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Application of reciprocity gap functional for elastostatic inverse problem of small well-separated defects identification

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Abstract

An inverse problem of identification of a finite number of small, well-separated defects in an anisotropic elastic body using the results of one static test is considered. It is supposed that the defects are cavities (in particular, cracks) or inclusions (rigid or linear elastic). If the defects are cavities then their boundaries are supposed unloaded. If the defects are inclusions it is supposed complete bonding between the matrix and inclusions. It is assumed also that in a static test the loads and displacements are measured on the external boundary of the body. Under these assumptions a method for determination of centers of the defects projections on an arbitrary plane is developed. In case of ellipsoidal defects their geometrical parameters (directions and magnitudes of the ellipsoids axes) are also determined.

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1. Introduction

An analytical-numerical method for identification of a single ellipsoidal defect in a linear elastic body was developed by the authors in a series of publications. In particular, a problem of identification of a single ellipsoidal defect in both an isotropic and anisotropic elastic body was solved in Shifrin (2010) and Shifrin and Shushpannikov (2013a). A method for identification of multiple defects in an isotropic elastic body was developed in Shifrin and

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Shushpannikov (2013b). The developed method was based on the application of the reciprocity gap functional (RGF) method first applied to inverse elastostatic problems in Andrieux et al. (1999). In the present paper we combine the ideas used in the publications of Shifrin and Shushpannikov (2013a, 2013b) to develop a method for identification multiple ellipsoidal defects in an anisotropic elastic body.

2. Statement of the problem

Let $V \subset R^3$ be a bounded domain with a boundary ∂V. $G_k \subset V$, $k = 1, 2, \ldots, n$ are small, simply connected subdomains. Suppose that $\bar{G}_i \cap \bar{G}_j = \varnothing$, $i \neq j$, where $\bar{G}_k$ is a closure of the subdomain G_k. Let us suppose that an anisotropic linear elastic body with the elastic moduli C^M_{ijkl} occupies the domain $\Omega = V \setminus \bigcup_{k=1}^{n} \bar{G}_k$. The defects G_k can be cavities (in particular, cracks) or inclusions (rigid or linear elastic). If G_k is a cavity we suppose that its boundary ∂G_k is unloaded. If G_k is an inclusion, it is supposed complete bonding between the matrix and inclusion. Assume that typical sizes of the defects have the same order and denote the typical size l. Assume also that typical distances between the defects have the same order and denote the typical distance L. We assume that the defects are small in the following sense

$$l \ll L \tag{1}$$

Let us introduce Cartesian coordinates $Ox_1x_2x_3$. We suppose that the loads t^d_i and displacements u^d_i are measured on ∂V in a single static test. We will mark with the superscript d the stress-strain state in the body Ω: σ^d_{ij} is the stress tensor, e^d_{ij} is the strain tensor and u^d_i is the displacement vector, $t^d_i = \sigma^d_{ij} n_j$, where n_i is a unit outward normal to the boundary ∂V and convention of summation for repeated indices is used. Below we will suppose that the defects are linear elastic inclusions. The cases of cavities and rigid inclusions can be considered as limit cases when the elastic moduli tend to zero or infinity, respectively. The stress-strain state in the inclusion G_k we will mark with the superscript Ik (σ^{Ik}_{ij}, e^{Ik}_{ij}, u^{Ik}_i). The elastic moduli of the inclusion G_k we denote by C^{Ik}_{ijpl}. According to our suppositions the following equalities are valid for $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$

$$C^M_{ijpl} u^d_{p,lj} = 0 \tag{2}$$

The elastic field with the superscript Ik satisfies in the domain G_k the equations analogical to Eq. (2) with the replacement of the values C^M_{ijpl} by the values C^{Ik}_{ijpl}.

We will call the elastic fields in the body V without defects as regular elastic fields and mark by a superscript r (σ^r_{ij}, e^r_{ij}, u^r_i, $t^r_i = \sigma^r_{ij} n_j$). The regular elastic fields satisfy the Eq. (2) in the domain V.

