Asymptotic Behavior of Volterra Difference Equations

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Abstract—In this paper, we investigate boundedness and stability properties of some classes of discrete Volterra equations. The main tool in this work is the use of a representation formula which allows us to express the solution of discrete Volterra equations in terms of the resolvent matrix of the corresponding system of Volterra difference equations. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Volterra equations with discrete time arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. For applications of the Volterra difference equations in combinatorics, see [1], and in epidemics, see [2].

Stability and boundedness are among the most important properties of the solutions of Volterra difference equations. In fact, error between the true and the numerical solution of a Volterra integral equation satisfies a Volterra discrete equation, and therefore, the boundedness of the solutions of this Volterra discrete equation assured the boundedness of the global error, that is, the stability of the considered numerical method, see [3].

Consider the Volterra difference equation

$$x(n + 1) = Ax(n) + \sum_{r=0}^{n} [B(n - r) + D(n - r)] x(r) + f(n), \quad x(0) = x_0,$$

where $n \in \mathbb{Z}^+$, the set of nonnegative integers, $x(n)$ and $f(n)$ are column vectors; $A$, $B$, and $D$ are square matrices.

Equation (1) will be treated as a perturbation of the equation

$$x(n + 1) = Ax(n) + \sum_{r=0}^{n} B(n - r)x(r) + f(n), \quad x(0) = x_0,$$

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In this paper, we will discuss the question under what conditions equation (1) preserves some properties of equation (2) related to the asymptotic behavior of the solutions. We will investigate the properties: boundedness, uniform asymptotic stability, and convergence.

Recently, Crisci et al. [3,4] established several results on the asymptotic behavior of the solutions of discrete Volterra equations, by means of the direct Lyapunov method and comparison theorems, thus establishing boundedness conditions of solutions for such equations.

Elaydi [5], Murakami [6], Agarwal and Pang [7], and Raffoul [8] studied the stability of some classes of Volterra difference equations.

In the present paper, we continue the research initiated in [9–11] addressing our study to establish new boundedness results for Volterra difference equations.

2. PRELIMINARIES

Let \( \mathbb{R}^k \) denote the real \( k \)-dimensional Euclidean space of column vectors with norm \( |x| = \sum_{i=1}^{k} |x_i| \), \( \mathcal{M}^k \) the space of all \( k \times k \) matrices \( Q = (q_{ij}) \) with norm \( |Q| = \max_{i} \sum_{j=1}^{k} |q_{ij}| \), \( Z^+ \) denotes the set of all nonnegative integers. \( S = S(Z^+, \mathbb{R}^k) \) denotes the space of all functions from \( Z^+ \) into \( \mathbb{R}^k \). Let \( S_1 = S_1(Z^+, \mathbb{R}^k) \) be the Banach space in \( S \) of all bounded functions from \( Z^+ \) to \( \mathbb{R}^k \). The norm in \( S_1 \) is defined by \( |x|_{S_1} = \sup \{ |x(n)| : n \in Z^+ \} \).

Let \( S_2 = S_2(Z^+, \mathbb{R}^k) \) denote the space of all functions in \( S_1 \) having a limit at infinity; \( S_3 = S_3(Z^+, \mathbb{R}^k) \) set of all functions in \( S_2 \) that have limit zero. \( S_2 \) and \( S_3 \) are Banach spaces with the supremum norm. For \( 1 \leq p < \infty \), \( \ell_p \) is the space of sequences \( u = \{u_i\}_{i=1}^{\infty} \) such that \( \sum_{i=1}^{\infty} |u_i|^p < \infty \) with norm \( |u|_p = (\sum_{i=1}^{\infty} |u_i|^p)^{1/p} < \infty \).

Now, we consider a linear difference system of the form

\[
R(n + 1, m) = A(n)R(n, m) + \sum_{r=0}^{n} B(n, r)R(r, m), \quad n > m,
\]

with \( R(m, m) = I \) for \( 0 \leq m \leq n \).

