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Robust network optimization under polyhedral demand uncertainty is *NP*-hard

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ABSTRACT

Minimum cost network design/dimensioning problems where feasibility has to be ensured w.r.t. a given (possibly infinite) set of scenarios of requirements form an important subclass of robust *LP* problems with right-hand side uncertainty. Such problems arise in many practical contexts such as Telecommunications, logistic networks, power distribution networks, etc. Though some evidence of the computational difficulty of such problems can be found in the literature, no formal NP-hardness proof was available up to now. In the present paper, this pending complexity issue is settled for all robust network optimization problems featuring polyhedral demand uncertainty, both for the single-commodity and multicommodity case, even if the corresponding deterministic versions are polynomially solvable as regular (continuous) linear programs. A new family of polynomially solvable instances is also discussed.

1. Introduction

The deterministic optimum link capacity expansion problem for a given network to satisfy a given set of point-to-point requirements, assuming that existing capacity is not sufficient to meet all the requirements, arises in connection with many applications such as telecommunication networks (see e.g. [18,19]), power networks (see e.g. [17]) or logistic networks (see e.g. [18,20]).

In the simple basic case where the cost of installing extra capacity on each arc is linear, and there is no additional restriction on how to route the flows corresponding to the various source–sink pairs, this problem can be formulated as minimum cost single-commodity or multicommodity network flow problems. For the single-commodity case, numerous efficient combinatorial (i.e. graph-based) minimum cost flow algorithms have been described in the literature (see e.g. [1]). In the multicommodity case those algorithms are not applicable, but efficient linear programming techniques taking advantage of the underlying network flow structures are available (see e.g. [1]).

In the present paper, we consider nondeterministic versions of such problems, which correspond to the case where the various flow requirements to be satisfied are subject to uncertainty. More specifically, we will consider the case of *polyhedral uncertainty*, i.e. the case where the uncertainty set for requirements is a given (bounded) polyhedron in $\mathbb{R}_+^K$ (K denotes the number of point-to-point requirements, and for each individual requirement $k \in [1, K]$, $s(k)$ and $t(k)$ denote the origin and destination nodes respectively).

However, let us insist on the fact that the class of network optimization problems addressed here is restricted to problems which are expressible as regular (continuous) linear programs in their deterministic version (i.e. in the case of a single scenario of requirements) and, as such, *solvable in polynomial time*. Consequently network optimization problems such as those featuring nonconvex and/or discontinuous link cost functions, or including complicating constraints of various types

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(e.g. imposing a maximum number of routes for each point-to-point requirement) which are already provably hard to solve, even in their deterministic versions, are out of the scope or our study. Our purpose here is mainly to show that considering polyhedral demand uncertainty alone is responsible for switching from P to NP-hard complexity status. Basic references on robust optimization are [5–8,12,14].

Robust versions of single-commodity and/or multicommodity network flow problems under demand uncertainty have been addressed by several authors, including [2,4,13,15,24–27].

Atamtürk & Zhang [2] investigate a class of 2-stage robust network flow problems with demand uncertainty in which two types of design variables are considered, corresponding to a partition (A, B) of the arc set of the network into two subsets. The capacities of the arcs in A are the first-stage decision variables ("here and now" decisions), the flows on the arcs in B are the second-stage variables ("wait-and-see" decisions).

It is shown that a linear description of the polyhedron of feasible solutions in the space of first-stage variables can be obtained via projection (close in spirit to Benders partitioning [3]), resulting in a large scale linear system for which the separation subproblem is NP-hard for various polyhedral uncertainty sets for requirements. Such complexity results for the separation problem indeed strongly suggest that the corresponding robust network design problems (concerning both the single-commodity and the multicommodity case) are computationally difficult, however they do not constitute a formal direct proof of NP-hardness for these problems.

