J. Differential Equations 246 (2009) 3126–3135

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On the cyclicity of weight-homogeneous centers

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ARTICLE INFO

Article history:
Received 12 December 2007
Revised 9 February 2009
Available online 24 February 2009

MSC:
34C07
34C05
34C14

Keywords:
Weight-homogeneous differential system
Poincaré–Pontryagin–Melnikov function
Cyclicity
Nilpotent center
Inverse integrating factor

ABSTRACT

Let W be a weight-homogeneous planar polynomial differential system with a center. We find an upper bound of the number of limit cycles which bifurcate from the period annulus of W under a generic polynomial perturbation. We apply this result to a particular family of planar polynomial systems having a nilpotent center without meromorphic first integral.

1. Statement of the results

The real planar differential system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y) \tag{1}$$

is said to be (α, β, ω)-*weight-homogeneous* if there exist weights $\alpha, \beta > 0$ and $\omega \in \mathbb{R}$ such that for all $\rho > 0$, $x, y \in \mathbb{R}$ holds $f(\rho^{\alpha} x, \rho^{\beta} y) = \rho^{\omega+\alpha} f(x, y)$ and $g(\rho^{\alpha} x, \rho^{\beta} y) = \rho^{\omega+\beta} g(x, y)$. Equivalently, the system (1) is weight-homogeneous if the foliation

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[1] Partially supported by a MCYT/FEDER grant number MTM2008-00694.
[2] Partially supported by a CIRIT grant number 2005SGR 00550.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.02.010

$$\mathcal{F}_0: \quad f(x,y)\,dy - g(x,y)\,dx = 0$$

is invariant under the dilatation

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto \Phi(x,y) = (\rho^\alpha x, \rho^\beta y). \tag{2}$$

In what follows we shall suppose that the weight-homogeneous system (1) *has a center at the origin.* Then the center is global and the open period annulus $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$ is a union of periodic orbits. The center needs to be global due to its invariance under the aforementioned dilatation. The purpose of the present paper is to study small perturbations of such global centers.

Consider a one-parameter analytic perturbation

$$\dot{x} = f(x,y) + \varepsilon Q(x,y,\varepsilon), \qquad \dot{y} = g(x,y) - \varepsilon P(x,y,\varepsilon) \tag{3}$$

of (1) and suppose that $P(x,y,\varepsilon)$, $Q(x,y,\varepsilon)$ are polynomials in x, y of degree d, depending analytically on the parameter ε. For every compact set K contained in the real positive half-axis $\mathbb{R}^+_* = \{(x,0){:}\ x > 0\}$, and for $|\varepsilon|$ small enough, there exists an open interval $\Delta \supset K$ on which the first return map

$$\Pi_\varepsilon : \Delta \to \mathbb{R}^+ : x \mapsto \Pi_\varepsilon(x) \tag{4}$$

is well defined and analytic

$$\Pi_\varepsilon(x) = x + \varepsilon^k M_k(x) + o(\varepsilon^k).$$

Its fixed points correspond to limit cycles of (3). We note that the function $M_k(x)$ is defined on the whole half-axis $\mathbb{R}^+_* = \{(x,0){:}\ x > 0\}$ and its zeros counted with multiplicity provide an upper bound for the number of fixed points of Π_ε on K. It is known that the so-called higher order Poincaré–Pontryagin–Melnikov function $M_k(x)$ allows an integral representation in terms of iterated path integrals [2–4] along the periodic orbits $\gamma(x)$ of the system (1). If the perturbation is generic, then $k = 1$ ($M_1 \not\equiv 0$). The first result of the paper is the following.

Theorem 1. *If the first Poincaré–Pontryagin–Melnikov function M_1 is not identically zero, then the perturbed system* (3) *has at most $(d+1)(d+4)/2 - 1$ limit cycles which tend to periodic orbits as ε tends to zero.*

In other words, the cyclicity of the open period annulus $\mathcal{O}$ with respect to generic (such that $M_1 \not\equiv 0$) perturbations of degree d is at most $(d+1)(d+4)/2 - 1$. This bound is certainly not exact, as one can easily check in the case of a linear center. The above theorem does not make any claim about the number of limit cycles which tend to the origin or to the "infinity".

In the case when the system (1) has a polynomial first integral and is moreover Hamiltonian, the computation is straightforward (as we have a well-known integral formula for M_1), see [6,8,11]. However, in general, a weight-homogeneous system with a global center is neither Hamiltonian, nor it has an analytic or even meromorphic first integral. Under the restrictions that the polynomials f, g have no common divisor in $\mathbb{R}[x,y]$ and that the origin has no characteristic directions, a result close to our Theorem 1 was recently announced in [7, Theorems A, B, C]. We note that our result is different, more general and with a much shorter proof.