Elaydi [5] showed that equation (3) has a unique solution \( x(n) \) which can be expressed as

\[
x(n) = x(n, 0, x_0) = R(n, 0)x_0 + \sum_{r=0}^{n-1} R(n, r + 1)f(r).
\]

In the case \( A(n) = A \) constant matrix (with \( A \) nonsingular) and \( B(n, r) = B(n - r) \), we obtain \( R(n, m) = R(n - m) \), and thus, equation (4) reduces to

\[
R(n + 1) = AR(n) + \sum_{r=0}^{n} B(n - r)R(r).
\]

Note 1. The unique solution of equation (2) is given by

\[
x(n, 0, x_0) = X(n)x_0 + \sum_{r=0}^{n-1} X(n - r - 1)f(r),
\]
where \( X(n) \) is called the fundamental matrix of system

\[
y(n + 1) = Ay(n) + \sum_{r=0}^{n} B(n - r)y(r), \quad y(0) = y_0. \tag{7}
\]

Notice that \( X(0) = I \) and \( y(n, 0, y_0) = X(n)y_0 \) is the unique solution of equation (7) with \( y(0, 0, y_0) = y_0 \).

The resolvent matrix associated with equations (1)–(3) will be denoted by \( R_1, R_2, \) and \( R_3 \), respectively.

**Theorem A.** (See [12].) If

1. \( \varphi(n) \) is a bounded sequence on \( \mathbb{Z}^+ \) and \( \lim_{n \to \infty} \varphi(n) \) exists,
2. \( \sum_{n=0}^{\infty} |\Psi(n)| < \infty, \)

then

\[
\lim_{n \to \infty} \sum_{j=0}^{n} \varphi(n - j)\Psi(j) = \varphi(\infty)\sum_{n=0}^{\infty} \Psi(n).
\]

3. **BOUNDEDNESS OF THE SYSTEM WITH DISTURBANCES**

We are now in a position to establish our main results dealing with the boundedness of equation (1).

**Theorem 1.** Assuming that

(i) \( f \in S_1, \)

(ii) \( B, D \in \ell_p \) for some \( 1 \leq p < \infty, \)

(iii) \( R_2 \in \ell_1, \) and

(iv) \( C(n) = \sum_{r=0}^{n} D(n - r) \in S_3, \) then the solution \( x(n) \) of equation (1) is in \( S_1. \)

**Proof.** By (5), the solution of equation (1) can be represented in the form

\[
x(n) = R_2(n)x_0 + \sum_{r=0}^{n-1} R_2(n - r - 1)f(r) + \sum_{r=0}^{n-1} R_2(n - r - 1)\sum_{j=0}^{r} D(r - j)x(j). \tag{8}
\]

Thus, for any subset \( (0, N) = \{0, 1, \ldots, N\}, \) with \( N \in \mathbb{Z}^+, \) we get

\[
|x|_{(0,N)} = \sup_{n \in (0,N)} |x(n)| \leq |R_2|s_1 \cdot |x_0| + |f|s_1 \sup_{n \in (0,N)} \sum_{r=0}^{n-1} |R_2(n - r - 1)|
\]

\[
+ |C|s_1 \cdot |x|_{(0,N)} \sup_{n \in (0,N)} \sum_{r=0}^{n-1} |R_2(n - r - 1)|
\]

\[
\leq |R_2|s_1 \cdot |x_0| + |R_2|1|f|s_1 + |C|s_1|R_2|1|x|_{(0,N)}.
\]

This yields

\[
|x|_{(0,N)} \leq (1 - |C|s_1|R_11|)^{-1}(|R_2|s_1|x_0| + |R_2|1|f|s_1|).
\]

Thus, we conclude that \( x(n) \) is in \( S_1, \) and the statement is valid for \( \sum_{r=0}^{n} D(n - r) |< |R_2|1|^{-1}, \)

that is, for small \( \sum_{r=0}^{n} D(n - r). \)

Now, we consider the case \( C(n) = \sum_{r=0}^{n} D(n - r) \in S_3. \) Then, there is a \( N \in \mathbb{Z}^+ \) such that

\[
|C_N(n)|s_1 < (|R_2|1)^{-1}, \)

where \( C_N(n) = C(n + N), n \in \mathbb{Z}^+. \)

