Journal of Pure and Applied Algebra 75 (1991) 75–83
North-Holland

Pure-injective modules over path algebras

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Communicated by A. Blass
Received 8 January 1990

Abstract

Okoh, F., Pure-injective modules over path algebras, Journal of Pure and Applied Algebra 75 (1991) 75–83.

The path algebra, R, over a field K, of a directed graph is the algebra with basis the paths and vertices of the graph, with multiplication given by path composition. In this paper the graphs are either Coxeter–Dynkin diagrams or extended Coxeter–Dynkin diagrams. All modules are unital right R-modules. The pure-injective R-modules, i.e., direct summands of direct products of finite-dimensional R-modules, are investigated in this paper. We show that—like the pure-projective modules—they are characterized by systems of cardinal invariants. Using these invariants we identify the pure-injective modules whose direct summands are direct products of finite-dimensional modules. It is also shown that an R-module is pure-projective and pure-injective if it has only finitely many isomorphism classes of finite-dimensional indecomposable submodules. This is a well-known result when R is the path algebra of a Coxeter–Dynkin diagram. The key lemma in the paper is a straightforward result on finite-dimensional modules. We also use it to show that an R-module always has a pure submodule of countable rank. Several properties of R-modules with no proper nonzero pure submodules are obtained.

1. Pure-injective and pure-projective modules

Let M be an R-module. A submodule N of M is *pure* if whenever N is contained in a submodule L of M with L/N finite-dimensional, then N is a direct summand of L. A module is *pure injective* if it is a direct summand of any module in which it is pure. The pure-injective modules are precisely the direct summands of direct products of finite-dimensional modules, see for example [8]. Dually, a module is *pure-projective* if and only if it is a direct sum of finite-dimensional modules. That is the end of the story for the structure of pure-projective modules—as long as one can describe the indecomposable finite-dimensional modules. If R is the path algebra of a Coxeter–Dynkin diagram, then every module is pure-projective and pure-injective. In that case, there are only finitely

0022-4049/91/$03.50

many isomorphism classes of indecomposable R-modules, i.e., R is of finite representation type, see [14]. The structure of pure-injective R-modules is a measure of the complexity of the module theory of R. If R is the path algebra of either a Coxeter–Dynkin diagram or an extended Coxeter–Dynkin diagram, then every pure-injective module has an indecomposable direct summand, see [8]. Otherwise, R always has a pure-injective module with no indecomposable direct summand (Theorem 13.7 of [13]). We refer to [13] for details on the pivotal role of pure-injective modules in the model theory of R-modules. It is also a good reference for the extensive work on pure-injective modules. The modules in this paper are unital right R-modules, where R is the path algebra of an extended Coxeter–Dynkin diagram. Such algebras are said to be *tame*. Path algebras that are neither of finite representation type nor tame are said to be *wild*; see [13] or [14] for a more precise definition. We shall indicate which of our results do not hold for wild algebras. This will show that these results are not special cases of results in the theory of modules over finite-dimensional hereditary algebras. The finite-dimensional indecomposable R-modules come in three types: *pre-injective*, *regular*, and *preprojective*. We cull the following description of the types from [14].

Let $0 \to P_1 \to P_2 \to M \to 0$ be a minimal projective resolution of a left or right R-module, M. The functor Hom(, R) denoted by $*$ yields a map $f^* : P_2^* \to P_1^*$ whose cokernel is denoted by $\operatorname{Tr} M$. We now apply to $\operatorname{Tr} M$ the functor Hom(, K) $= D$. The functor $*$ converts a right R-module to a left R-module and conversely, while the functor Hom(, K) reverses left and right R-modules. Denote $\operatorname{Tr} DM$ and $D\operatorname{Tr} M$ by $A^{-1}M$ and AM, respectively.

Let $P_1, P_2, \ldots, P_s$ be the indecomposable projective R-modules; let $P_{ns+i} = A^{-n}P_i$. This yields a sequence $(P_n)_{n=1}^{\infty}$ of indecomposable finite-dimensional modules. An arbitrary module is said to be *preprojective* if every nonzero submodule has a direct summand isomorphic to some P_n. As noted in [14, p. 350] the sequence $(P_n)_{n=1}^{\infty}$ has the following properties:

$$\operatorname{Hom}(P_i, P_j) \neq 0 \quad \text{implies that} \quad i \leq j\,. \tag{1}$$

Given any P_i there is a nonsplit sequence

$$0 \to P_i \xrightarrow{\alpha} X_i \xrightarrow{\beta} P_{i+s} \to 0\,, \tag{2}$$

where the indecomposable direct summands of X_i are of the form P_j, with $i < j < i + s$. Among other properties the sequence (2) has the property that if $g : P_i \to X'$ is a nonsplit monomorphism, then there exists a homomorphism

$$\gamma' : X_i \to X' \quad \text{with } \gamma = \gamma'\alpha\,. \tag{3}$$

The sequence $\{I_n = 1, 2, 3, \ldots\}$, the preinjective modules, are constructed in a similar fashion from the indecomposable injective R-modules.

