ELSEVIER

Journal of Pure and Applied Algebra 168 (2002) 407–423

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Relaxed multicategory structure of a global category of rings and modules

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Received 1 December 1999; accepted 4 May 2001

Abstract

In this paper we describe how to give a particular global category of rings and modules the structure of a relaxed multicategory, and we describe an algebra in this relaxed multicategory such that vertex algebras appear as such algebras. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 16B50; 17B69; 16D90; 16W30; 05C05

Our intention for this paper is to describe a method for giving a category of modules for a cocommutative Hopf algebra, the structure of a relaxed multicategory. Relaxed multicategories are a generalization of Lambek's multicategories [10], and were introduced by Borcherds in the paper [4], as the natural setting for vertex algebras. The idea was to give a category of modules for a Hopf algebra enough extra structure that vertex algebras would arise naturally as monoids.

It is worth mentioning here that although relaxed multicategories bear a strong resemblance to colored operads, they are nonetheless very different. Beilinson and Drinfeld have used multicategories/colored operads to investigate chiral algebras [2], and recently Soibelman and Kontsevich looked further into this approach [16], but it is fundamentally different from the relaxed multicategory treatment.

In the treatment that follows, we begin by reviewing the definition of a relaxed multicategory. We then define a global category of rings and modules and show that the forgetful functor to the category of rings defines a bifibration. Next, we describe

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PII: S0022-4049(01)00106-2

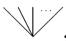
what types of singularities we will be working with, and we define binary singular multimaps. We then go on to define more general singular maps by using pushouts and pullbacks in our global category of rings and modules. Finally we show that suitable collections of these maps provide the structure of a relaxed multicategory, and we define an algebra in this setting. These algebras give a natural interpretation of the "locality" axiom of traditional vertex algebras, and they formalize the notion of operator product expansion.

1. Definition of a relaxed multicategory

In order to give the definition of a relaxed multicategory, we will need to work over a certain category of trees. The definition we take for our category of trees is due to Leinster [11,12] and seems to be a natural one arising from higher dimensional categorical considerations. Other categories of trees have been defined by Ginzburg and Kapranov [6], Manin [13] and Soibelman [15] which differ slightly from this definition in both their collections of objects and their allowed maps.

For each natural number, n, (including zero) we define, $T(n)$, the *set of n-leafed trees* by the recursion:

(1) For some formal symbol, $\bullet$, we have $\bullet \in T(1)$ and
(2) For natural numbers $n, k_1, \ldots, k_n$, and for $t_i \in T(k_i)$, we have $\langle t_1, \ldots, t_n \rangle \in T(k_1 + \cdots + k_n)$.

We may represent these trees pictorially as, for example, $\bullet = \bullet$, $\langle \bullet \rangle =$ , $\langle \bullet, \ldots, \bullet \rangle =$ , $\langle\langle \bullet, \bullet \rangle, \bullet \rangle =$. We have $\langle\ \rangle \in T(0)$, and we represent this empty tree by $\circ$. In $T(0)$ we also have trees of the form $\langle\langle\ \rangle, \langle\ \rangle\rangle$ which we consider to be trees with no leaves, and which are represented as .

In this pictorial representation of the trees, the bottom vertex (or node) is called the *root* of the tree, and among the other vertices, those which are joined to exactly one edge (excluding the root) are called the *external vertices* or the *leaves*. The remaining vertices are *internal vertices*. The *level of a vertex* is defined to be the number of edges separating that vertex and the root. If all the leaves of a tree have level one, then the tree is called *flat* or a *corolla*.

Trees compose by grafting a root to a specified leaf, and these compositions define associative maps $T(n) \times T(m) \to T(m + n - 1)$ for natural numbers n and m $(n > 0)$. For any two trees p, q with the same number of leaves, we say that p is a *refinement* of q if p arises after a succession (possibly zero) of the following moves:

(1) a vertex is replaced by an edge (i.e., any subtree t of a given tree can be replaced with $\langle t \rangle$),
(2) any proper subtree, $\langle t \rangle$, of a given tree is replaced with t (i.e., this mostly means that an edge which is not connected to a leaf can be shrunk down to a vertex).

We give $T(n)$ the structure of a category by defining a unique morphism $q \to p$ whenever p is a refinement of q.

Note 1. This is the same as the usual maps of trees, $Vert(T) \to Vert(T')$ respecting

$$Edge(T) \overset{s}{\underset{t}{\rightrightarrows}} Vert(T).$$

We are now ready to define a relaxed multicategory.