The next question addressed in the paper is: *are there non-Hamiltonian weight-homogeneous polynomial systems with a center*?

The answer turns out to be positive. Theorem 2 provides a large class of non-trivial weight-homogeneous systems with a global center to which Theorem 1 applies. An explicit example of such a system is given in (8) below, and the exact upper bound for the number of the limit cycles under a generic perturbation is given in Theorem 3.

To formulate Theorem 2, let $h_1, h_2 \in \mathbb{R}[x, y]$ be two (α, β, δ)-weight-homogeneous polynomials of the same weighted degree, that is, $h_i(\rho^\alpha x, \rho^\beta y) = \rho^\delta h_i(x, y)$, $i = 1, 2$, for any $x, y \in \mathbb{R}$ and any $\rho \in \mathbb{R}_+$. We recall that α, β, δ are real numbers with $\alpha, \beta > 0$. We also assume that h_1 and h_2 are such that

- $h_2(x, y) \geqslant 0$, $\forall (x, y) \in \mathbb{R}^2$,
- $h_1(x, y) = h_2(x, y) = 0$ if and only if $(x, y) = (0, 0)$.

Let $\sigma \in \mathbb{C}$ and $\Re\sigma \neq 0$ and put

$$H = (h_1 + ih_2)^\sigma (h_1 - ih_2)^{\bar{\sigma}}, \qquad V = \frac{1}{2}(h_1 + ih_2)^{1-\sigma}(h_1 - ih_2)^{1-\bar{\sigma}}.$$

Theorem 2. *The system*

$$\dot{x} = H_y V, \qquad \dot{y} = -H_x V \tag{5}$$

is a real polynomial $(\alpha, \beta, 2\delta - \alpha - \beta)$-weight-homogeneous planar differential system which has a global center.

The proof of this theorem is given in Section 2. Clearly some hypothesis can be relaxed. For instance, h_1, h_2 need not be polynomials.

Example. Put

$$h_1(x, y) = x^{2n} + y, \qquad h_2(x, y) = |\sqrt{1 + 4c}|x^{2n}/2, \qquad \sigma = 1 - 1/\sqrt{1 + 4c},$$

where n is a natural number $n \geqslant 1$ and c is a real number with $c < -1/4$. System (5) takes the form

$$\dot{x} = y + x^{2n}, \qquad \dot{y} = 2ncx^{4n-1}, \tag{6}$$

and it has a global center with a first integral

$$H = \left(y + \mu_+ x^{2n}\right)^{1-1/\sqrt{1+4c}} \left(y + \mu_- x^{2n}\right)^{1+1/\sqrt{1+4c}}, \tag{7}$$

where $\mu_\pm = (1 \pm \sqrt{1 + 4c})/2$.

We remark that system (6) is a $(1, 2n, 2n - 1)$-weight-homogeneous differential system.

To apply Theorem 1 to (6), we consider the following perturbed system

$$\dot{x} = y + x^{2n} + \varepsilon Q(x, y, \varepsilon), \qquad \dot{y} = 2ncx^{4n-1} - \varepsilon P(x, y, \varepsilon), \tag{8}$$

where $|\varepsilon| > 0$ is a small parameter, $P(x, y, \varepsilon)$ and $Q(x, y, \varepsilon)$ are polynomials in (x, y) and depend analytically on ε and $P(x, y, 0)$ and $Q(x, y, 0)$ are polynomials of degree at most $4n - 1$. Let $\{\gamma_h\}_h$ be the continuous family of periodic orbits surrounding the center. The first Poincaré–Pontryagin–Melnikov function M_1 can be written as follows

$$M_1(h) = \oint_{\gamma_h} \frac{P(x, y, 0)\,dx + Q(x, y, 0)\,dy}{V(x, y)}. \tag{9}$$

(This formula holds true for the system (3) too, where V is an appropriate integrating factor.) We deduce the following.