The RGF, depending on two stress-strain states with superscripts d and r, is defined as follows

$$RG(d,r) = \int_{\partial V} \left(t^d_i u^r_i - t^r_i u^d_i \right) dS \tag{3}$$

The problem is to reconstruct the defects G_k using the known loads t^d_i and displacements u^d_i on the boundary ∂V. Using the known data, the values $RG(d,r)$ can be calculated for all regular elastic fields r. So, the problem will be solved if we express the parameters of the domains G_k by means of the values $RG(d,r)$. According to the results of Shifrin (2010), Shifrin and Shushpannikov (2013a), Eq. (3) can be written in the following form

$$RG(d,r) = \sum_{k=1}^{n} \int_{G_k} \Delta\sigma_{ij}^{Ik} e_{ij}^{r}\, dx,\ \ \Delta\sigma_{ij}^{Ik} = \sigma_{ij}^{Ik} - \bar{\sigma}_{ij}^{Ik} \tag{4}$$

where, $\bar{\sigma}_{ij}^{Ik}$ are the stresses corresponding to the strains e_{ij}^{Ik} in the material with the elastic moduli C_{ijpl}^{M}.

Denote the centers of the defects G_k by $\mathbf{x}^k = \left(x_1^k, x_2^k, x_3^k\right)$ and the volumes of the domains G_k by $|G_k|$. Consider a regular elastic field in the body V without defects subjected to the loads t_i^d on the boundary ∂V. Let us mark the field with a superscript dr. Because the defects are small, the stress-strain state in the defect G_k is close to the state in the inclusion G_k located in an infinite elastic solid and subjected to the constant stresses $\sigma_{ij}^{dr}\left(\mathbf{x}^k\right)$ at the infinity. It follows from the Eshelby results that the stresses σ_{ij}^{Ik} are approximately constant in the ellipsoidal inclusion G_k, see for example Mura (1987). Finally, we assume that the values $\max_{i,j}\left|\sigma_{ij}^{Ik}\right|$ have the same order for different G_k.

3. Reduction of the problem to 2D problem of the centers of defects projections identification

According to our suppositions, formulated in Section 2, we will approximate the values of the RGF by the principal term of the asymptotic expansion of the Eq. (4) provided that $l/L \to 0$

$$RG(d,r) \approx \sum_{k=1}^{n} \Delta\sigma_{ij}^{Ik}\left(x^k\right) e_{ij}^{r}\left(x^k\right) |G_k| \tag{5}$$

Consider, for example, projections of defects on the plane x_1x_2. To determine projections of defects centers on this plane we will use the regular elastic fields depending on coordinates x_1 and x_2. Consider a regular elastic field $u_i^r\left(x_1,x_2\right)$. Let us search for the functions $u_i^r\left(x_1,x_2\right)$ in the form $u_i^r\left(x_1,x_2\right) = f_i\left(x_1 + sx_2\right)$. It follows from Eq. (2)

$$m_{ip}(s) f_p'' = 0,\ \ m_{ip}(s) = C_{i1p1}^{M} + \left(C_{i1p2}^{M} + C_{i2p1}^{M}\right)s + C_{i2p2}^{M} s^2 \tag{6}$$

Consider the matrix $\mathbf{M}(s) = \left(m_{ip}(s)\right)$. Eq. (6) has a nonzero solution if and only if $\det\left(\mathbf{M}(s)\right) = 0$. This equality leads to an algebraic equation of sixth order relative to s. Let us suppose for simplicity that the roots of the equation are simple. According to Lekhnitskii (1981) the imaginary parts of the roots are not zero. Because the coefficients of the algebraic equation are real, it has three pairs of complex-conjugated roots. Let us denote the not pair roots by $s_j = \alpha_j + i\beta_j$, $\beta_j > 0$, $j = 1,2,3$. Denote by $\gamma_j = \left(\gamma_{j1}, \gamma_{j2}, \gamma_{j3}\right)^T$ the corresponding normalized eigenvectors. The superscript T denotes transposition of a matrix. Let $g\left(x_1 + s_j x_2\right)$ be an arbitrary smooth function. Consider the following regular elastic fields: $u^{r_j} = \mathrm{Re}\left(\gamma_j g\left(z_j\right)\right)$ and $u^{\rho_j} = \mathrm{Re}\left(i\gamma_j g\left(z_j\right)\right)$, $z_j = x_1 + s_j x_2$. It follows from Eq. (5)

$$RG\left(d,r_j\right) - iRG\left(d,\rho_j\right) = \sum_{k=1}^{n} A_{kj}\varphi\left(z_{jk}\right),\ \ z_{jk} = x_1^k + s_j x_2^k,\ \ \varphi\left(z_j\right) = g'\left(z_j\right) \tag{7}$$

$$A_{kj} = \left[\Delta\sigma_{11}^{Ik}\gamma_{j1} + \Delta\sigma_{22}^{Ik} s_j\gamma_{j2} + \Delta\sigma_{12}^{Ik}\left(s_j\gamma_{j1} + \gamma_{j2}\right) + \Delta\sigma_{13}^{Ik}\gamma_{j3} + \Delta\sigma_{23}^{Ik} s_j\gamma_{j3}\right]|G_k| \tag{8}$$

