MATHEMATICS

CONNECTEDNESS AND CHAINABILITY IN METRIC SPACES

BY

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Let (X, d) be a metric space. An ε-chain from x to y is a finite sequence of points $x_1, \ldots, x_{n+1}$ such that $x_1 = x$, $x_{n+1} = y$ and $d(x_i, x_{i+1}) \leqslant \varepsilon$. If every pair of points in X is ε-chainable, then we define a new metric $d_\varepsilon(x, y) = \inf \{ \sum_{i=1}^{n} d(x_i, x_{i+1}) | x_1, \ldots, x_{n+1}$ is an ε-chain from x to $y\}$. A space is uniformly chainable if it is ε-chainable for each $\varepsilon > 0$ and there is a function $C\colon (0, \infty) \to (0, \infty)$ such that $\sup_{x \neq y} d_\varepsilon(x, y)/d(x, y) = C(\varepsilon)$.

Connectedness does not imply and is not implied by uniform chainability. However, for compact (X, d), X is ε-chainable for each $\varepsilon > 0$ if and only if it is connected. If $C(\varepsilon)$ is bounded, then $l(x, y) = \lim_{\varepsilon \to 0} d_\varepsilon(x, y)$ exists for all $x, y \in X$ and $l(x, y)$ is again a metric. Since $d(x, y) \leqslant l(x, y) \leqslant d(x, y) \cdot \sup C(\varepsilon)$, we have (X, d) is Lipschitz equivalent to (X, l). It is known [2] that (X, l) and hence (X, d) is connected and locally arcwise connected if (X, d) is complete.

In an attempt to relate connectedness and chainability, RAMER [1] states the following. If (X, d) is uniformly chainable and if $\liminf_{\varepsilon \to 0} \varepsilon C(\varepsilon) = 0$, then the metric completion $\tilde{X}$ of X is connected and locally arcwise connected. Below we give two counterexamples to this statement. Ramer's proof does give the following.

Theorem. Let (X, ϱ) be a uniformly chainable space. If there exists a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n \to 0$ and $\sum_{n=1}^{\infty} \varepsilon_n C(\varepsilon_{n+1}) < \infty$, then the metric completion $\tilde{X}$ of X is connected and locally arcwise connected.

Note. If $C(\varepsilon)$ is bounded, then Ramer's condition is satisfied.

Example 1. Let $X = \{(x, y) \in R^2 | x \geqslant 2$ and either $y = 0$ or $y = e^{-x^2}\}$. Set $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Of course, X is disconnected. To calculate $C(\varepsilon)$ for $\varepsilon < e^{-4}$, set $\varepsilon = e^{-t^2}$. One notices that $d_\varepsilon(P, Q)/d(P, Q)$ attains its maximum when $P = (x, 0)$ and $Q = (x, e^{-x^2})$ for some $t \geqslant x \geqslant 2$.

Notice that

$$d_\varepsilon((x, e^{-x^2}), (x, 0)) = e^{-x^2} + 2(t-x)$$

and

$$C(\varepsilon) = \max_{0 \leqslant x \leqslant t} 1 + 2(t-x)e^{x^2}.$$

Now $1+2(t-x)e^{x^2}$ attains its maximum at $x = (t+\sqrt{t^2-2})/2$. Thus

$$\varepsilon C(\varepsilon) \leqslant e^{-t^2} + t - \sqrt{t^2-2} \to 0 \text{ as } \varepsilon \to 0.$$

Example 2. Let $X = \{(\varrho, \theta) | \theta \geqslant 2$ and where $\varrho = 1$ or $\varrho = 1 - e^{-\theta^2}\}$. Define

$$d((\varrho_1, \theta_1), (\varrho_2, \theta_2)) = |\varrho_1 - \varrho_2| + \inf_{n \in Z} |\theta_1 - \theta_2 + 2\pi n|.$$

Calculations slightly more difficult than those above show that $\varepsilon C(\varepsilon) \to 0$ as $\varepsilon \to 0$. This space is neither locally connected nor arcwise connected, even though it is compact.

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REFERENCES

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2. Wilson, W. A. On rectifiability in metric spaces, Bull. Amer. Math. Soc. **38**, 419–426 (1932).