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Transformations of Families of Approximating Functions

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1. INTRODUCTION

In this paper we consider certain transformations of families of approximating functions which leave invariant some desirable properties of these families. These transformations appear to be useful, for example, in generating from linear Chebyshev families nonlinear families of types for which some characterization and uniqueness theorems are available. However, very little discussion of such transformations appears in the literature. Motzkin [4] mentions some transformations which preserve the unisolvence property. More recently, Moursund and Taylor [5] noted that generalized weight functions can be used to transform families of functions and that such a transformation preserves the varisolvence property, but they did not exploit this fact.

The transformations that we have studied are based essentially on Moursund's generalized weight functions. We find, however, that continuity and a monotonicity condition are sufficient for the transformation to preserve varisolvence. One additional hypothesis suffices to yield a unisolvence-preserving transformation. These results are presented in Section 2. Under slightly stronger hypotheses than are required for the varisolvency case, we obtain (Section 3) transformations preserving properties pertinent to the nonlinear approximation theory of Meinardus and Schwedt [1, 2]. Finally, Section 4 contains a theorem giving bounds on the error of approximation from a transformed family in terms of the error of approximation from the original family.

Usually varisolvent and unisolvent families are defined on a closed interval of the real line. Our results, however, will hold in a more general setting. We shall deal throughout with continuous real-valued functions defined on a compact metric space X. The norm to be used is the uniform (Chebyshev) norm.

2. Transformations of Varisolvent and Unisolvent Families

The theory of approximation by nonlinear unisolvent families dates back to Motzkin [3, 4] and Tornheim [9, 10]. For such families one has theorems quite analogous to the standard existence, uniqueness, and characterization theorems of linear Chebyshev theory. Rice [6, 7] has introduced the more general concept of varisolvent families and has obtained uniqueness and characterization theorems. We include here the key definitions for reference.

Definition 1. A family F of functions $f(a, x)$, where $x \in X$ and $a \in A$, A being a subset of the n-space R^n, is called unisolvent if for any prescribed $y = (y_1, y_2, ..., y_n) \in R^n$ and any n distinct points x_i $(i = 1, 2, ..., n)$ in X, there exists a unique $a \in A$ such that $f(a, x_i) = y_i$ for $i = 1, 2, ..., n$.

Definition 2. A family F of functions $f(a, x)$ is said to have property Z of degree m at $a_0 \in A$ if, for every $a \in A$, the function $f(a_0, x) - f(a, x)$ possesses at most $m - 1$ zeros on X or vanishes identically.

Definition 3. A family F of functions $f(a, x)$ is called (locally) solvent of degree m at $a_0 \in A$ if for every set of m distinct points x_i $(i = 1, 2, ..., m)$ in X and every prescribed $\epsilon > 0$ there exists a $\delta = \delta(a_0, \epsilon, x_1, ..., x_m) > 0$ such that

$$| y_i - f(a_0, x_i)| < \delta, \qquad i = 1, 2, ..., m,$$

implies the existence of an $a \in A$ satisfying

$$f(a, x_i) = y_i, \qquad i = 1, 2, ..., m,$$

and $\| f(a, x) - f(a_0, x)\| < \epsilon$.

Definition 4. A family F is called (locally) unisolvent of degree M at $a_0 \in A$ if M is the largest integer m for which F is solvent of degree m at a_0 and has property Z of degree m there.

Definition 5. A family F is called varisolvent if for each $a \in A$ there exists an integer $m(a)$ such that F is unisolvent of degree $m(a)$ at a.

We shall consider transformations of families F of functions $f(a, x)$ to families of functions of the form $W(x, f(a, x))$. Our first theorem deals with varisolvent families.