Let us take $\varphi\left(z_j\right) = \varphi_p\left(z_j\right) = L^{-p} z_j^p = w_j^p$, $p = 0,1,2,\ldots$. Let us denote by r_{jp} and ρ_{jp} the regular elastic fields corresponding to the function $\varphi_p\left(z_j\right)$. Eq. (7) for these regular elastic fields can be written in the form

$$\sum_{k=1}^{n} A_{kj} w_{jk}^{p} = b_{jp},\ w_{jk} = L^{-1} z_{jk},\ b_{jp} = RG\left(d, r_{jp}\right) - iRG\left(d, \rho_{jp}\right),\ p = 0,1,2,\ldots \tag{9}$$

Eq. (9) coincide with the equations arising in the problem of simple poles of a meromorphic function identification. In Shifrin and Shushpannikov (2013b), where an isotropic case was considered, the method, proposed in Kang and Lee (2004), was used to determine the number of defects and projections of their centers. The same method is used here. For known defects number n the method enables to determine the values of w_{jk} and A_{kj}, $k = 1,2,\ldots,n$. Usually we do not know the number of defects n, but we can suppose that we know an upper bound of the number N, $n \le N$. In this case using the method we obtain the values of w_{jk} and A_{kj}, $k = 1,2,\ldots,N$. Among the obtained values w_{jk} there are some values corresponding to real defects projections and some spurious values. In Shifrin and Shushpannikov (2013b) four criteria for excluding the spurious points were proposed.

In the anisotropic case, considered in the paper, it is possible to add one additional criterion 5:

- The coordinates $\left(x_{1j}^{k}, x_{2j}^{k}\right)$, obtained by means of the values w_{jk}, correspond to the real defects projections only in case when they are close to each other for different $j = 1,2,3$.

The value of w_{jk} is excluded if it should be deleted according at least one of the criteria.

Let us suppose that the spurious values of w_{jk} are excluded and we have found points $\left(x_{1}^{k}, x_{2}^{k}\right)$. The number of the points can exceed the number of projections of real defects because several points can correspond to one defect. To determine the exact number of defects projections it is possible to use Eq. (1). So, if we obtain, for example, three points 1, 2 and 3 and the distance between the points 1 and 2 is much less than the distances between the points 1, 3 and 2, 3 then we can suppose that the points 1 and 2 correspond to the projection of the same defect. After determination the number of defects we repeat the described above procedure for the obtained number of defects. As a result, we obtain points located close enough to the projections of defects centers.

4. Identification of small ellipsoidal defects

Let us suppose now that the defects G_k are ellipsoidal inclusions. To determine the geometrical parameters of the inclusions (the magnitudes and directions of their axes) we will use an approach developed in Shifrin (2010). According to our suppositions the stress state in the defect G_k are approximately constant and close to the stress state in the ellipsoidal inclusion G_k located in an infinite elastic solid and subjected to the stresses $\sigma_{ij}^{dr}\left(x^{k}\right)$ at the infinity. For definiteness, let us consider the defect G_1. To determine the geometrical parameters of the defect G_1, we construct a regular elastic field so that the contribution of the first term in the sum on the right side of the Eq. (4) was significantly greater than that of the remaining terms. First introduce Cartesian coordinates with the origin in the center of the defect G_1 : $x_i = \xi_i + x_i^1$. Denote coordinates of other defects centers in the coordinate system by $\left(\xi_1^k, \xi_2^k, \xi_3^k\right)$, $k = 2,3,\ldots,n$. Define a holomorphic functions $\chi_{mj}\left(\zeta_j\right)$, $\zeta_j = \xi_1 + s_j \xi_2$

$$\chi_{mj}\left(\zeta_j\right) = L^{-m(n-1)} \prod_{k=2}^{n} \left(\zeta_j - \zeta_{kj}\right)^{m},\ m \ge 3,\ \zeta_{kj} = \xi_1^k + s_j \xi_2^k,\ \chi_{mj}(0) = P_{mj} \tag{10}$$

