppose that both $e_1$ and $e_2$ are not fixed single edges. By Lemma 2 there exists a PM $M$ such that $c$ is an $M$-alternating cell, which contradicts that $e$ is a fixed single edge. □

**Lemma 7.** Let $G$ be a polyomino with PMs. If $G$ has a fixed single edge, then it must have a fixed single edge on the periphery of $G$.

**Proof.** Let $e_1$ be a fixed single edge of $G$ belonging to a cell $c_1$. By Lemma 6, we may assume that an edge $e_2$ of $c_1$ adjacent to $e_1$ is also a fixed single edge (see Fig. 6). Hence, we can find a zigzag chain $c_1c_2\cdots c_{t-1}$ ($t \geq 2$) with the central path $e_1e_2\cdots e_t$ such that the $e_i$'s are fixed single edges and $e_t$ is either a peripheral edge or a common edge of $c_{t-1}$ with another cell $s_1$ satisfying that an edge $h_1$ of $s_1$ adjacent to $e_t$ but not to $e_{t-1}$ is not a fixed single edge. For the former case, the lemma is proved. So we only consider the latter case. There exists a PM $M$ of $G$ such that $h_1 \in M$. Let $f_2$ denote an edge of $s_1$ parallel to $h_1$. By Lemma 6, $f_2$ must be a fixed single edge. We also can find a zigzag chain $Z = s_1s_2\cdots s_k$ with the central path $f_1(= e_t)f_2\cdots f_{k+1}$ ($k \geq 1$), where $f_{k+1}$ is a peripheral edge of $G$. We want to prove that the $f_i$'s are fixed single edges. If $k = 1$, it is trivial. In the following suppose that $k \geq 2$. Let $g_2(\neq f_2)$ denote the edge of $s_2$ adjacent to $e_i(= f_1)$. Since $e_{t-1}, e_t, f_2$ are fixed single edges, $g_2$ must
be a fixed double edge of $G$. So $h_1, g_2 \in M$. Obviously, $Z - \{f_1, f_2, \ldots, f_{k+1}\}$ consists exactly of two disjoint paths which are denoted by $g_2g_3 \cdots$ and $h_1h_2h_3 \cdots$ in the form of edge sequence, respectively. Since, $g_2$ is a fixed double edge, so an edge $g_3$ of $s_2$ adjacent to $g_2$ is a fixed single edge. By Lemma 6, it follows that $f_3$ is a fixed single edge. Repeating the above procedure finite times, we finally arrive at that the $f_i$'s are fixed single edges and $h_{2i-1} \in M$ for all $i \geq 1$ and the $g_{2i}$'s are fixed double edges. The proof is complete.  

As an immediate conclusion of Theorem 3 and Lemma 6, we have:

**Theorem 8.** The following statements are equivalent.

(a) A polyomino $G$ is normal,
(b) $G$ has a PM $M$ such that the periphery of $G$ is an $M$-alternating cycle,
(c) For each cell $c$ of $G$, there exists a PM $M$ such that $c$ is an $M$-alternating cell.

(Note that the same results of hexagonal systems was previously obtained by Zhang and Chen [9].)

**Proof.** (a) $\Rightarrow$ (b). It is obvious by Theorem 3.

(b) $\Rightarrow$ (c). Suppose that $G$ has a PM $M$ such that the periphery of $G$ is an $M$-alternating cycle. Then each edge of the periphery is not fixed. If $G$ has a fixed single edge, by Lemma 6 $G$ must have a fixed single edge on the periphery of $G$, a contradiction.

(c) $\Rightarrow$ (a). It follows by the definition.  

Assume that a polyomino $G$ has fixed edges. Deleting all fixed single edges of $G$, the resultant subgraph of $G$ consists of some normal plane bipartite graphs other than $K_2$ (called normal blocks of $G$) and some isolated $K_2$'s (factually fixed double edges of $G$) as its components.

**Lemma 9.** Let $G$ be a polyomino with fixed edges. Then $G$ has at least two normal blocks and each normal block is also a polyomino. For each PM $M$ of $G$, each normal block of $G$ contains an $M$-alternating cell.

