Existence and Uniqueness of a Strong Solution to Stochastic Differential Equations in the Plane with Stochastic Boundary Process

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Communicated by the Editors

Let $B$ be a 2-parameter Brownian motion on $\mathbb{R}_+^2$. Consider the non-Markovian stochastic differential equation in the plane

$$dX_z = a(z, X) dB(z) + \beta(z, X) dz$$

for $z \in \mathbb{R}_+^2$, i.e., $X_{s,t} = X_{s,0} + \int_{s}^{t} a(\zeta, X) dB(\zeta) + \int_{s}^{t} \beta(\zeta, X) d\zeta$ for $z \in \mathbb{R}_+^2$, where $R_z = [0, s] \times [0, t]$ for $z = (s, t) \in \mathbb{R}_+^2$. It is shown in this paper that a unique strong solution to the stochastic differential equation exists if and only if (1) for every probability measure $\mu$ on the space $\partial W$ of continuous real-valued functions on $\partial \mathbb{R}_+^2$, there exists a solution $(X, B)$ of the stochastic differential equation on some filtered probability space with $\mu$ as the probability distribution of $\partial X$, and (II) the pathwise uniqueness of solutions of the stochastic differential equation holds.


1. INTRODUCTION

The object of our study is a stochastic differential equation of non-Markovian type in the plane

$$dX_z = a(z, X) dB_z + \beta(z, X) dz,$$

Received August 20, 1987, revised November 9, 1987.

AMS 1980 subject classifications: primary 60H10, 60H20.

Key words and phrases: stochastic differential equations, strong solutions, uniqueness in probability distribution, pathwise uniqueness, regular conditional probability.

* Visiting the University of California, Irvine; partially supported by the CIRIT de la Generalitat de Catalunya.

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i.e.,

$$X_{s,t} - X_{0,t} - X_{s,0} + X_{0,0} = \int_{R_z} \alpha(\zeta, X) \, dB_\zeta + \int_{R_z} \beta(\zeta, X) \, d\zeta$$

(1.1)

for $z = (s, t) \in \mathbb{R}_+^2$ and $R_z = [0, s] \times [0, t]$, where $B$ is an \{ $\mathcal{F}_t$ \} Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_z)$ with $\partial B = 0$, $\partial B$ being the restriction of $B$ to the boundary $\partial \mathbb{R}_+^2$ of $\mathbb{R}_+^2$. Throughout this paper by a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_z)$ we mean a probability space $(\Omega, \mathcal{F}, P)$ with an increasing and right continuous family \{ $\mathcal{F}_z, z \in \mathbb{R}_+^2$ \} of sub-$\sigma$-fields of $\mathcal{F}$ each of which contains all the null sets in $(\Omega, \mathcal{F}, P)$. Let $\mathcal{B}(W)$ be the $\sigma$-field of the Borel sets in the space $W$ of all continuous real-valued functions on $\mathbb{R}^2_+$. With respect to this metric $W$ is a complete separable metric space and furthermore $\mathcal{B}(W)$ is equal to the $\sigma$-field generated by the cylinder sets in $W$. Parallel statements hold for $\mathcal{B}(\partial W)$, where $\partial W$ is the $\sigma$-field of all continuous real-valued functions on $\partial \mathbb{R}_+^2$. The notation $m_{W}$ designates the Wiener measure on $(W, \mathcal{B}(W))$ concentrated on those elements of $W$ which vanish on $\partial \mathbb{R}_+^2$. For $z \in \mathbb{R}_+^2$ we write $\mathcal{B}_z(W)$ for the $\sigma$-field of subsets of $W$ generated by the cylinder sets of the type \{ $w \in W; w(\zeta) \in E$ \} for some $\zeta \leq z$ and $E \in \mathcal{B}(\mathbb{R})$. The coefficients $\alpha$ and $\beta$ in (1.1) are assumed to be $\mathcal{B}(\mathbb{R}_+^2) \otimes \mathcal{B}(W)/\mathcal{B}(\mathbb{R})$ measurable real-valued functions on $\mathbb{R}_+^2 \times W$ such that $\alpha(z, \cdot)$ and $\beta(z, \cdot)$ are $\mathcal{B}_z(W)/\mathcal{B}(\mathbb{R})$ measurable real-valued functions on $W$ for every $z \in \mathbb{R}_+^2$. For two random variables $\xi$ and $\eta$ on a probability space we write $\xi = \eta$ to mean the almost sure equality of the two even when we do not add the qualifier “a.s.” Similarly we write $X = Y$ for two stochastic processes $X$ and $Y$ on a probability space when $X(\cdot, \omega) = Y(\cdot, \omega)$ for a.e. $\omega \in \Omega$. Finally, we write $m_\lambda$ for the Lebesgue measure on $\mathbb{R}_+$.

By an \{ $\mathcal{F}_t$ \} Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_z)$ we mean an \{ $\mathcal{F}_t$ \} adapted two-parameter Brownian motion $B$ on the filtered probability space satisfying the condition that $B((z, z')]$ and $\mathcal{F}^1_z \vee \mathcal{F}^2_z$ are independent for $z < z'$, or equivalently

$$E[\exp\{iuB((z, z')]\} \mid \mathcal{F}^1_z \vee \mathcal{F}^2_z] = \exp\left\{-\frac{u^2}{2} m_\lambda((z, z'])\right\}$$

for $u \in \mathbb{R}$,

where

$$\mathcal{F}^1_z = \sigma\left(\bigcup_{v \in \mathbb{R}_+} \mathcal{F}_{x,v}\right) \quad \text{and} \quad \mathcal{F}^2_z = \sigma\left(\bigcup_{u \in \mathbb{R}_+} \mathcal{F}_{u,t}\right)$$

for $z = (s, t) \in \mathbb{R}_+^2$.
DEFINITION 1.1. By a solution of (1.1) we mean a pair of 2-parameter stochastic processes \((X, B)\) on some filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)\) satisfying the following conditions:

1. \(B\) is an \(\{\mathcal{F}_t\}\)-Brownian motion with \(\partial B = 0\),
2. \(X\) is an \(\{\mathcal{F}_t\}\)-adapted process whose sample functions are all continuous on \(\mathbb{R}^2_+\),
3. for every \(T > 0\)
   \[
   \mathbb{E}_\mathbb{P}
   \left[
   \int_{[0, T] \times [0, T]} |\Phi(z, \cdot)|^2 m_t(dz)
   \right] < \infty
   \] (1.2)
   and
   \[
   \mathbb{E}_\mathbb{P}
   \left[
   \int_{[0, T] \times [0, T]} |\Psi(z, \cdot)| m_t(dz)
   \right] < \infty,
   \] (1.3)
   where \(\Phi\) and \(\Psi\) are defined by
   \[
   \Phi(z, \omega) = \alpha(z, X(\cdot, \omega)) \quad \text{for} \quad (z, \omega) \in \mathbb{R}^2_+ \times \Omega
   \] (1.4)
   and
   \[
   \Psi(z, \omega) = \beta(z, X(\cdot, \omega)) \quad \text{for} \quad (z, \omega) \in \mathbb{R}^2_+ \times \Omega,
   \] (1.5)
4. with probability 1, (1.1) holds for all \(z \in \mathbb{R}^2_+\).

DEFINITION 1.2. We say that the solution of (1.1) is unique in the sense of probability distribution if the following condition (U.1) is satisfied:

(U.1) Whenever \((X, B)\) and \((X', B')\) are two solutions of (1.1) on two possibly different filtered probability spaces and \(\partial X\) and \(\partial X'\) have the same probability distribution on \((\partial W, \mathcal{B}(\partial W))\), then \(X\) and \(X'\) have the same probability distribution on \((W, \mathcal{B}(W))\).

Consider the following condition:

(U.2) Whenever \((X, B)\) and \((X', B')\) are two solutions of (1.1) on two possibly different filtered probability spaces and \(\partial X = x\) and \(\partial X' = x\) also for some \(x \in \partial W\), then \(X\) and \(X'\) have the same probability distribution on \((W, \mathcal{B}(W))\).

Since the condition \(\partial X = x\) is equivalent to the condition that the probability distribution of \(\partial X\) on \((\partial W, \mathcal{B}(\partial W))\) is the unit mass at \(x \in \partial W\), (U.2) is implied by (U.1). In Theorem 2.3 we show that actually (U.1) and (U.2) are equivalent. That the uniqueness in probability distribution of solutions to a 1-parameter stochastic differential equation under deter-
ministic initial condition is equivalent to uniqueness in probability distribution of solutions under stochastic initial condition is mentioned in N. Ikeda and S. Watanabe [2] (see Remark 1.2 on p. 148). The proof of Theorem 2.3 is based on Theorem 2.2, which shows that if \((X, B)\) is a solution of (1.1) on a filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) with the probability distribution of \(\partial X\) denoted by \(\mu\) and if \(P^\mu_{\partial X, x, \epsilon} \in \partial W\) is a regular conditional probability of \(P\) given \(\partial X = x\) then \((X, B)\) is a solution of (1.1) with \(\partial X = x\) on the filtered probability space \((\Omega, \mathcal{F}, P^\mu_{\partial X, x}; \mathcal{F}_z)\) for \(\mu\) a.e. \(x\) in \(\partial W\). Theorem 2.2 will be used also in reducing the construction of a strong solution to (1.1) under stochastic boundary condition to that under deterministic boundary condition.

**Definition 1.3.** Let \(\mathcal{B}(W)\) be the completion of \(\mathcal{B}(W)\) with respect to \(m_w\). For every \(z \in \mathbb{R}^2_+\) we write \(\mathcal{B}_z(W)\) for the \(\sigma\)-field generated by \(\mathcal{B}_z(W)\) and the collection of all the null sets in \((W, \mathcal{B}(W), m_w)\). We write \(\mathcal{B}(\partial W \times W)^{\mu \times m_w}\) for the completion of \(\mathcal{B}(\partial W \times W)\) with respect to the product measure \(\mu \times m_w\).