Since \( x(n) \) is a solution of equation (1) on \( (0, N), \) then for \( n > N, \) \( x(n) \) still solves equation (1) and replacing \( n \) by \( n + N, \) we get

\[
x(n + N + 1) = Ax(n + N) + \sum_{r=0}^{n+N} [B(n + N - r) + D(n + N - r)]x(r) + f(n + N),
\]

and \( x(0 + N) = x(N). \)
Denoting $x(n + N)$ as $x_N(n)$, we have

$$x_N(n + 1) = Ax_N(n) + \sum_{r=0}^{n-N} [B(n + N - r) + D(n + N - r)]x(r) + f_N(n).$$

$$x_N(0) = x(N).$$

By performing the change of variable $r - N = u$, we get

$$x_N(n + 1) = Ax_N(n) + \sum_{u=0}^{n} [B(n - u) + D(n - u)]x_N(u) + F(n),$$

where

$$F(n) = \sum_{u=0}^{n} [B(n - u) + D(n - u)]x_N(u) + f_N(n).$$

By the assumptions, $F(n) \in S_1$.

Since $x_N(n)$ solves equation (9), and in this equation $C_N(n)$ is small, we infer that $x(n) \in S_1$ for all $n \in \mathbb{Z}^+$, concluding the proof.

**NOTE 2.**

(a) Theorems 1 remains valid if we replace Condition (iv) by the condition

$$\limsup_{n \to \infty} \left| \sum_{r=0}^{n} D(n - r) \right| < |R_2|^{-1}.$$

(b) We want to point out that for each $n \in \mathbb{Z}^+$, a function $x(n)$ solves equation (1), if and only if, $x(n)$ solves equation (8). Indeed, if $x(n)$ solves equation (1), then we define

$$h(n) = f(n) + \sum_{i=0}^{n} D(n - i)x(i)$$

and apply (4). This yields (8). The process may be reversed to see that (8) implies (1).

**COROLLARY 1.** Assuming that

(i) $B, D \in \ell_p$ for some $1 \leq p < \infty$,
(ii) $R_2 \in \ell_1$,
(iii) $C \in S_3$, and
(iv) $f \in S_2$,

then the solution $x(n)$ of equation (1) verifies

$$x(\infty) = \lim_{n \to \infty} x(n) = \sum_{r=0}^{\infty} R_2(r) \cdot f(\infty),$$

where $f(\infty) = \lim_{n \to \infty} f(n)$.

**PROOF.** From (8), the solution $x(n)$ of equation (1) satisfies

$$x(n) = R_2(n)x_0 + \sum_{r=0}^{n-1} R_2(n - r - 1)f(r) + \sum_{r=0}^{n-1} R_2(n - r - 1) \sum_{j=0}^{r} D(r - j)x(j),$$

and it is a bounded function.
Thus, by Theorem A, all the terms on the right-hand side have limits. From this, we infer that

$$\lim_{n \to \infty} x(n) = \sum_{r=0}^{\infty} R_2(r) \cdot f(\infty).$$

**Note 3.** If in Corollary 1, we suppose $f \in S_3$, then the solution $x(n)$ of equation (1) satisfies $\lim_{n \to \infty} x(n) = 0$.

**Corollary 2.** Assume that

(i) $B, D \in \ell_p$ for some $1 \leq p < \infty$,
(ii) $R_2 \in \ell_1$, and
(iii) $f \in \ell_p$,

then, if (iv) of Theorem 1 holds, the solution $x(n)$ of equation (1) is in $\ell_p$.

**Proof.** From (8), the solution $x(n)$ of equation (1) satisfies

$$x(n) = R_2(n)x_0 + \sum_{r=0}^{n-1} R_2(n - r - 1)f(r) + \sum_{r=0}^{n-1} R_2(n - r - 1) \sum_{j=0}^{r} D(r - j)x(j).$$

Since $R_2(n)x_0 \in \ell_1$, thus, $R_2(n)x_0 \in \ell_p$. Moreover, from Theorem 1, $x(n)$ is bounded, then $\sum_{r=0}^{n} D(n - r)x(r) \in \ell_p$. It is known that the convolution of a $\ell_1$ function with a $\ell_1$ function belongs to $\ell_p$. From this, we infer that $x(n) \in \ell_p$, concluding the proof.