Definition 1. A *relaxed multicategory* consists of a collection of objects, $\mathscr{C}$, together with a collection of *multimaps* from $A_1, \ldots, A_n$ to B for any $n+1$ objects, $A_1, \ldots, A_n, B$ in $\mathscr{C}$, and any n-leafed tree, p. This collection is denoted

$$Multi_p(A_1, \ldots, A_n; B).$$

This data satisfies the following axioms:
(1) *Identities*: For each object $A \in \mathscr{C}$, there is a unique identity map $1_A \in Multi_\bullet(A; A)$.
(2) *Composition*: Inherited from the grafting of trees, given trees $p \in T(n)$ and $q \in T(m)$ $(m \geqslant 1)$ and objects $A_1, \ldots, A_n, B_1, \ldots B_m, C \in \mathscr{C}$, we have a map

$$Multi_q(B_1, \ldots, B_m; C) \otimes Multi_p(A_1, \ldots, A_n; B_i)$$
$$\to Multi_{q \circ_i p}(B_1, \ldots, A_1, \ldots, A_n, \ldots, B_m; C),$$

where $q \circ_i p$ is the tree arrived at by grafting the root of the tree p to the ith leaf of the tree q. This composition is associative and must agree with identities on objects.
(3) *Refinement*: Maps between trees, $p \to q$, induce maps between multimaps in the opposite direction,

$$Multi_q \to Multi_p$$

which are natural with respect to composition.

A relaxed multicategory has an underlying category whose morphisms are given by $Hom(A, B) = Multi_\bullet(A; B)$.

2. The global category of rings and modules

The category which will provide the natural setting for working with vertex algebras will be a the global category of rings and modules. The intuitive idea is that we want to use the machinery of limits and colimits to give a certain category of modules some extra multicategory structure. We will need to complicate the situation slightly by giving our rings and modules the structure of modules for a cocommutative Hopf algebra.

Fix a commutative ring R with unit and take *R-Mod* to be the symmetric monoidal closed category of R-modules. We then take H to be a cocommutative Hopf algebra object in R-modules. Recall that a Hopf algebra is a module, H, over a commutative ring, R (with unit), that has both the structure of an algebra and a coalgebra where the algebra and coalgebra maps are compatible with one another (i.e. the maps giving H

the structure of an algebra are maps of coalgebras, and vice versa). A Hopf algebra also possesses a multiplication and comultiplication reversing bialgebra map, $S : H^{\mathrm{op}} \to H$, called antipode (see [1]).

Since H is a monoid, we can form the category, H-*Mod* $= [H, R\text{-}Mod]$, of H-modules. This category has tensor products inherited from R-modules, and the cocommutativity and coassociativity of H give H-*Mod* the structure of an enriched symmetric monoidal category (with unit R). It can be easily checked that the closed structure of the category of R-modules carries over to H-*Mod*. Because the category of R-modules is complete and cocomplete, it follows from some basic results of enriched category theory (see [9]) that the enriched presheaf category H-*Mod* is complete and cocomplete, and the limits and colimits are computed pointwise.

We now want to consider the global category, $\mathbb{E}$, of rings and modules built up from H-*Mod*. It has as objects, pairs (L, A) where A is a monoid in H-*Mod*, and L is an A-module object (because H-*Mod* is an abelian category, we may refer to the monoid A as a ring). The morphisms in this global category consist of pairs,

$$(\alpha, f) : (L, A) \to (M, B),$$

where $f : A \to B$ is a ring map and $\alpha : L \to f^*M$ is a map of A-modules. Note that the ring map, f, induces a bijection between the A-module maps and B-module maps:

$$\frac{L \to f^*M}{B \otimes_A L \to M}.$$

Recall that a functor $\pi : \mathbb{E} \to \mathbb{B}$ is said to be a *fibration* if for every map $f : A \to B$ in the base $\mathbb{B}$, and every object N in $\mathbb{E}$ which maps down to B, we get a (cartesian) lift of f to $\mathbb{E}$, $f' : M \to N$, such that given any map $k : M' \to N$ where M' is in the fiber over A, there exists a unique map from M' to M making the triangle commute, and which maps down to the identity morphism on A in $\mathbb{B}$. The composites of these (cartesian) liftings are also a (cartesian) lifting.

$$\begin{tikzcd} \mathbb{E} \arrow[ddd, "\pi"] & M \arrow[r, dashed, "f'"] & N \\ & M' \arrow[u, dashed] \arrow[ur, "k"'] & \\ & & \\ \mathbb{B} & A \arrow[r, "f"] & B \end{tikzcd}$$

A *cofibration* is defined dually. For more information see [3].

Lemma 2. *The functor $\pi : \mathbb{E} \to \mathbb{B}$, from the global category of rings and modules to the category of monoids (given by $\pi(M, A) = A$) is both a fibration and a cofibration. This is often called a* bifibration.

Proof. We see that the category of rings and modules is a fibration because given any map of rings, $f : A \to B$, and any B-module, M, the A-module, f^*M, gives us the lift of f to $\mathbb{E}$:

$$f^*:(f^*M,A)\to(M,B).$$

All maps to (M,B) which project down to $f:A\to B$ will be of the form $(\alpha,f):(N,A)\to(M,B)$, where $\alpha:N\to f^*M$ is just an A-module map, and hence α just gets mapped to the identity on A. The proof that this category is also a cofibration goes through similarly using the adjoint characterization of maps in $\mathbb{E}$. $\square$

Because we have a bifibration, we may deduce that the global category, $\mathbb{E}$, is complete and cocomplete if both the base category is complete and cocomplete, and each of the fibers is complete and cocomplete. But both the base and the fibers are algebras for a monad, and hence are complete and cocomplete. Note also that by construction, we have a forgetful functor from $\mathbb{E}$ to H-*Mod*.

