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Superinjective simplicial maps of complexes of curves and injective homomorphisms of subgroups of mapping class groups

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Abstract

Let S be a closed, connected, orientable surface of genus at least 3, $\mathscr{C}(S)$ be the complex of curves on S and Mod_S^* be the extended mapping class group of S. We prove that a simplicial map, $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$, preserves nondisjointness (i.e. if α and β are two vertices in $\mathscr{C}(S)$ and $i(\alpha, \beta) \neq 0$, then $i(\lambda(\alpha), \lambda(\beta)) \neq 0$) iff it is induced by a homeomorphism of S. As a corollary, we prove that if K is a finite index subgroup of Mod_S^* and $f : K \to Mod_S^*$ is an injective homomorphism, then f is induced by a homeomorphism of S and f has a unique extension to an automorphism of Mod_S^*.

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1. Introduction

Let R be a compact orientable surface possibly having nonempty boundary. The mapping class group, Mod_R, of R is the group of isotopy classes of orientation preserving homeomorphisms of R. The extended mapping class group, Mod_R^*, of R is the group of isotopy classes of all (including orientation reversing) homeomorphisms of R.

In Section 2, we give the notations and the terminology that we use in the paper.

In Section 3, we introduce the notion of superinjective simplicial maps of the complex of curves, $\mathscr{C}(S)$, of a closed, connected, orientable surface S and prove some properties of these maps.

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In Section 4, we prove that a superinjective simplicial map, λ, of the complex of curves induces an injective simplicial map on the complex of arcs, $\mathscr{B}(S_c)$, where c is a nonseparating circle on S and we prove that λ is induced by a homeomorphism of S.

In Section 5, we prove that if K is a finite index subgroup of Mod_S^* and $f: K \to Mod_S^*$ is an injective homomorphism, then f induces a superinjective simplicial map of $\mathscr{C}(S)$ and f is induced by a homeomorphism of S.

Our main results are motivated by the following theorems of Ivanov and the theorem of Ivanov and McCarthy.

Theorem 1.1 (Ivanov [3]). *Let S be a compact, orientable surface possibly with nonempty boundary. Suppose that the genus of S is at least 2. Then, all automorphisms of $\mathscr{C}(S)$ are given by elements of Mod_S^*. More precisely, if S is not a closed surface of genus 2, then $Aut(\mathscr{C}(S)) = Mod_S^*$. If S is a closed surface of genus 2, then $Aut(\mathscr{C}(S)) = Mod_S^*/Center(Mod_S^*)$.*

Theorem 1.2 (Ivanov [3]). *Let S be a compact, orientable surface possibly with nonempty boundary. Suppose that the genus of S is at least 2 and S is not a closed surface of genus 2. Let Γ_1, Γ_2 be finite index subgroups of Mod_S^*. Then, all isomorphisms $\Gamma_1 \to \Gamma_2$ have the form $x \to gxg^{-1}, g \in Mod_S^*$.*

Theorems 1.1 and 1.2 were extended to all surfaces of genus 0 and 1 with the exception of spheres with $\leqslant 4$ holes and tori with $\leqslant 2$ holes by M. Korkmaz. These extensions were also obtained by F. Luo independently.

Theorem 1.3 (Ivanov and McCarthy [4]). *Let S and S' be compact, connected, orientable surfaces. Suppose that the genus of S is at least 2, S' is not a closed surface of genus 2, and the maxima of ranks of abelian subgroups of Mod_S and $Mod_{S'}$ differ by at most one. If $h: Mod_S \to Mod_{S'}$ is an injective homomorphism, then h is induced by a homeomorphism $H: S \to S'$, (i.e. $h([G])=[HGH^{-1}]$, for every orientation preserving homeomorphism $G: S \to S$).*

The main results of the paper are the following two theorems:

Theorem 1.4. *Let S be a closed, connected, orientable surface of genus at least 3. A simplicial map, $\lambda: \mathscr{C}(S) \to \mathscr{C}(S)$, is superinjective if and only if λ is induced by a homeomorphism of S.*

Theorem 1.5. *Let S be a closed, connected, orientable surface of genus at least 3. Let K be a finite index subgroup of Mod_S^* and f be an injective homomorphism, $f: K \to Mod_S^*$. Then f is induced by a homeomorphism of the surface S (i.e. $f(k) = gkg^{-1}$ for some $g \in Mod_S^*$) and f has a unique extension to an automorphism of Mod_S^*.*

Theorem 1.5 is deduced from Theorem 1.4. Theorem 1.4 generalizes the closed case of Ivanov's Theorem 1.1 for surfaces of genus at least 3. Theorem 1.5 generalizes Ivanov's Theorem 1.2 in the case of closed surfaces and Ivanov and McCarthy's Theorem 1.3 in the case when the surfaces are the same and closed. In our proof, some advanced homotopy results about $\mathscr{C}(S)$ used by Ivanov are replaced by elementary surface topology arguments.

2. Notations and preliminaries

A *circle* on a surface, R, of genus g with b boundary components is a properly embedded image of an embedding $S^1 \to R$. A circle on R is said to be *nontrivial* (or *essential*) if it does not bound a disk and it is not homotopic to a boundary component of R. Let C be a collection of pairwise disjoint circles on R. The surface obtained from R by cutting along C is denoted by R_C. Two circles a and b on R are called *topologically equivalent* if there exists a homeomorphism $F: R \to R$ such that $F(a) = b$. The isotopy class of a Dehn twist about a circle a, is denoted by t_α, where $[a] = \alpha$.

Let $\mathscr{A}$ denote the set of isotopy classes of nontrivial circles on R. The *geometric intersection number* $i(\alpha, \beta)$ of $\alpha, \beta \in \mathscr{A}$ is the minimum number of points of $a \cap b$ where $a \in \alpha$ and $b \in \beta$.

A mapping class, $g \in Mod_R^*$, is called *pseudo-Anosov* if $\mathscr{A}$ is nonempty and if $g^n(\alpha) \neq \alpha$, for all α in $\mathscr{A}$ and any $n \neq 0$. g is called *reducible* if there is a nonempty subset $\mathscr{B} \subseteq \mathscr{A}$ such that a set of disjoint representatives can be chosen for $\mathscr{B}$ and $g(\mathscr{B}) = \mathscr{B}$. In this case, $\mathscr{B}$ is called a *reduction system* for g. Each element of $\mathscr{B}$ is called a *reduction class* for g. A reduction class, α, for g, is called an *essential reduction class* for g, if for each $\beta \in \mathscr{A}$ such that $i(\alpha, \beta) \neq 0$ and for each integer $m \neq 0$, $g^m(\beta) \neq \beta$. The set, $\mathscr{B}_g$, of all essential reduction classes for g is called the *canonical reduction system* for g. The correspondence $g \to \mathscr{B}_g$ is canonical. In particular, it satisfies $g(\mathscr{B}_h) = \mathscr{B}_{ghg^{-1}}$ for all g, h in Mod_R^*.

The *complex of curves*, $\mathscr{C}(R)$, on R is an abstract simplicial complex, as given in [8], with vertex set $\mathscr{A}$ such that a set of n vertices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ forms an $n-1$ simplex if and only if $\alpha_1, \alpha_2, \ldots, \alpha_n$ have pairwise disjoint representatives.

An arc i on R is called *properly embedded* if $\partial i \subseteq \partial R$ and i is transversal to ∂R. i is called *nontrivial* (or *essential*) if i cannot be deformed into ∂R in such a way that the endpoints of i stay in ∂R during the deformation. The *complex of arcs*, $\mathscr{B}(R)$, on R is an abstract simplicial complex. Its vertices are the isotopy classes of nontrivial properly embedded arcs i in R. A set of vertices forms a simplex if these vertices can be represented by pairwise disjoint arcs.

A nontrivial circle a on a closed, connected, orientable surface S is called *k-separating* (or a *genus k circle*) if the surface S_a is disconnected and one of its components is a genus k surface where $1 \leqslant k < g$. If S_a is connected, then a is called *nonseparating*. Two circles on S are topologically equivalent if and only if they are either both nonseparating or k-separating for some k.

We assume that S is a closed, connected, orientable surface of genus $g \geqslant 3$ throughout the paper.

3. Superinjective simplicial maps of complexes of curves

Definition 1. A simplicial map $\lambda: \mathscr{C}(S) \to \mathscr{C}(S)$ is called *superinjective* if the following condition holds: if α, β are two vertices in $\mathscr{C}(S)$ such that $i(\alpha, \beta) \neq 0$, then $i(\lambda(\alpha), \lambda(\beta)) \neq 0$.