**Definition 1.4.** Let \(\mathcal{S}(\partial W \times W)\) be the class of transformations \(F\) of \(\partial W \times W\) into \(W\) satisfying the condition that for every probability measure \(\mu\) on \((\partial W, \mathcal{B}(\partial W))\) there exists a transformation \(F_\mu\) of \(\partial W \times W\) into \(W\) such that

1. \(F_\mu\) is \(\mathcal{B}(\partial W \times W)^{\mu \times m_w}/\mathcal{B}(W)\) measurable.
2. for every \(x \in \partial W\), \(F_\mu[x, \cdot] = \mathcal{B}_z(W)/\mathcal{B}_z(W)\) measurable for every \(z \in \mathbb{R}^2_+\),
3. there exists a null set \(N_\mu\) in \((\partial W, \mathcal{B}(\partial W), \mu)\) such that \(F[x, w] = F_\mu[x, w]\) for a.e. \(w\) in \((W, \mathcal{B}(W), m_w)\) when \(x \in N_\mu\).

**Definition 1.5.** Let \((X, B)\) be a solution of the stochastic differential equation (1.1) on a filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) and let \(\mu\) be the probability distribution of \(\partial X\). We call \((X, B)\) a strong solution of (1.1) if there exists a transformation \(F_\mu\) of \(\partial W \times W\) into \(W\) satisfying (1) and (2) of Definition 1.4 such that

\[X = F_\mu[\partial X, B] \quad \text{a.s. on} \quad (\Omega, \mathcal{F}, P; \mathcal{F}_z)\].

**Definition 1.6.** We say that (1.1) has a unique strong solution if there exists \(F \in \mathcal{S}(\partial W \times W)\) such that

1. if \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) is a filtered probability space on which an \((\mathcal{F}_z)\)-Brownian motion \(B\) with \(dB = 0\) exists, then for every continuous \((\mathcal{F}_z)\)-adapted boundary process \(Z\) on \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) whose probability
distribution is denoted by \( \mu \), \((X, B)\) with \( X \equiv F_\mu[Z, B] \) is a solution of (1.1) with \( \partial X = Z \) on \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\).

(2) if \((X, B)\) is a solution of (1.1) on a filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) and the probability distribution of \( \partial X \) is denoted by \( \mu \), then

\[
X = F_\mu[\partial X, B] \quad \text{a.s. on } (\Omega, \mathcal{F}, P; \mathcal{F}_t).
\]

**Definition 1.7.** We say that the pathwise uniqueness of solutions to (1.1) holds if whenever \((X', B)\) and \((X'', B)\) with the same \( B \) are two solutions of (1.1) on the same filtered probability space and \( \partial X = \partial X' \), then \( X = X'' \).

Under the assumption of a Lipschitz condition and an order of growth condition on the coefficients \( \alpha \) and \( \beta \) in (1.1), one of the present authors showed the existence of a strong solution satisfying slightly different measurability conditions from those in Definition 1.4 (see Definition 3.6 and Theorem 3.12 of [9]). It was also shown that the Lipschitz condition implies the pathwise uniqueness of solutions. In Theorem 3.5 we show that a unique strong solution of (1.1) exists if and only if (I) for every probability measure \( \mu \) on \((\partial W, \mathcal{B}(\partial W))\) a solution \((X, B)\) of (1.1), in which \( \mu \) is the probability distribution of \( \partial X \), exists and (II) the pathwise uniqueness of solutions of (1.1) holds.

In Theorem in [12], we showed that if the pathwise uniqueness of solutions to the stochastic differential equation (1.1) holds and if for every \( x \in \partial W \) Eq. (1.1) has a solution \((X, B)\) satisfying \( \partial X = x \) on some filtered probability space, then there exists a transformation \( F \) of \( \partial W \times W \) into \( W \), with \( F[x, \cdot] \) uniquely determined up to a null set in \((W, \mathcal{B}(W), m_w)\) for each \( x \in \partial W \), such that

\[
(F.1) \quad \text{for every } x \in \partial W, \quad F[x, \cdot] \text{ is } \overline{\mathcal{B}_z(W)}/\mathcal{B}_z(W) \text{ measurable for every } z \in \mathbb{R}_+^d,
\]

\[
(F.2) \quad \text{if } (\Omega, \mathcal{F}, P; \mathcal{F}_t) \text{ is a filtered probability space on which there exists an } \{\mathcal{F}_s\}\text{-Brownian motion } B \text{ with } \partial B = 0, \text{ then } (X, B) \text{ with } X = F[x, B] \text{ is a solution of (1.1) satisfying } \partial X = x \text{ on the filtered probability space for each } x \in \partial W,
\]

\[
(F.3) \quad \text{any solution } (X, B) \text{ of (1.1) satisfying } \partial X = x \text{ for some } x \in \partial W \text{ satisfies } X = F[x, B].
\]

In Section 3 we show that our \( F \) is in fact a unique strong solution of (1.1) as defined by Definition 1.6.

Proof for the joint measurability of the strong solution seems to be missing in [2, 8]. We have an explicit proof of this joint measurability in the two-parameter case in Section 3. Our existence and uniqueness theorem will be applied in a subsequent paper. For properties of stochastic integrals
in the plane, we refer to [1, 7]. Existence and uniqueness of solutions to stochastic differential equations in the plane have been treated in [6, 9, 10, 11]. For related questions in one-parameter case, see [2, 3, 5].

2. THE EQUIVALENCE OF (U.1) AND (U.2)

We shall relate the condition (U.1) to the condition (U.2) through a regular conditional probability given \( \partial X = x \) for \( x \in \partial W \). For this to be possible we construct a solution \((X, B)\) of (1.1) on a probability space \((W \times W, \mathcal{B}(W \times W), P)\) which has the same probability distribution as a given solution \((\tilde{X}, \tilde{B})\) of (1.1). Since \( \mathcal{B}(W \times W) \) is the \( \sigma \)-field of Borel sets in the complete separable metric space \( W \times W \), a regular conditional probability of \( P \) given \( \partial X = x \) exists. See [4] for regular conditional probability.

Now let \((\tilde{X}, \tilde{B})\) be a solution of (1.1) on a filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_z)\). For \( i = 0 \) and 1 let \( W_i \) be copies of \( W \) and consider \( \tilde{B}(\cdot, \omega) \) and \( \tilde{X}(\cdot, \omega) \) for \( \omega \in \tilde{\Omega} \) as mappings of \( \tilde{\Omega} \) into \( W_0 \) and \( W_1 \), respectively. The probability distribution of \( \tilde{B} \) on \((W_0, \mathcal{B}(W_0))\) is then the Wiener measure \( m_W \). Let \( P \) be the probability distribution of \((\tilde{X}, \tilde{B})\) on \((W_1 \times W_0, \mathcal{B}(W_1 \times W_0))\), i.e.,

\[
P(A) = \tilde{P}((\tilde{X}, \tilde{B})^{-1}(A)) \quad \text{for} \quad A \in \mathcal{B}(W_1 \times W_0).
\]

A filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) is obtained if we set

\[
\Omega = W_1 \times W_0
\]

\[
\mathcal{F} = \mathcal{B}(W_1 \times W_0) = \mathcal{B}(W_1) \otimes \mathcal{B}(W_0)
\]

\[
\mathcal{F}_0 = \mathcal{B}_z(W_1 \times W_0) = \mathcal{B}_z(W_1) \otimes \mathcal{B}_z(W_0)
\]

\[
\mathcal{F}_z^* = \sigma(\mathcal{F}_0^* \cup \mathcal{R})
\]

\[
\mathcal{F}_z = \bigcap_{\varepsilon > 0} \mathcal{F}_{z + \varepsilon, t + \varepsilon}
\]

for \( z = (s, t) \in \mathbb{R}^2_+ \),

where \( \mathcal{B}_z(W_1 \times W_0) \) is the \( \sigma \)-field of subsets of \( W_1 \times W_0 \) generated by the cylinder sets of the type \( \{(w_1, w_0) \in W_1 \times W_0; w_i(\zeta_i) \in E_i \text{ for } i = 0, 1 \} \) with \( \zeta_i \leq z \) and \( E_i \in \mathcal{B}(W_i) \) and \( \mathcal{R} \) is the collection of the null sets in \((\Omega, \mathcal{F}, P)\).

**Lemma 2.1.** Let \((\tilde{X}, \tilde{B})\) be a solution of (1.1) on a filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_z)\) and let \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) be the filtered probability space defined on the function space \( W_1 \times W_0 \) by (2.0)–(2.5). Define a pair of 2-parameter stochastic processes \( B \) and \( X \) on \((\Omega, \mathcal{F}, P; \mathcal{F}_z)\) by setting

\[
B(z, \omega) = w_0(z) \quad \text{for} \quad \omega = (w_1, w_0) \in \Omega \quad \text{and} \quad z \in \mathbb{R}^2_+.
\]

\[
X(z, \omega) = w_1(z)
\]
Then $B$ is an $\{F_t\}$-Brownian motion with $\partial B = 0$ and $(X, B)$ is a solution of (1.1) on $(\Omega, \mathcal{F}, P; \mathbb{F}_\infty)$ in which $X$ has the same probability distribution as $\hat{X}$.