Ordoñez and Zhao [24] investigate minimum cost capacity expansion problems (for both single-commodity and multicommodity cases) under demand and travel time uncertainty. Both conic uncertainty sets and polyhedral uncertainty sets are considered. Apart from a number of polynomial special cases (discussed into details in the paper) it is shown that in the general case, the problem can be reduced to minimizing a convex function Φ of the design variables y (the vector of additional capacities on the arcs), but computing $\Phi(y)$ for any given y requires *maximizing a convex function*, an a priori difficult problem. Again, this strongly suggests that the robust network design problems considered in [24] are computationally difficult, but cannot be considered as a formal *NP*-hardness proof. Other similar robust network optimization models featuring demand uncertainty have been investigated in [25,26] where the authors mention the computational difficulties induced by the structural nonconvexity and non-smoothness present in the problems, but without providing any formal *NP*-hardness proof.

The present paper is organized as follows.

In Section 2, we show strong NP-hardness of robust feasibility testing even for a restricted subclass of single-source single-commodity flow problems with polyhedral demand uncertainty. Complexity issues concerning robust min-cost capacity expansion problems are discussed in Section 3, and, again, strong NP-hardness is shown even for a restricted subclass of single-source single-commodity flow problems. A whole subclass of polynomially solvable robust capacity expansion problems is proposed in Section 4. As a typical example, the special case of planar graphs with some special type of knapsack-constrained uncertainty sets is shown to belong to this subclass.

Let us stress upon the fact that, in what follows, only directed network models are considered, but it should be clear that this is not restrictive since any undirected network model can easily be reformulated as a directed one. Also it is worth pointing out that all the problems investigated in the present paper are special instances of the more general class R-LP-RHSU-PU of robust linear programming problems under right-hand side uncertainty, see e.g. [21]. Thus the NP-hardness results to be presented below can be viewed as a confirmation of NP-hardness of this general class, already derived as a consequence of the results in [22] and further investigated in [23].

2. The issue of feasibility testing and complexity results

In this section we consider the basic question of testing feasibility of a given capacitated network under polyhedral demand uncertainty. This class of problems will be denoted R-FCNP-PU (FCNP standing for "robust feasible capacitated network problem", and PU standing for "polyhedral uncertainty").

Note that the class R-FCNP-PU naturally decomposes into two subclasses depending on whether the requirements correspond to a single-commodity flow or to a multicommodity flow. Any instance is specified by a directed graph together with known capacities on the arcs, and a polyhedron $\mathcal{D} \subset \mathbb{R}_+^K$ representing the uncertainty set for various given point-to-point requirements $d_1, d_2, \ldots, d_K$. The question to be answered is whether the given capacities on the network are sufficient to guarantee the existence of a feasible flow for any $(d_1, d_2, \ldots, d_K) \in \mathcal{D}$.

In this section it will be shown that problems in R-FCNP-PU are strongly NP-hard by proving strong NP-hardness of the restricted subclass (denoted R-FSSCNP-KCU) corresponding to the single-commodity single-source case and to an uncertainty polyhedron in $1-1$ correspondence with the solution set of a (continuous) knapsack problem. More precisely, an instance of R-FSSCNP-KCU (FSSCNP stands for "Feasible Single-Source CNP" and KCU stands for "Knapsack-Constrained Uncertainty") is a decision problem defined as follows:

Given:

- a capacitated network $[G, c]$ where G is a directed graph with node set X ($|X| = n$), arc set U and $c = (c_u)_{u\in U}$ the vector of arc capacities;
- a single source node $s \in X$ with unlimited availability, the $n-1$ nodes in $T = X \setminus \{s\}$ being sink nodes;
- for each sink node $j \in T$, a nominal requirement $\bar{d}_j$ and a worst-case requirement $\hat{d}_j = \bar{d}_j + \delta_j$ (where $\delta_j > 0$);

- the right-hand side $\bar{\theta} \in]0, n-1[$ of the (continuous) knapsack constraint defining the uncertainty set $\mathcal{D}$ for requirements:

$$\mathcal{D} = \left\{ d \in \mathbb{R}_+^{n-1} / d_j = \bar{d}_j + \theta_j \delta_j;\ 0 \le \theta_j \le 1, \forall j \in T;\ \sum_{j \in T} \theta_j \le \bar{\theta} \right\}. \tag{1}$$

Question (Q): Is the given capacitated network feasible for any $d \in \mathcal{D}$?