In [9, Theorem 3], sufficient conditions are given in order to assure the solutions of the Volterra difference equation

$$\Delta(a_n\Delta x_n) + b_n\Delta x_n + c_n x_n = f(n, r_n, x_n, \Delta x_n),$$

where

$$f(n, r_n, x_n, \Delta x_n) = r_n \cdot \sum_{\ell=0}^{n-1} g_{\ell} x_{\ell} + h(n, x_n, \Delta x_n),$$

are in $\ell_p$, for $1 \leq p \leq \infty$.

Moreover, in [10, Theorem 6], sufficient conditions are obtained to the existence of bounded solutions of equation

$$\Delta x(n) = A(n)x(n) + B(n)x(h(n)) + \sum_{j=0}^{n-1} K(n,j)x(j),$$

$x(0) = 0$, $h : \mathbb{Z}^+ \to \mathbb{Z}^+$; $A, B$, and $K$ matrices.

Finally, in [11, Theorem 5–8], weighted norms are used to find sufficient conditions under which the discrete Volterra equation

$$u(n) = g(n) + \sum_{i=-\infty}^{n-1} f(n, i, u(i))$$

has unique solutions in $\ell_p$.

If in Theorem 1, we replace the condition $C(n) \in S_3$ by $C(n) \in \ell_1$, then we can establish the next boundedness theorem.

**Theorem 2.** Assuming that in equation (1) $f \in S_1$, $R_2 \in \ell_1$, and $\sum_{r=0}^{n} D(n - r) \in \ell_1$, then the solution $x(n)$ of equation (1) is in $S_1$.

**Proof.** From (8), the solution $x(n)$ of equation (1) satisfies

$$|x(n)| \leq |R_2| |s_1||x_0| + |f| |s_1||R_2| + |R_2| \cdot \sum_{j=0}^{n-1} |x(j)| \sum_{r=j}^{n-1} |D(j - r)|.$$
From here and the discrete version of the Gronwall-Bellman Lemma, it follows that

$$|x(n)| \leq M \exp \left[ |R_2| \sum_{j=0}^{n-2} \sum_{r=j}^{n-1} |D(j - r)| \right],$$

where $M = |R_2| |x_0| + |f| |s_1| \cdot |R_2|.$

We, therefore, have the boundedness of the solution $x(n)$ of equation (1).

Now, we will establish a new version of Theorem 2 for more general systems of Volterra difference equations. Consider the systems

$$x(n + 1) = A(n)x(n) + \sum_{r=0}^{n} [B(n,r) + D(n,r)]x(r) + f(n), \quad x(0) = x_0, \quad (10)$$

$$x(n + 1) = A(n)x(n) + B(n,r)x(r), \quad x(0) = x_0, \quad (11)$$

and

$$x(n + 1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + f(n), \quad x(0) = x_0. \quad (12)$$

**ASSUMPTION A.** (See [3].) $|C_{nj}| < M\alpha^{n-j}$ for some constants $M > 0$ and $\alpha \in (0, 1),$ where

$$C_{nj} = \begin{cases} A(n) + B(n,n), & n = j, \\ B(n,j), & i \neq j. \end{cases}$$

**THEOREM 3.** Assuming that

(i) $\sum_{n=0}^{\infty} |f(n)| < \infty,$

(ii) equation (11) is uniformly asymptotically stable, and Assumption A holds,

(iii) $\sum_{r=0}^{\infty} \sum_{j=0}^{\infty} |D(r,j)| < \infty,$

then the solution $x(n)$ of equation (10) is bounded.

**PROOF.** We observe that for all $n \in Z^+,$ a function $x(n)$ solves equation (10), if and only if $x(n)$ solves the equation

$$x(n) = R_{12}(n,0)x_0 + \sum_{r=0}^{n-1} R_{12}(n,r+1)f(r) + \sum_{r=0}^{n-1} R_{12}(n,r+1) \sum_{j=0}^{r} D(r,j)x(j), \quad (13)$$

where $R_{12}$ is the resolvent matrix of equation (12). It is straightforward to check the above statement, thus we will omit it.