Theorem 1. *Let $W(x, y)$ satisfy* (a) *$W(x, y)$ is a strictly increasing function of y for every $x \in X$;* (b) *$W(x, y)$ is continuous on $X \times (-\infty, \infty)$. Let F be a varisolvent family of functions. Then $\{W(\cdot, f) : f \in F\}$ is varisolvent on X, each $W(\cdot, f)$ having the same degree as the corresponding f.*

Proof. Suppose $f_1 \in F$ has degree m. It will suffice to prove property Z and solvency of $\{W(\cdot, f) : f \in F\}$ of degree m at f_1, for it will then follow that we cannot have solvency of degree $m + 1$. If this were not the case, then given any set of $m + 1$ distinct points $x_1, x_2, \ldots, x_{m+1} \in X$ we could find $f_2 \in F$ such that $W(x_i, f_1(x_i)) = W(x_i, f_2(x_i))$ for $i = 1, \ldots, m$ but $W(x_{m+1}, f_1(x_{m+1})) \neq W(x_{m+1}, f_2(x_{m+1}))$. However, property Z of degree m implies that $W(x, f_1(x))$ equals $W(x, f_2(x))$ identically, which is a contradiction.

In order to prove property Z, suppose there exist m distinct points $x_1, x_2, \ldots, x_m$ in X and a function $f_2 \in F$ such that $W(x_i, f_1(x_i)) = W(x_i, f_2(x_i))$ for $i = 1, 2, \ldots, m$. Then by (a) we must have $f_1(x_i) = f_2(x_i)$ for $i = 1, 2, \ldots, m$; but since the family F enjoys property Z, this implies that $f_1 = f_2$, and hence $W(\cdot, f_1) = W(\cdot, f_2)$. Thus $\{W(\cdot, f) : f \in F\}$ satisfies property Z.

In order to show that $\{W(\cdot, f) : f \in F\}$ is solvent of degree m at f_1, let $\epsilon > 0$ and distinct points $x_1, x_2, \ldots, x_m \in X$ be given. Let

$$I = [\min_{x \in X} f_1(x) - \epsilon, \max_{x \in X} f_1(x) + \epsilon].$$

Since $W(x, y)$ is uniformly continuous on the compact set $X \times I$, we can find an ϵ^* satisfying $0 < \epsilon^* \leqslant \epsilon$ such that $| l_1 - l_2 | < \epsilon^*$ implies $| W(x, l_1) - W(x, l_2)| < \epsilon$ for all $x \in X$ and all l_1, l_2 in I. Thus, for any function g on X.

$$\| g - f_1 \| < \epsilon^* \Rightarrow \| W(\cdot, g) - W(\cdot, f_1)\| < \epsilon. \tag{1}$$

By the varisolvence of F at f_1 we can find $\delta^* > 0$ such that

$$\begin{aligned} &| \alpha_i - f_1(x_i)| < \delta^* \text{ for all } i \Rightarrow \{\text{there exists an } f_2 \in F \\ &\text{such that } f_2(x_i) = \alpha_i \text{ for all } i \text{ and } \| f_1 - f_2 \| < \epsilon^*\}. \end{aligned} \tag{2}$$

Now, for each $x \in X$, $W_x(y) \equiv W(x, y)$ is a continuous, increasing function of y with a continuous, increasing inverse W_x^{-1}. From (a), (b) and the continuity of $W_{x_i}^{-1}$ for every i, we can find a number $\delta > 0$ such that for every i, $| y_i - W(x_i, f_1(x_i))| < \delta$ implies that y_i is in the range of W_{x_i} (i.e., $W_{x_i}^{-1}(y_i)$ exists) and $| W_{x_i}^{-1}(y_i) - f_1(x_i)| < \delta^*$. It follows from (2) that there is an $f_2 \in F$ such that $f_2(x_i) = W_{x_i}^{-1}(y_i)$ for all i and $\| f_1 - f_2 \| < \epsilon^*$. Therefore, $W(x_i, f_2(x_i)) = y_i$ for all i and, by (1), $\| W(\cdot, f_2) - W(\cdot, f_1)\| < \epsilon$. This completes the proof of the theorem.

The proof of Theorem 1 is essentially the same as that of Theorem 1 of Moursund and Taylor [5]. We have included it here to make it clear that our hypotheses are sufficient to allow the proof to go through. Now we shall show that, by adding one additional hypothesis, one obtains something stronger—namely, a unisolvency-preserving transformation.