A finite-dimensional module is *torsion* if it has no preprojective direct summand. Let tM be the submodule of an arbitrary module M generated by the finite-dimensional submodules of M that are torsion. If $tM = 0$, we say that M is *torsion-free*. In particular, preprojective modules are torsion-free. If $tM = M$, we say that M is *torsion*. Modules that are neither torsion nor torsion-free are *mixed*. The torsion modules with no pre-injective submodules are said to be *regular*. Regular torsion modules M over tame algebras behave like torsion modules over principal ideal domains—as detailed in Section 4 of [14]. In particular, M has a primary decomposition, $M = \sum_{t\in T} M_t$. Regular torsion modules over wild algebras have no tractable structure.

Lemmas 1.1 and 1.2—straightforward applications of Auslander–Reiten sequences—are crucial to the paper. Lemma 1.1 is contained in [2, Theorem 6.7] for Kronecker modules. It is in the latter form that it was used in [9] and [12].

Lemma 1.1. *Suppose P_i is an indecomposable preprojective submodule but not a direct summand of M, where M is torsion-free. Then P is contained in a finite-dimensional submodule N of M, where the indecomposable direct summands of N are of the form P_j, $i<j$.*

Proof. A finite-dimensional submodule of M is pure in M if and only if it is a direct summand of M [14, Theorem F]. Therefore, we may assume that we have a nonsplit exact sequence

$$0 \to P_i \to M_i \to M_i/P_i \to 0\,, \tag{4}$$

where M_i is a finite-dimensional submodule of M. Since M is torsion-free M_i is preprojective. It follows from the ordering of the P_n's in (1) that the endomorphism ring of P_i is a division ring. This fact coupled with the hypothesis that (4) does not split implies that P_i has no nonzero component in any direct summand P_j of M_i with $j<i$. The required submodule N is the submodule of M_i generated by all the direct summands of M_i of type P_j, $i<j$. □

Lemma 1.2. (a) $\mathrm{Ext}(P_{i+s}, P_i) \neq 0$.

(b) $\mathrm{Ext}(P_j, P_i) = 0$, $j<i+s$, *where the P's are indecomposable preprojective modules ordered as in* (1).

Proof. (a) follows from (2). (b) Suppose $j<i+s$ and

$$0 \to P_i \xrightarrow{g} X' \to P_j \to 0 \tag{5}$$

is a nonsplit sequence. From (3) we get the following diagram of exact sequences and commuting squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_i & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & P_{i+s} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & P_i & \underset{g}{\longrightarrow} & X' & \underset{h}{\longrightarrow} & P_j & \longrightarrow & 0 \end{array} \tag{6}$$

where $\chi(c) = h\gamma'(b)$, $b \in X_i$ and $\beta(b) = c$. Since $j < i + s$, χ is the zero map. Hence, $\gamma'(X) \subset P_i$. It follows from (1) and (2) that γ' is the zero map. This contradicts the fact that $\gamma'\alpha = g$. Therefore, the bottom row splits. Hence $\mathrm{Ext}(P_j, P_i) = 0$. □

Theorem 1.3. *Let M be a module over a tame hereditary finite-dimensional algebra R. If M has only finitely many isomorphism classes of indecomposable submodules, then M is pure-projective and pure-injective.*

Proof. Let tM be the submodule of M generated by the finite-dimensional submodules of M that are torsion. By Theorem 4.1 of [14], tM is a pure submodule of M. The hypothesis implies that tM is bounded in the sense of [5] and hence is pure-injective and pure-projective as can be deduced from Sections 17, 30, and 100 of [5]. Therefore,

$$M = tM \dotplus M' ,$$

where M' is a torsion-free module. We now show that M' is preprojective.