Theorem 3. *Suppose that $M_1(h)$ is not identically zero. Then*

(a) *the perturbed system* (8) *has at most $n(3n+1)-1$ limit cycles which tend to period orbits as ε tends to zero*;
(b) *for suitable polynomials $P(x,y,0)$, $Q(x,y,0)$, and for all sufficiently small $|\varepsilon|$, the perturbed system* (8) *has at least $n(2n+1)-1$ limit cycles which tend to period orbits.*

The bounds are written in terms of the number n associated to system (6) which is not its degree. The degree $d=4n-1$, and hence the upper bound for the number of limit cycles is $(d+1)(3d+7)/16-1$, which is better than the one given in Theorem 1, due to the symmetries of (6). The lower bound can also be written in terms of the degree and it is $(d+1)(d+3)/8-1$. It is obtained by a direct study of the function M_1. The symmetries of system (6) imply that the lower bound for the number of limit cycles ($(d+1)(d+3)/8-1$) is strictly lower than the corresponding lower bound ($d(d+1)/2$) for a degree d perturbation of a generic Hamiltonian system of degree d [5].

2. Proofs

Proof of Theorem 1. We shall estimate the number of isolated zeros (counted with multiplicity) of the first Poincaré–Pontryagin–Melnikov function $I(h)=M_1(h)$, see (9). Consider first the special case $Q(x,y,0)=0$ and $P(x,y,0)=x^iy^j$ with $i,j\geqslant 0$ and $i+j\leqslant d$. Denote by $\mathcal{F}_\varepsilon$ the real foliation defined by

$$\mathcal{F}_\varepsilon\colon\quad -g(x,y)\,dx+f(x,y)\,dy+\varepsilon x^iy^j\,dx=0 \tag{10}$$

and let Π_ε be the corresponding first return map. The dilatation Φ defined in (2) transforms $\mathcal{F}_\varepsilon$ to the foliation $\Phi^*\mathcal{F}_\varepsilon$

$$\Phi^*\mathcal{F}_\varepsilon\colon\quad -g(x,y)\,dx+f(x,y)\,dy+\varepsilon\rho^\ell x^iy^j\,dx=0,\quad \ell=\omega+\beta-i\alpha-j\beta. \tag{11}$$

The first return map $\Pi_{\varepsilon\rho^\ell}$ of the foliation $\mathcal{F}_{\varepsilon\rho^\ell}=\Phi^*\mathcal{F}_\varepsilon$ is therefore conjugated to Π_ε

$$\Pi_\varepsilon=\phi^{-1}\circ\Pi_{\varepsilon\rho^\ell}\circ\phi, \tag{12}$$

where $\phi(x)=\Phi(x,0)$ is the restriction of Φ to the cross-section $\mathbb{R}^+_*=\{(x,0)\colon\ x>0\}$.

Given a foliation $\mathcal{F}$ with an associated Poincaré map Π and a diffeomorphism Φ, then the Poincaré map associated to $\Phi^*\mathcal{F}$ is conjugated to Π. This is due to the fact that the Poincaré map of a foliation is defined by its flow, see for instance Theorem 1 in page 207 of [9], and the flows of the foliations $\mathcal{F}$ and $\Phi^*\mathcal{F}$ are conjugated, see for instance Lemma 11 in page 217 of [10].

We have

$$\Pi_\varepsilon(x)=x+\varepsilon I(x)+o(\varepsilon), \tag{13}$$

$$\Pi_{\varepsilon\rho^\ell}(x)=x+\varepsilon\rho^\ell I(x)+o(\varepsilon), \tag{14}$$

$$\phi^{-1}\circ\Pi_{\varepsilon\rho^\ell}\circ\phi(x)=\rho^{-\alpha}\big(\rho^\alpha x+\varepsilon\rho^\ell I\big(\rho^\alpha x\big)+o(\varepsilon)\big) \tag{15}$$

$$=x+\varepsilon\rho^{\ell-\alpha}I\big(\rho^\alpha x\big)+o(\varepsilon). \tag{16}$$

Therefore, equating the first order terms in ε in (13) and (16) we get

$$I(x)=\rho^{\ell-\alpha}I\big(\rho^\alpha x\big),$$

for any positive real numbers ρ, x. This implies, choosing $\rho=x^{-1/\alpha}$, that

$$I(x)=I(1)x^{(\alpha-\ell)/\alpha}=x^{1-(\omega+\beta)/\alpha}I(1)x^{(\alpha i+\beta j)/\alpha}. \tag{17}$$

In a similar way, if we suppose that $Q(x,y,0)=x^iy^j$ and $P(x,y,0)=0$ with $i,j\geqslant 0$ and $i+j\leqslant d$, we deduce

$$I(x)=x^{1-(\omega+\beta)/\alpha}I(1)x^{(\alpha(i-1)+\beta(j+1))/\alpha} \tag{18}$$