Lemma 3.1. *A superinjective simplicial map, $\lambda: \mathscr{C}(S) \to \mathscr{C}(S)$, is injective.*

Proof. Let α and β be two distinct vertices in $\mathscr{C}(S)$. If $i(\alpha, \beta) \neq 0$, then $i(\lambda(\alpha), \lambda(\beta)) \neq 0$, since λ preserves nondisjointness. So, $\lambda(\alpha) \neq \lambda(\beta)$. If $i(\alpha, \beta) = 0$, we choose a vertex γ of $\mathscr{C}(S)$ such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$. Then, $i(\lambda(\gamma), \lambda(\alpha)) = 0$, $i(\lambda(\gamma), \lambda(\beta)) \neq 0$. So, $\lambda(\alpha) \neq \lambda(\beta)$. Hence, λ is injective. $\square$

Lemma 3.2. *Let α, β be two distinct vertices of $\mathscr{C}(S)$, and let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Then, α and β are connected by an edge in $\mathscr{C}(S)$ if and only if $\lambda(\alpha)$ and $\lambda(\beta)$ are connected by an edge in $\mathscr{C}(S)$.*

Proof. Let α, β be two distinct vertices of $\mathscr{C}(S)$. By Lemma 3.1, λ is injective. So, $\lambda(\alpha) \neq \lambda(\beta)$. Then we have, α and β are connected by an edge $\Leftrightarrow i(\alpha, \beta) = 0 \Leftrightarrow i(\lambda(\alpha), \lambda(\beta)) = 0$ (by superinjectivity) $\Leftrightarrow \lambda(\alpha)$ and $\lambda(\beta)$ are connected by an edge. $\square$

Let P be a set of pairwise disjoint circles on S. P is called a *pair of pants decomposition* of S, if S_P is a disjoint union of genus zero surfaces with three boundary components, pairs of pants. A pair of pants of a pants decomposition is the image of one of these connected components under the quotient map $q : S_P \to S$ together with the image of the boundary components of this component. The image of the boundary of this component is called the *boundary of the pair of pants*. A pair of pants is called *embedded* if the restriction of q to the corresponding component of S_P is an embedding.

Lemma 3.3. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Let P be a pair of pants decomposition of S. Then, λ maps the set of isotopy classes of elements of P to the set of isotopy classes of elements of a pair of pants decomposition, P', of S.*

Proof. The set of isotopy classes of elements of P forms a top dimensional simplex, Δ, in $\mathscr{C}(S)$. Since λ is injective, it maps Δ to a top dimensional simplex Δ' in $\mathscr{C}(S)$. Pairwise disjoint representatives of the vertices of Δ' give a pair of pants decomposition P' of S. $\square$

By Euler characteristic arguments it can be seen that there exist exactly $3g-3$ circles and $2g-2$ pairs of pants in a pair of pants decomposition of S. An ordered set $(a_1, \ldots, a_{3g-3})$ is called an *ordered pair of pants decomposition* of S if $\{a_1, \ldots, a_{3g-3}\}$ is a pair of pants decomposition of S. Let P be a pair of pants decomposition of S. Let a and b be two distinct elements in P. Then, a is called *adjacent* to b w.r.t. P iff there exists a pair of pants in P which has a and b on its boundary.

Remark. Let P be a pair of pants decomposition of S. Let $[P]$ be the set of isotopy classes of elements of P. Let $\alpha, \beta \in [P]$. We say that α is adjacent to β w.r.t. $[P]$ if the representatives of α and β in P are adjacent w.r.t. P. By Lemma 3.3, λ gives a correspondence on the isotopy classes of elements of pair of pants decompositions of S. $\lambda([P])$ is the set of isotopy classes of a pair of pants decomposition which corresponds to P, under this correspondence.

Lemma 3.4. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Let P be a pair of pants decomposition of S. Then, λ preserves the adjacency relation for two circles, i.e. if $a, b \in P$ are two circles which are adjacent w.r.t. P and $[a] = \alpha$, $[b] = \beta$, then $\lambda(\alpha), \lambda(\beta)$ are adjacent w.r.t. $\lambda([P])$.*

Proof. Let P be a pair of pants decomposition of S. Let a, b be two adjacent circles in P and $[a] = \alpha$, $[b] = \beta$. Let P_o be a pair of pants of P, having a and b on its boundary. By Lemma 3.3, we can choose a pair of pants decomposition, P', such that $\lambda([P]) = [P']$.

Either P_o is embedded or nonembedded. In the case P_o is embedded, either a and b are the boundary components of another pair of pants or not. In the case P_o is nonembedded, either a or

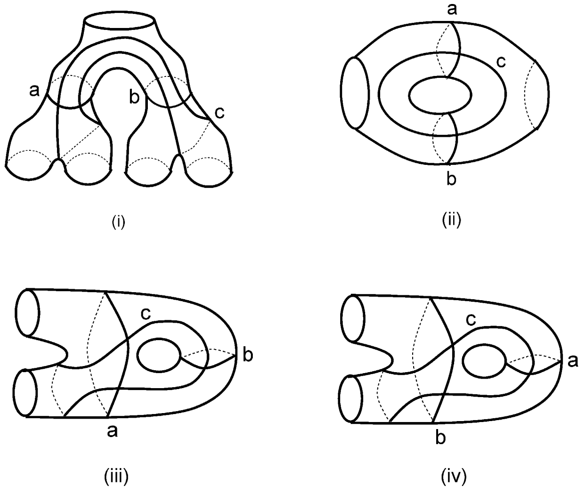

Fig. 1. Four possible cases for P_o.

b is a separating curve on S. So, there are four possible cases for P_o. For each of these cases, in Fig. 1, we show how to choose a circle c which essentially intersects a and b and does not intersect any other circle in P.

Let $\gamma = [c]$. Assume that $\lambda(\alpha)$ and $\lambda(\beta)$ do not have adjacent representatives. Since $i(\gamma, \alpha) \neq 0 \neq i(\gamma, \beta)$, we have $i(\lambda(\gamma), \lambda(\alpha)) \neq 0 \neq i(\lambda(\gamma), \lambda(\beta))$ by superinjectivity. Since $i(\gamma, [e]) = 0$ for all e in $P \setminus \{a, b\}$, we have $i(\lambda(\gamma), \lambda([e])) = 0$ for all e in $P \setminus \{a, b\}$. But this is not possible because $\lambda(\gamma)$ has to intersect geometrically essentially with some isotopy class other than $\lambda(\alpha)$ and $\lambda(\beta)$ in the image pair of pants decomposition to be able to make essential intersections with $\lambda(\alpha)$ and $\lambda(\beta)$. This gives a contradiction to the assumption that $\lambda(\alpha)$ and $\lambda(\beta)$ do not have adjacent representatives. □

Let P be a pair of pants decomposition of S. A curve $x \in P$ is called a 4-*curve* in P, if there exist four distinct circles in P, which are adjacent to x w.r.t. P.

Lemma 3.5. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Then, λ sends the isotopy class of a k-separating circle to the isotopy class of a k-separating circle, where* $1 \leqslant k \leqslant g - 1$.

Proof. Let $\alpha = [a]$ where a is a k-separating circle where $1 \leqslant k \leqslant g - 1$. Since the genus of S is at least 3, a is a separating curve of genus at least 2. So, it is enough to consider the cases when $k \geqslant 2$.

Case I: Assume that $k = 2$. Let S_2 be a subsurface of S of genus 2 having a as its boundary. We can choose a pair of pants decomposition $Q = \{a_1, a_2, a_3, a_4\}$ of S_2 as shown in Fig. 2(i). Then, we can complete $Q \cup \{a\}$ to a pair of pants decomposition P of S in any way we like. By Lemma 3.3, we can choose a pair of pants decomposition, P', of S such that $\lambda([P]) = [P']$.

Let a_i' be the representative of $\lambda([a_i])$ which is in P' for $i = 1, \ldots, 4$ and a' be the representative of $\lambda([a])$ which is in P'. Since a_1 and a_3 are 4-curves in P, by Lemmas 3.4 and 3.1, a_1' and a_3' are

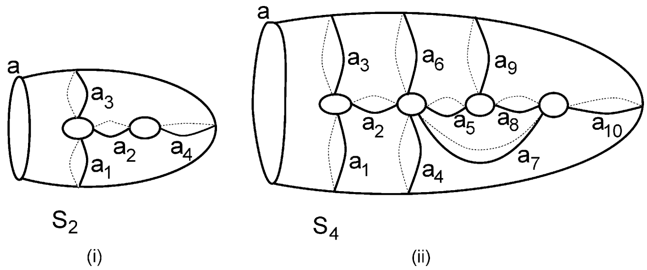

Fig. 2. Pants decompositions with separating circles.

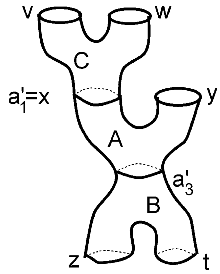

Fig. 3. Adjacent circles.