**Proof.** Clearly $B$ is an $\{F_t\}$-adapted stochastic process with continuous sample functions with $\partial B = 0$ on the filtered probability space $(\Omega, \mathcal{F}, P; \mathbb{F}_\infty)$. To show that $B$ is an $\{F_t\}$-Brownian motion it remains to verify that for $z, z' \in \mathbb{R}_+, z < z'$, and $u \in \mathbb{R}$,

$$
\mathbb{E}_P[\exp\{iuB((z, z'))\} | F_z^1 \vee F_z^2] = \exp \left\{-\frac{u^2}{2} m_L((z, z')) \right\}, \quad (2.8)
$$

that is,

$$
\int_A \exp\{iuB((z, z'))\} \, dP = \exp \left\{-\frac{u^2}{2} m_L((z, z')) \right\} P(A) \quad \text{for} \quad A \in F_z^1 \vee F_z^2.
$$

Now since $\hat{B}$ and $\hat{X}$ are $\{F_t\}$-adapted we have $(\hat{X}, \hat{B})^{-1}(\mathbb{F}_\infty^0) \subset \mathbb{F}_\infty^z$. From this it follows that

$$(\hat{X}, \hat{B})^{-1}(\mathbb{F}_\infty^0 \vee \mathbb{F}_\infty^{z, 2}) \subset \mathbb{F}_\infty^1 \vee \mathbb{F}_\infty^2.
$$

Therefore by the image probability law and by the fact that $\hat{B}$ is an $\{F_t\}$-Brownian motion on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}; \hat{\mathbb{F}}_\infty^z)$, we have for every $A \in \mathbb{F}_\infty^{z, 1} \vee \mathbb{F}_\infty^{z, 2}$ that

$$
\int_A \exp\{iuB((z, z'))\} \, dP
$$

$$
= \int_{(\hat{X}, \hat{B})^{-1}(A)} \exp\{iu\hat{B}((z, z'))\} \, d\hat{P}
$$

$$
= \exp \left\{-\frac{u^2}{2} m_L((z, z')) \right\} \hat{P}[(\hat{X}, \hat{B})^{-1}(A)]
$$

$$
= \exp \left\{-\frac{u^2}{2} m_L((z, z')) \right\} P(A).
$$

(2.9)

This equality then holds for every $A \in \mathbb{F}_\infty^{z, 1} \vee \mathbb{F}_\infty^{z, 2}$ also. Thus

$$
\mathbb{E}_P[\exp\{iuB((z, z'))\} | \mathbb{F}_\infty^{z, 1} \vee \mathbb{F}_\infty^{z, 2}] = \exp \left\{-\frac{u^2}{2} m_L((z, z')) \right\}. \quad (2.10)
$$

Let $z_n = (s + 1/n, t + 1/n)$ for $z = (s, t)$, where $n$ is so large that $z_n < z'$. Then

$$
\mathbb{E}_P[\exp\{iuB((z_n, z'))\} | F_z^1 \vee F_z^2]
$$

$$
= \mathbb{E}_P[\mathbb{E}_P[\exp\{iuB((z_n, z'))\} | \mathbb{F}_\infty^{z_n, 1} \vee \mathbb{F}_\infty^{z_n, 2}] | F_z^1 \vee F_z^2]
$$

$$
= \exp \left\{-\frac{u^2}{2} m_L((z_n, z')) \right\}
$$
by (2.10). Letting $n \to \infty$ we have (2.8), completing the proof that $B$ is an \{\mathcal{F}_s\}-Brownian motion on \((\Omega, \mathcal{F}, P, \mathcal{F}_s)\).

By (2.6) and (2.7), the probability distribution of \((X, B)\) on \((W_1 \times W_0, \mathcal{B}(W_1 \times W_0))\) is equal to $P$, i.e., the probability distribution of \((\hat{X}, \hat{B})\). Therefore $X$ has the same probability distribution as $\hat{X}$ and furthermore it can be shown that \((X, B)\) is a solution of (1.1) on \((\Omega, \mathcal{F}, P, \mathcal{F}_s)\).

Let \((X, B)\) be the solution of (1.1) defined by (2.6) and (2.7) on the filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F}_s)\) constructed on the function space \(\mathcal{F}_s = W_1 \times W_0\) by (2.0)-(2.5). Let $\mu$ be the probability distribution on \((\partial W, \mathcal{B}(\partial W))\) of the mapping $\partial X$ of $\Omega$ into $\partial W$ defined by

\[ \partial X(\cdot, \omega) = w_1 \big|_{\partial W} \quad \text{for} \quad \omega = (w_1, w_0) \in \Omega. \]

Since $\Omega = W_1 \times W_0$ is a complete separable metric space and $\mathcal{F} = \mathcal{B}(W_1 \times W_0)$ is the $\sigma$-field of the Borel sets in $\Omega$ and since $\partial X$ is $\mathcal{F}/\mathcal{B}(\partial W)$ measurable, there exists a unique regular conditional probability of $P$ given $\partial X = x$ for $x \in \partial W$, i.e., there exists a real-valued function

\[ P_{\partial X} = \{P_{\partial X}^x(A); A \in \mathcal{F}, x \in \partial W\} \]

such that

- (P.1) for every $x \in \partial W$, $P_{\partial X}^x$ is a probability measure on $(\Omega, \mathcal{F})$,
- (P.2) for every $A \in \mathcal{F}$, $P_{\partial X}^x(A)$ is $\mathcal{B}(\partial W)/\mathcal{B}(\partial W)$ measurable,
- (P.3) for every $A \in \mathcal{F}$ and $E \in \mathcal{B}(\partial W)$,

\[ P(A \cap \{w \in \Omega; \partial X(\cdot, \omega) \in E\}) = \int_E P_{\partial X}^x(A) \mu(dx), \]

(1.4) if $\{p(x, A); A \in \mathcal{F}, x \in \partial W\}$ is a real-valued function satisfying the conditions (P.1)-(P.3) then there exists a null set $N$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

\[ p(x, A) = P_{\partial X}^x(A) \quad \text{for all} \quad A \in \mathcal{F} \quad \text{when} \quad x \in N. \]

For an integrable random variable $\xi$ on $(\Omega, \mathcal{F}, P)$, (P.3) has an integral form

\[ (P.3') \int_{(\partial X)^{-} \setminus \{E\}} \xi(\omega) \, P(d\omega) = \int_E \{\int_\Omega \xi(\omega) \, P_{\partial X}^x(d\omega)\} \mu(dx) \quad \text{for} \quad E \in \mathcal{B}(\partial W) \]

from which we obtain

\[ (P.3'') E_\mu(\xi|\partial X)(x) = \int_\Omega \xi(\omega) \, P_{\partial X}^x(d\omega) \quad \text{for a.e.} \quad x \in (\partial W, \mathcal{B}(\partial W), \mu). \]

Since $\mathcal{B}(\partial W)$ is countably determined (i.e., there exists a countable subset of $\mathcal{B}(\partial W)$ such that whenever two probability measures on $\mathcal{B}(\partial W)$
agree on the subset then they agree on the entire $\mathcal{B}(\partial W)$, and since $\{x\} \in \mathcal{B}(\partial W)$ for every $x \in \partial W$, our regular condition probability $P_{\partial x}$ has the additional property

(P.5) there exists a null set $N$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

$$P^{\partial x}_{\partial x} \{ \omega \in \Omega; \partial X(\cdot, \omega) = x \} = 1 \quad \text{for} \quad x \in N^c.$$  

For each $x \in \partial W$, let $\mathcal{N}_x$ be the collection of the null sets in $(\Omega, \mathfrak{F}, P_{\partial x}^\times)$ and let

$$\mathfrak{F}^x_{x^+.x} = \sigma(\mathfrak{F}^\times_0 \cup \mathcal{N}_x) \quad (2.11)$$

$$\mathfrak{F}^x_z = \bigcap_{\varepsilon > 0} \mathfrak{F}^x_{x^+.x^+.\varepsilon} \quad \text{for} \quad z = (s, t) \in \mathbb{R}^2_z. \quad (2.12)$$

Then $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$ is a filtered probability space. As we shall show in Theorem 2.2, the $\{\mathfrak{F}^x_z\}$-Brownian motion $B$ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}^x_z)$ defined by (2.6) is also an $\{\mathfrak{F}^x_z\}$-Brownian motion on $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$ for a.e. $x$ in $(\partial W, \mathcal{B}(\partial W), \mu)$. We write $(P_{\partial x}^\times) \int_{R_z} \alpha(\zeta, X) \, dB_\zeta$ and $(P_{\partial x}^\times) \int_{R_z} \beta(\zeta, X) \, d \zeta$ for the stochastic integrals of $\alpha$ and $\beta$ on $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$ if they exist. The stochastic integrals $\int_{R_z} \alpha(\zeta, X) \, dB_\zeta$ and $\int_{R_z} \beta(\zeta, X) \, d \zeta$ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}^x_z)$ will be at times written as $(P) \int_{R_z} \alpha(\zeta, X) \, dB_\zeta$ and $(P) \int_{R_z} \beta(\zeta, X) \, d \zeta$.