**Proof.** By Dulmage–Mendelsohn Decomposition of bipartite graphs [6, Ch. 4] and the minimum degree of $G$ with larger than one, it follows immediately that $G$ has at least two normal blocks. Let $G_i$ be a normal block of $G$. Let $C$ denote the periphery of $G_i$. By Theorem 3, $G_i$ has a PM $F$ such that $C$ is an $F$-alternating cycle. Let $G(C)$ denote the subgraph of $G$ consisting of $C$ and the interior. Since $G$ is a polyomino, $G(C)$ is also a polyomino. Hence by Theorem 8 $G(C)$ is normal, i.e. each edge of $G(C)$ is not a fixed edge of $G(C)$ and thus of $G$. Moreover, $G_i = G(C)$, i.e. each normal block of $G$ is still a normal polyomino. For each PM $M$ of $G$, the
restriction of $M$ on $G_i$ is a PM of $G_i$. Since $Z(G_i)$ is connected and $G_i$ has at least two PMs, the minimum degree of $Z(G_i) \geq 1$. Hence, $G_i$ contains at least one $M$-alternating cell. □

Suppose that a polyomino $G$ contains fixed single edges. Let $G_1, G_2, \ldots, G_k$ be the normal blocks of $G(k \geq 2)$. From Lemma 9, each $G_i$ is a normal polyomino. Hence each finite face of $G_i$ is also that of $G$. By the definition we have that

$$Z(G) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_k).$$

Therefore, Theorem 5 is valid for general polyominos with PMs.

Let $G$ be a polyomino with a PM $M$ and $G_0$ a subgraph of $G$. Suppose that the restriction of $M$ on $G_0$, i.e. $M \cap E(G_0)$, is also a PM of $G_0$. We call an $M$-alternating path $P$ an ear of $G_0$ with respect to $M$, if $G_0$ contains only the two end vertices of $P$. It is obvious that the end edges of $P$ are $M$-single edges and end vertices are of different colors and $P$ is of odd length, which is illustrated in Fig. 7.

**Lemma 10.** Let $G$ be a normal polyomino and $G_0 \subset G(G_0 \neq G)$ be also a normal polyomino. Suppose that the restriction of a PM $M$ of $G$ in $G_0$ is a PM of $G_0$. Then $G_0$ has an ear $P$ with respect to $M$ such that

- (a) The end vertices $u$ and $v$ of $P$ are of distance 1,
- (b) If $uv \notin E(G_0)$, $P = uv$, and
- (c) If $uv \in E(G_0)$, the end edges are parallel and thus belong to the same cell of $G$.

**Proof.** By the hypothesis of the lemma, $G$ has an edge $e$ not belonging to $G_0$ with at least one end vertex lying on the periphery of $G_0$. Since $G$ is normal, it has a PM $M'(\neq M)$ containing $e$. Then the symmetric difference $M \oplus M'$ must have an $M(M')$-alternating cycle $C$ containing $e$. Let $P$ be a path in $C$ such that $G_0$ shares only the end vertices (say $u$ and $v$) of $P$. Then $P$ is an ear of $G_0$ with respect to $M$. Let $P_0$ denote a path on the periphery of $G_0$ sharing end vertices with $P$ and satisfying that the interior of a cycle $C' = P \cup P_0$ lies in the exterior of $G_0$. Obviously, $C'$ and the interior form a polyomino and $P_0$ is of odd length. Without loss of generality, suppose that $P$ is a minimum ear of $G_0$ with respect to $M$ in the sense that $G_0$ has no such an ear other than $P$ in the interior of $C'$. If the internal vertices of $P_0$ are of degree 4, it is easily shown that $P_0$ is of length 3 and $P = uv$ (see Fig. 8(a)). Otherwise we assert that $P_0 = uv$. If $P_0$ has at least one internal vertex (say $w$) of degree $\leq 3$ in $G_0$, then $G$ has an edge $e'$ incident with $w$ not belonging to $G_0$. Similar to the argument at the beginning of the proof, we have that $G_0 \cup P$ has an ear $P'$, the end vertices of which is denoted by $w$ and $w'$, with respect to $M$ such that $P'$ lies in the interior of $C'$. If $w'$ lies on $P_0$, $P'$ is an ear of $G_0$ with respect to $M$; otherwise $P' \cup P(u,w')$ or $P' \cup P(w',v)$ is an ear of $G_0$ with respect to $M$ according as $v$ and $w$ having the same color or different colors, where $P(u,w')$ and $P(w',v)$ denote the subpaths of $P$ with end vertices $u,w'$ and $w',v$, respectively. These contradict the minimality of $P$. 