Let X be a nonzero submodule of M'. We have to show that X has a direct summand isomorphic to some P_i. For $0 \neq x \in X$, xR is a finite-dimensional submodule of X. Hence it is preprojective. Let P_r be an indecomposable preprojective submodule of xR. If X has no finite-dimensional preprojective direct summand, we apply Lemma 1.1 to P_r to get other preprojective finite-dimensional indecomposable submodules of X. An inductive application of Lemma 1.1 gives infinitely many isomorphism classes of such submodules. Since this contradicts the hypothesis, X must have a finite-dimensional preprojective direct summand, as required. With M' preprojective, it follows from the hypothesis and Corollary 2.3 of [12] that it is pure-projective and pure-injective. □

Remark 1.4. If the module M in Theorem 1.3 has no infinite-dimensional pre-injective submodule, then the converse of Theorem 1.3 is also true.

2. Cardinal invariants for pure-injective modules

A module M is said to be *divisible* if $\mathrm{Ext}(S, M) = 0$ for all simple regular torsion modules S. A module with no nonzero divisible submodule is said to be *reduced*.

The following proposition is Proposition 3 in [9]:

Proposition 2.1. *A pure-injective module M can be put in the form $M = M_1 \dotplus M_2 \dotplus M_3$, where M_1 is divisible, M_2 is reduced and torsion, and M_3 is preprojective.* □

The summands M_1 and M_2 can be characterized by complete and independent systems of invariants, see Section 40 of [5], and Sections 4 and 5 of [14]. We now turn our attention to M_3. Let $\{P_n\}_{n=1}^{\infty}$ be the set of indecomposable preprojective modules ordered as in (1). For a fixed positive integer k and for J_k an indexing set, possibly empty, let $M_k = \bigoplus_{J_k} P_k$, $M = \prod_{k=1}^{\infty} M_k$.

We can now prove the following lemma about M. Lemma 2.2(b) was called Condition (1) in [9].

Lemma 2.2. (a) *Every direct summand of M_k is isomorphic to P_k.*

(b) *Every indecomposable direct summand of M is isomorphic to a direct summand of M_k for some positive integer k.*

Proof. (a) is immediate from Azumaya's theorem; see, e.g., Theorem 12.6 of [1].

(b) Let N be an indecomposable direct summand of M. Since M is preprojective, by Proposition 2.7 of [14], N is isomorphic to P_r for some positive integer r. By the ordering in (1), N has no nonzero component in any M with $k<r$.

So, $N \subset \prod_{k=r}^{\infty} M_k$. If $M_r \neq 0$, then we are done. Suppose $M_r = 0$. Then $N \subset \prod_{k=r+1}^{\infty} M_k$. Let $\prod_{k=r+1}^{\infty} M_k = N \oplus N'$. By Lemma 1.4(b), $\mathrm{Ext}(P_{r+s}, \prod_{k=r+1}^{\infty} M_k) = 0$, while

$$\mathrm{Ext}(P_{r+s}, P_r \oplus N') = \mathrm{Ext}(P_{r+s}, P_r) \oplus \mathrm{Ext}(P_{r+s}, N') \neq 0$$

by Lemma 1.2(a), a contradiction. Therefore, $M_r \neq 0$ and N is isomorphic to P_r, as required. □

Let M be a torsion-free module. For any P_k there is a submodule N_k of M with N_k isomorphic to M_k for some indexing set J_k, possibly empty, such that M/N_k has no direct summand of type P_k. Let $M' = \bigoplus_{k=1}^{\infty} N_k$. It is the maximal pure-projective submodule of M that is also pure in M. In particular, M/M' is torsion-free and has no finite-dimensional direct summand. The submodule M' is unique up to isomorphism. For proofs see Section G of [14].

The following theorem is Theorem 1 of [9] with the proviso 'provided Condition (1) is satisfied' deleted. Lemma 2.2(b) ensures that 'Condition (1)' is always satisfied. Below, $\bigoplus P_k$ stands for a direct sum of P_k over an arbitrary indexing set, P_k fixed.

Theorem 2.3. *Let M be a preprojective pure-injective module and for k a natural number let $M_k = \bigoplus_{J_k} P_k$ be maximal among pure submodules of M of type $\bigoplus P_k$. Then M is isomorphic to $\prod_{k=1}^{\infty} M_k$.* □

Corollary 2.4 [9, Corollary 1]. *The set $\{\mathrm{Card}(J_k)\colon k = 1, 2, 3, \ldots\}$ is a complete independent system of invariants for preprojective pure-injective modules.* □

Example 2.5. The set $\{1, 1, 1, \ldots\}$ is the invariant for $\prod_{n=1}^{\infty} P_n$.