Let us observe that the above definition of R-FSSCNP-KCU is general enough to include cases where the set T of sink nodes is only a strict subset of $X \setminus \{s\}$; indeed, if $j \in X \setminus \{s\}$ is not a sink node it is enough to set $\bar{d}_j = \delta_j = 0$ in the above general model.

Also we note that knapsack-constrained uncertainty sets typically correspond to the uncertainty model investigated by Bertsimas and Sim [7,8], though in a quite different context: indeed their approach concerns robust LP (or MILP) problems with row-wise uncertainty, whereas we address here LP problems with right-hand side uncertainty (a special case of columnwise uncertainty). The big difference in structure between row-wise uncertainty and RHS uncertainty when polyhedral uncertainty sets are considered was already pointed out in [22,23]. This difference becomes obvious just by observing that for continuous LP's the robust counterpart is polynomially solvable in the former case, and strongly NP-hard in the latter case.

We now prove:

Proposition 1. *Problem R-FSSCNP-KCU is strongly NP-hard.*

Proof. The proof is based on a reduction from the problem R-MAX-FLOW-KCU (see [22]) which consists in determining a maximum $s-t$ flow value in a capacitated network with uncertain arc capacities and uncertainty set in $1-1$ correspondence with the solution set of a continuous knapsack problem (KCU stands for "knapsack-constrained uncertainty set"). An instance of R-MAXFLOW-KCU is specified as follows.

A directed graph $G = [V, U]$ is given together with $s \in V$, $t \in V$ two distinguished vertices, and a target value W for a $s-t$ flow. Moreover, associated with each arc $u = (i, j) \in U$, there are two real numbers $\bar{c}_u \ge 0$ and ε_u ($0 \le \varepsilon_u \le \bar{c}_u$), $\bar{c}_u$ being interpreted as the nominal capacity, and $\bar{c}_u - \varepsilon_u \ge 0$ as the reduced (or "worst-case") capacity. Finally, the uncertainty set for capacities is defined as:

$$\mathcal{C} = \left\{ c \in \mathbb{R}_+^U / c_u = \bar{c}_u - \theta_u \varepsilon_u, \forall u \in U;\ \theta_u \in [0, 1], \forall u \in U;\ \sum_{u \in U} \theta_u \le \bar{\theta} \right\}$$

where $0 < \bar{\theta} < |U|$ is a given constant.

The question is then: is the maximum $s-t$ flow value $\ge W$ for all $c \in \mathcal{C}$?

Problem R-MAXFLOW-KCU has been shown to be strongly NP-hard in [22] based on a reduction of MIN-CUT-INTO-BOUNDED-SETS (see [10], page 210). Indeed, looking carefully at the proof given in [22], strong NP-hardness is observed to hold even for the subclass R-MAXFLOW-KCU' of R-MAXFLOW-KCU for which the only arcs having $\varepsilon_u > 0$ are those in $\omega^+(s) = \{(s, j)/(s, j) \in U\}$, and $\varepsilon_u = \varepsilon > 0$, $\forall u \in \omega^+(s)$ (uniform capacity reduction). Also, U does not contain the direct arc (s, t).