THEOREM 2. *Let* $W(x, y)$ *satisfy* (a) *and* (b) *of Theorem* 1 *and also* (c): $\lim_{|y|\to\infty} |W(x, y)| = \infty$. *Let F be a family of functions unisolvent of degree n. Then* $\{W(\cdot, f) : f \in F\}$ *is unisolvent of degree n.*

Proof. Given any $y = (y_1, y_2, ..., y_n)$ and any n distinct points x_i ($i = 1, 2, ..., n$) in X, it follows from the assumptions on $W(x, y)$ that $W_{x_i}^{-1}(y_i)$ is defined for all i. By the unisolvence of F we can find an $f \in F$ such that $f(x_i) = W_{x_i}^{-1}(y_i)$, or $W(x_i, f(x_i)) = y_i$, for all i. Since property Z has already been established in the proof of Theorem 1, this f must be unique. Thus unisolvency is proved.

From the theorems of this section we can deduce, for example, the varisolvence on any closed and bounded interval of the set of functions $\{\exp(p(x)) : p \in P_n\}$, where P_n is the family of polynomials of degree n or less. As another example, we see that the family $\{(g(x) + p(x))^k : p \in P_n\}$, for any odd integer k and any fixed continuous function g, must be unisolvent of degree $n + 1$.

3. LOCAL HAAR CONDITION AND ASYMPTOTIC CONVEXITY

Meinardus and Schwedt [1, 2] have developed a theory of nonlinear approximation (including the usual uniqueness and characterization theorems) based on the concepts of asymptotic convexity and the local Haar condition. A differentiability condition is imposed on the approximating functions, but otherwise the theory is very similar to that based on varisolvency. One might expect, therefore, that transformations of the type discussed in Section 2, but with perhaps additional smoothness conditions, will preserve the key properties pertinent to this theory. We shall show that this is indeed the case, but first we mention the definitions of the concepts under consideration.

DEFINITION 6. An n-dimensional linear space F of functions on X is said to satisfy the Haar condition if every f in F either vanishes identically or has at most $n - 1$ zeros on X.

We note for future reference that, if $\{\varphi_i\}_{i=1}^n$ is a basis for F, the Haar condition is equivalent to the condition that the determinant $\det(\varphi_i(x_j)) \neq 0$ for any distinct points $x_1, x_2, ..., x_n$ of X.

Now let A be an open subset of R^n, so that any $a \in A$ has the form $a = (\alpha_1, \alpha_2, ..., \alpha_n)$.

DEFINITION 7. A set of functions $\{f(a, x) : a \in A\}$ is said to satisfy the local Haar condition if each $f(a, x)$ is continuously differentiable with respect to α_ν ($\nu = 1, 2, \ldots, n$) and $\overline{W}(a)$, the linear span of the set $\{\partial f(a, x)/\partial\alpha_1, \ldots, \partial f(a, x)/\partial\alpha_n\}$, satisfies the Haar condition. We denote the dimension of $\overline{W}(a)$ by $d(a)$.

DEFINITION 8. A set of functions $\{f(a, x) : a \in A\}$ is asymptotically convex if for each a, b in A and each $t \in [0, 1]$ there exist a parameter-value $a(t) \in A$ and a continuous real-valued function $g(x, t)$, defined on $X \times [0, 1]$ and satisfying $g(x, 0) > 0$ for all $x \in X$, such that

$$\|(1 - tg(x, t))f(a, x) + tg(x, t)f(b, x) - f(a(t), x)\| = o(t) \qquad \text{as} \quad t \to 0.$$

THEOREM 3. *Let $W(x, y)$ satisfy* (a′) *$W(x, y)$ is continuously differentiable with respect to y on $X \times (-\infty, \infty)$, and* (b′) *$\partial W(x, y)/\partial y > 0$ for all (x, y) in $X \times (-\infty, \infty)$. Then if $\{f(a, x) : a \in A\}$ satisfies the local Haar condition, so does $\{W(x, f(a, x)) : a \in A\}$, and the transformation leaves $d(a)$ unchanged.*