On the other hand, because of Assumption A, there exist constants $\lambda > 0$ and $\gamma \in (0, 1)$ such that

$$|R_{12}(n,m)| \leq \lambda \gamma^{n-m}, \quad n \geq m,$$

where $R_{12}$ is the resolvent of equation (12). Then, taking in account the assumptions, we get

$$|x(n)| \leq \lambda |x_0| + \sum_{r=0}^{n-1} \gamma^{n-r-1} f(r) + \lambda \sum_{r=0}^{n-1} \gamma^{n-r-1} \sum_{j=0}^{r} |D(r,j)||x(j)|$$

$$\leq \lambda (|x_0| + |f| + \sum_{j=0}^{n-1} |x(j)| \sum_{r=j}^{n-1} |D(r,j)|).$$

From the discrete version of the Gronwall-Bellman Lemma, if follows that

$$|x(n)| \leq (\lambda |x_0| + \lambda |f|) \exp \left( \lambda \sum_{j=0}^{n-1} \sum_{r=j}^{n-1} |D(r,j)| \right).$$

From this, we infer the boundedness of solution $x(n)$ of equation (10), concluding the proof. \[\blacksquare\]
Now, we will discuss whether Hypothesis (iii) of Theorem 3 can be weakened. Crisci et al. [3], showed that the weaker hypothesis \( \lim_{j \to \infty} |D(i,j)| = 0, \ i > j, \) is not sufficient to assure the boundedness of equation (10), with \( A(n) = 0. \) Suppose \( f(n) = 0 \) for all \( n \in \mathbb{Z}^+ \) in equation (10), thus obtaining

\[
x(n + 1) = A(n)x(n) + \sum_{r=0}^{n} [B(n,r) + D(n,r)]x(r), \quad x(0) = x_0. \tag{14}
\]

Thus, in the next theorem, we will show how Condition (iii) of Theorem 3 can be weakened.

**Theorem 4.** Assume that equation (11) is uniformly asymptotically stable and Assumption A holds. If \( \sum_{r=0}^{\infty} |D(r,j)| \gamma^{-r} < \infty \) and \( \lim_{j \to \infty} \sum_{r=j}^{\infty} |D(r,j)| \gamma^{-r} = 0, \) then the solution \( x(n) \) of equation (14) is bounded.

**Proof.** We can represent the solution \( x(n) \) of equation (14) in the form

\[
x(n) = R_{14}(n,0)x_0 + \sum_{r=0}^{n-1} R_{14}(n,r+1) \sum_{j=0}^{r} D(r,j)x(j),
\]

where \( R_{14} \) is the resolvent matrix of equation (14). Thus, the proof of this theorem can be given along the lines of Theorem 4.2 [3], and therefore, we will omit it.

Now, we will consider the following system of Volterra difference equations:

\[
x(n + 1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + F(n,x(n)), \quad x(0) = x_0, \tag{15}
\]

where \( A \) and \( B \) are square matrices of order \( k \) on the sets \( \mathbb{Z}^+ \) and \( \mathbb{Z}^+ \times \mathbb{Z}^+, \) respectively, \( F \) is a vector function on \( \mathbb{Z}^+ \times \Omega_0 \) continuous to the second argument, \( \Omega_0 \) denotes the ball with center zero and radius \( r_0 \) in the space \( \mathbb{R}^k, \) and we denote by \( R_{15} \) the resolvent matrix of equation (15).

**Theorem 5.** Assume that

(i) \( \lim_{n \to \infty} R_{15}(n,r) = R_0(r) \) on every interval \((0,b)\) of \( \mathbb{Z}^+, \)

(ii) \( \sum_{r=0}^{\infty} |R_0(r+1)| |\lambda(r) < \infty \) and \( \sum_{r=0}^{\infty} |R_0(r+1)||F(r,0)| < \infty, \) where \( \lambda : \mathbb{Z}^+ \to \mathbb{R} \) is a positive function,

(iii) \( \lim_{n \to \infty} \sum_{r=0}^{n-1} |R_{15}(n,r)| |\lambda(r) = \sum_{r=0}^{\infty} |R_0(r+1)||\lambda(r), \)

(iv) \( R_{15} \) is bounded and \( \lim_{n \to \infty} R_{15}(n,0) = L \) exist, and