So let us consider an arbitrary instance of R- MAXFLOW-KCU' on $G = [V, U]$ with uncertainty set for capacities of the arcs in $\omega^+(s)$:

$$\mathcal{C}' = \left\{ c \in \mathbb{R}_+^{|\omega^+(s)|} / c_u = \bar{c}_u - \theta_u \varepsilon, \forall u \in \omega^+(s);\ \theta_u \in [0, 1], \forall u \in \omega^+(s);\ \sum_{u \in \omega^+(s)} \theta_u \le \bar{\theta} \right\}$$

and $\bar{\theta} \in]0, \left|\omega^+(s)\right|[$. The question to be answered is whether for any $c \in \mathcal{C}'$ there exists a feasible $s-t$ flow with value $\ge W$ in G.

We show that this problem can be reduced to an instance of R-FSSCNP-KCU on the capacitated graph $[G, \bar{c}]$ (i.e. graph G with each arc $u \in U$ having capacity $\bar{c}_u$). To achieve this, we specify the uncertain demands at the various nodes $j \in V \setminus \{s\}$ as follows. For any $j \in \Gamma^+(s) = \{j/(s, j) \in U\}$ we take the nominal requirement $\bar{d}_j = 0$ and worst-case requirement $\delta_j = \varepsilon$. For $j = t$ we consider the nominal requirement $\bar{d}_t = W$, and worst-case requirement W (so, $\delta_t = 0$ in that case).

The uncertainty set for requirements is defined as

$$\mathcal{D} = \left\{ d \in \mathbb{R}_+^{|\Gamma^+(s)|+1} / d_j = \theta_j \varepsilon, \forall j \in \Gamma^+(s), d_t = W, \theta_j \in [0, 1] \forall j \in \Gamma^+(s), \sum_{j \in \Gamma^+(s)} \theta_j \le \bar{\theta} \right\}.$$

Now, the fact that the answer to the above-defined instance of R-FSSCNP-KCU provides the right answer to the instance of R-MAXFLOW-KCU follows from the fact that $[G, \bar{c}]$ is feasible for d if and only if there exists in G a $s-t$ flow of value $\ge W$ compatible with the capacities $\bar{c}_{sj} - d_j$ for all $j \in \Gamma^+(s)$. This completes the proof. □

As an immediate consequence of the above we can state the following:

Corollary 1. *R-FCNP-PU is strongly NP-hard in the single-commodity flow version as well as in the multicommodity flow version.*

3. Complexity of robust Min-Cost capacity expansion problems

Robust minimum cost capacity expansion problems arise when the answer to the feasibility question addressed in the previous section is negative, i.e. when the existing capacities on the various links of the network are not sufficient for meeting all possible requirements d in the uncertainty set $\mathcal{D}$. Then extra capacity has to be added on some (possibly all) links of the network, the problem being to determine where and how much extra capacity should be installed in order to minimize the corresponding cost.

As already mentioned, we assume that the cost of installing an amount x_u of extra capacity on link $u \in U$ is linear of the form $\gamma_u x_u$, where $\gamma_u > 0$ is a given unit cost coefficient. This class of robust minimum cost capacity expansion problems with polyhedral uncertainty set $\mathcal{D}$ will be denoted R-CEP-PU. The subclass formed by instances for which the uncertainty set $\mathcal{D}$ is in $1 - 1$ correspondence with the solution set of a (continuous) knapsack problem will be denoted R-CEP-KCU. In the special case involving single-commodity and single-source network flows, the subclasses corresponding to R-CEP-PU and R-CEP-KCU will be denoted R-SSCEP-PU and R-SSCEP-KCU respectively.

An instance of R-CEP-PU is obtained by specifying a directed graph $G = [X, U]$, a list of source–sink pairs, and an uncertainty polyhedron $\mathcal{D}$ for requirements. Let $F(G, \mathcal{D})$ denote the set of all robust feasible capacity assignments, in other words for all $y \in F(G, \mathcal{D})$ the capacitated network $[G, y]$ is feasible for all $d \in \mathcal{D}$. Given existing capacities $c = (c_u)_{u\in U} \in \mathbb{R}_+^U$ on the arcs and an arc cost vector $\gamma = (\gamma_u)_{u\in U} \in \mathbb{R}_+^U$, the robust minimum cost capacity expansion problem can be stated as:

$$P(G, \mathcal{D}) \begin{cases} \text{Minimize } \displaystyle\sum_{u\in U} \gamma_u x_u \\ \text{subject to:} \\ \quad c + x \in F(G, \mathcal{D}) \\ \quad x \geq 0. \end{cases}$$