**Theorem 2.2.** Let $(\Omega, \mathfrak{F}, P, \mathfrak{F}_x)$ and $(X, B)$ be as in Lemma 2.1. Let $\mu$ be the probability distribution of $\partial X$ and let

$$P^{\partial x}_0 = \{ P_{\partial x}^\times(A); A \in \mathfrak{F}, x \in \partial W \}$$

be a regular conditional probability of $P$ given $\partial X = x$. Then for a.e. $x$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ we have

1. $B$ is an $\{\mathfrak{F}^x_z\}$-Brownian motion with $\partial B = 0$ on the filtered probability space $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$,

2. the stochastic integrals $(P_{\partial x}^\times) \int_{R_z} \alpha(\zeta, X) \, dB_\zeta$ and $(P_{\partial x}^\times) \int_{R_z} \beta(\zeta, X) \, d \zeta$ for $z \in \mathbb{R}^2_z$ exist on $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$ and

   $$(P_{\partial x}^\times) \int_{R_z} \alpha(\zeta, X) \, dB_\zeta = (P) \int_{R_z} \alpha(\zeta, X) \, dB_\zeta \quad (2.13)$$

   $$(P_{\partial x}^\times) \int_{R_z} \beta(\zeta, X) \, d \zeta = (P) \int_{R_z} \beta(\zeta, X) \, d \zeta, \quad (2.14)$$

3. $(X, B)$ is a solution of (1.1) on $(\Omega, \mathfrak{F}, P_{\partial x}^\times; \mathfrak{F}^x_z)$,

4. $P_{\partial x}^\times \{ \omega \in \Omega; \partial X(\cdot, \omega) = x \} = 1$, that is, $\partial X = x, P_{\partial x}^\times$-a.s.
Proof. (1) Clearly $B$ is a continuous stochastic process with $\partial B = 0$ on $(\Omega, \mathcal{F}, P_{\partial x})$ for every $x \in \partial W$. By (2.6), $B$ is $\{\mathcal{F}^0_z\}$-adapted and thus $\{\mathcal{F}^x_z\}$-adapted for every $x \in \partial W$. To show that $B$ is an $\{\mathcal{F}^x_z\}$-Brownian motion on $(\Omega, \mathcal{F}, P_{\partial x}, \mathbb{P}^x)$ for a.e. $x \in (\partial W, \mathcal{B}(\partial W), \mu)$ it remains to show that for a.e. $x$ we have for all $z, z' \in \mathbb{R}^d_+$, $z < z'$, and $u \in \mathbb{R}$,

$$E_{\mathcal{P}^x_z}[\exp\{iuB((z, z'))\} | \mathcal{F}^{x, 1}_z \vee \mathcal{F}^{x, 2}_z] = \exp\left\{-\frac{u^2}{2} m_L((z, z'))\right\}, \quad (2.15)$$

that is, for all $A \in \mathcal{F}^{x, 1}_z \vee \mathcal{F}^{x, 2}_z$

$$\int_A \exp\{iuB((z, z'))\} dP_{\mathcal{P}^x_z} = \exp\left\{-\frac{u^2}{2} m_L((z, z'))\right\} P_{\mathcal{P}^x_z}(A). \quad (2.16)$$

Consider first $A \in \mathcal{F}^{0, 1}_z \vee \mathcal{F}^{0, 2}_z$. Then for every $E \in \mathcal{B}(\partial W)$ we have

$$\int_E E_{\mathcal{P}^x_z}[\exp\{iuB((z, z'))\}] 1_A] \mu(dx)$$

$$= E_{\mathcal{P}}[\exp\{iuB((z, z'))\} 1_{A \cap (\partial X)^{-1}(E)}]$$

$$= \exp\left\{-\frac{u^2}{2} m_L((z, z'))\right\} P[A \cap (\partial X)^{-1}(E)]$$

$$= \exp\left\{-\frac{u^2}{2} m_L((z, z'))\right\} \int_E P_{\mathcal{P}^x_z}(A) \mu(dx)$$

by (P.3), the fact that $B$ is an $\{\mathcal{F}^0_z\}$-Brownian motion on $(\Omega, \mathcal{F}, P)$ as established by (2.9), and finally (P.3). From the arbitrariness of $E \in \mathcal{B}(\partial W)$, there exists a null set $N(z, z', u, A)$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

$$E_{\mathcal{P}^x_z}[\exp\{iuB((z, z'))\}] 1_A]$$

$$= \exp\left\{-\frac{u^2}{2} m_L((z, z'))\right\} P_{\mathcal{P}^x_z}(A) \quad \text{for } x \in N(z, z', u, A). \quad (2.17)$$

Since $\mathcal{F}^0_z$ is countably determined so is $\mathcal{F}^{0, 1}_z \vee \mathcal{F}^{0, 2}_z$. Let $\mathcal{E}$ be a countable determining subset of $\mathcal{F}^{0, 1}_z \vee \mathcal{F}^{0, 2}_z$. Let $\mathcal{Q}$ be the collection of rational numbers and let $\mathcal{Q}_+$ be that of the nonnegative rational numbers and consider the null set $N$ in $(\partial W, \mathcal{B}(\partial W) \mu)$ defined by

$$N = \bigcup_{z, z' \in \mathcal{Q}_+, z < z', u \in \mathcal{Q}, A \in \mathcal{E}} N(z, z', u, A).$$

Then for $x \in N$, (2.17) or equivalently (2.16) holds for all $z, z' \in \mathcal{Q}_+, z < z'$,
\(u \in Q\), and \(A \in \mathfrak{A}^{0.1}_2 \cup \mathfrak{A}^{0.2}_2\). Then by the continuity of \(B\), (2.16) holds for all \(z, z' \in \mathbb{R}^d_+\), \(z < z'\), \(u \in \mathbb{R}\), and \(A \in \mathfrak{A}^{0.1}_2 \cup \mathfrak{A}^{0.2}_2\) when \(x \in N^c\). From this follows (2.15) for \(x \in N^c\) in the same way (2.8) followed from (2.9). This completes (1).

(2) The continuous stochastic process \(X\) on \((\Omega, \mathfrak{F}, P)\) defined by (2.7) is \(\{\mathfrak{F}^x\}\)-adapted and is thus an \(\{\mathfrak{F}^x\}\)-adapted process on \((\Omega, \mathfrak{F}, P^\xi_{\mathbf{x}}; \mathfrak{F}^x)\) for every \(x \in \partial W\). To show that for a.e. \(x\) in \((\partial W, \mathfrak{B}(\partial W), \mu)\) the stochastic integrals \((P^x_{\mathbf{x}}) \int_{R^d} \alpha(\zeta, X) \, dB_\zeta\) and \((P^x_{\mathbf{x}}) \int_{R^d} \beta(\zeta, X) \, d\zeta\) for \(z \in \mathbb{R}^d_+\) exist on \((\Omega, \mathfrak{F}, P^\xi_{\mathbf{x}}; \mathfrak{F}^x)\) it remains to verify the integrability conditions (1.2) and (1.3) with respect to the probability measure \(P^\xi_{\mathbf{x}}\) for a.e. \(x\) for the stochastic processes \(\Phi\) and \(\Psi\) defined by (1.4) and (1.5). Now

\[
E_x \left[ E_{P^\xi_{\mathbf{x}}} \left[ \int_{[0, T] \times [0, T]} |\Phi(z, \cdot)|^2 \, m_L(dz) \right] \right] = E_x \left[ \int_{[0, T] \times [0, T]} |\Phi(z, \cdot)|^2 \, m_L(dz) \right] < \infty
\]

by (P.3)' and (1.2). Thus for a.e. \(x\) in \((\partial W, \mathfrak{B}(\partial W), \mu)\)

\[
E_{P^\xi_{\mathbf{x}}} \left[ \int_{[0, T] \times [0, T]} |\Phi(z, \cdot)|^2 \, m_L(dz) \right] < \infty,
\]

and similarly by (P.3)' and (1.3)

\[
E_{P^\xi_{\mathbf{x}}} \left[ \int_{[0, T] \times [0, T]} |\Psi(z, \cdot)| \, m_L(dz) \right] < \infty.
\]

This establishes the existence of \((P^x_{\mathbf{x}}) \int_{R^d} \alpha(\zeta, X) \, dB_\zeta\) and \((P^x_{\mathbf{x}}) \int_{R^d} \beta(\zeta, X) \, d\zeta\) for \(z \in \mathbb{R}^d_+\).

To prove (2.13) let \(\Phi^n, n = 1, 2, \ldots\), be a sequence of \(\{\mathfrak{A}^0\}\)-adapted step stochastic processes on \((\Omega, \mathfrak{F}, P)\) such that

\[
\lim_{n \to \infty} E_x \left[ \int_{R^d} |\Phi^n_\zeta - \Phi_\zeta|^2 m_L(d\zeta) \right] = 0.
\]

(2.18)

Then by (P.3)' we have

\[
\lim_{n \to \infty} \int_{\partial W} E_{P^\xi_{\mathbf{x}}} \left[ \int_{R^d} |\Phi^n_\zeta - \Phi_\zeta|^2 m_L(d\zeta) \right] \mu(dx) = 0
\]

so that there exist a subsequence \(\{\Phi'^n\}\) of \(\{\Phi^n\}\) and a null set \(N\) in \((\partial W, \mathfrak{B}(\partial W), \mu)\) such that

\[
\lim_{m \to \infty} E_{P^\xi_{\mathbf{x}}} \left[ \int_{R^d} |\Phi'^m_\zeta - \Phi_\zeta|^2 m_L(d\zeta) \right] = 0 \quad \text{for} \quad x \in N^c.
\]

(2.19)
From (2.18) and (2.19) we have
\[ \lim_{m \to \infty} \int_{R_1} \Phi^m \cdot dB = (P) \int_{R_1} \Phi \cdot dB \quad \text{in} \quad L^2(P) \]
and
\[ \lim_{m \to \infty} \int_{R_2} \Phi^m \cdot dB = (P) \int_{R_2} \Phi \cdot dB \quad \text{in} \quad L^2(P) \]
for \( x \in N^c \), respectively. Thus, for fixed \( x \in N^c \), there exists a subsequence \( \{ \Phi' \} \) of \( \{ \Phi^m \} \) such that
\[ \lim_{l \to \infty} \int_{R_1} \Phi' \cdot dB = (P) \int_{R_1} \Phi \cdot dB \quad \text{a.e. on} \quad (\Omega, \mathcal{F}, P) \]
and
\[ \lim_{l \to \infty} \int_{R_2} \Phi' \cdot dB = (P) \int_{R_2} \Phi \cdot dB \quad \text{a.e. on} \quad (\Omega, \mathcal{F}, P) \]
Since our stochastic integrals are defined up to null sets in respective filtered probability spaces, we have
\[ (P) \int_{R_1} \Phi \cdot dB = (P) \int_{R_2} \Phi' \cdot dB \quad \text{when} \quad x \in N^c, \]
proving (2.13). Similarly (2.14) holds.