(v) \( |F(n,x) - F(n,y)| \leq \lambda(n)||x - y||, \) for every \( n \in \mathbb{Z}^+ \) and \( x,y \in \Omega_0. \)

Then, the solution \( x(n) \) of equation (15) satisfies the relation

\[
x(\infty) = Lx_0 + \sum_{n=0}^{\infty} R_0(n+1)F(n,x(n)),
\]

where \( x(\infty) = \lim_{n \to \infty} x(n). \)

**Proof.** By representing the solution \( x(n) \) of equation (15) in the form

\[
x(n) = R_{15}(n,0)x_0 + \sum_{r=0}^{n-1} R_{15}(n,r+1)F(r,x(r)),
\]

we get

\[
x(n) = R_{15}(n,0)x_0 + \sum_{r=0}^{n-1} R_{15}(n,r+1)(F(r,x(r)) - F(r,0)) + \sum_{r=0}^{n-1} R_{15}(n,r+1)F(r,0).
\]
To prove the convergence of \( x(n) \), we consider
\[
\sum_{r=0}^{n-1} |R_{15}(n, r + 1)(F(r, x(r)) - F(r, 0))| \leq \sum_{r=0}^{n-1} |R_{15}(n, r + 1)||F(r, x(r)) - F(r, 0)|
\]
\[
\leq \sum_{r=0}^{n-1} |R_{15}(n, r + 1)|\lambda(r)|x(r)| \leq r_0 \cdot \sum_{r=0}^{n-1} |R_{15}(n, r + 1)|\lambda(r).
\]
Thus, we infer from the assumptions the convergence of the term
\[
\sum_{r=0}^{n-1} R_{15}(n, r + 1)(F(r, x(r)) - F(r, 0)).
\]

On the other hand,
\[
\sum_{r=0}^{n-1} R_{15}(n, r + 1)F(r, 0) = \sum_{r=0}^{n-1} [R_{15}(n, r + 1) - R_0(n + 1)]F(r, 0) + \sum_{r=0}^{n-1} R_0(r + 1)F(r, 0).
\]

Then, observing that
\[
\sum_{r=0}^{n-1} |R_0(r + 1)F(r, 0)| \leq \sum_{r=0}^{n-1} |R_0(r + 1)||F(r, 0)|,
\]
by (ii), we deduce the convergence of
\[
\sum_{r=0}^{n-1} R_0(r + 1)F(r, 0).
\]

It remains to prove that \( \sum_{r=0}^{n-1} |R_{15}(n, r + 1) - R_0(n + 1)||F(r, 0)| \) converges.

From (i)–(iii), we infer that for each \( \varepsilon > 0 \), there is \( N_1 > 0 \) such that
\[
\sum_{r=0}^{n-1} |R_{15}(n, r + 1)|\lambda(r) < \varepsilon, \quad \text{for all } n - 1 > N_1,
\]
and there is \( N_2 > 0 \) such that
\[
\sum_{r=0}^{n-1} |R_0(r + 1)|\lambda(r) < \varepsilon.
\]

Thus, if we define \( N_0 = \max\{N_1, N_2\} \), then for all \( n > N_0 \),
\[
\sum_{r=0}^{n-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r) = \sum_{r=0}^{N_0-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r)
\]
\[
+ \sum_{r=N_0}^{n-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r)
\]
\[
\leq \sum_{r=0}^{N_0-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r)
\]
\[
+ \sum_{r=N_0}^{n-1} |R_{15}(n, r + 1)|\lambda(r) + \sum_{r=N_0}^{n-1} |R_0(r + 1)|\lambda(r)
\]
\[
\leq \sum_{r=0}^{N_0-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r) + 2\varepsilon.
\]

Hence, from Condition (i), it follows that
\[
\lim_{n \to \infty} \sum_{r=0}^{n-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r) \leq 2\varepsilon.
\]

From this, we infer that
\[
\lim_{n \to \infty} \sum_{r=0}^{n-1} |R_{15}(n, r + 1) - R_0(r + 1)|\lambda(r) = 0.
\]

We, therefore, have that \( \sum_{r=0}^{n-1} R_{15}(n, r + 1)F(r, x(r)) \) is convergent, concluding the proof.
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