Note that if $\mathcal{D}$ is a bounded polyhedron, it has a finite set of extreme points $\text{ext}(\mathcal{D}) = \left\{d^{(1)}, d^{(2)}, \dots, d^{(P)}\right\}$ then the above problem can be formulated as a (possibly large scale) linear program. This stems from the fact that $y \in F(G, \mathcal{D})$ if an only if the capacitated network $[G, y]$ is feasible for each $d^{(p)} \in \text{ext}(\mathcal{D})$, and it is standard that the latter condition can be expressed as a linear equality/inequality system in both the single-commodity and multicommodity case. However the number of extreme points can grow exponentially large with K (dimension of the vector of demands) making the solution of the linear program $P(G, \mathcal{D})$ computationally difficult.

The following result confirms this analysis and settles the complexity status of the various classes of robust min-cost capacity expansion problems.

Proposition 2. *R-SSCEP-KCU is strongly NP-hard.*

Proof. It proceeds by exhibiting a reduction from R-FSSCNP-KCU which is strongly NP-hard (Proposition 1). Consider any instance of the latter problem specified by a capacitated directed graph $[G, c]$ and a knapsack-constrained uncertainty set $\mathcal{D}$ for requirements w.r.t. a given list of K source–sink pairs assuming that all have a common source. From this we can readily build an instance of R-SSCEP-KCU where we set $\gamma_u = 1$ for all $u \in U$, so that the objective function merely consists in minimizing $\sum_{u\in U} x_u$, the total extra capacity to be installed in order to satisfy the feasibility condition:

$$c + x \in F(G, \mathcal{D}), \quad x \geq 0.$$

Now it is easily seen that the answer to the instance of R-FSSCNP-KCU is YES if and only if the above instance of R-SSCEP-KCU has optimal solution value 0, and this completes the proof. □

As an immediate consequence of Proposition 2 we get:

Corollary 2. *R-SSCEP-PU and R-CEP-PU are strongly NP-hard.*

4. Polynomial special cases

Various polynomial special cases for robust min-cost capacity expansion problems under uncertain demand have been suggested in the literature. An obvious situation for which R-CEP-PU instances are practically tractable (in both the single-commodity and multicommodity case) is when the uncertainty polytope $\mathcal{D}$ has a number of extreme points bounded by a fixed (and not too big) constant. An example of this for the multicommodity case can be found in [4]. Other polynomial special cases involving uncertainty polytopes for requirements defined as intervals, or products of intervals, are discussed in [24]. Also we mention the special case of arborescences with knapsack-constrained uncertainty sets which is shown to be polynomially solvable in [2]. In this section, we first state a fairly general sufficient condition for polynomial solvability of instances of R-CEP-PU. Next we exhibit a new polynomial special case of R-SSCEP-KCU satisfying this condition and involving planar graphs.

Let $[G, c]$ be a given directed capacitated graph and a list of K source–sink pairs $s(k), t(k)$ with corresponding demands $d_1, d_2, \ldots, d_K$ (we only consider the most general case of a multicommodity flow model, while observing that the whole discussion to follow remains valid for the single-commodity flow case).

A classical necessary and sufficient condition for feasibility of a given capacity assignment $y = (y_u)_{u\in U}$ is that the following inequality (called: metric inequality) holds:

$$\sum_{u\in U} \lambda_u y_u - \sum_{k=1}^{K} \mu_k d_k \geq 0 \tag{2}$$

for all $(\lambda, \mu) \in \mathcal{R}$, the (finite) set of extreme rays of the cone Γ defined by the linear inequality system:

$$\begin{cases} \mu_k \leq \sum\limits_{u\in\pi} \lambda_u, & \forall k,\ \forall \pi \in P_k \\ \lambda_u \geq 0, & \forall u \in U \\ \mu_k \geq 0, & \forall k = 1, \ldots, K \end{cases}$$

where, $\forall k$, P_k denotes the set of elementary $s(k) - t(k)$ paths in the graph.