(this computation is omitted). Finally, taking into consideration the additivity of the Poincaré–Pontryagin–Melnikov function (9) with respect to the monomials of $P(x,y,0)$, $Q(x,y,0)$ we conclude

$$I(x)=x^{1-(\omega+\beta)/\alpha}\sum_{(i,j)\in\mathcal{J}}c_{ij}x^{(\alpha i+\beta j)/\alpha},$$

where $\mathcal{J}=\{(i,j)\in\mathbb{Z}^2\colon\ -1\leqslant i\leqslant d,\ 0\leqslant j\leqslant d+1,\ 0\leqslant i+j\leqslant d\}$ and $c_{ij}\in\mathbb{R}$. The number $(d+1)(d+4)/2$ is the cardinal of the set $\mathcal{J}$. Obviously the function $I(x)$ has at most $(d+1)(d+4)/2-1$ zeros (counted with multiplicity), on the interval $(0,\infty)$. This completes the proof of Theorem 1. $\square$

Remark 4. Clearly, the above bound is exact if and only if α and β are not commensurable. The weights α and β of any weight-homogeneous polynomial system (1) with a center at the origin are necessarily commensurable. To show this fact just take any monomial of $f(x,y)$ with a nonzero coefficient, $a_{ij}x^iy^j$, and any monomial of $g(x,y)$ with a nonzero coefficient: $b_{i'j'}x^{i'}y^{j'}$. Since $f(\rho^\alpha x,\rho^\beta y)=\rho^{\omega+\alpha}f(x,y)$ and $g(\rho^\alpha x,\rho^\beta y)=\rho^{\omega+\beta}g(x,y)$, for any $x,y,\rho\in\mathbb{R}$, we deduce the identities: $\alpha i+\beta j=\omega+\alpha$ and $\alpha i'+\beta j'=\omega+\beta$. We subtract them to deduce that $\alpha(i-i'-1)+\beta(j-j'+1)=0$, which gives that α and β are commensurable. In the case when both $f(x,y)$ and $g(x,y)$ have only one monomial with nonzero coefficient ($f(x,y)=a_{ij}x^iy^j$, $g(x,y)=b_{i'j'}x^{i'}y^{j'}$) and such that $i=i'+1$ and $j'=j+1$ the weights α and β are not commensurable, but the origin is a linear node instead of a center.

Remark 5. The same result directly follows from the formula (9) for $I(h)$; it suffices to note that H,H_x,H_y,V are weight-homogeneous functions of appropriate degree.

Proof of Theorem 2. The weight-homogeneous degree of the system $(\alpha,\beta,2\delta-\alpha-\beta)$ follows from straightforward computations.

The condition that $h_2(x,y)\geqslant 0$ implies that the variation of the argument of $h_1(x,y)+ih_2(x,y)$ along any closed path $l\subset\mathbb{R}^2\setminus\{(0,0)\}$ is zero. Therefore for every fixed $\sigma\in\mathbb{C}$ the function $(h_1+ih_2)^\sigma$ has a single valued analytic continuation on $\mathbb{R}^2\setminus\{(0,0)\}$. From now on we fix some determination of $(h_1+ih_2)^\sigma$. We note that functions H,V defined above, as well the associated differential system, do not depend on this particular determination. We may suppose without loss of generality that $\Re\sigma>0$ (otherwise we just replace H by $1/H$). Then H has a continuous limit at $(0,0)$ and we may put $H(0,0)=0$. We claim that each level set $\{(x,y)\colon\ H(x,y)=\varepsilon\}$, $\varepsilon>0$, is a smooth closed curve containing the origin. Indeed, the restriction of H on a half-line l starting at the origin is again a positive weight-homogeneous function. It follows that $\{(x,y)\colon\ H(x,y)=\varepsilon\}\cap l$ consists of a single point and therefore $\{(x,y)\colon\ H(x,y)=\varepsilon\}$ is a closed curve. Suppose that $dH(x_0,y_0)=0$. Then the differential of H is zero at any point belonging to the half-line l_0 starting at the origin and containing (x_0,y_0). It follows that $H|l_0$ is a constant and moreover this constant equals to $0=H(0,0)$ which is impossible. Therefore the level set $\{(x,y)\colon\ H(x,y)=\varepsilon\}$, $\forall\varepsilon>0$, is a closed periodic orbit. The system has a global center. $\square$

Proof of Theorem 3. According to Theorem 2 the origin of system (6) is a center. As noted in Remark 5, the first integral H and the inverse integrating factor V are weight-homogeneous. More precisely

Lemma 6. *Given any $\rho\in\mathbb{C}-\{0\}$ we have*

$$V\big(\rho x,\rho^{2n}y\big)=\rho^{4n}V(x,y),\qquad H\big(\rho x,\rho^{2n}y\big)=\rho^{4n}H(x,y),$$

for any $(x,y)\in\mathbb{R}^2-\{(0,0)\}$.