Proof. Let $a \in A$ be fixed, and, renumbering if necessary, let $\{\partial f(a, x)/\partial\alpha_\nu\}_{\nu=1}^{d(a)}$ be a basis in $\overline{W}(a)$. We note that $\partial W(x, f(a, x))/\partial\alpha_k$, $k = d(a) + 1, \ldots, n$, are linearly dependent on $\{\partial W(x, f(a, x))/\partial\alpha_\nu\}_{\nu=1}^{d(a)}$, since

$$\begin{aligned} \frac{\partial W(x, f(a, x))}{\partial\alpha_k} &= \left.\frac{\partial W(x, y)}{\partial y}\right|_{y=f(a,x)} \cdot \frac{\partial f(a, x)}{\partial\alpha_k} \\ &= \left.\frac{\partial W(x, y)}{\partial y}\right|_{y=f(a,x)} \cdot \sum_{\nu=1}^{d(a)} \gamma_\nu \frac{\partial f(a, x)}{\partial\alpha_\nu} \quad (\text{for some } \gamma_1, \ldots, \gamma_{d(a)}) \\ &= \sum_{\nu=1}^{d(a)} \gamma_\nu \frac{\partial W(x, f(a, x))}{\partial\alpha_\nu}. \end{aligned}$$

Therefore, the transformation can not increase $d(a)$.

Now let distinct points $x_1, \ldots, x_{d(a)}$ in X be given, and consider the following matrices. Let

$$B = (b_{ij}), \qquad \text{where} \quad b_{ij} = \frac{\partial W(x_j, f(a, x_j))}{\partial\alpha_i},$$

$$C = (c_{ij}), \qquad \text{where} \quad c_{ij} = \frac{\partial f(a, x_j)}{\partial\alpha_i},$$

and let $D = (d_{ij})$ be the diagonal matrix with

$$d_{ii} = \left.\frac{\partial W(x_i, y)}{\partial y}\right|_{y=f(a,x_i)}.$$

We note that $B = CD$, so $\det(B) = \det(C) \cdot \det(D)$, and it follows from our assumptions that this product is nonzero. It is then clear that the $d(a)$

functions $\partial W(x, f(a, x))/\partial \alpha_\nu$ are linearly independent, and that the local Haar condition is satisfied (with degree $d(a)$ at each a) by the set $\{W(x, f(a, x)) : a \in A\}$.

THEOREM 4. *If $W(x, y)$ satisfies conditions* (a′) *and* (b′) *of Theorem* 3, *and if $\{f(a, x) : a \in A\}$ is asymptotically convex, then $\{W(x, f(a, x)) : a \in A\}$ is also asymptotically convex.*

Proof. From the hypotheses it follows that

$$f(a(t), x) = f(a, x) + tg(x, t)(f(b, x) - f(a, x)) + r(x, t),$$

where $\| r(x, t)\| = o(t)$ as $t \to 0$. Now consider any fixed $x \in X$. We have

$$\left.\frac{\partial W(x, y)}{\partial y}\right|_{y=f(a,x)} = \lim_{\Delta y \to 0} \frac{W(x, f(a, x) + \Delta y) - W(x, f(a, x))}{\Delta y}.$$

Taking $\Delta y = tg(x, t)(f(b, x) - f(a, x)) + r(x, t)$, which approaches zero in norm as $t \to 0$, we find

$$W(x, f(a(t), x)) = W(x, f(a, x)) + [tg(x, t)(f(b, x) - f(a, x)) + r(x, t)] \left.\frac{\partial W(x, y)}{\partial y}\right|_{y=f(a,x)} + q(x, t), \tag{3}$$

where $\| q(x, t)\| = o(t)$ as $t \to 0$. By the mean-value theorem we have

$$W(x, f(b, x)) - W(x, f(a, x)) = [f(b, x) - f(a, x)] \left.\frac{\partial W(x, y)}{\partial y}\right|_{y=f(a,x) + \theta[f(b,x) - f(a,x)]} \tag{4}$$

for some θ (depending on x) in (0, 1). Substituting (4) into (3), we obtain an equality of the form

$$W(x, f(a(t), x)) = W(x, f(a, x)) + tG(x, t)[W(x, f(b, x)) - W(x, f(a, x))] + E(x, t),$$

where, as is easily seen, $G(x, t)$ is continuous, $G(x, 0) > 0$, and $\| E(x, t)\| = o(t)$ as $t \to 0$.