Theorem 2.6. *Let $M = \prod_{j\in J} M_j$, M_j in $\{P_n\}_{n=1}^{\infty}$ for each j in J. Every direct summand of M is again a direct product of modules in $\{P_n\}_{n=1}^{\infty}$ if and only if for each P_n the set $S_n = \{j \in J\colon M_j$ is isomorphic to $P_n\}$ is finite.*

Proof. Suppose S_n is infinite for some n. Let $N = \prod_{j\in S_n} M_j$. Then N is a direct summand of M. Then from [15] or Section 1 of [12] we deduce that N, hence M, has a direct summand that is not a direct product of modules in $\{P_n\}_{n=1}^{\infty}$.

Suppose S_n is finite for each n. Then, in the notation of Theorem 2.3, M_k is finite-dimensional for each k by Lemma 2.2. Since M is a direct product of finite-dimensional modules it, and its direct summands, are pure-injective; see, for example, [8]. Therefore, if N is a direct summand of M, then $N = \prod_{k=1}^{\infty} N_k$, $N_k \subset M_k$. Hence, N_k is finite-dimensional. So N is a direct product of modules in $\{P_n\}_{n=1}^{\infty}$. □

Let $M = \prod_{j\in J} M_j$ be a direct product of finite-dimensional indecomposable modules. Then $M = M_1 \dotplus M_2 \dotplus M_3$, where $M_1 = \prod_{j\in J_1} M_j$, M_j pre-injective for all $j \in J_1$, $M_2 = \prod_{j\in J_2} M_j$, M_j preprojective for all $j \in J_2$, $M_3 = \prod_{t\in T}\prod_{j\in J_{3t}} M_j$, M_j finite-dimensional indecomposable torsion regular for all $j \in J_{3t}$. Let $I_n : n = 1, 2, 3, \ldots$ be the set of isomorphism classes of indecomposable pre-injective modules. Then $\bigoplus_{n=1}^{\infty} I_n$ is a direct summand of $\prod_{n=1}^{\infty} I_n$, see [14, Theorem 3.7]. Using this, Theorem 2.6, and Section 40 of [5] we get the following theorem about M—which is specific to tame algebras because of the use of properties of regular torsion modules.

Theorem 2.7. *Every direct summand of M is a direct product of finite-dimensional indecomposable modules if and only if the following conditions are simultaneously satisfied:*

(a) *J_1 and J_{3t}, for each $t \in T$, are finite.*
(b) *the summand M_2 satisfies the conditions of Theorem 2.6.*

As a general rule indecomposable modules over tame algebras can be arbitrarily large, see [4] which extends [3]. In [7] it is shown that there is a bound on the size of a special class of indecomposable modules over the tame algebra, called a *Kronecker algebra*, that arises from pairs of linear maps. Lemma 1.1 makes the extension of this result to all tame algebras a mere formality. However, we need to recall some definitions.

There is a unique (up to isomorphism) torsion-free indecomposable divisible module Q. Every torsion-free module M can be embedded in $L = \bigoplus_r Q$ with L/M torsion-regular. The cardinal number r is an invariant of M and is called the *rank* of M [14, Theorem 5.5]. The preprojective modules P_n have finite rank. If X is a subset of a module M, the smallest submodule N of M with $X \subset N$ and M/N

torsion-free is called the *torsion-closure* of X in [12] M and is denoted by tc X. A pure submodule of a torsion-free module M is torsion-closed in M. We shall be using other properties of tc stated in [12]. A module M is *purely simple* if it has no proper nonzero pure submodule. A mixed module cannot be purely simple because its torsion submodule is a pure submodule (Theorem 4.1 of [14]).

Theorem 2.8. *Every torsion-free M has a pure submodule of countable rank. In particular, a module of uncountable rank is not purely simple.*

Proof. If M has a direct summand of type P_k, then we are done. So let us suppose that M has no direct summand of type P_k for any k.