The proof is a straightforward computation.

We define the following functions:

$$I_{ij}(h) = \oint\limits_{H=h} \frac{x^i y^j}{V(x,y)}\,dx, \qquad J_{ij}(h) = \oint\limits_{H=h} \frac{x^i y^j}{V(x,y)}\,dy,$$

where i, j are nonnegative integer numbers. By the expression of $I(h)$ and taking into account that $P(x,y)$ and $Q(x,y)$ are polynomials with real coefficients of degree at most $4n-1$, we have that

$$I(h) = \sum_{0\leqslant i+j\leqslant 4n-1} \alpha_{ij} I_{ij}(h) + \beta_{ij} J_{ij}(h), \tag{19}$$

with α_{ij} and β_{ij} real numbers and i, j are nonnegative integer numbers.

Lemma 7. *Given any $\rho, h \in \mathbb{R}$ with $h > 0$ and $\rho \neq 0$, we have*

$$I_{ij}\big(\rho^{4n} h\big) = |\rho|^{i+1+2nj-4n} I_{ij}(h), \qquad J_{ij}\big(\rho^{4n} h\big) = |\rho|^{i+2n(j+1)-4n} J_{ij}(h),$$

where $|\rho|$ stands for the absolute value of ρ.

Proof. The change of variables $x \mapsto |\rho| x$, $y \mapsto \rho^{2n} y$ in the integrals I_{ij} and J_{ij} (which preserves the orientation of the oval $\{H = h\}$) implies

$$I_{ij}\big(\rho^{4n} h\big) = \oint\limits_{H=\rho^{4n}h} \frac{x^i y^j}{V(x,y)}\,dx = \oint\limits_{H=h} |\rho|^{i+1+2nj-4n} \frac{x^i y^j}{V(x,y)}\,dx = |\rho|^{i+1+2nj-4n} I_{ij}(h),$$

$$J_{ij}\big(\rho^{4n} h\big) = \oint\limits_{H=\rho^{4n}h} \frac{x^i y^j}{V(x,y)}\,dy = \oint\limits_{H=h} |\rho|^{i+2n(j+1)-4n} \frac{x^i y^j}{V(x,y)}\,dy = |\rho|^{i+2n(j+1)-4n} J_{ij}(h). \qquad \square$$

From the previous lemma we give the form of the functions $I_{ij}(h)$ and $J_{ij}(h)$. The same argument given in the proof of Theorem 1 holds: we choose $h = 1$ and $\rho = h^{\frac{1}{4n}}$ in the expressions given in Lemma 7.

Lemma 8. *Given any $h > 0$ we have*

$$I_{ij}(h) = h^{\frac{i+1+2nj-4n}{4n}} I_{ij}(1), \qquad J_{ij}(h) = h^{\frac{i+2n(j+1)-4n}{4n}} J_{ij}(1).$$

We get that $I_{ij}(h)$ and $J_{ij}(h)$ are monomials of h up to a fractional power. Hence, we only need to determine how many of these functions are linearly independent, so as to know how many zeroes can have $I(h)$. We first determine which of the functions $I_{ij}(h)$ and $J_{ij}(h)$ are identically zero.

Lemma 9. *If i is odd, then $I_{ij}(h) \equiv 0$. If i is even, then $J_{ij}(h) \equiv 0$.*

Proof. Let us consider the change of variables $x \mapsto -x$ and $y \mapsto y$ in the integrals I_{ij} and J_{ij}. We note that this change of coordinates reverses the orientation of the oval $\{H = h\}$. We denote the oval with reversed orientation by $-\{H = h\}$. Moreover, from Lemma 6 we have that this change of coordinates leaves $V(x,y)$ and $H(x,y)$ invariant.

$$I_{ij}(h) = \oint\limits_{H=h} \frac{x^i y^j}{V(x, y)}\, dx = \oint\limits_{-\{H=h\}} (-1)^{i+1} \frac{x^i y^j}{V(x, y)}\, dx = (-1)^i I_{ij}(h),$$

$$J_{ij}(h) = \oint\limits_{H=h} \frac{x^i y^j}{V(x, y)}\, dy = \oint\limits_{-\{H=h\}} (-1)^i \frac{x^i y^j}{V(x, y)}\, dy = (-1)^{i+1} J_{ij}(h).$$

Taking i odd, we deduce that $I_{ij}(h)$ needs to be identically zero, and the same is true for $J_{ij}(h)$ taking i even. □

Lemma 9 can also be proved using Green's Theorem and analogous reasonings.