3. Maps with singularities

3.1. Binary singular maps

Now that we have the setting of this global category of rings and modules, we are ready to use its complete and cocomplete structure to form a relaxed multicategory. We begin by making precise the notion of singularity we will be using.

Definition 3. An *elementary vertex structure* associated to a (cocommutative) Hopf algebra H is defined to be an R-module, K, which has the structure of a commutative algebra over H^* as well as that of a 2-sided H-module. We require the natural map from H^* to K to be an H-module morphism. The algebra structure of K is H-linear, and the antipode defined on H^* extends to a map from K^{op} to K.

This definition is due to Borcherds, and can be found together with a number of examples in [4, Definition 3.2]. Intuitively we think of K as some collection of singular maps defined on H. The following example motivates the treatment which follows.

Example 4. Take $H=R[D^{(0)},D^{(1)},D^{(2)},\dots]$ to be a module over a commutative ring, R, with unit. We give it the structure of a monoid by defining multiplication

$$D^{(i)}D^{(j)}=\binom{i+j}{i}D^{(i+j)}$$

and unit $D^{(0)}$, and we make it into a Hopf algebra by defining comultiplication $\Delta D^{(i)}=\sum_{p+q=i}D^{(p)}\otimes D^{(q)}$, counit $\varepsilon(D^{(i)})=\delta_{i,0}$, and antipode $S(D^{(i)})=(-1)^iD^{(i)}$. Then $H^*\cong R[\![x]\!]$ and we can take $K=R[\![x]\!][x^{-1}]$ as our elementary vertex structure. Then for all $j\in\mathbb{Z}$ we have $D^{(i)}x^j=\binom{j}{i}D^{(j-i)}$ and $S(x^j)=(-1)^jx^j$.

For any H-module, D, the collection of linear maps $Hom_R(H^{\otimes 2}, D)$ has the structure of an $H^{\otimes 2^*}$ module just as H^* has the structure of an H^*-module. Using the dual of the map

$$H \otimes H \xrightarrow{f} H,$$

$$h_1 \otimes h_2 \mapsto h_1 S(h_2),$$

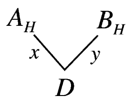
we consider K as an $H^{\otimes 2^*}$-module, and we can tensor over the dual, f^*, to form the module, $Hom_R(H^{\otimes 2}, D) \otimes_{f^*} K$. Throughout the rest of this paper, all tensors with K will be over f^*, so we will leave them from the subscript. This is an $H^{\otimes 2}$ module, and so for H-modules A and B, we can consider the collection of $H^{\otimes 2}$-linear maps

$$A \otimes B \to Hom_R(H^{\otimes 2}, D) \otimes_{f^*} K. \tag{1}$$

which we call the *singular maps* from $A \otimes B$ to D.

Note 2. We are interested in this collection of maps because a linear map from $A \otimes B$ to D can be regarded as an $H^{\otimes 2}$-linear map from $A \otimes B$ to $Hom_R(H^{\otimes 2}, D)$. Hence we have just "added singularities" to linear maps in a way that depends on the underlying Hopf algebra.

In order to simplify the notation we will be using to describe these singular maps, we use labelled trees. The collection of singular maps from $A \otimes B$ to D in Eq. (1) will be denoted by either of the following:

$Multi_{\vee}(A_H, B_H; D) =$ 

On the right-hand side, the leaves of the tree are labelled by the domain of our singular maps and the root is labelled by the codomain of the singular maps. The singularity can be thought of as appearing at the root, and depending on the inputs above. The edges are labelled with dummy variables which act as placeholders for the two copies of H. We use two distinct dummy variables in order to be able to distinguish the different actions of H. The $H^{\otimes 2}$ linearity of our maps is designated by the subscripts on the A and B, and with example 4 in mind, we could emphasize this linearity by saying $\partial_A = \partial_x$ and $\partial_B = \partial_y$.

Note 3. Notice that the symmetry of the tensor product implies that the given tree is isomorphic to its vertical reflection (in terms of the functions they represent), and $Multi_{\vee}(A, B; D) \cong Multi_{\vee}(B, A; D)$.

This collection of maps has a number of associated structures which we will use for working in the global category of rings and modules. Firstly, there is the collection of *nonsingular maps*, $Hom_{H^{\otimes 2}}(A \otimes B, Hom_R(H^{\otimes 2}, D))$ sitting inside the collection of singular maps. Secondly, we can remove the requirement that the singular and non-singular maps be $H^{\otimes 2}$-invariant, giving *proto-singular* and *proto-nonsingular* maps from $A \otimes B$ to D. And finally, these proto-singular and proto-nonsingular maps are modules for the rings $Hom_R(H^{\otimes 2}, R) \otimes K$ and $Hom_R(H^{\otimes 2}, R)$, respectively. These are called the *associated rings*.