Therefore, the assertion follows. Similarly, we can show that the end edges of $P$ are parallel and thus belong to the same cell of $G$. □

Let $G$ be a polyomino with a PM $M$. We use notation $d(M)$ and $\delta$ representing the degree of $M$ and the minimum degree of $Z(G)$, respectively. Obviously, $d(M)$ is the number of $M$-alternating cells of $G$.

**Lemma 11.** Let $G$ be a polyomino with a PMs. For each PM $M$ of $G$, $Z(G)$ has a 2-path (of length 2) starting at $M$ with only two exceptions: (1) $G$ has exactly one cell, and (2) $G$ has exactly two cells such that the three $M$-double edges are parallel to each other.

**Proof.** Suppose that $Z(G)$ has no 2-path with an end vertex $M$. Then $G$ has no disjoint $M$-alternating cells. Otherwise, suppose that $c_1$ and $c_2$ are disjoint $M$-alternating cells. Then $M(M \oplus c_1)(M \oplus c_1 \oplus c_2)$ is a 2-path, a contradiction. Hence by Lemma 9 $G$ is normal. By the connectedness of $Z(G)$, it is a star with a center $M$. Hence the number of PMs of $G$ is $d(M)+1$. It is obvious that $1 \leq d(M) \leq 2$. We distinguish the following two cases.

Case I: $d(M) = 2$. Let $c_1$ and $c_2$ be two $M$-alternating cells sharing an edge $u'v'$. Let $G_0$ be a polyomino consisting of $c_1$ and $c_2$. Suppose that $G_0 \neq G$. Then, by Lemma 10 $G_0$ has an ear $P$ (as described in Lemma 10) with respect to $M$. If the end vertices of $P$ are $u$ and $v$ (see Fig. 9), then $M \oplus E(P+uv)$ is a PM of $G$ different from $M,M \oplus c_1$ and $M \oplus c_2$, a contradiction. If the end vertices of $P$ are $v$ and $v'$, $M \oplus E(P+v'v'uv)$ is a new PM of $G$, and the same contradiction occurs. For other cases, the same fact holds. Hence $G = G_0$.

Case II: $d(M) = 1$. Similar to Case I, we can prove that $G$ has exactly one cell.

Let $G$ be a polyomino with a PM $M$ and $Z$ a zigzag chain of $G$. Let $c$ and $c'$ be two cells of $Z$ having a common edge. Then $Z$ is called an $M$-zigzag chain with
Fig. 9.

Fig. 10. An $M$-zigzag chain with respect to $(c, c')$.

respect to $(c, c')$ if the restriction of $M$ in the $Z$ is also a PM of $Z$ and $M$ contains three mutually parallel edges of $c$ and $c'$ (in fact $M$ contains only one edge of each cell of $Z$ instead of $c$ and $c'$) (see Fig. 10). Furthermore, if $Z$ is not a proper subchain of any $M$-zigzag chain with respect to $(c, c')$, we call $Z$ a maximal $M$-zigzag chain with respect to $(c, c')$.

Lemma 12. Let $G$ be a polyomino with a PM $M$, and $G$ has a maximal $M$-zigzag chain. Then $\delta \leq d(M) - 1$.

Proof. Let $Z = c_1c_2 \cdots c_t (t \geq 2)$ be a maximal $M$-zigzag chain with respect to $(c_i, c_{i+1})$ for some $1 \leq i \leq t - 1$. We first consider two special cases.

Case 1: $t = 2$. Let $M_1 = M \oplus c_1$. We claim that neither $g$ nor $h$ (see Fig. 11(a)) are $M_1$-alternating cells. Otherwise, $g$ (or $h$) is an $M_1$-alternating cell, then $gc_1c_2$ is an $M$-zigzag chain with respect to $(c_1, c_2)$, which contradicts the maximality of $Z$. So $g, h$ and $c_2$ are not $M_1$-alternating cells, and thus $d(M_1) \leq d(M) - 1$.

Case 2: $t \geq 3$ and $i = 1$. Let $M_1 = M \oplus c_1$. Similar to Case 1, we can show that $h$ is not an $M_1$-alternating cell (see Fig. 11(b)). Obviously, $g$ is not $M_1$-alternating and $d(M_1) \leq d(M) - 1$.