We are going to characterize some of the functions $I_{ij}(h)$ and $J_{ij}(h)$ which are not zero at any point of $h > 0$.

Lemma 10. *For any k, ℓ nonnegative integers we have*

$$I_{2k,2\ell+1}(h) \not\equiv 0, \qquad J_{2k+1,2\ell}(h) \not\equiv 0,$$

where we recall that

$$I_{2k,2\ell+1}(h) := \oint\limits_{H=h} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx, \qquad J_{2k+1,2\ell}(h) := \oint\limits_{H=h} \frac{x^{2k+1} y^{2\ell}}{V(x, y)}\, dy,$$

with $H(x, y)$ as given in (7) *and $V(x, y) := y^2 + x^{2n} y - cx^{4n}$.*

Proof. We note that the orientation of system (6) over the oval $H = h$ is clockwise and that $V(x, y)$ is strictly positive over all the oval $H = h$.

Let us denote by $x_1(h)$ and $x_2(h)$ the intersections of the oval $H = h$ with the horizontal axis ($y = 0$) with $x_1(h) < 0$ and $x_2(h) > 0$. We denote by $\{H = h\}_{y<0}$ the half part of the oval below the horizontal axis oriented from $x_2(h)$ to $x_1(h)$ and by $\{H = h\}_{y>0}$ the half part of the oval above the horizontal axis oriented from $x_1(h)$ to $x_2(h)$. We have

$$\begin{aligned} I_{2k,2\ell+1}(h) &= \oint\limits_{H=h} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx = \int_{x_2(h)}^{x_1(h)} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx + \int_{x_1(h)}^{x_2(h)} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx \\ &= -\int_{x_1(h)}^{x_2(h)} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx + \int_{x_1(h)}^{x_2(h)} \frac{x^{2k} y^{2\ell+1}}{V(x, y)}\, dx, \end{aligned}$$

where the first integral is done over the path $\{H = h\}_{y<0}$ and the second integral is done over the path $\{H = h\}_{y>0}$. Therefore, the first integral is strictly negative and the second integral is strictly positive and we are adding two positive values, due to the minus sign. Hence, $I_{2k,2\ell+1}(h) > 0$ and it cannot be zero at any point.

Analogously for $J_{2k+1,2\ell}(h)$, we define $y_1(h)$ and $y_2(h)$ the intersections of the oval $H = h$ with the vertical axis ($x = 0$) with $y_1(h) < 0$ and $y_2(h) > 0$. We denote by $\{H = h\}_{x<0}$ the half part of the oval at the left of the vertical axis oriented from $y_1(h)$ to $y_2(h)$ and by $\{H = h\}_{x>0}$ the half part of the oval at the right of the vertical axis oriented from $y_2(h)$ to $y_1(h)$, and we have

$$J_{2k+1,2\ell}(h) = \oint_{H=h} \frac{x^{2k+1}y^{2\ell}}{V(x,y)}\,dy = \int_{y_2(h)}^{y_1(h)} \frac{x^{2k+1}y^{2\ell}}{V(x,y)}\,dy + \int_{y_1(h)}^{y_2(h)} \frac{x^{2k+1}y^{2\ell}}{V(x,y)}\,dy$$
$$= -\int_{y_1(h)}^{y_2(h)} \frac{x^{2k+1}y^{2\ell}}{V(x,y)}\,dy + \int_{y_1(h)}^{y_2(h)} \frac{x^{2k+1}y^{2\ell}}{V(x,y)}\,dy,$$

where the first integral is done over the path $\{H=h\}_{x>0}$ and the second integral is done over the path $\{H=h\}_{x<0}$. Therefore, the first integral is strictly positive and the second integral is strictly negative and we are adding two negative values, due to the minus sign. Hence $J_{2k+1,2\ell}(h) < 0$ and it cannot be zero at any point. $\square$