4-curves in P'. Notice that a curve in a pair of pants decomposition can be adjacent to at most four curves w.r.t. the pair of pants decomposition.

a'_3 is a 4-curve. So, there are two pairs of pants, A and B, of P' such that each of them has a'_3 on its boundary. Let x, y, z, t be as shown in Fig. 3. Since a'_3 is a 4-curve and x, y, z, t are the only curves which are adjacent to a'_3 w.r.t. P', x, y, z, t are four distinct curves.

a_3 is adjacent to a_1 w.r.t. P implies that a'_3 is adjacent to a'_1 w.r.t. P'. Then, W.L.O.G. we can assume that $x = a'_1$. So, a'_1 is a boundary component of A. Since a'_1 is a 4-curve in P', a'_1 is also a boundary component of a pair of pants different from A. Since $a'_1 \neq z, a'_1 \neq t$ and $a'_1 \neq a'_3, a'_1$ is not a boundary component of B. So, there is a new pair of pants, C, which has a'_1 on its boundary. Let v, w be as shown in Fig. 3. Since a'_1 is a 4-curve and y, v, w, a'_3 are the only curves which are adjacent to a'_1 w.r.t. P', y, v, w, a'_3 are four distinct curves. Since a_1 is adjacent to each of a, a_2, a_3, a_4 w.r.t. P, a'_1 is adjacent to each of a', a'_2, a'_3, a'_4 w.r.t. P'. Then, $\{y, v, w, a'_3\} = \{a', a'_2, a'_3, a'_4\}$ and, so, $\{y, v, w\} = \{a', a'_2, a'_4\}$. Similarly, since a_3 is adjacent to each of a, a_1, a_2, a_4 w.r.t. P, a'_3 is adjacent to each of a', a'_1, a'_2, a'_4 w.r.t. P'. Then, since each of x, y, z, t is adjacent to a'_3, we have $\{x, y, z, t\} = \{a', a'_1, a'_2, a'_4\}$. Since $x = a'_1, \{y, z, t\} = \{a', a'_2, a'_4\} = \{y, v, w\}$. Then $\{z, t\} = \{v, w\}$. Hence, $A \cup B \cup C$ is a genus 2 subsurface of S having y as its boundary.

Since $\{y, v, w\} = \{a', a'_2, a'_4\}, y$ is a' or a'_2 or a'_4. Assume that $y = a'_2$. Then, a'_2 cannot be adjacent to a'_4 w.r.t. P' as it can be seen from Figure 3. But this gives a contradiction since a_2 is adjacent to a_4 w.r.t. P, a'_2 must also be adjacent to a'_4 w.r.t. P'. Similarly, if $y = a'_4$, we get a contradiction since a'_4 would not be adjacent to a'_2 w.r.t. P', but it should be as a_4 is adjacent to a_2 w.r.t. P. Hence, $y = a'$. This proves that a' is a genus 2 curve.

Case II: Assume that $k \geqslant 3$. Let S_k be a subsurface of S of genus k having a as its boundary. We can choose a pair of pants decomposition $Q = \{a_1, a_2, \ldots, a_{3k-2}\}$ of S_k, and then complete $Q \cup \{a\}$ to a pair of pants decomposition P of S such that each of a_i is a 4-curve in P for $i = 1, 2, \ldots, 3k-2$ and a, a_1, a_3 are the boundary components of a pair of pants of Q. In Fig. 2(ii), we show how to choose Q when $k = 4$. In the other cases, when $k = 3$ or $k \geqslant 5$, a similar pair of pants decomposition of S_k can be chosen.

Let P' be a pair of pants decomposition of S such that $\lambda([P]) = [P']$. Let a'_i be the representative of $\lambda([a_i])$ which is in P', for $i = 1, \ldots, 3k-2$, and a' be the representative of $\lambda([a])$ which is in P'. Let $Q' = \{a'_1, a'_2, \ldots, a'_{3k-2}\}$.

For every 4-curve x in P', there exist two pairs of pants $A(x)$ and $B(x)$ of P' having x as one of their boundary components. Let $C(x) = A(x) \cup B(x)$. The boundary of $C(x)$ consists of four distinct curves, which are adjacent to x w.r.t. P'. Since all the curves $a_1, \ldots, a_{3k-2}$ in Q are 4-curves in P, all the corresponding curves $a'_1, \ldots, a'_{3k-2}$ are 4-curves in P' by Lemmas 3.4 and 3.1. Let $N' = C(a'_1) \cup C(a'_2) \cup \cdots \cup C(a'_{3k-2})$.

Claim. *N' is a genus k subsurface of S having a' as its boundary.*

Proof. Each boundary component of $C(a'_i)$ is either a' or a'_j for some $j = 1, \ldots, 3k-2$. Since a'_i is in the interior of $C(a'_i), a'_i$ is in the interior of N' for $i = 1, 2, \ldots, 3k-2$. So, all the boundary components of $C(a'_i)$ which are different from a' are in the interior of N'. Hence, N' has at most one boundary component, which could be a'. We know that a' is adjacent to two distinct curves, a'_1, a'_3, w.r.t. P' since a is adjacent to a_1, a_3 w.r.t. P. Suppose that a' is in the interior of N'. Then a' is only adjacent to curves w.r.t. P' which are in Q'. On the other hand, being adjacent to a'_1, a'_3, it has to be adjacent to a third curve, a'_j in Q', w.r.t. P'. But each of such curves has four distinct adjacent curves w.r.t. P' which are different from a' already since for $j \neq 1, 3$, each of a_j has four distinct adjacent curves w.r.t. P, which are different from a by our choice of P. So, a' cannot be adjacent to a'_j w.r.t. P' when $j \neq 1, 3$. Hence, a' is not in the interior of N'. It is on the boundary of N'. So, N' is a subsurface of S having a' as its boundary. To see that N' has genus k, it is enough to realize that $\{a'_1, \ldots, a'_{3k-2}\}$ is a pair of pants decomposition of N'. Hence, a' is a k-separating circle. □

Lemma 3.6. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Let t be a k-separating circle on S, where $1 \leqslant k \leqslant g-1$. Let S_1, S_2 be the distinct subsurfaces of S of genus k and $g-k$, respectively, which have t as their boundary. Let $t' \in \lambda([t])$. Then t' is a k-separating circle and there exist subsurfaces S'_1, S'_2 of S of genus k and $g-k$, respectively, which have t' as their boundary such that $\lambda(\mathscr{C}(S_1)) \subseteq \mathscr{C}(S'_1)$ and $\lambda(\mathscr{C}(S_2)) \subseteq \mathscr{C}(S'_2)$.*

Proof. Let t be a k-separating circle where $1 \leqslant k \leqslant g-1$. Since the genus of S is at least 3, t is a separating curve of genus at least 2. So, it is enough to consider the cases when $k \geqslant 2$.

Let S_1, S_2 be the distinct subsurfaces of S of genus k and $g-k$, respectively, which have t as their boundary. Let $t' \in \lambda([t])$. By Lemma 3.5, t' is a k-separating circle. As we showed in the proof of Lemma 3.5, there is a pair of pants decomposition P_1 of S_1, and $P_1 \cup \{t\}$ can be completed to a pair of pants decomposition P of S such that a set of curves, P'_1, corresponding (via λ) to the curves in P_1, can be chosen such that P'_1 is a pair of pants decomposition of a subsurface that has t' as its boundary. Let S_1' be this subsurface. Let S_2' be the other subsurface of S which has t' as its boundary. A pairwise disjoint representative set, P', of $\lambda([P])$ containing $P_1' \cup \{t'\}$ can be chosen. Then, by Lemma 3.3, P' is a pair of pants decomposition of S. Let $P_2 = P \setminus (P_1 \cup t)$ and $P_2' = P' \setminus (P_1' \cup t')$. Then P_2, P_2' are pair of pants decompositions of S_2, S_2', respectively, as P_1, P_1' are pair of pants decompositions of S_1, S_1', respectively.

Now, let α be a vertex in $\mathscr{C}(S_1)$. Then, either $\alpha \in [P_1]$ or α has a nonzero geometric intersection with an element of $[P_1]$. In the first case, clearly $\lambda(\alpha) \in \mathscr{C}(S_1')$ since elements of $[P_1]$ correspond to elements of $[P_1'] \subseteq \mathscr{C}(S_1')$. In the second case, since λ preserves zero and nonzero geometric intersection (since λ is superinjective) and α has zero geometric intersection with the elements of $[P_2]$ and $[t]$ and nonzero intersection with an element of $[P_1], \lambda(\alpha)$ has zero geometric intersection with elements of $[P_2']$ and $[t']$ and nonzero intersection with an element of $[P_1']$. Then, $\lambda(\alpha) \in \mathscr{C}(S_1')$. Hence, $\lambda(\mathscr{C}(S_1)) \subseteq \mathscr{C}(S_1')$. The proof of $\lambda(\mathscr{C}(S_2)) \subseteq \mathscr{C}(S_2')$ is similar. □

Lemma 3.7. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Then λ preserves topological equivalence of ordered pairs of pants decompositions of S, (i.e. for a given ordered pair of pants decomposition $P = (c_1, c_2, \dots, c_{3g-3})$ of S, and a corresponding ordered pair of pants decomposition $P' = (c_1', c_2', \dots, c_{3g-3}')$ of S, where $[c_i'] = \lambda([c_i])$ $\forall i = 1, 2, \dots, 3g-3$, there exists a homeomorphism $F : S \to S$ such that $F(c_i) = c_i'$ $\forall i = 1, 2, \dots, 3g-3$).*