(3) For \( x \in \partial W \) let
\[ Y_z = X((0, z]) - (P) \int_{R_1} \alpha(\zeta, X) \, dB = (P) \int_{R_1} \beta(\zeta, X) \, d\zeta \quad \text{for} \quad z \in \mathbb{R}^2_+ \]
(2.20)
To show that \( (X, B) \) is a solution of (1.1) on \( (\Omega, \mathcal{F}, P^x_{\partial X}; \mathcal{G}_x') \) for a.e. \( x \) in \( (\partial W, \mathcal{B}(\partial W), \mu) \) we show that \( Y_z = 0 \) a.e. on \( (\Omega, \mathcal{F}, P^x_{\partial X}) \) for a.e. \( x \). Let
\[ Y_z = X((0, z]) - (P) \int_{R_1} \alpha(\zeta, X) \, dB = (P) \int_{R_1} \beta(\zeta, X) \, d\zeta \quad \text{for} \quad z \in \mathbb{R}^2_+ \]
(2.21)
Then, since \( (X, B) \) is a solution of (1.1) on \( (\Omega, \mathcal{F}, P; \mathcal{G}_x') \), we have \( Y_z = 0 \) for all \( z \in \mathbb{R}^2_+ \) a.e. on \( (\Omega, \mathcal{F}, P) \). Thus there exists a null set \( A \) in \( (\Omega, \mathcal{F}, P) \) such that
\[ \omega \in A^c \Rightarrow Y_z(\omega) = 0 \quad \text{for all} \quad z \in \mathbb{R}^2_+ \]
(2.22)
Now by (P.3)

$$\int_{\partial W} P^x_{\partial X}(A) \mu(dx) = P(A) = 0$$

so that there exists a null set $N_1$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

$$P^x_{\partial X}(A) = 0 \quad \text{for} \quad x \in N_1.$$  \hfill (2.23)

By (2) there exists a null set $N_2$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that (2.13) and

$$P^x_{\partial X}(A) = 0 \quad \text{for} \quad x \in N_2.$$  \hfill (2.24)

Theorem 2.3. The conditions (U.1) and (U.2) are equivalent.

Proof. Clearly (U.1) implies (U.2). It remains to show that (U.2) implies (U.1).

Let $(\tilde{X}_i, \tilde{B}_i)$ be a solution of (1.1) on a filtered probability space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_i; \tilde{\mathcal{F}}_{i,z})$$

for $i = 1$ and 2 and assume that $\partial \tilde{X}_1$ and $\partial \tilde{X}_2$ have the same probability distribution $\mu$ on $(\partial W, \mathcal{B}(\partial W))$. Let $v_i$ be the probability distribution of $\tilde{X}_i$ on $(W, \mathcal{B}(W))$ for $i = 1$ and 2. We proceed to show that under the assumption of (U.2) we have $v_1 = v_2$.

By Lemma 2.1 there exists a solution $(X_i, B_i)$ of (1.1) on the filtered probability space

$$(\Omega, \mathcal{F}, P; \mathcal{F}_{i,z})$$

constructed on the function space

$$\Omega = W_1 \times W_2$$

by (2.0)–(2.5) such that $X_i$ has the same probability distribution on $(W, \mathcal{B}(W))$ as $\tilde{X}_i$ for $i = 1$ and 2. Then $\partial X_1$ and $\partial X_2$ have the same probability distribution $\mu$ on $(\partial W, \mathcal{B}(\partial W))$ as $\partial \tilde{X}_1$ and $\partial \tilde{X}_2$.

According to Theorem 2.2 for a.e. $x$ in $(\partial W, \mathcal{B}(\partial W), \mu)$, $(X_i, B_i)$ is a solution of (1.1) on the filtered probability space

$$(\Omega, \mathcal{F}, P^x_{\partial X}; \mathcal{F}_{i,z})$$

for $i = 1$ and 2. Thus, for a.e. $x$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ let $v^x_i$ be the probability distribution on $(W, \mathcal{B}(W))$ of $X_i$ in the solution $(X_i, B_i)$ of (1.1) with $\partial X_i = x$ on $(\Omega, \mathcal{F}, P^x_{\partial X}; \mathcal{F}_{i,z})$. Then by (U.2) there exists a null set $N$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

$$v^x_1 \text{ and } v^x_2 \text{ exist and } v^x_1 = v^x_2 \text{ for } x \in N_c.$$  \hfill (2.24)
Since \( v_i \) is the probability distribution on \((W, \mathcal{B}(W))\) of the stochastic process \( X_i \) on \((\Omega, \mathfrak{F}, P_i, \delta\chi_i)\) we have
\[
 v_i(A) = P_i^{x_i, \delta\chi_i}(X_i^{-1}(A)) \quad \text{for} \quad A \in \mathcal{B}(W), i = 1 \text{ and } 2, \text{ when } x \in \mathbb{N}^c. \tag{2.25}
\]
By (P.3) and (2.25) we have for \( A \in \mathcal{B}(W) \) and \( i = 1 \text{ and } 2 \) that
\[
v_i(A) = P_i(X_i^{-1}(A)) = \int_{\partial W} P_i^{x_i, \delta\chi_i}(X_i^{-1}(A)) \mu(dx) = \int_{\partial W} v_i(A) \mu(dx),
\]
so that by (2.24)
\[
v_1(A) = v_2(A).
\]
This proves the equality of \( v_1 \) and \( v_2 \) on \((W, \mathcal{B}(W))\).

**COROLLARY 2.4.** The pathwise uniqueness of the solution of the stochastic differential equation (1.1) implies its uniqueness in the sense of probability distribution.

**Proof.** In view of the equivalence of the two conditions \((U.1)\) and \((U.2)\) by Theorem 2.3, it suffices to show that the pathwise uniqueness condition implies \((U.2)\).

For fixed \( x \in \partial W \), if \((\hat{X}_i, \hat{B}_i)\) is a solution of (1.1) under the boundary condition \( \delta\hat{X}_i = x \) on a filtered probability space \((\Omega_i, \mathfrak{F}_i, \hat{P}_i; \delta\hat{\chi}_i)\) for \( i = 1 \text{ and } 2 \), then by (F.3)
\[
\hat{X}_i = F[x, \hat{B}_i] \quad \text{for} \quad i = 1 \text{ and } 2
\]
so that \((\hat{X}_1, \hat{B}_1)\) and \((\hat{X}_2, \hat{B}_2)\) have the same probability distribution \( \hat{P} \) on \((W_1 \times W_0, \mathcal{B}(W_1 \times W_0))\). If we let \((X_i, B_i)\) be the representation of \((\hat{X}_i, \hat{B}_i)\) on the filtered probability space \((\Omega, \mathfrak{F}, \hat{P}; \delta\hat{\chi}_i)\) as defined by (2.0)-(2.5) for \( i = 1 \text{ and } 2 \), then \( B_1 = B_2 \) and \( \partial X_1 = x = \partial X_2 \) so that by the pathwise uniqueness we have \( X_1 = X_2 \). In particular \( X_1 \) and \( X_2 \) have the same probability distribution and consequently so do \( \hat{X}_1 \) and \( \hat{X}_2 \). From the arbitrariness of \( x \in \partial W \), the condition \((U.2)\) is now satisfied.

3. **Existence and Uniqueness of a Strong Solution**

Suppose the stochastic differential equation (1.1) has a solution \((\hat{X}, \hat{B})\) on a filtered probability space \((\Omega_i, \mathfrak{F}_i, \hat{P}_i; \delta\hat{\chi}_i)\) and let \( \mu \) be the probability distribution of \( \delta\hat{X} \). According to Lemma 2.1, \((X, B)\) defined by (2.6) and (2.7) is a solution of (1.1) on the filtered probability space \((\Omega, \mathfrak{F}, P; \delta\chi)\)
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defined on $\Omega = W_1 \times W_0$ by (2.0)–(2.5). Since $X$ has the same probability distribution on $\hat{X}$, the probability distribution of $\partial X$ is given by $\mu$.