All of the collections given are H-modules, and so we could consider the corresponding H-invariant modules. We sum this up by pointing out that the collection of proto-singular maps,

$$Hom_R(A \otimes B, Hom_R(H^{\otimes 2}, D) \otimes K),$$

has an action of H at A, B, and D, and what we have been calling the singular maps are just those proto-singular maps which are invariant under the action of H at both A and B. Similarly, the H-invariant singular maps are just those maps invariant under the action of H at A, B, and D. Using the notation from above to emphasize this H-action, the proto-singular maps are denoted $Multi_{\vee}(A, B; D)$, where we have removed the H subscript from the A and B as expected.

Note 4. Because we will be working in a category of rings and modules, we focus on the proto-singular and proto-nonsingular maps, since they are modules for their associated rings. We recover our original singular maps by taking sufficiently H-invariant subcollections.

We now consider composing two proto-singular maps. Since our motivation was provided by ordinary multilinear composition, we would like our treatment to reduce to ordinary linear composition when $K = H^*$. It is easy to check that this means requiring the proto-singular maps to be H-invariant at the point of composition. So we compose and element of $Multi_{\vee}(A_1, A_2; B_{1H})$ with an element of $Multi_{\vee}(B_{1H}, B_2; D)$:

$$A_1 \otimes A_2 \to Hom_H(H^{\otimes 2}, B_1) \otimes K,$$

$$B_1 \otimes B_2 \to Hom_R(H^{\otimes 2}, D) \otimes K,$$

gives an element of

$$Hom_R(A_1 \otimes A_2, Hom_{H \otimes R}(H^{\otimes 2} \otimes B_2, Hom_R(H \otimes H, D) \otimes K) \otimes K), \tag{2}$$

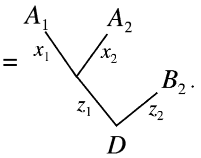
where the inner subscript, $H \otimes R$, emphasizes that the only H-linearity is between the copy of $H^{\otimes 2}$ tensored with B_2 and the first of the innermost H's. The associated ring is

$$Hom_{H \otimes R}(H^{\otimes 2} \otimes H, Hom_R(H \otimes H, R) \otimes K) \otimes K.$$

Notice that in Eq. (2), we have taken special care to emphasize that one of the singularities depends only on inputs A_1 and A_2, while the other singularity depends on all inputs. This dependence of singularities on inputs will be important for the relaxed multicategory structure we are constructing. The collection of proto-singular functions in Eq. (2) also has a corresponding collection of nonsingular maps, and together with its ring, defines an inclusion map in the global category.

The collection of proto-singular maps given in Eq. (2) will be denoted

$$Multi_{\vee}(A_1, A_2, B_2; D) =$$



The advantage of this notation is that we can see where we have actions of H. As before we have it between our three inputs, and their corresponding copies of H (marked with dummy variables x_1, x_2 and z_2). The requirement that the maps from $A_1 \otimes A_2$ to B_1 be H-invariant at B_1 means that the action of H at z_1 passes through to an action on $H^{\otimes 2}$ at x_1 and x_2. We use the following suggestive notation to denote this linearity: $\partial_{x_1} + \partial_{x_2} = \partial_{z_1}$. Isomorphic collections of maps appear if we reflect the tree at the internal nodes.

From the discussion of composition and labelled tree notation, it is clear that by taking suitably H-invariant proto-singular maps we could compose another binary proto-singular map at either A_1 or A_2. Repeating this process, we see directly how to build up a tower of these proto-singular maps with only one internal node at each level. But we should also be able to compose at B_2. Composing a proto-singular map, $Multi_{\vee}(A_3, A_4, B_{2H})$, say, with a map from $Multi_{\vee}(A_1, A_2, B_{2H}; D)$, we end up with an element of

$$Hom(A_3 \otimes A_4, K \otimes Hom(A_1 \otimes A_2, K \otimes Hom(H^{\otimes 4}, K \otimes Hom(H^{\otimes 2}, D)))). \tag{3}$$

(We have removed the subscripts denoting H-linearity in order to focus the discussion on the singularities. We return to the question of linearity at the end of this section.) But this collection contains maps which do not appear as composites of binary proto-singular functions, because there is no singularity here which depends only on inputs A_1 and A_2. Reversing the order of composition, the map we end up with is an element of

$$Hom(A_1 \otimes A_2, K \otimes Hom(A_3 \otimes A_4, K \otimes Hom(H^{\otimes 4}, K \otimes Hom(H^{\otimes 2}, D)))). \tag{4}$$

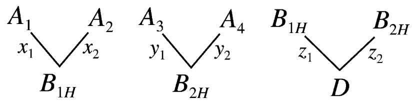
These two collections of maps differ only in their dependency of singularities on inputs. In particular, the corresponding collection of nonsingular functions is the same, and so we have inclusion maps in the global category from the collection of nonsingular functions to both of (3) and (4).

Note 5. We also see that the associated rings are isomorphic and are given by

$$K^{\otimes 2} \otimes Hom(H^{\otimes 4}, K \otimes Hom(H^{\otimes 2}, R)), \tag{5}$$

where the first of the outer two copies of K is tensored over f^* with the first two copies of H in $H^{\otimes 4}$, and similarly for the second copy.