In fact, we have been able to numerically prove that the integrals $I_{2k,2\ell}(h) \not\equiv 0$ and $J_{2k+1,2\ell+1}(h) \not\equiv 0$ for some particular fixed values of the integers k, ℓ, with $k \geqslant 0$ and $\ell \geqslant 0$, in some fixed cases of the function $H(x,y)$ defined in (7). To show this fact, we have parameterized the oval $H(x,y)=h$ by $(x_+(\tau), y(\tau))$ when $x>0$ and $(x_-(\tau), y(\tau))$ when $x<0$, with

$$x_\pm(\tau) = \pm h^{\frac{1}{4n}} (\tau+\mu_+)^{-\frac{\sigma}{4n}} (\tau+\mu_-)^{-\frac{\bar\sigma}{4n}}, \qquad y(\tau) = h^{\frac{1}{2}} \tau (\tau+\mu_+)^{-\frac{\sigma}{2}} (\tau+\mu_-)^{-\frac{\bar\sigma}{2}},$$

where $\mu_\pm = (1 \pm \sqrt{1+4c})/2$, $\sigma = 1 - 1/\sqrt{1+4c}$ and the rank of the parameter τ is all the real line $\tau \in (-\infty, +\infty)$ in both parts of the oval. When we write $h^{\frac{1}{4n}}$ or $h^{\frac{1}{2}}$ we mean the positive real root.

Next lemma shows that the possible nonzero values of the integrals $J_{ij}(h)$ are redundant, since there is an integral of the form $I_{ij}(h)$ which corresponds to the same monomial.

Lemma 11. *The first Melnikov function can be expressed by*

$$I(h) = \sum_{(i,j)\in\mathcal{I}} \alpha_{ij} I_{ij}(h),$$

where $\alpha_{ij} \in \mathbb{R}$ *and* $\mathcal{I} = \{(i,j) \in \mathbb{Z}^2\colon\ 0 \leqslant i, j \leqslant 4n-1,\ i \text{ is even},\ i+j \leqslant 4n-1\}$.

Proof. The expression of $I(h)$ given in (19) ensures that this function is a linear combination of the integrals $I_{ij}(h)$ and $J_{ij}(h)$. We are going to show that any possible monomial expressed by a $J_{ij}(h)$ can also be got by a monomial of $I_{i'j'}(h)$. Lemma 9 gives that only the integrals $J_{ij}(h)$ with i odd need to be considered, hence we take any two nonnegative integers (i,j) such that i is odd and $i+j \leqslant 4n-1$. We define $i' = i-1$ and $j' = j+1$ and we have that both i' and j' are nonnegative integers strictly lower than $4n$ and $i'+j' = i+j \leqslant 4n-1$. Moreover, i' is even in accordance with Lemma 9 and $(i', j') \in \mathcal{I}$. Hence, any monomial given by a $J_{ij}(h)$ is also expressed by an $I_{ij}(h)$ with i even. $\square$

To end with the proof of Theorem 3, we first count the cardinal of $\mathcal{I}$ and we are going to show that $\sharp\mathcal{I} = 2n(2n+1)$. We take any even nonnegative integer i from 0 to $4n-2$, that is, we take $2n$ possible values of k with $i=2k$ and given a fixed $0 \leqslant k \leqslant 2n-1$ we can take any value of j from 0 to $4n-1-2k$. Hence,

$$\sharp\mathcal{I} = \sum_{k=0}^{2n-1} \sum_{j=0}^{4n-1-2k} 1 = \sum_{k=0}^{2n-1} (4n-2k) = 2+4+6+\cdots+4n = 2n(2n+1).$$

We note that the cardinal of $\mathcal{I}$ is an upper bound for the number of independent monomials given by the nonzero $I_{ij}(h)$ because it may happen that two elements of $I_{ij}(h)$ give rise to the same monomial, that is, it may happen that $(i, j) \in \mathcal{I}$, $(i', j') \in \mathcal{I}$ with $(i, j) \neq (i', j')$ but $i + 2nj = i' + 2nj'$. To end with the proof of part (a) of Theorem 3, we need to characterize the number of repeated exponents corresponding to different indexes $(i, j) \in \mathcal{I}$, $(i', j') \in \mathcal{I}$. We note that if two such pairs give $i + 2nj = i' + 2nj'$, then j and j' can differ at most by one, because if they differ by two or more then i and i' differ by $4n$ or more which is not possible since $0 \leqslant i, i' \leqslant 4n - 1$. We assume that $j' = j + 1$ and we have that $i' = i - 2n$. Hence, fixing (i, j), the condition to have a repeated exponent is that $i \geqslant 2n$ and we already have that $j < 4n - 1$ because $i + j \leqslant 4n - 1$. Let us count how many of these indexes we do have in $\mathcal{I}$. We fix k such that $i = 2k$ and k goes from n to $2n - 1$. Given such a k we can take, as before, any value of j from 0 to $4n - 1 - 2k$. Therefore, the number of repeated exponents is

$$\sum_{k=n}^{2n-1} \sum_{j=0}^{4n-1-2k} 1 = \sum_{k=n}^{2n-1} (4n - 2k) = 2 + 4 + \cdots + 2n = n(n+1).$$

We conclude that the number of different exponents associated to the indexes of $\mathcal{I}$ is $2n(2n + 1) - n(n + 1) = n(3n + 1)$.