Consider the mapping $(\partial X, X, B)$ of $\Omega$ into $\partial W \times W_1 \times W_0$ defined by

$$ (\partial X, X, B)(\omega) = (w_1 |_{\partial R^+}, w_1, w_0) \quad \text{for} \quad \omega = (w_1, w_0) \in \Omega. \quad (3.0) $$

Let $Q$ be the probability distribution of $(\partial X, X, B)$ on $(\partial W \times W_1 \times W_0, \mathcal{B}(\partial W \times W_1 \times W_0))$, i.e.,

$$ Q(A) = P((\partial X, X, B)^{-1}(A)) \quad \text{for} \quad A \in \mathcal{B}(\partial W \times W_1 \times W_0). \quad (3.1) $$

Since $\partial X$ and $B$ are independent random vectors on $(\Omega, \mathcal{F}, P)$, the probability distribution of $(\partial X, B)$ on $(\partial W \times W_0, \mathcal{B}(\partial W \times W_0))$ is given by the product measure $\mu \times m_w$. Let $Q \cdot \cdot$ be a regular condition probability of $Q$ under the projection of $\partial W \times W_1 \times W_0$ onto $\partial W \times W_0$, i.e.,

$$ Q \cdot \cdot = \{Q^{\cdot \cdot}(A_1) ; A_1 \in \mathcal{B}(W_1), (x, w_0) \in \partial W \times W_0 \} \quad (3.2) $$

satisfying the following conditions:

(Q.1) for every $(x, w_0) \in \partial X \times W_0$, $Q^{x,w_0}$ is a probability measure on $(W_1, \mathcal{B}(W_1))$,

(Q.2) for every $A_1 \in \mathcal{B}(W_1)$, $Q \cdot \cdot (A_1)$ is $\mathcal{B}(\partial W \times W_0)/\mathcal{B}(R)$ measurable,

(Q.3) for every $A_1 \in \mathcal{B}(W_1)$, $E \in \mathcal{B}(\partial W)$, and $A_0 \in \mathcal{B}(W_0)$,

$$ Q(E \times A_1 \times A_0) = \int_{E \times A_0} Q^{x,w_0}(A_1)(\mu \times m_w)(d(x, w_0)). $$

Consider $P_{\partial X}^x$, a regular conditional probability of $P$ given $\partial X = x$ for $x \in \partial W$ as introduced in Section 2. For each $x \in \partial W$, let $(P_{\partial X}^x)^{\cdot \cdot}$ be a regular conditional probability of the probability measure $P_{\partial X}^x$ under the projection $B$ of $W_1 \times W_0$ onto $W_0$, i.e.,

$$ (P_{\partial X}^x)^{\cdot \cdot} = \{(P_{\partial X}^x)^{w_0}(A_1) ; A_1 \in \mathcal{B}(W_1), w_0 \in W_0 \} \quad (3.3) $$

satisfying the following conditions:

(R.1) for every $w_0 \in W_0$, $(P_{\partial X}^x)^{w_0}$ is a probability measure on $(W_1, \mathcal{B}(W_1))$,

(R.2) for every $A_1 \in \mathcal{B}(W_1)$, $(P_{\partial X}^x)^{\cdot \cdot}(A_1)$ is $\mathcal{B}(W_0)/\mathcal{B}(R)$ measurable,

(R.3) for every $A_1 \in \mathcal{B}(W_1)$ and $A_0 \in \mathcal{B}(W_0)$,

$$ (P_{\partial X}^x)(A_1 \times A_0) = \int_{A_0} (P_{\partial X}^x)^{w_0}(A_1) m_w(dw_0). $$
Our $Q^{-\cdot}$ and $(P_{\partial X})^{-\cdot}$ are related by the following lemma. In each of the four lemmas below we assume as above, without repeating the statement each time, that (1.1) has a solution $(\tilde{X}, \tilde{B})$ on some filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; \tilde{\mathcal{F}}_\omega)$ with the probability distribution of $\partial \tilde{X}$ denoted by $\mu$. The notations $(\Omega, \mathcal{F}, P, \mathcal{F}_\omega), (X, B, Q, Q^{-\cdot}, P_{\partial X}^\omega)$, and $(P_{\partial X}^\omega)^{-\cdot}$ denote the same as defined above.

**Lemma 3.1.** There exist a null set $N$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ and a collection \{\(M(x), x \in N^-\)\} of null sets in \((W_0, \mathcal{B}(W_0), m_\omega)\) such that

\[ Q^{x, w_0} = (P_{\partial X}^\omega)^{w_0} \text{ on } (W_1, \mathcal{B}(W_1)) \text{ when } w_0 \in M(x)^c \text{ and } x \in N^c. \quad (3.4) \]

**Proof.** Let $E \in \mathcal{B}(\partial W)$, $A_1 \in \mathcal{B}(W_1)$, and $A_0 \in \mathcal{B}(W_0)$. Then

\[
\int_E \left\{ \int_{A_0} Q^{x, w_0}(A_1) m_\omega(dw_0) \right\} \mu(dx) = Q(E \times A_1 \times A_0) \\
= P((\partial X, X, B)^{-1}(E \times A_1 \times A_0)) = P(\partial X^{-1}(E) \cap A_1 \times A_0) \\
= \int_E P_{\partial X}^\omega(A_1 \times A_0) \mu(dx)
\]

by (Q.3), (3.1), (3.0), and (P.3). Thus, corresponding to our $A_1$ and $A_0$ there exists a null set $N(A_1, A_0)$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ such that

\[
\int_{A_0} Q^{x, w_0}(A_1) m_\omega(dw_0) = P_{\partial X}^\omega(A_1 \times A_0) \quad \text{when } x \in N(A_1, A_0)^c.
\]

Let $\mathcal{E}$ be a countable determining set for $\mathcal{B}(W_1 \times W_2)$ so that for the null set $N$ in $(\partial W, \mathcal{B}(\partial W), \mu)$ defined by

\[ N = \bigcup_{A \in \mathcal{E}} N(A), \]

we have the last equality holding for all $A_0 \in \mathcal{B}(W_0)$ and $A_1 \in \mathcal{B}(W_1)$ whenever $x \in N^c$. Then by (R.3)

\[
\int_{A_0} Q^{x, w_0}(A_1) m_\omega(dw_0) = \int_{A_0} (P_{\partial X}^\omega)^{w_0}(A_1) m_\omega(dw_0)
\]

for all $A_0 \in \mathcal{B}(W_0), A_1 \in \mathcal{B}(W_1)$, and $x \in N^c$. Thus corresponding to each $x \in N^c$ and $A_1 \in \mathcal{B}(W_1)$, there exists a null set $M(x, A_1)$ in $(W_0, \mathcal{B}(W_0), m_\omega)$ such that

\[ Q^{x, w_0}(A_1) = (P_{\partial X}^\omega)^{w_0}(A_1) \quad \text{when } w_0 \in M(x, A_1). \]
Since $\mathfrak{B}(W_1)$ is countably determined, corresponding to our $x \in N^c$ there exists a null set $M(x)$ in $(W_0, \mathfrak{B}(W_0), m_w)$ such that

$$Q^{x, w_0}(A_1) = (P^x_{\delta})(A_1) \quad \text{for all } A_1 \in \mathfrak{B}(W_1) \text{ when } w_0 \in M(x)^c \text{ and } x \in N^c.$$

This completes the proof.

**Lemma 3.2.** Assume further that the pathwise uniqueness of solutions to the stochastic differential equation (1.1) holds and that for every $x \in \partial W$ there exists a solution $(\hat{X}, \hat{B})$ of (1.1) satisfying $\partial \hat{X} = x$ on some filtered probability space. Let $F$ be a transformation of $\partial W \times W_0$ into $W_1$ satisfying the conditions (F.1)–(F.3). Then there exist a null set $N^c_\mu$ in $(\partial W, \mathfrak{B}(\partial W), \mu)$ and a collection of null sets $\{M(x), x \in N^c_\mu\}$ in $(W_0, \mathfrak{B}(W_0), m_w)$ such that

$$Q^{x, w_0} = \delta_{F[x, w_0]} \quad \text{for } w_0 \in M(x)^c \text{ and } x \in N^c_\mu. \quad (3.5)$$

**Proof.** Let $N^c_\mu$ be the union of the two null sets in $(\partial W, \mathfrak{B}(\partial W), \mu)$, one in the statement of Theorem 2.2 and the other in that of Lemma 3.1. Let $x \in N^c_\mu$ be fixed for the rest of the proof.

Let $C_1 = \{W_1, \phi\}$ and $A_1 \in \mathfrak{B}(W_1)$. Let us show that

$$1_{w_1 \times F[x, \cdot]^{-1}(A_1)}(w_1, w_0) \quad \text{for } (w_1, w_0) \in W_1 \times W_0 \quad (3.6)$$

is a version of

$$F_{\pi_2}1_{A_1 \times W_0}[C_1 \otimes \mathfrak{B}(W_0)](w_1, w_0) \quad \text{for } (w_1, w_0) \in W_1 \times W_0. \quad (3.7)$$

Now by (F.1), $F[x, \cdot]$ is $\mathfrak{B}(W_0)/\mathfrak{B}(W_1)$ measurable so that $F[x, \cdot]^{-1}(A_1) \in \mathfrak{B}(W_0)$ and $W_1 \times F[x, \cdot]^{-1}(A_1) \in C_1 \otimes \mathfrak{B}(W_0)$. Thus it remains only to show that for every $A_0 \in \mathfrak{B}(W_0)$ we have

$$\int_{W_1 \times A_0} 1_{w_1 \times F[x, \cdot]^{-1}(A_1)}(w_1, w_0) P^x_{\delta X}(d(w_1, w_0)) = \int_{W_1 \times A_0} 1_{A_1 \times w_0}(w_1, w_0) P^x_{\delta X}(d(w_1, w_0)).$$