Let us take $\varphi_{mj}^{*}\left(\zeta_j\right) = L^{-2} \zeta_j^{2} \chi_{mj}\left(\zeta_j\right)$. Consider regular elastic fields r_{mj}^{*} and ρ_{mj}^{*} constructed by means of the function $\varphi\left(\zeta_j\right) = \varphi_{mj}^{*}\left(\zeta_j\right)$ in a way similar to that used in Section 3 for constructing the fields r_{jp} and ρ_{jp}. It follows from Eq. (4), (8) and suppositions formulated in Section 2

$$RG\left(d,r_{mj}^{*}\right)-iRG\left(d,\rho_{mj}^{*}\right)\approx\sum_{k=1}^{n}A_{kj}\left|G_{k}\right|^{-1}\int_{G_{k}}\varphi_{mj}^{*}\left(\zeta_{j}\right)d\xi_{1}d\xi_{2} \tag{11}$$

Using arguments similar to those given in Shifrin and Shushpannikov (2013b), one can show that for sufficiently large m, the contribution of the first term in Eq. (11) is much greater than the sum of contributions of the remaining terms. So, using the definition of function $\varphi_{mj}^{*}\left(\zeta_{j}\right)$, it is possible to obtain the following equation

$$RG\left(d,r_{mj}^{*}\right)-iRG\left(d,\rho_{mj}^{*}\right)\approx A_{k1}P_{mj}\left|G_{1}\right|^{-1}L^{-2}\int_{G_{1}}\left[\xi_{1}^{2}+2s_{j}\xi_{1}\xi_{2}+s_{j}^{2}\xi_{2}^{2}\right]d\xi,\ d\xi=d\xi_{1}d\xi_{2}d\xi_{3} \tag{12}$$

According to Shifrin (2010), to identify the ellipsoidal defect G_1, it is sufficiently to construct a matrix $\mathbf{Z}^1=\left(Z_{ij}^1\right),\ Z_{ij}^1=5\left|G_1\right|^{-1}\int_{G_1}\xi_i\xi_j\,d\xi$. The eigenvalues of the matrix $\mathbf{Z}^1$ equal $\left(a_1^1\right)^2$, $\left(a_2^1\right)^2$ and $\left(a_3^1\right)^2$, where a_j^1, $j=1,2,3$ are the semiaxes of the ellipsoid G_1. The corresponding eigenvectors are directed along the axes of the ellipsoid. Since the plane elliptic crack (with unloaded surfaces) is a degenerate ellipsoidal cavity, the defect G_1 can also be the elliptic cracks. In this case one of the eigenvalues of the matrix $\mathbf{Z}^1$ should be zero. The eigenvector corresponding to zero eigenvalue is directed normal to the crack plane. Note that the errors in the measured data lead to the fact that none of the eigenvalues is not zero, but one of the eigenvalues is small relative to the others.

Eq. (12) is a complex-valued linear equation with respect to three unknowns $Z_{pq}^1,\quad p=1,2,\quad q=1,2$. Using the described procedure for different s_j, $j=1,2,3$, we obtain three linear complex-valued equations

$$Z_{11}^1+2s_jZ_{12}^1+s_j^2Z_{22}^1=5L^2A_{k1}^{-1}P_{mj}^{-1}\left[RG\left(d,r_{mj}\right)-iRG\left(r,\rho_{mj}\right)\right],\ j=1,2,3 \tag{28}$$

Because $s_i\neq s_j$ for $i\neq j$ the determinant of the system is not zero and the unknowns can be uniquely determined. Considering projections on the planes x_1x_3 and x_2x_3 we obtain all elements of the matrix $\mathbf{Z}^1$.