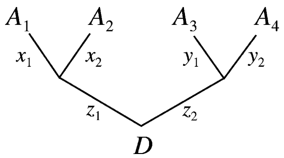
In order to give an exact description of the collection of composites of these three proto-singular maps,



we first take the pushout in the global category of (3) and (4) along the maps including nonsingular functions, and then we take the pullback over that pushout. Explicitly, let T denote the ring in Eq. (5). Then in the the fiber of T-modules, we are taking the pullback of the pushout of the inclusion of

$$T \otimes_S Hom(A_1 \otimes A_2 \otimes A_3 \otimes A_4, Hom(H^{\otimes 4}, Hom(H^{\otimes 2}, D)))$$

in (3) and (4), where S is the ring associated to the nonsingular functions. We denote this collection by the following labelled tree:



(6)

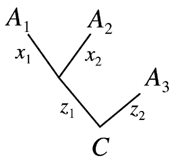
Throughout this treatment, we have deliberately postponed the discussion of any H-linearity. With this labelled tree, it becomes much simpler to see the actions of our Hopf algebra. As usual, we have H-actions between the inputs and the corresponding copies of H. The H-invariance at B_1 and B_2 adds a further $H^{\otimes 2}$-invariance which we denote $\partial_{x_1} + \partial_{x_2} = \partial_{z_1}$ and $\partial_{y_1} + \partial_{y_2} = \partial_{z_2}$ as above.

3.2. Multimaps parameterized by binary trees

Using the fact that we can represent our proto-singular maps as labelled trees, we may describe all possible composites of the binary proto-singular maps by defining proto-singular maps for each binary tree. The general situation will be similar to the situation for the tree in (6). What will follow will be an algorithmic procedure for describing the proto-singular maps associated to any binary labelled tree.

For an arbitrary binary tree, we consider its collection of internal vertices. We will assume that these include the root, but they do not include the leaves. Considering them as a set, this set inherits a partial order from the tree, where the root is the least element. We know that any partial order can be extended to at least one total ordering, possibly many.

Up to this point, our trees have been labeled with H-modules at their leaves and root. It will be useful for the explanation which follows to assume that every internal node is also labelled. For any internal node, q, connected to n *incoming* nodes (i.e., nonempty nodes whose height is equal to the height of q plus one and connected to q by a single edge), we will label q by $H^{\otimes n}$. We can also associate to q the tensor product of the labels of the incoming nodes, and denote it X_q. Thus, the following labelled tree has two internal nodes,



and we have $X_{\mathrm{root}} = H^{\otimes 2} \otimes A_3$, and $X_{\mathrm{internal}} = A_1 \otimes A_2$.

We put our tree in *augmented* form by adding an additional edge and vertex to the root of our tree. The root of this new tree is labelled by the output of the original tree, and the original root is labelled, just as any internal node, by its inputs. For binary trees, the original root is labelled $H^{\otimes 2}$ because it had two incoming nodes. We denote the new vertex $\perp$, and we therefore have $X_\perp = H^{\otimes 2}$.

Definition 5. Let p be a binary tree with n leaves, and let t denote a total ordering, $\perp < \mathrm{root} < p_1 < \cdots < p_l$, of the internal vertices of augmented p, compatible with the the partial ordering inherited from the tree structure of p. We define an operator on $H^{\otimes 2}$-modules:

$$Sing_{p_i} = Hom(X_{p_i}, K \otimes \cdot). \tag{7}$$

Iterating this operator and imposing H-linearity at all internal nodes (but not the output node), we have

$$Ord_t(A_1, \ldots, A_n; C) = Sing_{p_l} \cdots Sing_{p_1} Sing_{\mathrm{root}} Hom(X_\perp, C). \tag{8}$$

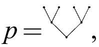
Notice that for all total orderings, t, the collections of nonsingular functions associated to the Ord_t are isomorphic, so we are led to the following definition of $Multi_p$.

Definition 6. $Multi_p(A_1, \ldots, A_n; C)$ is defined to be the pullback of the pushout of each Ord_t for all possible total orderings, t, of the internal vertices of augmented p, over the nonsingular functions in the global category of rings and modules.

Note 6. As above, the symmetry of the tensor product implies that vertical reflection of labelled (sub) trees induces isomorphisms of multimaps.

Example 7. If p is a tree with only one binary vertex at each level, then there is only one total ordering, t, of internal vertices of the tree, and so

$$Multi_p(A_1, \ldots, A_n; C) = Ord_t(A_1, \ldots, A_n; C).$$

Example 8. When $p =$ , there are exactly two total orderings of internal vertices of this tree, and the corresponding $Ord_t(A_1, \ldots, A_4, C)$ functions are given by

$$Ord_{t_1} = Hom(A_1 \otimes A_2, K \otimes Hom(A_3 \otimes A_4, K \otimes Hom(H^{\otimes 4}, K \otimes Hom(H^{\otimes 2}, C))))$$

$$Ord_{t_2} = Hom(A_3 \otimes A_4, K \otimes Hom(A_1 \otimes A_2, K \otimes Hom(H^{\otimes 4}, K \otimes Hom(H^{\otimes 2}, C)))),$$

as in Eqs. (3) and (4). The pullback is exactly the one described explicitly in the previous section.