But according to (3) and (4) of Theorem 2.2 and (F.3),

$$P^x_{\delta X}\{(w_1, w_0) \in W_1 \times W_0; w_1 = F[x, w_0]\} = 1.$$

Therefore each one of the two integrals above is equal to

$$\int_{W_1 \times A_0} 1_{A_1 \times F[x, \cdot]^{-1}(A_1)}(w_1, w_0) P^x_{\delta X}(d(w_1, w_0)),$$

proving the equality of the two.
Next, note that an arbitrary version of
\[ E_{P^x_{\omega}}[1_{A \times \omega_0} | \mathcal{C}_1 \otimes \mathcal{B}(W_0)](w_1, w_0) \quad \text{for} \quad (w_1, w_0) \in W_1 \times W_0 \] (3.8)
is also a version of the conditional expectation (3.7). Now since a regular conditional probability is a version of the conditional expectation,
\[ (P^x_{\omega})_{\omega_0}(A_1) \quad \text{for} \quad \omega_0 \in W_0 \] is a version of the conditional expectation (3.8). But according to Lemma 3.1
\[ (P^x_{\omega})_{\omega_0}(A_1) = Q^{x, \omega_0}(A_1) \quad \text{for a.e.} \quad \omega_0 \quad \text{in} \quad (W_0, \mathcal{B}(W_0), m_\omega). \]
Therefore the characteristic function in (3.6) and \( Q^{x, \omega_0}(A_1) \) for \( \omega_0 \in W_0 \) are two versions of the conditional expectation (3.7) so that there exists a null set \( M(x) \) in \( (W_0, \mathcal{B}(W_0), m_\omega) \) such that
\[ Q^{x, \omega_0}(A_1) = 1_{F^x_{\omega_0} \setminus \omega[A_1]}(w_0) \quad \text{for} \quad \omega_0 \in M(X). \]
Let \( a \in M(x)^c \) and \( b = F^x_{\omega_0} \setminus \omega[A_1] \). Then
\[ Q^{x, \omega_0}(\{b\}) - 1_{F^x_{\omega_0} \setminus \omega[A_1]}(w_0) \quad \text{for} \quad \omega_0 \in M(x)^c \]
so that in particular with \( \omega_0 = a \) we have
\[ Q^{x, \omega_0}(\{b\}) = 1_{F^x_{\omega_0} \setminus \omega[A_1]}(a) = 1. \]
Therefore we have shown that
\[ Q^{x, \omega_0}(\{F^x_{\omega_0} \setminus \omega[A_1]\}) = 1 \quad \text{i.e.,} \quad Q^{x, \omega_0} = \delta_{F^x_{\omega_0} \setminus \omega[A_1]} \quad \text{for every} \quad \omega_0 \in M(x)^c, \]
completing the proof. 1

LEMA 3.3. Under the same assumption on (1.1) as in Lemma 3.2, let
\[ A = \{ (x, w_0) \in \partial W \times W_0 ; Q^{x, \omega_0} \text{ is not a unit mass} \}. \] (3.9)
Then \( A \) is a null set in \( (\partial W \times W_0, \mathcal{B}(\partial W \times W_0), \mu \times m_\omega). \)

Proof. Since \( W_1 \) is a separable metric space, for every positive integer \( n \) there exist countably many closed balls \( B_{n,i}, i = 1, 2, \ldots \), each with diameter \( n^{-1} \), such that \( W_1 = \bigcup_{i=1}^{\infty} B_{n,i} \). Let us show that for an arbitrary probability measure \( v \) on \( (W_1, \mathcal{B}(W_1)) \) we have
\[ v \text{ is a unit mass} \iff v(\mathcal{B}(B_{n,i})) = 0 \text{ or } 1 \text{ for every } n \text{ and } i. \] (3.10)
Now if \( v \) is a unit mass, clearly \( v(B_{n,i}) = 0 \) or \( 1 \) for every \( n \) and \( i \). Assume conversely that \( v(B_{n,i}) = 0 \) or \( 1 \) for every \( n \) and \( i \). For fixed \( n \), no two closed balls \( B_{n,i} \) with \( v \) measure \( 1 \) can be disjoint for otherwise we would have
v(W_1) \geq 2$. Let $K_n$ be the closed set which is the intersection of those closed balls in the collection $\{B_{n,i}, i = 1, 2, \ldots\}$ with $v$ measure 1. Then $v(K_n) = 1$ and the diameter $\delta(K_n) \leq n^{-1}$. For the sequence of closed sets $K_n, n = 1, 2, \ldots$, we have $K_n \cap K_m \neq \emptyset$ by the same reason as above. Then $C_n, n = 1, 2, \ldots$, where $C_n = \bigcap_{m=1}^{n} K_m$ is a decreasing sequence of closed sets with $v(C_n) = 1$ and $\delta(C_n) \leq n^{-1}$ for every $n$. Since $W_1$ is a complete metric space and $\delta(C_n) \to 0$ as $n \to \infty$ there exists $w_1 \in W_1$ such that $\bigcap_{n=1}^{\infty} C_n = \{w_1\}$. Then $v(\{w_1\}) = \lim_{n \to \infty} v(C_n) = 1$. This proves (3.10).

By (3.10) we have

$$A^c = \{(x, w_0) \in \partial W \times W_0; Q^{x, w_0} \text{ is a unit mass}\}$$

$$= \{(x, w_0) \in \partial W \times W_0; Q^{x, w_0}(B_{n,i}) = 0 \text{ or } 1 \text{ for every } n \text{ and } i\}.$$ Since $B_{n,i} \in \mathcal{B}(W_1), Q^{x, w_0}(B_{n,i})$ is a $\mathcal{B}(\partial W \times W_0)$ measurable function of $(x, w_0) \in \partial W \times W_0$ by (Q2) for every $n$ and $i$ so that $A^c \in \mathcal{B}(\partial W \times W_0)$. Thus $A \in \mathcal{B}(\partial W \times W_0)$.

To show that $(\mu \times m_w)(A) = 0$, let $N_\mu$ and $M(x)$ for $x \in N_\mu^c$ be as in the statement of Lemma 3.2. For $x \in N_\mu^c$ and $w_0 \in M(x)^c$, we have $Q^{x, w_0} = \delta_{F[x, w_0]}$ by Lemma 3.2 so that $(x, w_0) \in A^c$ and thus $w_0 \in (A^c)_x$, the section of $A^c \subset \partial W \times W_0$ at $x \in \partial W$. Therefore $x \in N_\mu^c$ implies $M(x)^c \subset (A^c)_x$, i.e., $A_x \subset M(x)$. Thus

$$(\mu \times m_w)(A) = \int_{\partial W} m_w(A_x) \mu(dx) = \int_{N_\mu^c} m_w(A_x) \mu(dx)$$

$$\leq \int_{N_\mu^c} m_w(M(x)) \mu(dx) = 0$$

since $m_w(M(x)) = 0$ for $x \in N_\mu^c$. Therefore $(\mu \times m_w)(A) = 0$. $

**Lemma 3.4.** Under the same assumptions on the stochastic differential equation (1.1) as in Lemma 3.2 and Lemma 3.3 let $G$ be a transformation of $\partial W \times W_0$ into $W_1$ defined by

$$\delta_{G[x, w_0]} = Q^{x, w_0} \quad \text{for } (x, w_0) \in A^c \quad (3.11)$$

$$G[x, w_0] = 0 \in W_1 \quad \text{for } (x, w_0) \in A \quad (3.12)$$

and define a transformation $F_\mu$ of $\partial W \times W_0$ into $W_1$ by

$$F_\mu[x, w_0] = G[x, w_0] \quad \text{for } (x, w_0) \in N_\mu^c \times W_0, \quad (3.13)$$

$$F_\mu[x, w_0] = F[x, w_0] \quad \text{for } (x, w_0) \in N_\mu \times W_0. \quad (3.14)$$
Then

(1) \( G \) is \( \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) measurable,

(2) \( F_\mu \) satisfies the conditions (1) and (2) of Definition 1.4,

(3) \( F_\mu \) and \( F \) satisfy the condition (3) of Definition 1.4.

Proof. Equation (3.11) defines \( G[x, w_0] \) for \((x, w_0) \in A^c\) as the point in \( W_1 \) on which the unit mass \( Q^{x, w_0} \) concentrates. To prove the \( \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) measurability of \( G \) note that \( \mu, \mu' \in \mathcal{B}(aWxW_0) \) by Lemma 3.3 and for \( A, A' \in \mathcal{B}(W) \) we have on one hand

\[
G^{-1}(A) \cap A^c = \{(x, w_0) \in A^c; G[x, w_0] = w_1 \text{ for some } w_1 \in A_1\}
\]

by (Q.2) while on the other hand \( G^{-1}(A_1) \cap A = \emptyset \) or \( \phi \) depending as \( 0 \in A_1 \) or \( 0 \in A^c_1 \) so that in any case \( G^{-1}(A_1) \cap A \in \mathcal{B}(\partial W \times W_0) \). Therefore \( G^{-1}(A_1) \in \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) proving the \( \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) measurability of \( G \).

Since \( G \) is a \( \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) measurable transformation defined on \( \partial W \times W_0 \) and \( F_\mu = G \) on \( N_\mu \times W_0 \) by (3.13) and since \( N_\mu \times W_0 \) is a null set in \( (\partial W \times W_0, \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1)) \), \( F_\mu \) is \( \mathcal{B}(\partial W \times W_0) \cap \mathcal{B}(W_1) \) measurable, satisfying (1) in Definition 1.4.

To verify (2) of Definition 1.4 for \( F_\mu \), let \( x \in \partial W \) be fixed. If \( x \in N_\mu \) then by (3.14) and (F.1), \( F_\mu[x, \cdot] \) is \( \mathcal{B}(W_0) \cap \mathcal{B}(W_1) \) measurable for every \( z \in \mathbb{R}_+^2 \). If \( x \in N_\mu^c \) then \( F_\mu[x, \cdot] = G[x, \cdot] \) by (3.13). Now for fixed \( x \in \partial W \) and \( A \in \mathcal{B}(W) \), \( Q^{x, w_0}(A) \) is \( \mathcal{B}(W_0) \cap \mathcal{B}(W_1) \) measurable for every \( z \in \mathbb{R}_+^2 \) as can be shown by the same argument as in Lemma 1 of [12]. Also for \( x \in N_\mu^c \), \( A_x \) is a null set in \( (W_0, \mathcal{B}(W_0), m_W) \) as we saw in the proof of Lemma 3.3 so that \( A_x \in \mathcal{B}(W_0) \) for every \( z \in \mathbb{R}_+^2 \) by Definition 1.3. Therefore

\[
G[x, \cdot]^{-1}(A) \cap A^c = \{ w_0 \in A^c; Q^{x, w_0}(A_1) = 1 \} \in \mathcal{B}(W_0)
\]

while \( G[x, \cdot]^{-1}(A) \cap A_x = A_x \) or \( \phi \) depending as \( 0 \in A_1 \) or \( 0 \in A^c_1 \). Thus \( G[x, \cdot]^{-1}(A_1) \in \mathcal{B}(W_0) \) for every \( z \in \mathbb{R}_+^2 \) when \( A_1 \in \mathcal{B}(W_1) \) proving the \( \mathcal{B}(W_0) \cap \mathcal{B}(W_1) \) measurability of \( G[x, \cdot] \) and hence that of \( F_\mu[x, \cdot] \) for every \( z \in \mathbb{R}_+^2 \) when \( x \in N_\mu^c \). This verifies (2) of Definition 1.4 for our \( F_\mu \).