Proof. Let P be a pair of pants decomposition of S and A be a nonembedded pair of pants in P. The boundary of A consists of the circles x, y where x is a 1-separating circle on S and y is a nonseparating circle on S. Let R be the subsurface of S of genus $g-1$ which is bounded by x and let T be the subsurface of S of genus 1 which is bounded by x. Let P_1 be the set of elements of $P \setminus \{x\}$ which are on R and P_2 be the set of elements of $P \setminus \{x\}$ which are on T. Then, P_1, P_2 are pair of pants decompositions of R, T, respectively. So, $P_2 = \{y\}$ is a pair of pants decomposition of T. By Lemma 3.6, there exists a 1-separating circle $x' \in \lambda([x])$ and subsurfaces T', R', of S, of genus 1 and $g-1$, respectively, such that $\lambda(\mathscr{C}(R)) \subseteq \mathscr{C}(R')$ and $\lambda(\mathscr{C}(T)) \subseteq \mathscr{C}(T')$. Since $[P_1] \subseteq \mathscr{C}(R)$, we have $\lambda([P_1]) \subseteq \mathscr{C}(R')$. Since $[P_2] \subseteq \mathscr{C}(T)$, we have $\lambda([P_2]) \subseteq \mathscr{C}(T')$. Since λ preserves disjointness, we can see that a set, P_1', of pairwise disjoint representatives of $\lambda([P_1])$ disjoint from x' can be chosen. By counting the number of curves in P_1', we can see that P_1' is a pair of pants decomposition of R'. Similarly, a set, P_2', of pairwise disjoint representatives of $\lambda([P_2])$ disjoint from x' can be chosen. By counting the number of curves in P_2', we can see that P_2' is a pair of pants decomposition of T'. Since P_2 has one element, y, P_2' has one element. Let $y' \in P_2'$. Since x', y' correspond to x, y, respectively, and y and y' give pair of pants decompositions on T and T' (which are both nonembedded pairs of pants) and x and x' are the boundaries of R and R', we see that λ "sends" a nonembedded pair of pants to a nonembedded pair of pants.

Let B be an embedded pair of pants of P. Let $x, y, z \in P$ be the boundary components of B. We consider two cases:

(i) At least one of x, y or z is a separating circle.
(ii) All of x, y, z are nonseparating circles.

In the first case, W.L.O.G assume that x is a k-separating circle for $1 \leqslant k < g$. Let S_1, S_2 be the distinct subsurfaces of S of genus k and $g-k$, respectively, which have x as their boundary.

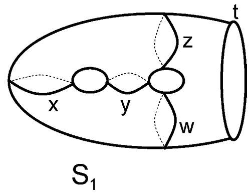

Fig. 4. Nonseparating circles, x, y, z, bounding a pair of pants.

W.L.O.G. assume that y, z are on S_2. Let $x' \in \lambda([x])$. By Lemma 3.6, there exist subsurfaces, S'_1, S'_2, of S of genus k and $g-k$, respectively, which have x' as their boundary such that $\lambda(\mathscr{C}(S_1)) \subseteq \mathscr{C}(S'_1)$ and $\lambda(\mathscr{C}(S_2)) \subseteq \mathscr{C}(S'_2)$. Then, since $y \cup z \subseteq S_2, \lambda(\{[y],[z]\}) \subseteq \mathscr{C}(S'_2)$. Let $y' \in \lambda([y]), z' \in \lambda([z])$ such that $\{x', y', z'\}$ is pairwise disjoint. Let P' be a set of pairwise disjoint representatives of $\lambda([P])$ which contains x', y', z'. P' is a pair of pants decomposition of S. Then, since x is adjacent to y and z w.r.t. P, x' is adjacent to y' and z' w.r.t. P' by Lemma 3.4. Then, since $x' \cup y' \cup z' \subseteq S'_2$, and x' is the boundary of S'_2, there is an embedded pair of pants in S'_2 which has x', y', z' on its boundary. So, λ “sends” an embedded pair of pants bounded by x, y, z to an embedded pair of pants bounded by x', y', z' in this case.

In the second case, we can find a nonseparating circle w and a 2-separating circle t on S such that $\{x, y, z, w\}$ is pairwise disjoint and x, y, z, w are on a genus 2 subsurface, S_1, that t bounds as shown in Fig. 4. Let $P_1 = \{x, y, z, w\}$. P_1 is a pair of pants decomposition of S_1. We can complete $P_1 \cup \{t\}$ to a pants decomposition P of S.

Let S_2 be the subsurface of S of genus $g-2$ which is not equal to S_1 and that is bounded by t. By Lemma 3.6, there exist a 2-separating circle $t' \in \lambda([t])$ and subsurfaces, S'_1, S'_2, of S of genus 2 and $g-2$, respectively, such that $\lambda(\mathscr{C}(S_1)) \subseteq \mathscr{C}(S'_1)$ and $\lambda(\mathscr{C}(S_2)) \subseteq \mathscr{C}(S'_2)$. Since $P_1 \subseteq S_1, \lambda([P_1]) \subseteq \mathscr{C}(S'_1)$. We can choose a set, P'_1, of pairwise disjoint representatives of $\lambda([P_1])$ on S'_1. Then, $P'_1 \cup \{t'\}$ is a pair of pants decomposition of S'_1. We can choose a pairwise disjoint representative set, P', of $\lambda([P])$ containing P'_1. P' is a pair of pants decomposition of S. Let $x', y', z', w' \in P'_1$ be the representatives of x, y, z, w, respectively. Then, since t is adjacent to z and w w.r.t. P, t' is adjacent to z' and w' w.r.t. P' by Lemma 3.4. Then, since $t' \cup z' \cup w' \subseteq S'_1$ and t' is the boundary of S'_1, there is an embedded pair of pants in S'_1 which has t', z', w' on its boundary. Since z is a 4-curve in P, z' is a 4-curve in P'. Since z is adjacent to x, y w.r.t. P, z' is adjacent to x', y' w.r.t. P'. Since z' is on the boundary of a pair of pants which has w', t' on its boundary, and z' is adjacent to x', y', there is a pair of pants having x', y', z' on its boundary. So, λ “sends” an embedded pair of pants bounded by x, y, z to an embedded pair of pants bounded by x', y', z' in this case too.

Assume that $P = (c_1, c_2, \ldots, c_{3g-3})$ is an ordered pair of pants decomposition of S. Let $c'_i \in \lambda([c_i])$ such that the elements of $\{c'_1, c'_2, \ldots, c'_{3g-3}\}$ are pairwise disjoint. Then, $P' = (c'_1, c'_2, \ldots, c'_{3g-3})$ is an ordered pair of pants decomposition of S. Let $(B_1, B_2, \ldots, B_{2g-2})$ be an ordered set containing the connected components of S_P. By the arguments given above, there is a corresponding, “image”, ordered collection of pairs of pants $(B'_1, B'_2, \ldots, B'_{2g-2})$. Nonembedded pairs of pants correspond to nonembedded pairs of pants and embedded pairs of pants correspond to embedded pairs of pants. Then, by the classification of surfaces, there exists an orientation preserving homeomorphism $h_i : B_i \to B'_i$, for all $i = 1, \ldots, 2g-2$. We can compose each h_i with an orientation preserving homeomorphism r_i which switches the boundary components, if necessary, to get $h'_i = r_i \circ h_i$ to agree with the

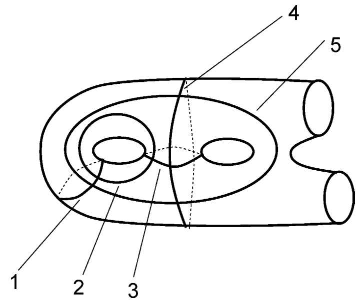

Fig. 5. Circles intersecting once.

correspondence given by λ on the boundary components, (i.e. for each boundary component a of B_i for $i=1,\ldots,2g-2, \lambda([q(a)])=[q'(h'_i(a))]$ where $q:S_P \to S$ and $q':S_{P'} \to S$ are the natural quotient maps). Then for two pairs of pants with a common boundary component, we can glue the homeomorphisms by isotoping the homeomorphism of the one pair of pants so that it agrees with the homeomorphism of the other pair of pants on the common boundary component. By adjusting these homeomorphisms on the boundary components and gluing them we get a homeomorphism $F:S\to S$ such that $F(c_i)=c'_i$ for all $i=1,2,\ldots,3g-3$. □

Remark. Let $\mathscr{E}$ be an ordered set of vertices of $\mathscr{C}(S)$ having a pairwise disjoint representative set E. Then, E can be completed to an ordered pair of pants decomposition, P, of S. We can choose an ordered pairwise disjoint representative set, P', of $\lambda([P])$ by Lemma 3.3. Let E' be the elements of P' which correspond to the elements of E. By Lemma 3.7, P and P' are topologically equivalent as ordered pants decompositions. Hence, the set E and E' are topologically equivalent. So, λ gives a correspondence which preserves topological equivalence on a set which has pairwise disjoint representatives. In particular, λ "sends" the isotopy class of a nonseparating circle to the isotopy class of a nonseparating circle.

We use the following lemma to understand some more properties of superinjective simplicial maps.