4. UPPER AND LOWER BOUNDS FOR $\rho_{M \circ F}(g)$

Let g be an arbitrary function defined on X, and let F be a family of functions on X. Define $\rho_F(g) = \inf_{f \in F} \| f - g \|$, the error of approximation to g from F. It is clearly of interest to determine how a transformation of the family F will affect ρ. In this section we study this problem for transformations which are a special case of those considered in Section 2.

Let M be a real-valued function defined on the real line R and such that on any closed interval it is (a) strictly increasing and (b) absolutely continuous. Furthermore, assume that (c) M^{-1} is absolutely continuous on any closed and bounded interval contained in $M(R)$. For a given F, we shall consider the family $\{M \circ f : f \in F\}$, denoted by $M \circ F$. Now let g be continuous on X, and assume that $M^{-1} \circ g$ exists at x, for all $x \in X$. Then the quantity for which we wish to obtain bounds is $\rho_{M \circ F}(g)$. We agree to exclude the case where $\rho_F(M^{-1} \circ g) = 0$; it is easily shown that this occurs if and only if $\rho_{M \circ F}(g) = 0$.

Taking any $\delta > 0$, let $f_1 \in F$ be such that

$$\| f_1 - M^{-1} \circ g \| \leqslant (1 + \delta)\, \rho_F(M^{-1} \circ g).$$

Using a standard theorem from real analysis [8], we see that M is differentiable a.e. and, for any $x' \in X$,

$$g(x') - M \circ f_1(x') = \int_{f_1(x')}^{M^{-1} \circ g(x')} M'(t)\, dt. \tag{5}$$

Now $\rho_{M \circ F}(g) \leqslant \| g - M \circ f_1 \| = \sup_{x' \in X} | g(x') - M \circ f_1(x')|$. Introducing the notation

$$\langle a, b \rangle = \begin{cases} [a, b], & a < b, \\ \{a\}, & a = b, \\ [b, a], & a > b, \end{cases}$$

it follows from (5) that

$$\rho_{M \circ F}(g) \leqslant \sup_{x' \in X} \{| M^{-1} \circ g(x') - f_1(x')| \cdot \sup_{t \in T} | M'(t)|\},$$

where $T = \langle f_1(x'), M^{-1} \circ g(x') \rangle$.

From this follows

$$\begin{aligned} \rho_{M \circ F}(g) &\leqslant \| f_1 - M^{-1} \circ g \| \cdot \sup_{x' \in X} \sup_{t \in T} | M'(t)| \\ &\leqslant (1 + \delta)\, \rho_F(M^{-1} \circ g) \sup_{t \in G(\delta)} | M'(t)|, \end{aligned} \tag{6}$$

where

$$G(\delta) \equiv [\inf_{x' \in X} M^{-1} \circ g(x') - (1 + \delta)\, \rho_F(M^{-1} \circ g), \sup_{x' \in X} M^{-1} \circ g(x') + (1 + \delta)\, \rho_F(M^{-1} \circ g)].$$

Since the inequality (6) holds for all $\delta > 0$, we may let $\delta \to 0$ in the first factor, obtaining

$$\rho_{M \circ F}(g) \leqslant \rho_F(M^{-1} \circ g) \sup_{t \in G(\delta)} |M'(t)| \qquad \text{for any} \quad \delta > 0. \tag{7}$$

Note that assumption (c) was not used in getting this upper bound.