In what follows, suppose that $2 \leq i \leq t - 2$. Let $M_1 = M \oplus c_i \oplus c_{i-1} \oplus \cdots \oplus c_1$. By the same reason as Case 1, we have that $h$ is not an $M_1$-alternating cell (see Fig. 12). In $Z$ only one cell $c_1$ is an $M_1$-alternating cell. In addition to $c_1$, no new $M_1$-alternating cell appears. Thus, $d(M_1) \leq d(M) - 1$. The Lemma is proved. □
We now state our main result in the following.

**Theorem 13.** Let $G$ be a polyomino with PMs. Then the connectivity of $Z$-transformation graph $Z(G)$ of perfect matchings of $G$ is equal to its minimum degree $\delta$ except for $Z(T)$ and $Z(T) \times K_2$ shown in Figs. 3 and 4.

Before proving the main theorem, we will deduce the following crucial lemma.

**Lemma 14.** Let $G$ be a polyomino with PMs. Suppose that $G$ is neither a crossed polyomino $T$ nor a polyomino having exactly two normal blocks $T$ and one cell. For any 2-path $M_1M_2$ of $Z(G)$, then $ZG$ has at least $\delta$ internally disjoint paths joining $M_1$ and $M_2$. 

4. The connectivity of $Z(G)$

We now state our main result in the following.
Proof. We first make a convention. Let $F_1F_2\cdots F_t$ be a path of $Z(G)$. Put $c_i = F_i \oplus F_{i+1} (1 \leq i \leq t-1)$. Then $c_i$ is $F_i$- and $F_{i+1}$-alternating cell of $G$. For the sake of simplicity, we may denote the path $F_1F_2\cdots F_t$ by

$$F_1c_1c_2\cdots c_{t-1}F_t.$$ 

Suppose that $M_1M'M_2$ is any given 2-path of $Z(G)$. Let $s_1 = M_1 \oplus M'$ and $s_2 = M' \oplus M_2$. Then $p(s_1) := M_1 \oplus \Delta M_2 = M_1M'M_2$.

Let $A$ denote the set of $M_1$-alternating cells except for $s_1$ and $s_2$. Let $B = \{s_i \in A : s_i$ and $s_1 \cup s_2$ are disjoint$\}$. For each $s_i \in B$, let $p(s_i) := M_1 \oplus \Delta \Delta M_2$. Let $P(B) = \{p(s_i) : s_i \in B\}$.

It is easy to see that $P(B) \cup \{p(s_1)\}$ is the set of $|B| + 1$ internally disjoint paths joining $M_1$ and $M_2$. We distinguish the following two cases.

Case 1: $E(s_1) \cap E(s_2) \neq \emptyset$. Without loss of generality, suppose that $d(M_1) \leq d(M_2)$. Obviously $s_2$ is not an $M_1$-alternating cell. Thus, $d(M_1) = |A| + 1$. If $A = B$, the lemma follows immediately. In what follows, we assume that $A \neq B$.

There is at most one $M_1$-alternating cell adjacent to $s_1$; otherwise $d(M_1 \cup s_1 \cup s_2) = d(M_2) \leq d(M_1) - 1$, a contradiction.

Subcase 1.1: Let $x_1 \in A \setminus B$ be adjacent to $s_1$. We can find a maximal $M_1$-zigzag chain $x_r \cdots x_1s_1s_2x'_1 \cdots x'_k$ with respect to $(s_1,x_1)$ (see Fig.13(a)). Let $M^* = M_1 \oplus x_1 \oplus \cdots \oplus x_r$. By Lemma 12, we have that $\delta \leq d(M^*) \leq d(M_1) - 1$. If $y_1 \notin A \setminus B$, $\delta \leq d(M_1) - 1 = |B| + 1$ and the result follows. If $y_1 \in A \setminus B$, from the assumption $d(M_1) \leq d(M_2)$ we have that $y_2$ is an $M_1 \oplus s_1$-alternating cell, $d(M_1) = |B| + 3$ and $r = k = 1$ (see Fig. 13(b)). We can construct a path as follows:

$$p^* := M_1 \frac{y_1}{s_1} \frac{s_2}{y_2} M_2$$

Then $P(B) \cup \{p(s_1), p^*\}$ is the set of $d(M_1) - 1 = |B| + 2$ internally disjoint paths.