Note that in the single-commodity case, condition (2) can be shown to reduce to Gale's theorem [9]; in this case, the extreme rays in $\mathcal{R}$ are exactly those $0 - 1$ vectors corresponding to the various cuts in the graph. Also we mention that, in the case of *undirected* networks, the set $\mathcal{R}$ of extreme rays corresponds to the extreme points of the so-called *semi-metric polytope* (see e.g. [16]).

In the context of a robust capacity expansion problem, where the demand vector $d = (d_1, d_2, \ldots, d_K)$ can be arbitrarily taken in a given uncertainty polytope $\mathcal{D}$, a necessary and sufficient condition for a given capacity assignment y to be robust feasible is therefore that:

$$\sum_{u\in U} \lambda_u y_u \geq \rho^*(\mu) = \operatorname*{Max}_{d\in\mathcal{D}} \left\{ \sum_{k=1}^{K} \mu_k d_k \right\} \tag{3}$$

for all $(\lambda, \mu) \in \mathcal{R}$.

In view of this, the robust minimum cost capacity expansion problem R-CEP-PU can be reformulated as the (large scale) linear program:

$$(\mathrm{LP}) \quad \begin{cases} \text{Minimize } \sum\limits_{u\in U} \gamma_u x_u \\ \text{subject to:} \\ \quad c + x = y \\ \quad \sum \lambda_u y_u \geq \rho^*(\mu) \quad \forall (\lambda, \mu) \in \mathcal{R} \\ \quad x \geq 0. \end{cases}$$

Now we introduce the following general condition (C):

$$(\mathrm{C}) \quad \begin{cases} \text{given any capacity assignment } y \in \mathbb{R}_+^U \text{, checking} \\ \text{whether (3) holds for any} (\lambda, \mu) \in \mathcal{R}\text{, or} \\ \text{exhibiting } \left(\bar{\lambda}, \bar{\mu}\right) \in \mathcal{R} \text{ such that} \\ \sum\limits_{u\in U} \bar{\lambda}_u y_u < \rho^*\left(\bar{\mu}\right) \\ \text{can be carried out in polynomial time.} \end{cases}$$

Then we can state:

Proposition 3. *Under condition* (C)*, the linear programming formulation* (LP) *of R-CEP-PU can be solved to optimality in polynomial time.*

Proof. Condition (C) amounts to assuming polynomial solvability of the *separation subproblem* for the polyhedron of robust feasible capacity assignments defined by (3). From the well-known result by Grötschel et al. [11] stating equivalence (w.r.t. complexity) between *separation* and *optimization*, we can conclude that when (C) holds, (LP) can be solved in polynomial time. □

We now show that condition (C) holds for a special subclass of R-SSCEP-KCU defined as follows:

The graph G under consideration is *planar*, there is a single-source node s, a set of sink nodes T $(s \notin T)$ and the graph G' deduced from G by adding an extra node t (representing a super-sink) and $|T|$ arcs of the form (j, t) for all $j \in T$ is *planar* too.

In addition to this, a single-commodity flow model is assumed, with knapsack-constrained uncertainty set for requirements defined as:

$$\mathcal{D} = \left\{ d \in \mathbb{R}_+^T / d_j = \bar{d}_j + \theta_j \delta, \forall j \in T; \theta_j \in [0, 1], \forall j \in T; \sum_{j\in T} \theta_j \leq \bar{\theta} \right\} \tag{4}$$

for given *integer* $\bar{\theta} \in]0, |T|[$. (thus, the difference between nominal and worst-case requirement is equal to the same constant $\delta > 0$ for all $j \in T$). The above special subclass of problems will be denoted R-SSCEP-KCU*.