Lemma 3.8 (Ivanov [3]). *Let α_1 and α_2 be two vertices in $\mathscr{C}(S)$. Then, $i(\alpha_1,\alpha_2)=1$ if and only if there exist isotopy classes $\alpha_3,\alpha_4,\alpha_5$ such that*

(i) *$i(\alpha_i,\alpha_j)=0$ if and only if the ith and jth circles on Fig. 5 are disjoint.*
(ii) *if α_4 is the isotopy class of a circle C_4, then C_4 divides S into two pieces, and one of these is a torus with one hole containing some representatives of the isotopy classes α_1,α_2.*

Lemma 3.9. *Let $\lambda:\mathscr{C}(S)\to\mathscr{C}(S)$ be a superinjective simplicial map. Let α,β be two vertices of $\mathscr{C}(S)$. If $i(\alpha,\beta)=1$, then $i(\lambda(\alpha),\lambda(\beta))=1$.*

Proof. Let α,β be two vertices of $\mathscr{C}(S)$ such that $i(\alpha,\beta)=1$. Then, by Ivanov's Lemma, there exist isotopy classes $\alpha_3,\alpha_4,\alpha_5$ such that $i(\alpha_i,\alpha_j)=0$ if and only if ith, jth circles in Fig. 5 are disjoint and if α_4 is the isotopy class of a circle C_4, then C_4 divides S into two pieces, and one of these is a torus with one hole containing some representative of the isotopy classes α_1, α_2. Then, since λ is superinjective $i(\lambda(\alpha_i), \lambda(\alpha_j)) = 0$ if and only if ith, jth circles in Fig. 5 are disjoint, and by Lemma 3.6, if $\lambda(\alpha_4)$ is the isotopy class of a circle C'_4, then C'_4 divides S into two pieces, and one of these is a torus with one hole containing some representative of the isotopy classes $\lambda(\alpha_1), \lambda(\alpha_2)$. Then, by Ivanov's Lemma, $i(\lambda(\alpha), \lambda(\beta)) = 1$. $\square$

4. Induced map on complex of arcs

In this section, we assume that $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ is a superinjective simplicial map, c, d are nonseparating circles on S, and $[d] = \lambda([c])$. Let $\mathscr{V}(S_c)$ and $\mathscr{V}(S_d)$ be the sets of vertices of $\mathscr{B}(S_c)$ and $\mathscr{B}(S_d)$, respectively. We prove that λ induces a map $\lambda_* : \mathscr{V}(S_c) \to \mathscr{V}(S_d)$ with certain properties. Then we prove that λ_* extends to an injective simplicial map $\lambda_* : \mathscr{B}(S_c) \to \mathscr{B}(S_d)$.

First, we prove some lemmas which we use to see some properties of λ.

Lemma 4.1. *Let a and b be two disjoint arcs on S_c connecting the two boundary components, ∂_1, ∂_2, of S_c. Let N be a regular neighborhood of $a \cup b \cup \partial_1 \cup \partial_2$ in S_c. Then, $(N, a, b) \cong (S^2_4, a_o, b_o)$ where S^2_4 is a standard sphere with four holes and a_o, b_o are arcs as shown in* Fig. 6.

Proof. Let $\Gamma = \partial_1 \cup \partial_2 \cup a \cup b$. Let N be a regular neighborhood of Γ in S_c. N deformation retracts onto Γ. So, N and Γ have the same Euler characteristics. Let m be the genus of N and n be the number of boundary components of N. It is easy to see that $\chi(\Gamma) = -2$ (Fig. 7). Then, $-2 = \chi(N) = 2 - 2m - n$. So, $2m + n = 4$.

There are three possibilities for m and n.

(i) $m = 0,\ n = 4$, (ii) $m = 1,\ n = 2$, (iii) $m = 2,\ n = 0$.

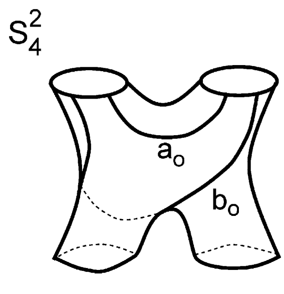

Fig. 6. Unlinked disjoint arcs of type 2.

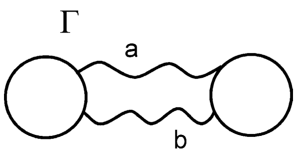

Fig. 7. The graph of a, b and the boundary components.

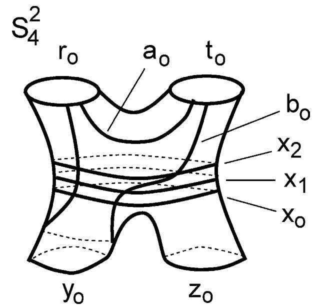

Fig. 8. Disjoint arcs on pairs of pants.

Since there are 2 boundary components corresponding to ∂_1, ∂_2, none of which is the boundary component on a given side of the arc a, n is at least 3. So, only (i) can hold. Therefore, N is homeomorphic to S_4^2, a sphere with four holes.

By using Euler characteristic arguments we can see that a regular neighborhood of $\partial_1 \cup \partial_2 \cup a$ in N is homeomorphic to a pair of pants, P. Let w be the boundary component of P different from ∂_1, ∂_2.

Let Γ' be the graph that we get when we contract ∂_1 and ∂_2 to two points. If a is isotopic to b in N, the two arcs corresponding to a and b on Γ' should be homotopic to each other relative to these two points. But this gives a contradiction since the arcs intersect only at these end points and the union of the arcs is γ' which is a circle and two such arcs cannot be homotopic relative to their end points on a circle. So, a is not isotopic to b in N. Since b connects the two boundary components ∂_1 and ∂_2 and $\partial_1, \partial_2 \in P, b \cap P$ is nonempty. W.L.O.G. assume that b intersects the boundary components of P transversely and does not intersect a. Since N is a regular neighborhood of $a \cup b \cup \partial_1 \cup \partial_2$ and b is a properly embedded essential arc which is nonisotopic to a in $N, b \cap P$ contains exactly two essential properly embedded arcs, let us call them b_1, b_2. One of them starts on ∂_1 and ends on w and the other one starts on ∂_2 and ends on w. Let P' be a regular neighborhood of $\partial_1 \cup \partial_2 \cup a$ in P such that $P' \cap b = b_1 \cup b_2$. Let x be the boundary component of P', which is different from ∂_1, ∂_2. We choose this neighborhood to get rid of the possible inessential arcs of b in P. P' is a pair of pants. Since N is a sphere with four holes, the complement of P' in N is a pair of pants, R. Let y and z be the boundary components of R which are different from x. Then, we have a homeomorphism $\phi : (N, \partial_1, \partial_2, x, y, z) \to (S_4^2, r_o, t_o, x_o, y_o, z_o)$ where r_o, t_o, x_o, y_o, z_o are as shown in Fig. 8.

Let P_o be the pair of pants bounded by r_o, t_o, x_o. Let x_1, x_2 be two parallel copies of x_o in P_o as shown in Fig. 8 such that each of them intersects $\phi(b)$ transversely at exactly 2 points and none of x_1, x_2 intersects $\phi(a)$. Let A_o be the annulus which is bounded by x_1 and x_2, and B_o be the annulus which is bounded by x_o, x_1. Let Q_o be the pair of pants bounded by r_o, t_o, x_2.

By the classification of properly embedded essential arcs on a pair of pants, there exists an isotopy on Q_o carrying $\phi(a)$ and $Q_o \cap \phi(b)$ to $Q_o \cap a_o$ and $Q_o \cap b_o$, respectively, where a_o, b_o are as shown in the figure. Let $\kappa : Q_o \times I \to Q_o$ be such an isotopy. We can extend κ to $\tilde{\kappa} : (Q_o \cup A_o) \times I \to (Q_o \cup A_o)$ so that $\tilde{\kappa}_t$ is id on x_1 for all $t \in I$.

Let R_o be the pair of pants bounded by x_o, y_o, z_o. Suppose that $R_o \cap \phi(b)$ is an inessential arc in R_o. Then it can be deformed into the interior of P_o and then we get $\phi(a)$ is isotopic to $\phi(b)$ in P_o

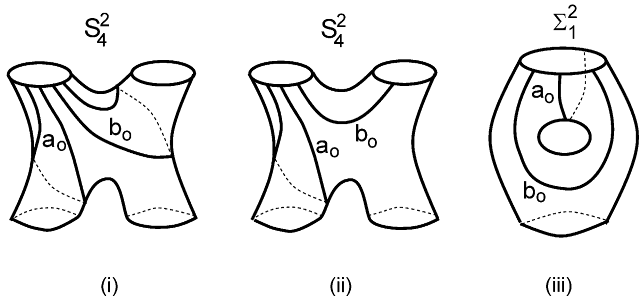

Fig. 9. Disjoint arcs and neighborhoods.

which implies that a is isotopic to b in N. This gives a contradiction. So, $b \cap R_o$ is an essential arc in R_o. Then, by the classification of properly embedded essential arcs on a pair of pants there exists an isotopy carrying $R_o \cap \phi(b)$ to $R_o \cap b_o$. Let $\tau : R_o \times I \to R_o$ be such an isotopy. We can extend τ to $\tilde{\tau} : (R_o \cup B_o) \times I \to (R_o \cup B_o)$ so that $\tilde{\tau}_t$ is *id* on x_1 for all $t \in I$. Then, by gluing the extensions $\tilde{\kappa}$ and $\tilde{\tau}$ we get an isotopy ϑ on $S_4^2 \times I$ which fixes each of r_o, t_o, x_o, y_o, z_o. By the classification of isotopy classes of arcs relative to the boundary on an annulus, $\vartheta_1(\phi(b)) \cap (A_o \cup B_o)$ can be isotoped to $t_{x_o}^k(b_o) \cap (A_o \cup B_o)$ for some $k \in \mathbb{Z}$. Let us call this isotopy μ. Let $\tilde{\mu}$ denote the extension by id to N. Then we have, $t_{x_o}^{-k}(\tilde{\mu}_1(\vartheta_1(\phi(b)))) = b_o$. Clearly, $t_{x_o}^{-k} \circ \tilde{\mu}_1 \circ \vartheta_1$ fixes each of r_o, t_o, x_o, y_o, z_o. Hence, we get a homeomorphism, $t_{x_o}^{-k} \circ \tilde{\mu}_1 \circ \vartheta_1 \circ \phi : (N, a, b) \to (S_4^2, a_o, b_o)$. □

Let a and b be two disjoint arcs connecting a boundary component of S_c to itself. Then, a and b are called *linked* if their end points alternate on the boundary component. Otherwise, they are called *unlinked*.