Subcase 1.2. There is no cell of $A \setminus B$ adjacent to $s_1$. Since $A \neq B$, $y_1 \in A \setminus B$. Then $d(M_1) = |A| + 1 = |B| + 2$. Assume that $y_2$ is not an $M_1 \oplus s_1$-alternating cell (otherwise,
let \( p^* \) be the same as \((*)\), the result follows). If \( h_1 \) and \( h_2 \) are not \( M_2 \)-alternating, 
\[ \delta \leq d(M_1 + s_1 + s_2) \leq d(M_1) - 1; \] if only one (say \( h_1 \)) of \( h_1 \) and \( h_2 \) is an \( M_2 \)-alternating cell, 
\[ \delta \leq d(M_1 + s_1 + s_2 + h_1) \leq d(M_1) - 1. \] So we may assume that both \( h_1 \) and \( h_2 \) 
are \( M_2 \)-alternating cells (see Fig. 14). Let \( G_0 \) be a polyomino consisting of \( s_1, s_2, h_1, h_2 \) 
and \( y_1 \). If \( Z(G) \) has a 2-path \( M_1 \cong M_2 \) such that cells \( t_1 \) and \( t_2 \) are disjoint with \( G_0 \). 
We can construct a path as follows:

\[
p^{**} := M_1 \frac{y_1}{t_1} \frac{t_2}{y_1} \frac{s_1}{s_2} \frac{h_1}{t_1} T \frac{h_1}{M_2}
\]

and \( \{ p^{**}, p(s_1) \} \cup P(B) \) is the set of \(|B| + 2 = d(M_1) \geq \delta \) internally disjoint paths. In 
the latter of Case 1.2, we assume that \( Z(G) \) has no 2-path \( M_1 \cong M_2 \) such that \( t_1 \) and 
\( t_2 \) are disjoint with \( G_0 \). We now need to consider the following two subcases.

Subcase 1.2.1: \( G \) is a normal polyomino. Then \( G_0 \neq G \) by the assumption of the 
Lemma. By Lemma 10, \( G_0 \) has an ear \( P \) with respect to \( M_1 \) such that the end vertices 
\( u \) and \( v \) are adjacent in \( G_0 \) and the end edges \( uv' \) and \( vu' \) are parallel. If \( P \) is of length 
3, by the assumption of Subcase 1.2 we know that \( uv \) belongs to \( y_1, h_1 \) or \( h_1 \) and 
easily deduce that \( d(M_1) - 1 \geq \delta \). Hence the desired follows. Otherwise, by Lemma 11 
and the above assumption it is not difficult to see that \( P \) is of length 5 and thus 
\( P + uv + u'v' \) consists of two cells \( z_1 = uvu'v' \) and \( z_2 = (P - u - v) + u'v' \) (see Fig. 15).
(a) Let $uv$ be an edge of a cell $y_1$. Put

$$p^*(z_2) := M_1 \frac{z_2}{s_1} \frac{z_1}{s_2} h_1 \frac{z_1}{z_2} M_2,$$

$$p^*(y_1) := M_1 \frac{y_1}{s_1} \frac{z_2}{s_2} h_1 \frac{z_1}{y_2} \frac{h_1}{z_2} M_2.$$ 

Then $\{p^*(z_2), p^*(y_1), p(s_1)\} \cup (P(B) \setminus \{p(z_2)\})$ is the set of $|B| + 1 \geq \delta$ internally disjoint paths.

(b) Let $uv$ be an edge of $h_1$ or $h_2$ (say $h_2$). Let $t_1 = z_2$ and $t_2 = z_1$. Then we can construct a path $p^{**}$ as (**). The result follows.

(c) Let $uv$ be an edge of $s_1$. Set

$$p^* := M_1 \frac{y_1}{s_1} \frac{z_2}{s_2} h_1 \frac{z_1}{y_2} \frac{h_1}{z_2} M_2.$$ 

Then $P(B) \cup \{p^*, p(s_1)\}$ is the set of $|B| + 2$ internally disjoint paths.

Subcase 1.2.2: $G$ is not normal. Then $G$ has exactly two normal blocks. In fact, if $G$ has more than two normal blocks, by Lemma 9 we can take two disjoint $M_1$-alternating cells which are disjoint with $G_0$, which contradicts our assumption. By the proof of Subcase 1.2.1, we only need to consider the case $G_0$ being one normal block of $G$.