Proposition 4. *R-SSCEP-KCU* is in P.*

Proof. First we observe that an equivalent instance of R-SSCEP-KCU* is obtained by replacing $\mathcal{D}$ with $\mathcal{D}'$, the polytope deduced from $\mathcal{D}$ by replacing in (4) the condition $\sum_{j\in T} \theta_j \leq \bar{\theta}$ with $\sum_{j\in T} \theta_j = \bar{\theta}$. So in what follows, it is not restrictive to assume that the uncertainty set is $\mathcal{D}'$. As a consequence, we note that, for all $d \in \mathcal{D}'$, the total demand $\sum_{j\in T} d_j$ is equal to a constant, namely $\bar{D} = \sum_{j\in T} \bar{d}_j + \bar{\theta}\delta$.

Now, it is well-known that, for any $d \in \mathcal{D}'$, the feasibility problem on G with arc capacities y_u $(u \in U)$ can be transformed into a maximum $s - t$ flow problem on the graph G', when each arc (j, t) is assigned an upper capacity bound equal to d_j. Then the problem is feasible if and only if the maximum $s - t$ flow value on G' is equal to $\sum_{j\in T} d_j$.

Now suppose that d is uncertain and belongs to the uncertainty set $\mathcal{D}'$. In the above transformation into a maximum $s - t$ flow problem on G', uncertain demands induce uncertain capacities on the various arcs (j, t), $(\forall j \in T)$. More precisely assign to each arc (j, t) a nominal capacity $\bar{d}_j + \delta$ and a reduced capacity $\bar{d}_j$, and consider the following uncertainty set for capacities:

$$\mathcal{C}' = \left\{ c_{jt} \in \mathbb{R}_+^T / c_{jt} = \bar{d}_j + \delta - \alpha_j\delta, \forall j \in T; \alpha_j \in [0, 1], \forall j \in T; \sum_{j\in T} \alpha_j = \bar{\alpha} = |T| - \bar{\theta} \right\}. \tag{5}$$

Then the robust feasibility problem for R-SSCEP-KCU* turns out to be reformulated as testing whether the maximum robust $s-t$ flow value on G' with uncertainty set $\mathcal{C}'$ for the capacities of the arcs (j, t) $(j \in T)$ is equal to $\bar{D} = \sum_{j\in T} \bar{d}_j + \bar{\theta}\delta$ (for more details on robust maximum flow under uncertain capacities see [22]). Since G' is planar, we can use the polynomial-time algorithm described in [22] to solve this problem in polynomial time $\mathcal{O}\left(\bar{\alpha} m' \log_2 m'\right)$ where $\bar{\alpha} = |T| - \bar{\theta}$ and $m' = |U| + |T|$ is the number of arcs of G'. Moreover, in case infeasibility is detected, the algorithm exhibits an $s-t$ cut in G' with worst-case capacity strictly less than $\bar{D}$, which corresponds to one of the extreme rays of the cone $\mathcal{R}$. So condition (C) is satisfied in this case and the polynomial-time solvability of R-SSCEP-KCU* follows. □

5. Conclusion

Robust minimum cost single-commodity or multicommodity network design problems under uncertain customer requirements are key to many practical applications. We have investigated here various complexity issues related to such problems, including a direct proof of NP-hardness, not only for the general case, but also for some fairly restricted cases (such as R-SSCEP-KCU, i.e. for single-source single-commodity models with knapsack-constrained uncertainty set). A new class of polynomially solvable instances on planar networks has also been exhibited. Clearly, identifying further polynomial special cases would be an interesting direction for future research. For more general (*NP*-hard) cases, strong MIP formulations, possibly leading to practically efficient Branch-and-Bound-based exact solution procedures, would certainly be worth investigating.

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