We finish this section with a proof that composition holds for binary trees. In order to give this proof, we first need a lemma about evaluation in symmetric monoidal categories.

Lemma 9. *In any symmetric monoidal category, $\mathscr{C}$, the following diagram commutes*:

$$\begin{tikzcd}
Hom(A_1, B_1 \otimes C_1) \otimes Hom(A_2, B_2 \otimes C_2) \arrow[r] \arrow[d] & Hom(A_1, C_1 \otimes Hom(A_2, B_1 \otimes B_2 \otimes C_2)) \arrow[d] \\
Hom(A_2, C_2 \otimes Hom(A_1, B_1 \otimes C_1 \otimes B_2)) \arrow[r] & Hom(A_1 \otimes A_2, B_1 \otimes C_1 \otimes B_2 \otimes C_2).
\end{tikzcd}$$

Proof. The proof follows immediately from the fact that the evaluation of

$$Hom(A_1, B_1 \otimes C_1) \otimes Hom(A_2, B_2 \otimes C_2)$$

on $A_1 \otimes A_2$ gives the same result when carried out by either first evaluating A_1, or by first evaluating A_2 or by evaluating both together. $\square$

Proposition 10. *There exists an associative composition map*

$$Multi_q(B_{1H}, \ldots, B_m; C) \otimes Multi_p(A_1, \ldots, A_n;\ B_{1H})$$

$$\to Multi_{q \circ p}(A_1, \ldots, A_n, B_1, \ldots, B_m; C)$$

for all H-modules A_i, B_j, C and all binary trees p, q.

Proof. Given two proto-singular maps, $f \in Multi_p(A_1, \ldots, A_n;\ B_{1H})$ and $g \in Multi_q(B_{1H}, \ldots, B_m; C)$ we prove that they compose to give an element of $Multi_{q \circ p}(A_1, \ldots, A_n, B_1, \ldots, B_m; C)$. We know that $Multi_{q \circ p}$ is defined to be a pullback over objects of the form $Ord_t(A_1, \ldots, A_n, B_2, \ldots, B_m; C)$, so we first show that f and g compose to give an element of any such Ord_t. This follows from the fact that the linear ordering, t, of the internal nodes of $q \circ p$, induces linear orderings on the internal nodes of p and q. Denote these linear orderings t_p and t_q. Regarding f and g as elements of Ord_{t_p} and Ord_{t_q}, we know that $f \circ g$ is an element of Ord_t because of the H-invariance at B_1. By Lemma 9, we know that each image of $f \circ g$ in Ord_t gets mapped to the same point in the pushout, and hence they compose to give an element of $Multi_{q \circ p}(A_1, \ldots, A_n, B_1, \ldots, B_m; C)$ as desired. Associativity of this map is clear. $\square$

3.3. Nonbranching trees

In order to build up our relaxed multicategory, we want to extend our definition of multimaps to more general trees. We begin by expanding Definition 5 to allow for nonbranching trees:

Definition 11. If the tree p in Definition 5 is allowed to also have nonbranching subtrees, then $Multi_p(A_1,\dots,A_n;C)$ is defined exactly as in that definition except that when an internal vertex, p_i has only one incoming edge, we define an operator to act on H-modules,

$$Sing_{p_i} = Hom_R(X_{p_i},\cdot),$$

where X_{p_i} is the label of the incoming node as in Section 3.2.

The first and most obvious such tree consists of just a root, $\bullet$. Composing with such a tree leaves the tree unchanged. So we hope that this definition defines $Multi_\bullet$ so that it composes with a proto-singular map of type p (for some tree, p) to give a proto-singular map of type p. Indeed, labelling the tree $\bullet$ with input and output H-modules, the definition gives

$$Multi_\bullet(A;B) = Hom_R(A,B).$$

Thus we have included all maps from the underlying category. If we apply Definition 11 to the augmented version of the tree, $\bullet$, we find

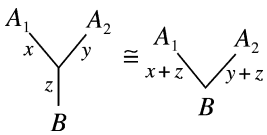
$$Multi_\bullet(A;B) = Hom_H(A, Hom_R(H,B)),$$

and this process can be repeated for any non-branching tree. Note that because of the internal H-invariance, we have $Multi_{\overset{\circ}{\bullet}}(A;B) \cong Multi_\bullet(A;B)$.

Example 12. Consider the proto-singular maps associated to the tree, Y. From the definition we have

$$\begin{aligned} Multi_{Y}(A_1,A_2;B) &= Hom_R(A_1\otimes A_2, K\otimes Hom_H(H^{\otimes 2}, Hom(H,B))) \\ &\cong Hom_R(A_1\otimes A_2, K\otimes Hom(H^{\otimes 2},B)) \\ &= Multi_{\vee}(A_1,A_2;B). \end{aligned}$$

When H is the classical vertex group this says that there is a bijection between the collection of proto-singular multimaps associated to the following trees:



(9)

This isomorphism follows immediately from the H-invariance at the internal node, where it provides the relation $\partial_x + \partial_y = \partial_z$.