By Lemma 11 and our assumption the other block has exactly one cell or two cells in which three $M_i$-double edges are parallel to each other. The former case does not occur. For the latter, it is easy to see that $d(M_1) - 1 \geq \delta$. So the proof of Case I is complete.

Case II: $E(s_1) \cap E(s_2) = \emptyset$. Both $s_1$ and $s_2$ are disjoint $M_1$-alternating cells. Let $p(s_2) := M_1 \frac{z_2}{s_1} \frac{z_1}{s_2} M_2$. Thus, $P(B) \cup \{p(s_1), p(s_2)\}$ are $|B| + 2$ internally disjoint paths. We aim at constructing at least $\delta$ internally disjoint paths joining $M_1$ and $M_2$. It needs to consider these cases depending on the numbers of $M_1$-alternating cells adjacent to $s_1$ and $s_2$, which are denoted by $a$ and $b$, respectively. If $a, b \leq 1$, similar to Case 1.1, we can find maximal $M_1$-zigzag chain(s). By Lemma 12, we can show that $G$ has a PM $M^*$ such that $d(M^*) \leq d(M_1) - |A \setminus B| = |B| + 2(a + b - |A \setminus B|)$.

Without loss of generality, assume that $a = 2$. Let $y_1$ and $y_2$ be distinct $M_1$-alternating cells sharing the edges of $s_1$. The cells adjacent to $s_1$ and $s_2$ are represented in Fig. 16.
Subcase 2.1: \( b = 2 \). Let \( y'_1, y'_2 \in A \setminus B \) intersecting \( s_2 \). If neither \( h_1 \) nor \( h_2 \) is an \( M_1 \oplus s_1 \)-alternating cell, then \( d(M_1 \oplus s_1) \leq d(M_1) - 2 \); if only one (say \( h_1 \)) of \( h_1 \) and \( h_2 \) is an \( M_1 \oplus s_1 \)-alternating cell, then \( d(M_1 \oplus s_1 \oplus h_1) \leq d(M_1) - 2 \). For \( s_2 \), the same result holds. Hence it only needs to consider the following two cases.

(a) \( h_1, h_2, h'_1 \) and \( h'_2 \) are all \( M_2 \)-alternating cells. Obviously, \( \delta \leq d(M_1) - 2 \). If one of \( y_1 \) and \( y_2 \) is the same as one of \( y'_1 \) and \( y'_2 \) (say \( y_2 = y'_1 \)), the required path is

\[
p^* := M_1 \frac{y_1 s_2}{y_1 s_1 h'_1} \frac{y_1 s_1}{h'_1 M_2};
\]

Otherwise, without loss of generality, we may assume that \( y_1 \) and \( h'_1 \) are disjoint and \( y_2 \) and \( h'_2 \) are disjoint. The required two paths are

\[
p^*(y_1) := M_1 \frac{y_1 s_2}{y_1 s_1} \frac{h'_1}{y_1 s_1 h'_1 M_2},
\]

\[
p^*(y_2) := M_1 \frac{y_2 s_2}{y_2 s_1} \frac{h'_2}{y_2 s_1 h'_2 M_2}.
\]

(b) \( h_1, h_2 \) and \( h'_1 \) are all \( M_2 \)-alternating cells, but \( h'_2 \) is not. We may assume that \( y_1 \) is different from \( y'_2 \) and \( y'_1 \), and \( h'_1 \) and \( y'_1 \) are disjoint. Then \( \delta \leq d(M_1 \oplus y_1 \oplus s_2 \oplus h'_1) \leq d(M_1) - 3 \) and the desired path is

\[
p^* := M_1 \frac{y_1 s_2}{y_1 s_1 h'_1} \frac{y_1 s_1}{h'_1 M_2}.
\]

So we complete the proof of Subcase 2.1.

Subcase 2.2: \( b = 1 \). We may assume that \( h_1 \) and \( h_2 \) are \( M_2 \)-alternating cells and \( y'_2 \neq y_1, y_2 \), which are referred to Fig. 17. Obviously, \( d(M_1) - 2 \geq \delta \). Furthermore, we may assume that \( y_2 \) and \( h_1 \) are disjoint. Set

\[
p^* := M_1 \frac{y'_2 s_1}{y'_2 s_1} \frac{h_1}{y'_2 s_1 h_1 M_2}.
\]