The proofs of Lemmas 4.2–4.4 are similar to the proof of Lemma 4.1. So, we do not prove these lemmas here. We only state them.

Lemma 4.2. *Let a and b be two disjoint arcs which are unlinked, connecting one boundary component ∂_k of S_c to itself for $k = 1, 2$. Let N be a regular neighborhood of $a \cup b \cup \partial_k$ on S_c. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where a_o, b_o are the arcs drawn on a standard sphere with four holes, S_4^2, as shown in* Fig. 9(i).

Lemma 4.3. *Let a and b be two disjoint arcs on S_c such that a connects one boundary component ∂_k of S_c to itself for some $k = 1, 2$ and b connects the boundary components ∂_1 and ∂_2 of S_c. Let N be a regular neighborhood of $a \cup b \cup \partial_1 \cup \partial_2$. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where a_o, b_o are the arcs drawn on a standard sphere with four holes, S_4^2, as shown in Fig.* 9(ii).

Lemma 4.4. *Let a and b be two disjoint, linked arcs connecting one boundary component ∂_k of S_c to itself for $k = 1, 2$. Let N be a regular neighborhood of $a \cup b \cup \partial_k$. Then, $(N, a, b) \cong (\Sigma_1^2, a_o, b_o)$ where Σ_1^2 is a standard surface of genus one with two boundary components, and a_o, b_o are as shown in Fig.* 9(iii).

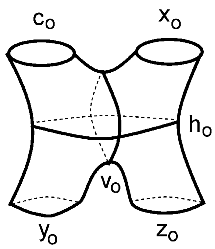

Fig. 10. “Horizontal” and “vertical” circles.

Let M be a sphere with k holes and $k \geqslant 5$. A circle a on M is called an *n-circle* if a bounds a disk with n holes on M where $n \geqslant 2$. If a is a 2-circle on M, then there exists up to isotopy a unique nontrivial embedded arc a' on the two-holed disk component of M_a joining the two holes in this disc. If a and b are two 2-circles on M such that the corresponding arcs a', b' can be chosen to meet exactly at one common end point, and $\alpha = [a]$, $\beta = [b]$, then $\{\alpha, \beta\}$ is called a *simple pair*. A *pentagon* in $\mathscr{C}(\mathscr{M})$ is an ordered 5-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, defined up to cyclic permutations, of vertices of $\mathscr{C}(\mathscr{M})$ such that $i(\alpha_j, \alpha_{j+1}) = 0$ for $j = 1, 2, \ldots, 5$ and $i(\alpha_j, \alpha_k) \neq 0$ otherwise, where $\alpha_6 = \alpha_1$. A vertex in $\mathscr{C}(\mathscr{M})$ is called an *n-vertex* if it has a representative which is an n-circle on M. Let M' be the interior of M. There is a natural isomorphism $\chi : \mathscr{C}(M') \to \mathscr{C}(M)$ which respects the above notions and the corresponding notions in [5]. Using this isomorphism, we can restate a theorem of Korkmaz as follows:

Theorem 4.5 (Korkmaz [5]). *Let M be a sphere with n holes and $n \geqslant 5$. Let α, β be two 2-vertices of $\mathscr{C}(\mathscr{M})$. Then $\{\alpha, \beta\}$ is a simple pair iff there exist vertices $\gamma_1, \gamma_2, \ldots, \gamma_{n-2}$ of $\mathscr{C}(M)$ satisfying the following conditions*:

(i) $(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ *is a pentagon in* $\mathscr{C}(M)$,
(ii) γ_1 *and* γ_{n-2} *are* 2-*vertices*, γ_2 *is a* 3-*vertex and* γ_k *and* γ_{n-k} *are* k-*vertices for* $3 \leqslant k \leqslant \frac{n}{2}$,
(iii) $\{\alpha, \gamma_3, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}, \{\alpha, \gamma_2, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}, \{\beta, \gamma_3, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$, *and* $\{\gamma_1, \gamma_2, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$ *are codimension-zero simplices.*

By using the following lemmas, we will see some more properties of λ.

Lemma 4.6. *Let c, x, y, z, h, v be essential circles on S. Suppose that there exists a subsurface N of S and a homeomorphism $\varphi : (N, c, x, y, z, h, v) \to (N_o, c_o, x_o, y_o, z_o, h_o, v_o)$ where N_o is a standard sphere with four holes having c_o, x_o, y_o, z_o on its boundary and h_o, v_o (horizontal, vertical) are two circles which have geometric intersection 2 and algebraic intersection 0 as indicated in* Fig. 10. *Then, there exist $c' \in \lambda([c])$, $x' \in \lambda([x])$, $y' \in \lambda([y])$, $z' \in \lambda([z])$, $h' \in \lambda([h])$, $v' \in \lambda([v])$, $N' \subset S$ and a homeomorphism $\chi : (N', c', x', y', z', h', v') \to (N_o, c_o, x_o, y_o, z_o, h_o, v_o)$.*

Proof. Let $A = \{c, x, y, z, v\}$. Any two elements in A which are isotopic in S bound an annulus on S. Let B be a set consisting of a core from each annulus which is bounded by elements in A, circles in A which are not isotopic to any other circle in A, and v. We can extend B to a pants decomposition P of S. Since the genus of S is at least 3, there are at least four pairs of pants of P. Note that $\{v\}$ is

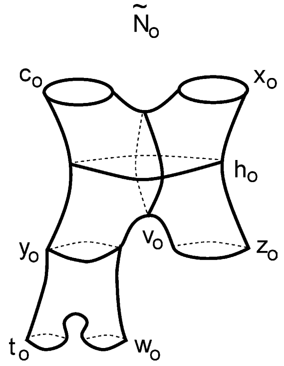

Fig. 11. Sphere with five holes.

a pair of pants decomposition of N. Each pair of pants of this pants decomposition of N is contained in exactly one pair of pants in P. Hence, there is a pair of pants R of P whose interior is disjoint from N and has at least one of c, x, y, z as one of its boundary components. W.L.O.G. assume that R has y on its boundary. Let T be a regular neighborhood, in R, of the boundary components of R other than y. Let t, w be the boundary components of T which are in the interior of R. Then, y, t, w bound an embedded pair of pants Q in R. Let $\tilde{N} = N \cup Q$. Then, we can extend N_o to $\tilde{N}_o$ and extend φ to a homeomorphism $\tilde{\varphi}:(\tilde{N}, c, x, y, z, h, v, t, w) \to (\tilde{N}_o, c_o, x_o, y_o, z_o, h_o, v_o, t_o, w_o)$, where $\tilde{N}_o$ is as shown in Fig. 11.

Using Lemma 3.7, we can choose pairwise disjoint representatives $c', x', y', z', v', t', w'$ of $\lambda([c])$, $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([v]), \lambda([t]), \lambda([w])$, respectively, s.t. there exists a subsurface $\tilde{N}'$ of S and a homeomorphism $\chi:(\tilde{N}', c', x', y', z', v', t', w') \to (\tilde{N}_o, c_o, x_o, y_o, z_o, v_o, t_o, w_o)$. Clearly, we have, $i([h]),$ $([c]) = i([h],[x]) = 0, i([h],[y]) = i([h],[z]) = 0$. Since c, x, y, z are all essential circles on S, we have $i([h],[v]) \neq 0$. Then, since λ is superinjective, we have, $i(\lambda([h]), \lambda([c])) = 0, i(\lambda([h]), \lambda([x])) =$ $i(\lambda([h]), \lambda([y])) = i(\lambda([h]), \lambda([z])) = 0$ and $i(\lambda([h]), \lambda([v])) \neq 0$. Then, a representative h' of $\lambda([h])$ can be chosen such that h' is transverse to v', h' does not intersect any of c', x', y', z', and $i(\lambda([h])), \lambda([v]) =$ $|h' \cap v'|$. Since $i(\lambda([h]), \lambda([v])) \neq 0, h'$ intersects v'. Hence, h' is in the sphere with four holes bounded by c', x', y', z' in $\tilde{N}'$.