Finally, to verify (3) of Definition 1.4 recall that by (3.5), (3.9), and (3.11)

\[
\begin{align*}
x \in N_\mu^c \text{ and } w_0 \in M(x)^c \Rightarrow Q^{x, w_0} &= \delta_{F[x, w_0]} \\
\Rightarrow (x, w_0) \in A^c \\
\Rightarrow \delta_{G[x, w_0]} &= Q^{x, w_0}
\end{align*}
\]

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so that

$$F[x, w_0] = G[x, w_0] \quad \text{for} \quad x \in N_\mu^c \text{ and } w_0 \in M(x)^c.$$  \hspace{1cm} (3.15)

Then by (3.15) and (3.13)

$$F[x, w_0] = F_\mu[x, w_0] \quad \text{for} \quad x \in N_\mu^c \text{ and } w_0 \in M(x)^c$$

verifying (3) of Definition 1.4 for our $F$ and $F_\mu$. \[\square\]

**Theorem 3.5.** The stochastic differential equation (1.1) has a unique strong solution if and only if the following two conditions are satisfied:

(I) for every probability measure $\mu$ on $(\partial W, \mathcal{B}(\partial W))$, a solution $(\hat{X}, \hat{B})$ of (1.1) with $\mu$ as the probability distribution of $\partial \hat{X}$ exists on some filtered probability space $(\hat{\mathcal{F}}, \hat{\mathcal{P}}; \hat{\mathcal{F}}_\text{z})$,

(II) the pathwise uniqueness of solutions of (1.1) holds.

**Proof:** Necessity. Suppose a unique strong solution $F$ of (1.1) exists. Consider the filtered probability space $(\hat{\mathcal{F}}, \hat{\mathcal{P}}; \hat{\mathcal{F}}_\text{z})$, where $(\hat{\mathcal{F}}, \hat{\mathcal{P}}) = (W, \mathcal{B}(W), m_W)$ and $\hat{\mathcal{F}}_\text{z}$ is defined in the usual way (see (2.4) and (2.5)). Define an $\{\hat{\mathcal{F}}_\text{z}\}$-Brownian motion $\hat{B}$ with $\partial \hat{B} = 0$ on this filtered probability space by setting $\hat{B}(z, w) = w(z)$ for $(z, w) \in \mathbb{R}^3 \times W$. For an arbitrary probability measure $\mu$ on $(\partial W, \mathcal{B}(\partial W))$ let $\hat{Z}$ be a continuous $\{\hat{\mathcal{F}}_\text{z}\}$-adapted boundary process on the filtered probability space with $\mu$ as its probability distribution. If we set $\hat{X} = F[\hat{Z}, \hat{B}]$ then by (1) of Definition 1.6, $(\hat{X}, \hat{B})$ is a solution of (1.1) with $\partial \hat{X} = \hat{Z}$ on $(\hat{\mathcal{F}}, \hat{\mathcal{P}}; \hat{\mathcal{F}}_\text{z})$ so that (I) is satisfied.

To verify (II) suppose $(\hat{X}, \hat{B})$ and $(\hat{X}', \hat{B}')$ are two solutions on the same filtered probability space with $\hat{B} = \hat{B}'$ and $\partial \hat{X} = \partial \hat{X}'$. Then by (2) of Definition 1.6

$$\hat{X} = F[\partial \hat{X}, \hat{B}] = F[\partial \hat{X}', \hat{B}'] = \hat{X}',$$

which verifies the pathwise uniqueness of solutions of (1.1).

**Sufficiency.** Assume (I) and (II). Part (I) implies in particular that for every $x \in \partial W$ there exists a solution $(\hat{X}, \hat{B})$ to (1.1) with $\partial \hat{X} = x$ on some filtered probability space. This and (II) together imply according to Theorem in [12] the existence of a transformation $F$ of $\partial W \times W$ into $W$ satisfying (F.1)-(F.3). According to Lemma 3.4, $F$ and $F_\mu$ defined by (3.13) for every probability measure $\mu$ on $(\partial W, \mathcal{B}(\partial W))$ satisfy (1), (2), and (3) in Definition 1.4 so that $F \in S(\partial W \times W)$. It remains to show that $F$ satisfies (1) and (2) of Definition 1.6.

To verify (2) of Definition 1.6, let $(\hat{X}, \hat{B})$ be a solution of (1.1) on a filtered probability space $(\hat{\mathcal{F}}, \hat{\mathcal{P}}; \hat{\mathcal{F}}_\text{z})$ and let $\mu$ be the probability dis-
tribution of $\partial \hat{X}$. Let $(X, B)$ be the representation of $(\hat{X}, \hat{B})$ on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_2)$ constructed on $\Omega = W_1 \times W_0$ as in Lemma 2.1 so that $(X, B)$ is a solution of (1.1) on this filtered probability space. To show that $X = F_\mu[\partial X, B]$ recall that for $x \in N_\mu$ defined in the proof of Lemma 3.2, $(X, B)$ is a solution of (1.1) with $\partial X = x$ on the filtered probability space $(\Omega, \mathcal{F}, P_{\partial X}; \mathcal{F}_2)$ according to Theorem 2.2. Thus by (F.3)

$$X = F_\mu[x, B], P^x_{\partial X} \text{ almost surely for } x \in N_\mu. \quad (3.16)$$

Let

$$A = \{(w_1, w_0) \in W_1 \times W_0; F[\partial X, B] = X\}.$$

Since $F_\mu[\partial X, B]$ and $X$ are $\mathcal{B}(\overline{W_1 \times W_0})^{\mu \times m_w}/\mathcal{B}(W_1)$ measurable we have

$$A \in \mathcal{B}(\overline{W_1 \times W_0})^{\mu \times m_w}. \text{ Then by (P.3) and (3.16)}$$

$$P(A) = \int_{\partial W} P^x_{\partial X}(A) \mu(dx) = 0.$$

Thus $X = F_\mu[\partial X, B]$, a.s. (P). Since $(\hat{X}, \hat{B})$ has the same probability distribution as $(X, B)$ we have $\hat{X} = F_\mu[\partial \hat{X}, \hat{B}]$, a.s. (P). This verifies (2) of Definition 1.6.

To verify (1) of Definition 1.6, let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}; \hat{\mathcal{F}}_2)$ be a filtered probability space on which an $(\hat{\mathcal{F}}_2)$-Brownian motion with $\partial \hat{B} = 0$ exists. Let $\hat{Z}$ be a continuous $(\hat{\mathcal{F}}_2)$-adapted boundary process on this filtered probability space with a probability distribution $\mu$. By (I) there exists a solution $(\hat{X}, \hat{B})$ of (1.1) with $\mu$ as the probability distribution of $\partial \hat{X}$ on some filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}; \hat{\mathcal{F}}_2)$. Thus by Lemma 2.1 there exists a solution $(X, B)$ of (1.1) with $\mu$ as the probability distribution of $\partial X$ on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_2)$ constructed on $\Omega = W_1 \times W_0$ by (2.0)–(2.5). Then by (2) of Definition 1.6, which we have verified above, we have

$$X = F_\mu[\partial X, B] \quad \text{a.s. (P).} \quad (3.17)$$

Define

$$\hat{X} = F_\mu[\hat{Z}, \hat{B}]. \quad (3.18)$$

Since $(\partial X, B)$ and $(\hat{Z}, \hat{B})$ have the same probability distribution $\mu \times m_w$ on $(\partial W \times W_0, \mathcal{B}(\partial W \times W_0)), (X, B)$ and $(\hat{X}, \hat{B})$ have the same probability distribution on $(W_1 \times W_0 \mathcal{B}(W_1 \times W_0))$ so that $(\hat{X}, \hat{B})$ is a solution of (1.1) on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}; \hat{\mathcal{F}}_2)$. Finally, to show that $\partial \hat{X} = \hat{Z}$, let $\rho$ be the mapping of $W$ onto $\partial W$ defined by $\rho w = w|_{\epsilon \mathbb{R}_+^2}$ for $w \in W$. Then from (3.17)

$$(\rho \circ F_\mu)[\partial X, B] = \rho(X) = \partial X \quad \text{a.s. (P).} \quad (3.19)$$
Since \((\hat{X}, \hat{B})\) and \((X, B)\) have the same probability distribution which is our \(P\) by construction, we have for any null set \(A\) in \((W_1 \times W_0, \mathcal{B}(W_1 \times W_0), P)\)

\[
\hat{P}[(\hat{X}, \hat{B})^{-1}(A)] = P(A) = 0. 
\]  
(3.20)

Thus by (3.18), (3.19), and (3.20)

\[
\hat{X}(\cdot, \omega) = \rho(\hat{X}(\cdot, \hat{w})) = (\rho \circ F_\omega)[\hat{Z}(\cdot, \omega), \hat{B}(\cdot, \omega)] \\
= \hat{Z}(\cdot, \omega) \quad \text{for a.e. } \omega \text{ in } \hat{\Omega}, \hat{\mathcal{G}}, \hat{\mathcal{F}}.
\]

This completes the verification of (1) of Definition 1.6. [1]

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