Subcase 2.3: \( b = 0 \). It suffices to consider the case that \( h_1 \) and \( h_2 \) are \( M_2 \)-alternating (see Fig. 18). It is obvious that \( d(M_1) - 1 \geq \delta \). Let \( G_0 \) be a polyomino consisting of cells \( y_1, y_2, h_1, h_2 \) and \( s_1 \). Without loss of generality, assume that \( s_2 \) and \( h_1 \cup y_2 \) are disjoint. Assume that \( Z(G) \) has a path \( M_1 \oplus M_1^* \) such that \( t_1 \) and \( G_0 \) are disjoint. If \( t_1 \) is not an \( M_2 \)-alternating cell, the required path is

\[
p^* := M_1 \frac{y'_2 s_1}{y'_2 s_1} \frac{h_1}{y'_2 s_1 h_1 M_2}.
\]

Otherwise, both \( s_2 \) and \( t_1 \) are disjoint \( M_1 \)-alternating cells. Set

\[
p^*(y_1) := M_1 \frac{y_1 t_1}{y_1 t_1} \frac{s_2}{y_1 s_1 t_1 M_2},
\]

\[
p^*(t_1) := M_1 \frac{t_1 s_1}{t_1 s_1} \frac{h_1}{t_1 s_1 h_1 M_2}.
\]

Then \( (P(B) \setminus \{ p(t_1) \}) \cup \{ p^*(y_1), p^*(t_1), p(s_1), p(s_2) \} \) is the set of \( |B| + 3 = d(M_1) - 1 \geq \delta \) internally disjoint paths.

So we now can assume that \( Z(G) \) has no such a 2-path as mentioned above. Obviously, \( G \) has at most two normal blocks. If \( G_0 \) is a normal block, by Lemma 11
and our assumption we know that the other normal block must be $s_2$, a contradiction. Hence, $G_0$ is not a normal block of $G$. By Lemma 10, $G_0$ has an ear $P$ with respect to $M_1$ such that the end vertices $u$ and $v$ are adjacent in $G_0$ and the end edges $uu'$ and $vv'$ are parallel and belong to the same cell $z_1$. Obviously, $uv \in M_1$ and $uu', vv' \notin M_1$. If $P$ is of length 3, we easily deduce that $d(M_1) - 2 \geq \delta$, the result follows. Otherwise, $(P - u - v) + u'v'$ is an $M_1$-alternating cycle. Thus $z_1$ is not $M_1$-alternating and $z_1$ and one of $y_1$ and $y_2$ (say $y_2$) are disjoint. From our assumption and Lemma 11 we easily know that $s_2 = (P - u - v) + u'v'$. The desired path is

$$P^* := M_1 \frac{y_2}{s_2} \frac{s_2}{z_1} \frac{y_2}{s_1} \frac{s_1}{z_1} M_2.$$ 

Now the entire proof of the lemma is complete. \qed

**Proof of Theorem 13.** Let $C$ be a minimal cut of $Z(G)$. Then $Z(G)$ has a 2-path $M_1M'M_2$ such that $M' \in C$ and $M_1$ and $M_2$ belong to different components of $Z(G) - C$. If $G$ is either a crossed polyomino $T$ or has exactly two normal blocks $T$ and one cell, the connectivity of $Z(G)$ is equal to $\delta - 1$ (see Figs. 3 and 4). Otherwise, by Lemma 14, $Z(G)$ has $\delta$ internally disjoint paths joining $M_1$ and $M_2$. Thus, $|C| \geq \delta$. Furthermore, the connectivity of $Z(G)$ is equal to $\delta$. \qed

As an immediate consequence, we have:

**Corollary 15.** Let $G$ be a 2-connected polyomino with PMs. The connectivity of $Z(G)$ is equal to its minimum degree except for $Z(T)$.

**Proof.** Assume that $T'$ is a polyomino with PMs having exactly two normal blocks $T$ and one cell $c$. By Theorem 13, it suffices to prove that $T'$ has a cut-vertex. By Dulmage-Mendelsohn Decomposition, it follows easily that only vertices of the same color (say black vertices) of the normal block $c$ are incident with fixed single edges; that is, the white vertices of $c$ lie on the periphery of $T'$. Hence, one of the two black vertices must be a cut-vertex of $T'$. \qed

**Acknowledgements**

The author is grateful to Professor Fuji Zhang and the referees for their many helpful comments and suggestions. This work is supported by the National Natural Foundation of China and Youth Foundation of Lanzhou University.
References