$\tilde{N}$ and $\tilde{N}'$ are spheres with five holes in S. Since c, x, y, z are essential circles in S, the essential circles on $\tilde{N}$ are essential in S. Similarly, since c', x', y', z' are essential circles in S, the essential circles on $\tilde{N}'$ are essential in S. Furthermore, we can identify $\mathscr{C}(\tilde{N})$ and $\mathscr{C}(\tilde{N}')$ with two subcomplexes of $\mathscr{C}(S)$ in such a way that the isotopy class of an essential circle in $\tilde{N}$ or in $\tilde{N}'$ is identified with the isotopy class of that circle in S. Now, suppose that α is a vertex in $\mathscr{C}(\tilde{N})$. Then, with this identification, α is a vertex in $\mathscr{C}(S)$ and α has a representative in $\tilde{N}$. Then, $i(\alpha,[c]) = i(\alpha,[x]) = i(\alpha,[t]) = i(\alpha,[w]) = i(\alpha,[z]) = 0$. Then there are two possibilities: (i) $i(\alpha,[v]) = 0$ and $i(\alpha,[y]) = 0$ (in this case $\alpha = [v]$ or $\alpha = [y]$), (ii) $i(\alpha,[v]) \neq 0$ or $i(\alpha,[y]) \neq 0$. Since λ is injective, $\lambda(\alpha)$ is not equal to any of $[c'],[x'],[t'],[w'],[z']$. Since λ is superinjective, we have, $i(\lambda(\alpha),[c']) = i(\lambda(\alpha),[x']) = i(\lambda(\alpha),[t']) = i(\lambda(\alpha),[w']) = i(\lambda(\alpha),[z']) = 0$. Then, there are two possibilities: (i) $i(\lambda(\alpha),[v']) = 0$ and $i(\lambda(\alpha),[y']) = 0$ (in this case $\lambda(\alpha) = [v']$ or $\lambda(\alpha) = [y']$), (ii) $i(\lambda(\alpha),[v']) \neq 0$ or $i(\lambda(\alpha),[y']) \neq 0$. Then, a representative of $\lambda(\alpha)$ can be chosen in $\tilde{N}'$. Hence, λ maps the vertices of $\mathscr{C}(S)$ that have essential representatives in $\tilde{N}$ to the vertices of $\mathscr{C}(S)$ that have

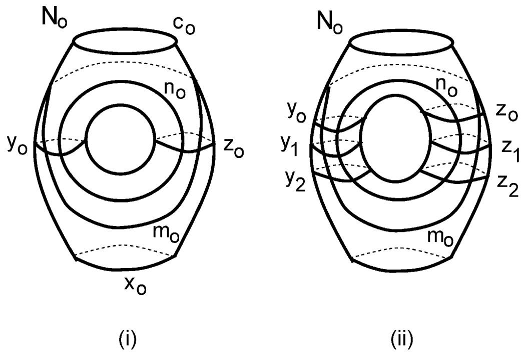

Fig. 12. Circles on torus with two boundary components.

essential representatives in $\tilde{N'}$, (i.e. λ maps $\mathscr{C}(\tilde{N}) \subseteq \mathscr{C}(S)$ to $\mathscr{C}(\tilde{N'}) \subseteq \mathscr{C}(S)$). Similarly, λ maps $\mathscr{C}(N) \subseteq \mathscr{C}(S)$ to $\mathscr{C}(N') \subseteq \mathscr{C}(S)$.

It is easy to see that $\{[h],[v]\}$ is a simple pair in $\tilde{N}$. Then, by Theorem 4.5, there exist vertices $\gamma_1,\gamma_2,\gamma_3$ of $\mathscr{C}(\tilde{N})$ such that $(\gamma_1,\gamma_2,[h],\gamma_3,[v])$ is a pentagon in $\mathscr{C}(\tilde{N}),\gamma_1$ and γ_3 are 2-vertices, γ_2 is a 3-vertex, and $\{[h],\gamma_3\}$, $\{[h],\gamma_2\},\{[v],\gamma_3\}$ and $\{\gamma_1,\gamma_2\}$ are codimension-zero simplices of $\mathscr{C}(\tilde{N})$.

Since λ is superinjective and c,x,y,z are essential circles, we can see that $(\lambda(\gamma_1),\lambda(\gamma_2), \lambda([h]), \lambda(\gamma_3),\lambda([v]))$ is a pentagon in $\mathscr{C}(\tilde{N'})$. By Lemma 3.7, $\lambda(\gamma_1)$ and $\lambda(\gamma_3)$ are 2-vertices, and $\lambda(\gamma_2)$ is a 3-vertex in $\mathscr{C}(\tilde{N'})$. Since λ is an injective simplicial map $\{\lambda([h]),\lambda(\gamma_3)\},\{\lambda([h]),\lambda(\gamma_2)\}$, $\{\lambda([v]), \lambda(\gamma_3)\}$ and $\{\lambda(\gamma_1),\lambda(\gamma_2)\}$ are codimension-zero simplices of $\mathscr{C}(\tilde{N'})$. Then, by Theorem 4.5, $\{\lambda([h]), \lambda([v])\}$ is a simple pair in $\tilde{N'}$. Since $\lambda([h])$ has a representative, h', in N', such that $i(\lambda([h])),\lambda([v])= |h'\cap v'|$ and $\{\lambda([h]),\lambda([v])\}$ is a simple pair in $\tilde{N'}$, there exists a homeomorphism $\chi:(N',c',x',y',z', h',v') \to (N_o,c_o,x_o,y_o,z_o,h_o,v_o)$. $\square$

Lemma 4.7. *Let c,x,y,z,m,n be essential circles on S. Suppose that there exists a subsurface N of S and a homeomorphism $\varphi:(N,c,x,y,z,m,n) \to_\varphi (N_o,c_o,x_o,y_o,z_o,m_o,n_o)$ where N_o is a standard torus with two boundary components, c_o,x_o, and y_o,z_o,m_o,n_o are circles as shown in Fig. 12. Then, there exist $c'\in\lambda([c]),x'\in\lambda([x]),y'\in\lambda([y]),z'\in\lambda([z]),m'\in\lambda([m]),n'\in\lambda([n]),N'\subseteq S$ and a homeomorphism χ such that $(N',c',x',y',z',m',n') \to_\chi (N_o,c_o, x_o,y_o,z_o,m_o,n_o)$.*

Proof. Since the genus of S is at least 3, c cannot be isotopic to x in S. So, we can complete $\{c,x,y,z\}$ to a pair of pants decomposition, P, of S. Since $\{y,z\}$ gives a pair of pants decomposition on N, by Lemma 3.7, there exists a subsurface $N'\subseteq S$ which is homeomorphic to N_o and there are pairwise disjoint representatives c',x',y',z' of $\lambda([c]),\lambda([x]),\lambda([y]),\lambda([z])$, respectively, and a homeomorphism ϕ such that $(N',c',x',y',z') \to_\phi (N_o,c_o,x_o,y_o,z_o)$. Then by Lemma 3.9, we have the following:

$$i([m],[z])=1 \Rightarrow i(\lambda([m]),\lambda([z]))=1, \quad i([n],[z])=1 \Rightarrow i(\lambda([n]),\lambda([z]))=1,$$

$$i([m],[y])=1 \Rightarrow i(\lambda([m]),\lambda([y]))=1, \quad i([n],[y])=1 \Rightarrow i(\lambda([n]),\lambda([y]))=1,$$

$$i([m],[c])=0 \Rightarrow i(\lambda([m]),\lambda([c]))=0, \quad i([n],[c])=0 \Rightarrow i(\lambda([n]),\lambda([c]))=0,$$

$$i([m],[x])=0 \Rightarrow i(\lambda([m]),\lambda([x]))=0, \quad i([n],[x])=0 \Rightarrow i(\lambda([n]),\lambda([x]))=0,$$

$$i([m],[n])=0 \Rightarrow i(\lambda([m]),\lambda([n]))=0.$$

There are representatives $m_1 \in \lambda([m]), n' \in \lambda([n])$ such that $|m_1 \cap y'| = |m_1 \cap z'| = 1, |m_1 \cap c'| = |m_1 \cap x'| = 0, |n' \cap y'| = |n' \cap z'| = 1,\ |n' \cap c'| = |n' \cap x'| = |m_1 \cap n'| = 0$ with all intersections transverse. Since ϕ is a homeomorphism, we have $|\phi(m_1) \cap \phi(n')| = 0,\ |\phi(n') \cap y_o| = 1,\ |\phi(n') \cap z_o| = 1, |\phi(n') \cap c_o| = |\phi(n') \cap x_o| = 0 = |\phi(m_1) \cap c_o| = |\phi(m_1) \cap x_o| = 0, |\phi(m_1) \cap y_o| = |\phi(m_1) \cap z_o| = 1.$

Let us choose parallel copies y_1, y_2 of y_o and z_1, z_2 of z_o as shown in Fig. 12 so that each of them has transverse intersection one with $\phi(m^1)$ and $\phi(n')$. Let P_1, P_2 be the pair of pants with boundary components c_o, y_o, z_o, and x_o, y_2, z_2, respectively. Let Q_1, Q_2, R_1, R_2 be the annulus with boundary components $\{y_o, y_1\}, \{y_1, y_2\},\ \{z_o, z_1\}, \{z_1, z_2\}$, respectively. By the classification of isotopy classes of families of properly embedded disjoint arcs in pairs of pants, $\phi(m^1) \cap P_1, \phi(m^1) \cap P_2,\ \phi(n') \cap P_1$ and $\phi(n') \cap P_2$ can be isotoped to the arcs $m_o \cap P_1,\ m_o \cap P_2,\ n_o \cap P_1,\ n_o \cap P_2$, respectively. Let $\kappa : P_1 \times I \to P_1, \tau : P_2 \times I \to P_2$ be such isotopies. By a tapering argument, we can extend κ and τ and get $\tilde{\kappa} : (P_1 \cup Q_1 \cup R_1) \times I \to (P_1 \cup Q_1 \cup R_1)$ and $\tilde{\tau} : (P_2 \cup Q_2 \cup R_2) \times I \to (P_2 \cup Q_2 \cup R_2)$ so that $\tilde{\kappa}_t$ is *id* on $y_1 \cup z_1$ and $\tilde{\tau}_t$ is *id* on $y_1 \cup z_1$ for all $t \in I$. Then, by gluing these extensions we get an isotopy ϑ on $N_o \times I$.