We have given an upper bound for the number of independent functions in which $I(h)$ can be split as a linear combination. Since this upper bound is $n(3n + 1)$, we have that an upper bound for the number of isolated zeroes of $I(h)$ is $n(3n + 1) - 1$.

The proof of part (b) in Theorem 3 comes from Lemma 10, which ensures that only half of the $2n(2n+1)$ functions $I_{ij}(h)$ are ensured to be different from zero, the ones with an even i and an odd j. To end with, we only need to see that two such functions give place to two independent monomials. Let $(2k, 2\ell + 1) \in \mathcal{I}$ and $(2k', 2\ell' + 1) \in \mathcal{I}$ with $(2k, 2\ell + 1) \neq (2k', 2\ell' + 1)$ be such that $2k + 2n(2\ell + 1) = 2k' + 2n(2\ell' + 1)$ and we will get a contradiction. If $k = k'$ then $\ell = \ell'$, so we can assume that $k < k'$. We have $2n(\ell - \ell') = k' - k$ which gives that $\ell \geqslant \ell' + 1$. This inequality gives that $k' \geqslant 2n + k$ and since $k \geqslant 0$, we have that $k' \geqslant 2n$, which is impossible since $2k' \leqslant 4n - 1$. Hence, any two exponents given by different functions $I_{2k,2\ell+1}(h)$ are independent. Choosing adequate parameters in system (6) we get that $I(h)$ can always be given as a sum of at least $n(2n + 1)$ independent functions and, adapting parameters, it can always be chosen with at least $n(2n + 1) - 1$ different isolated zeroes. $\square$

Acknowledgment

We thank I.D. Iliev for the critical remarks.

Appendix A. The Andreev theorem

We already noted that Theorem 2 implies that the origin of the system (6) is a global center. It is interesting to note that the same result follows also from a classical theorem of Andreev [1] which we briefly illustrate. This approach can be useful in other cases of nilpotent centers, which are not covered by Theorem 2.

Theorem 12 *(Andreev's theorem). (See [1].) Let $F(x, y)$ and $G(x, y)$ be analytic functions in a neighborhood of the origin of order $\geqslant 2$ and let the origin be an isolated singularity of the differential system: $\dot{x} = y + F(x, y)$, $\dot{y} = G(x, y)$. Let $y = \phi(x)$ be the solution of the equation $y + F(x, y) = 0$ such that $\phi(0) = 0$. We denote by $\xi(x) = G(x, \phi(x))$ and $\Delta(x) = \operatorname{div}(x, \phi(x))$ and we develop them in a neighborhood of $x = 0$:*

$$\xi(x) = \alpha_1 x^{k_1} + \mathcal{O}\big(x^{k_1+1}\big), \qquad \Delta(x) = \alpha_2 x^{k_2} + \mathcal{O}\big(x^{k_2+1}\big),$$

where $\alpha_1 \neq 0$, $k_1 \geqslant 2$ and $\alpha_2 \neq 0$, $k_2 \geqslant 1$ or $\Delta(x) \equiv 0$. The origin of the differential system is monodromic if, and only if, $\alpha_1 < 0$, k_1 is an odd number, and one of the following three conditions holds:

(a) $k_1 = 2k_2 + 1$ *and* $\alpha_2^2 + 4\alpha_1(k_2 + 1) < 0$,
(b) $k_1 < 2k_2 + 1$,
(c) $\Delta(x) \equiv 0$.

The function $\xi(x)$ associated to (6) is $\xi(x) = 2ncx^{4n-1}$, so $\alpha_1 = 2nc$ and $k_1 = 4n-1$. The divergence of system (6) is $\operatorname{div}(x, y) = 2nx^{2n-1}$, so $\Delta(x) = 2nx^{2n-1}$ and we have $\alpha_2 = 2n$ and $k_2 = 2n-1$. Since n is a natural number and $c < -1/4$, we already have that $\alpha_1 < 0$ and k_1 is an odd integer. The condition (a) of Theorem 12 is satisfied because $2k_2 + 1 = 4n - 1 = k_1$ and $\alpha_2^2 + 4\alpha_1(k_2 + 1) = 4n^2(1 + 4c) < 0$. We conclude that the origin of system (6) is monodromic. Finally, the origin of the system (6) is time-reversible because it is invariant by the change $(x, y, t) \to (-x, y, -t)$, which implies that it is a center. Moreover the center is global, as the system is weight-homogeneous.

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