Now that we have a description of unary proto-singular maps, it is natural to consider the nullary type multimaps. Applying the algorithmic definition for associating proto-singular maps to trees, we first augment the empty tree, giving $\overset{\circ}{\underset{\bullet}{|}}$. As with the tree $\bullet$, the only internal node is $\perp$, and we take $X_\perp = R$ since we do not consider the empty node as a leaf. This gives:

Definition 13. For any H-module, A, the proto-singular multimaps parameterized by the empty tree are given by

$$Multi_\circ(R;A) = Hom_R(R,A) \cong A.$$

What happens when we compose an element of this collection with a binary proto-singular multimap?

Lemma 14. *We have a composition map*

$$Multi_{H,\circ}(R;A_{1H}) \otimes Multi_{\vee}(A_{1H},A_2;B) \to Multi_{\overset{\circ}{\underset{\bullet}{|}}}(A_2;B). \tag{10}$$

In fact, we have such a composition map for composition with any $Multi_p$.

Proof. Given any binary proto-singular map $f \in Multi_{\vee}(A_{1H},A_2;B)$ and an element $a \in H\text{-inv }(A_1) = Multi_\circ(R;A_{1H})$, we have a map

$$f(a \otimes \cdot)\colon A_2 \to K \otimes Hom_R(H^{\otimes 2},B),$$

such that $\varepsilon(h)f(a \otimes \cdot) = f(ha \otimes \cdot)$. But f is H-linear at A_1, so the map f must factor through

$$f(a \otimes \cdot)\colon A_2 \to Hom_R(H,B)$$

and so we have an element of $Multi_{\overset{\circ}{\underset{\bullet}{|}}}(A_2;B)$.

3.4. Ternary maps and beyond

We would now like to define proto-singular maps associated to more general trees. With the goal of forming a relaxed multicategory, we would like to give a definition for $Multi_{\vee\!\!\!\vee}(A_1,A_2,A_3;B)$ together with maps to each of the multimaps $Multi_{\vee\!\!\!/}(A_1,A_2,A_3;B)$, $Multi_{\vee\!\!\!/}(A_2,A_3,A_1;B)$ and $Multi_{\vee\!\!\!/}(A_3,A_1,A_2;B)$. We know each of these three modules has the same associated collection of nonsingular functions, together with inclusion maps into each of them, so we can pullback the pushout of these three objects over the nonsingular functions in the global category. This gives us an object which we take as $Multi_{\vee\!\!\!\vee}(A_1,A_2,A_3;B)$, together with the desired maps. More generally, we have the following definition:

Definition 15. For H-modules, $A_1,\dots,A_n,B$, the collection of proto-singular maps associated to the flat tree with n leaves, $\vee\!\!\!\vee\!\!\!\vee$, is denoted $Multi^{\vee\!\!\!\vee\!\!\!\vee}(A_1,\dots,A_n;B)$, and is defined by first taking the pushout of

$$Ord_t(A_{\sigma(1)},\ldots,A_{\sigma(n)};B)$$

for all permutations, σ, and for each total ordering, t, of the internal vertices of binary trees, p. with n leaves, height $n-1$ or less, and no nonbranching nodes, over the corresponding collection of nonsingular functions

$$Hom(A_1 \otimes \cdots \otimes A_n, Hom(H^{\otimes 2}, \cdots Hom(H^{\otimes 2}, B)\cdots))$$

and then pulling back over this pushout.

The idea of this definition is that we take all possible (nontrivial) collections of proto-singular maps associated to trees which refine to the flat n-leafed tree, we take the pushout in order to patch the singularities along the nonsingular maps, and we pullback to give an "intersection" of the the modules of singularities. In fact, this definition suggests a general definition for proto-singular maps associated to arbitrary trees which generalizes Definitions 5, 11 and 15.

Note 7. Here we again have isomorphisms of multimaps induced by permutation of input labels.

Definition 16. For H-modules, $A_1,\ldots,A_n,B$, the collection of proto-singular maps associated to an arbitrary tree, q with n leaves, $Multi_q(A_1,\ldots,A_n;B)$ is as in Definition 15 except that we pushout and pullback only those Ord which can be mapped to from $Multi_q(A_1,\ldots,A_n;B)$ by the refinement maps of the relaxed multicategory.

Note 8. We have symmetries of these general multimaps induced by the symmetries of the subtrees.

4. Relaxed multicategory structure

Now that we know how to define $Multi_p(A_1,\ldots,A_n;B)$ for objects A_i,B, and each n-leafed tree p, we have a relaxed multicategory by taking the fully H-invariant elements of each collection. In other words, the multimaps are $Multi_p(A_{1H},\ldots,A_{nH};B_H)$.