As in the proof of Theorem 1.3, M has a submodule N_1 of type P_k for some k. Apply Lemma 1.1 to N_1 to get a submodule N_2 of M with $N_1 \subset N_2$, and if N_2 has a direct summand of type P_r, then $r > k$. Now apply Lemma 1.1 to each of the summands of N_2. Continuing in this way we get an ascending union of finite-dimensional submodules, $N_1 \subset N_2 \subset \cdots \subset N_k \cdots$, with the following property: If N_k has a direct summand of type P_r, then $r > \max\{n\colon N_{k-1}$ has a direct summand of type $P_n\}$. The submodule $N = \bigcup_{k=1}^{\infty} N_k$ has no direct summand of type P_k for any positive integer k: Suppose $N = L_1 \dotplus L_2$, where L_1 is of type P_k. By the ordering in (1) and the last sentence there exists a positive integer k_0 such that $N_{k_0+j} \subset L_2$ for all $j \geq 1$. Therefore, L_2 is 0. Since N is an ascending union of modules of finite rank its rank is at most countable. The torsion closure N' of N in M has no direct summand of type P_k for any k: Suppose $N' = L_1 \dotplus L_2$, where L_1 is of type P_k. As in the proof of the same statement for N, $N \subset L_2$. Since L_2 is torsion-closed in N', i.e. it is its own torsion closure in N', $N' \subset L_2$. So L_1 is 0. By Corollary 2.3 of [14], N' is a pure submodule of M. Since its rank is at most that of N it is our desired pure submodule. □

Remark 2.9. It can be shown, see [6], that if S is any ring, then any S-module of cardinality greater than that of S is not purely simple. From that we can deduce Theorem 2.8 for countable path algebras.

Proposition 2.10. *If a torsion-free module M of infinite rank is purely simple, then it is an ascending union of finite-dimensional torsion-closed submodules and each nonzero proper torsion-closed submodule of M has a finite-dimensional direct summand.*

Proof. Suppose that M is purely simple. If M has a nonzero proper torsion-closed submodule N with no finite-dimensional direct summand, then N is a pure submodule of M, by Corollary 2.3 of [14]. The submodule N' in the proof of Theorem 2.8 is an ascending union of finite-dimensional torsion-closed submodules of M. Since it is a nonzero pure submodule of M it is equal to M. □

Remark 2.11. By adding an extra hypothesis in Proposition 2.10, as in Theorem 2 of [7], we can get a characterization of purely simple R-modules of infinite rank. However, the existence of such modules is moot.

We have seen that the specificity to Kronecker algebras of the results and proofs in [7] is only apparent. We conclude the paper with results that show that the same statement applies to the results in [11]. The following facts should be borne in mind: rank is additive on extensions of torsion-free modules by torsion-free modules; an infinite-dimensional torsion-free module that is not purely simple has an infinite-dimensional proper pure submodule; and Corollary 2.3 of [14]. In particular, we have the following proposition:

Proposition 2.12. *A torsion-free infinite-dimensional module M of finite rank is purely simple if and only if every proper torsion-closed submodule N of M is finite-dimensional.* □

Corollary 2.13. *Let M be a torsion-free module of finite rank n. Suppose M is infinite-dimensional and purely simple.*

(a) *Then every torsion-free quotient of M is purely simple.*

(b) *M has a finite-dimensional torsion-closed submodule, L, of rank $n-1$. In particular, M is an extension of L by a rank one torsion-free module.*

(c) *Every nonzero endomorphism of M is monic.*

Proof. (a) Let N be a submodule of M with M/N torsion-free. By Proposition 2.12, N is finite-dimensional. So M/N is infinite-dimensional. If N' is a proper torsion-closed infinite-dimensional submodule of M/N, the torsion-closure in M of its inverse image under the natural projection would contradict Proposition 2.12. So N' does not exist. Again, by Proposition 2.12, M/N is purely simple.

(b) The proof is by induction on rank. By Lemma 6.3 of [14], M has a finite-dimensional submodule, M', of rank one. (We note that the rank one hypothesis is not used in the proof of that lemma in [14].) The torsion-closure, N, of M' in M is still of rank one. By Proposition 2.12, N is finite-dimensional. By Corollary 2.13(a), M/N is purely simple. Its rank is $n-1$. Hence, by the induction hypothesis, it has a finite-dimensional torsion-closed submodule, N', of rank $n-2$. Its inverse image, L, is a finite-dimensional torsion-closed submodule of M of rank $n-1$.

(c) Let f be a nonzero endomorphism of M. Then $N=$ the kernel of f is a torsion-closed submodule of M. Applying Proposition 2.12 to the submodules N, and the image of f, isomorphic to M/N, gives (c). □

Remark 2.14. There is a list in [8] of the modules that are both purely simple and pure-injective. The endomorphism rings of the modules in the list are readily determined. Let M be a module satisfying the hypotheses of Corollary 2.13. By

Corollary 2.13(c) the endomorphism ring of M is an integral domain. Except when rank of M is one we do not yet know which integral domains can occur.

Acknowledgment

We thank the referee for pointing out the redundancy of assuming that K is algebraically closed. Professor Istvan Agostan informed us in a letter that he also proved Theorem 2.8 of this paper.

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