5. A numerical example: identification of two elliptic cracks

Let us consider a numerical example illustrating the efficiency of the developed method. Analogously to the examples considered in Shifrin and Shushpannikov (2013a, 2013b), we assume below that elastic body V is a cube $\left\{x:\left|x_i\right|\leq 10,\ i=1,2,3\right\}$. The loads applied to the boundary ∂V correspond to uniaxial tension in the direction of the axis x_3: $t_1^d\left(\mathbf{x}\right)=t_2^d\left(\mathbf{x}\right)=0,\ t_3^d\left(\mathbf{x}\right)=\sigma n_3\left(\mathbf{x}\right),\ \mathbf{x}\in\partial V$, where $\sigma=200$ MPa. The elastic moduli C_{ijkl}^M of the matrix Ω are chosen correspond to orthotropic topaz. The orientation of the crystallographic coordinate system is specified in terms of the Euler angles $\left(\beta_1,\beta_2,\beta_3\right)=\left(30^\circ,45^\circ,60^\circ\right)$ (for details, see Shifrin and Shushpannikov (2013a)).

Let us consider a case of two elliptic cracks with the same geometrical parameters as in the example considered in Shifrin and Shushpannikov (2013b) for isotropic case.

The direct Neumann problem for the considered example is solved using FEM and displacements u_i^d are determined on the boundary ∂V. After that the values of the RGF are calculated. Using the calculated values of the RGF the defects number and their geometrical parameters are determined by means of the developed method.

The identified centers of the defects projections obtained for one of the roots s_j, $j=1,2,3$ (see Section 4) in the assumption that the defects number $n=10$ are presented on Fig. 1. The projections of the given elliptic cracks are grey dashed on the figures ((a) – plane x_1x_2; (b) – plane x_2x_3; (c) – plane x_1x_3). The projections of the body V are marked with the solid lines. The identified centers are marked with the thick points ($\bullet$). The arrows ($\rightarrow$) denote the centers located outside the figures bounds.

Let us consider for example the projection on the plane x_1x_2. The points marked by the symbols $\times$, $\bigcirc$, $\square$, $\Diamond$, $\triangle$, satisfy criteria $1,2,\ldots,5$, respectively, and should be excluded. The distance between the points 1, 2 is much less than the distances between the points 1, 3 and 2, 3 (or 1, 4 and 2, 4). So, accounting for Eq. (1), one can conclude that the points 1, 2 correspond to one defect. The same holds for the points 3, 4. So, the number of defects $n = 2$.

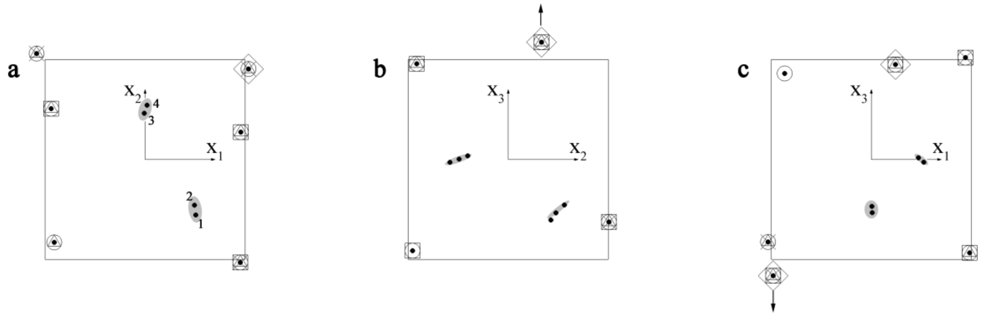

Fig. 1. Identification of the projections of defects centers.

After determination of the defects number, their centers and geometrical parameters are determined using the formulae presented in Sections 3, 4. The identification results obtained for $m = 3$ (see Section 4) are presented on Fig. 2. Here the boundaries of the identified elliptic cracks projections are marked with the solid lines. The results presented on Fig. 4 show that for chosen value of the parameter m the identified defects projections are in good agreement with the projections of given elliptic cracks.

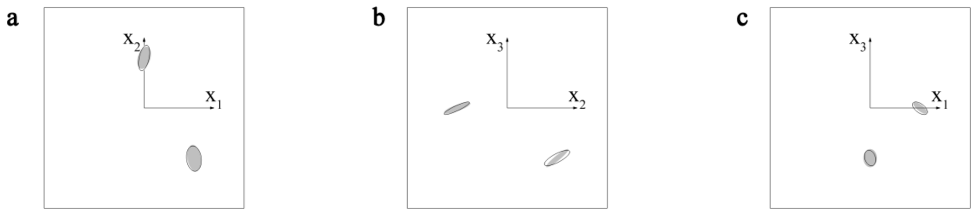

Fig. 2. Identification of two elliptic cracks.

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