In order to prove that we have defined a relaxed multicategory, we need to check that we have satisfied the axioms given in Definition 1. We have satisfied the identity and naturality axioms by drawing on the underlying categorical structure of H-Mod, so we only need to show that composition and refinement axioms are satisfied. We sketch the proof here.

Theorem 17. *There exists an associative composition map*

$$Multi_q(B_{1H},\ldots,B_{mH};C_H)\otimes Multi_p(A_{1H},\ldots,A_{nH};B_{1H})$$
$$\to Multi_{q\circ p}(A_{1H},\ldots,A_{nH},B_{1H},\ldots,B_{mH};C_H)$$

for all H-modules A_i,B_j,C and all trees p,q.

Note 9. Keep in mind that we are composing the trees p and q and not the augmented trees. We only use augmented trees for the purpose of describing their associated proto-singular multimaps.

Proof. We defined $Multi_{q\circ p}(A_1,\dots,A_n,B_2\dots,B_m;C)$ to be the pullback of all collections of multimaps associated to trees which refine to $q\circ p$. So choosing an arbitrary such tree we have refinements of both p and q which map to the corresponding subtrees. Thus we are left with showing that binary trees compose appropriately, which we saw in Proposition 10. It takes a little work to see that each of these composites is mapped to the same element of the pushout, but is a straightforward calculation. So we see that the proto-singular maps do compose to give an element of $Multi_{p\circ q}(A_{1H},\dots,A_{nH},B_{2H}\dots,B_{mH};C_H)$.

From the construction of $Multi_p$, we already have nearly all our refinement maps. The only refinement maps we excluded were those which mapped to trees with non-branching internal vertices. By suitable composition with the following refinement map, we have all the required refinement maps.

Definition 18. A *refinement for a singularity* is the map,

$$K \to Hom_H(H,K)$$

which takes any $k\in K$ to the map $f\in Hom_H(H,K)$ defined by $f(g)=g\cdot k$ for any $g\in H$. For any other H-module, A, and H-invariant map $\alpha\colon A\to H$, we define a refinement for K by composition

$$K \to Hom_H(H,K)\xrightarrow{\alpha} Hom_H(A,K).$$

5. Algebra in the relaxed multicategory

Definition 19. An (*associative*) *algebra* in a relaxed multicategory, $\mathscr{B}$, consists of an object $B\in\mathscr{B}$ and a collection of maps

$$\{f_p\}=\{f_p\in Multi_{H,p}(B,\dots,B,B)\,p\in T(n),\ n\in\mathbb{N}\}.$$

These maps must satisfy the following axioms:

(1) *Composition*: If $q\circ(p_1,\dots,p_n)$ is the tree formed by gluing the root of each tree p_i to the ith external edge of an n leafed tree, q, (p_i possibly empty), then

$$f_{q\circ(p_1,\dots,p_n)}=f_q\circ(f_{p_1},\dots,f_{p_n}). \tag{11}$$

(2) *Unit*: The map $f_\circ : R\to B$ (where $\circ$ is the empty tree) defines a unit for the algebra in the sense that for any n leafed tree, p, and any $1\leqslant k\leqslant n$,

$$f_p\circ_k f_\circ = f_{p'}$$

where $\circ_k$ denotes composition at the kth leaf of p, and p' is the $n-1$ leaved tree arrived at by removing the kth leaf from p.

(3) *Refinement*: If $p, q \in T(n)$ and p is a refinement of q, then $r_{p,q}(f_p) = f_q$ where $r_{p,q}$ is the refinement map given by the refinement axiom for a relaxed multicategory.

This is an algebra in the sense that each map f_p defines an "n-fold multiplication" for elements of B. For all $n \in \mathbb{N}$ we denote the multimap associated to the flat tree with n leaves by f_n. Since composition of multimaps in $\mathscr{B}$ is associative, the associativity of $(B, \{f_p\})$ is a consequence of the composition axiom. Considering $\bullet$, the 1 leafed tree with zero edges, then since $f_p \circ_k f_\bullet = f_p$ and $f_\bullet \circ f_p = f_p$, we see that $f_\bullet = 1_B$. The algebra defined by $(B, \{f_p\})$ is said to be *commutative* if the multimaps, $\{f_p\}$ are invariant under an appropriate action of the symmetric group. This notion makes sense because our relaxed multicategory is symmetric.

Note 10. This definition of an algebra is just a functor from the opposite of the category of trees to $\mathscr{B}$ where each object $p \in \mathscr{T}$ is mapped to an element of $Multi_p$.

Traditional vertex algebras, as found in the literature (e.g., [5,7,8]), arise as exactly algebras for the Hopf algebra and elementary vertex structure defined in Example 4, over the ring $\mathbb{C}$. The "locality" axiom is summed up by the refinement map $f \to f$, the vacuum is given by $f_\circ$, and operator product expansions can be deduced from f. For more details about the relation to these axiomatic vertex algebras see [14].

Acknowledgements

I would like to thank Martin Hyland for his help and guidance. I would also like to thank the conference organizers for a very interesting and beautifully run conference.

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