To obtain a lower bound, take $\delta > 0$ and let $f_2 \in F$ be such that $\| g - M \circ f_2 \| \leqslant (1 + \delta) \rho_{M \circ F}(g)$. Then, using (c), we see that for any $x' \in X$

$$M^{-1} \circ g(x') - f_2(x') = \int_{M \circ f_2(x')}^{g(x')} (M^{-1})'(t)\, dt.$$

Noting that $\rho_F(M^{-1} \circ g) \leqslant \| M^{-1} \circ g - f_2 \|$, and using the same procedure by which we arrived at (7), we find

$$\rho_F(M^{-1} \circ g) \leqslant \rho_{M \circ F}(g) \sup_{t \in H(\delta)} |(M^{-1})'(t)| \qquad \text{for any} \quad \delta > 0, \tag{8}$$

where

$$H(\delta) \equiv [\inf_{x' \in X} g(x') - (1 + \delta)\rho_{M \circ F}(g), \sup_{x' \in X} g(x') + (1 + \delta)\rho_{M \circ F}(g)] \cap M(R).$$

Since $\rho_F(M^{-1} \circ g) \neq 0$ by assumption, $\sup_{t \in H(\delta)} |(M^{-1})'(t)| > 0$ for all $\delta > 0$. Therefore, we obtain the lower bound

$$\rho_{M \circ F}(g) \geqslant \rho_F(M^{-1} \circ g) \inf_{t \in H(\delta)} | 1/(M^{-1})'(t)|. \tag{9}$$

Combining (7) and (9) yields our final result:

$$\sup_{t \in G(\delta)} | M'(t)| \geqslant \frac{\rho_{M \circ F}(g)}{\rho_F(M^{-1} \circ g)} \geqslant \inf_{t \in H(\delta)} | 1/(M^{-1})'(t)| \tag{10}$$

for any $\delta > 0$.

If M and M^{-1} are continuously differentiable, then $G(\delta)$ and $H(\delta)$ can be replaced by $G(0)$ and $H(0)$, respectively. If $\rho_F(M^{-1} \circ g)$ can be calculated, one can get an upper bound on $\rho_{M \circ F}(g)$; using this bound to get a set $H \supseteq H(\delta)$ (for some δ), one can then get a lower bound on $\rho_{M \circ F}(g)$.

As a simple example, let $M(y) = e^y$; we then have $M \circ F = \{e^f : f \in F\}$, $M^{-1}(x) = \ln x$, and $M(R) = (0, +\infty)$. Let g be the function to be approximated. We have assumed that $M^{-1}(g(x)) = \ln(g(x))$ exists for all $x \in X$, i.e., $g > 0$ on X. This is a reasonable restriction since $e^{f(x)}$ is always positive. Also we assume that $\rho_F(\ln g) \neq 0$. Let α, β be such that $0 < \alpha \leqslant g(x) \leqslant \beta$, for all $x \in X$. Then $\ln \alpha \leqslant M^{-1} \circ g \leqslant \ln \beta$. It follows that

$$G(0) \subseteq [\ln \alpha - \rho_F(\ln g), \ln \beta + \rho_F(\ln g)] \equiv G,$$

and

$$H(0) \subseteq [\alpha - \rho_{\exp\circ F}(g), \beta + \rho_{\exp\circ F}(g)] \equiv H.$$

Finally, we compute

$$\sup_{t \in G} | M'(t)| = \sup_{t \in G} | e^t | = \beta \exp[\rho_F(\ln g)],$$

and

$$\inf_{t \in H} | 1/(M^{-1})'(t)| = \inf_{t \in H} | t | = \max(0, \alpha - \rho_{\exp\circ F}(g)).$$

From (10) it then follows immediately that

$$(\alpha - \rho_{\exp\circ F}(g))\, \rho_F(\ln g) \leqslant \rho_{\exp\circ F}(g) \leqslant \beta \exp[\rho_F(\ln g)]\, \rho_F(\ln g),$$

and, rearranging, we find

$$\frac{\alpha}{\rho_F(\ln g) + 1} \leqslant \frac{\rho_{\exp\circ F}(g)}{\rho_F(\ln g)} \leqslant \beta \exp[\rho_F(\ln g)].$$

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