By the classification of isotopy classes of arcs (relative to the boundary) on an annulus, $\vartheta_1(\phi(n')) \cap (R_1 \cup R_2)$ can be isotoped to $t_{z_o}^k(n_o) \cap (R_1 \cup R_2)$ for some $k \in \mathbb{Z}$. Let us call this isotopy μ. Let $\tilde{\mu}$ denote the extension by id to N_o. Similarly, $\vartheta_1(\phi(n')) \cap (Q_1 \cup Q_2)$ can be isotoped to $t_{y_o}^l(n_o) \cap (Q_1 \cup Q_2)$ for some $l \in \mathbb{Z}$. Let us call this isotopy v. Let $\tilde{v}$ denote the extension by id to N_o. Then, "gluing" the two isotopies $\tilde{\mu}$ and $\tilde{v}$, we get a new isotopy, ε, on N_o. Then we have, $t_{y_o}^{-l}(t_{z_o}^{-k}(\varepsilon_1(\vartheta_1(\phi(n'))))) = n_o$. Clearly, $t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \varepsilon_1 \circ \vartheta_1$ fixes c_o, x_o, y_o, z_o. So, we get $t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \varepsilon_1 \circ \vartheta_1 \circ \phi : (N', c', x', y', z', n') \to (N_o, c_o, x_o, y_o, z_o, n_o)$. Let $\chi = t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \varepsilon_1 \circ \vartheta_1 \circ \phi$. Then, we also get, $\chi(m_1)$ is isotopic to m_o because of the intersection information. Let $m' = \chi^{-1}(m_o)$. Then we get, $\chi : (N', c', x', y', z', m', n') \to (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$. $\square$

Let i be an essential properly embedded arc on S_c. Let A be a boundary component of S_c which has one end point of i and B be the boundary component of S_c which has the other end point of i. Let N be a regular neighborhood of $i \cup A \cup B$ in S_c. By Euler characteristic arguments, N is a pair of pants. It can be seen that the boundary components of N correspond to nontrivial circles, *encoding circles*, on S. We call the set of isotopy classes of the boundary components of N to be the *encoding simplex*, Δ_i, of i (and of $[i]$). Note that $[c]$ is a vertex in the encoding simplex of i.

An essential properly embedded arc i on S_c is called *type* 1.1 if it joins one boundary component ∂_k of S_c to itself for $k = 1, 2$ and if $\partial_k \cup i$ has a regular neighborhood N in S_c which has only one circle on its boundary which is inessential w.r.t. S_c. If N has two circles on its boundary which are inessential w.r.t. S_c, then i is called *type* 1.2. We call i to be *type 2*, if it joins the two boundary components ∂_1 and ∂_2 of S_c to each other. An element $[i] \in \mathscr{V}(S_c)$ is called *type* 1.1 (1.2, 2) if it has a type 1.1 (1.2, 2) representative.

Let ∂^1, ∂^2 be the boundary components of S_d. We prove the following lemmas in order to show that λ induces a map $\lambda_* : \mathscr{V}(S_c) \to \mathscr{V}(S_d)$ with certain properties.

Lemma 4.8. *Let $\partial_k \subseteq \partial S_c$ for some $k \in \{1,2\}$. Then, there exists a unique $\partial^l \subseteq \partial S_d$ for some $l \in \{1,2\}$ such that if i is a properly embedded essential arc on S_c connecting ∂_k to itself, then there exists a properly embedded arc j on S_d connecting ∂^l to itself such that $\lambda(\Delta_i) = \Delta_j$.*

Proof. Assume that each of ∂^1 and ∂^2 satisfies the hypothesis. Let i be a properly embedded, essential, type 1.1 arc connecting ∂_k to itself. Then, there exist properly embedded arcs, j_1, connecting ∂^1 to itself, and j_2, connecting ∂^2 to itself, such that $\lambda(\Delta_i) = \Delta_{j_1}$ and $\lambda(\Delta_i) = \Delta_{j_2}$. Then, we have $\Delta_{j_1} = \Delta_{j_2}$. Note that a properly embedded essential arc i is type 1.1 iff Δ_i has exactly 3 elements. Otherwise Δ_i has 2 elements. Since i is type 1.1 and $\lambda(\Delta_i) = \Delta_{j_1}$ and $\lambda(\Delta_i) = \Delta_{j_2}$ and λ is injective, j_1 and j_2 are type 1.1. We can choose a pairwise disjoint representative set $\{a,b,d\}$ of Δ_{j_1} on S. Since $\Delta_{j_1} = \Delta_{j_2}, \{a,b,d\}$ is a pairwise disjoint representative set for Δ_{j_2} on S. Then, the curves $\tilde{a}, \tilde{b}$ on S_c which correspond to a, b on S and ∂^1 bound a pair of pants, P, on S_c containing an arc, j_1', isotopic to j_1. Similarly, $\tilde{a}, \tilde{b}, \partial^2$ bound a pair of pants, Q, on S_c containing an arc, j_2', isotopic to j_2. Let us cut S_c along $\tilde{a}$ and $\tilde{b}$. Then, P is the connected component of $S_{c \cup a \cup b}$ containing ∂^1 and Q is the connected component of $S_{c \cup a \cup b}$ containing ∂^2. $P \neq Q$ since ∂^2 is not in P and ∂^2 is in Q. Then P and Q are distinct connected components meeting along $\tilde{a}$ and $\tilde{b}$. Hence, S_c is $P \cup Q$, a torus with two holes. This implies that S is a genus 2 surface which gives a contradiction since the genus of S is at least 3. So, only one boundary component of S_d can satisfy the hypothesis.

Since i is type 1.1, Δ_i contains $[c]$ and two other isotopy classes of non-trivial circles which are not isotopic to c in S. Let P' be a pairwise disjoint representative set of $\lambda([\Delta_i])$, containing d. By the proof of Lemma 3.7, P' bounds a pair of pants on S. Since the genus of S is at least 3, P' bounds a unique pair of pants on S, which corresponds to a unique pair of pants, Q, in S_d which has only one inessential boundary component. Let $\partial^{l(i)}$ be this inessential boundary component. Let j be an essential properly embedded arc connecting $\partial^{l(i)}$ to itself in Q. Then, we have $\lambda(\Delta_i) = \Delta_j$.

Now, to see that $\partial^{l(i)}$ is independent of the type 1.1 arc i connecting ∂_k to itself, we prove the following claim:

Claim 1. *If we start with two type 1.1 arcs i and j starting and ending on ∂_k, then $\partial^{l(i)} = \partial^{l(j)}$.*

Proof. Let $[i], [j]$ be type 1.1 and i, j connect ∂_k to itself. W.L.O.G. we can assume that i and j have minimal intersection. First, we prove that there is a sequence $j = r_0 \to r_1 \to \cdots \to r_{n+1} = i$ of essential properly embedded arcs joining ∂_k to itself so that consecutive pair is disjoint, i.e. the isotopy classes of these arcs define a path in $\mathscr{B}(S_c)$, between i and j.

If $|i \cap j| = 0$, then take $r_0 = j$, $r_1 = i$. We are done. Assume that $|i \cap j| = m > 0$. We orient i and j arbitrarily. Then, we define two arcs in the following way: Start on the boundary component ∂_k, on one side of the beginning point of j and continue along j without intersecting j, till the last intersection point along i. Then we would like to follow i, without intersecting j, until we reach ∂_k. So, if we are on the correct side of j we do this; if not, we change our starting side from the beginning and follow the construction. This gives us an arc, say j_1. We define j_2, another arc, by changing the orientation of j and following the same construction. It is easy to see that j_1, j_2 are disjoint properly embedded arcs connecting ∂_k to itself as i and j do. One can see that j_1, j_2 are essential arcs since i, j intersect minimally. In Fig. 13, we show the beginning and the end points of i, the essential intersections of i, j, and j_1, j_2 near the end point of i on ∂_k.

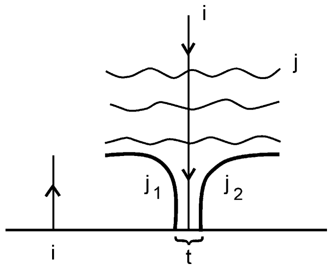

Fig. 13. Splitting the arc j along the end of i.

$|i \cap j_1| < m$, $|i \cap j_2| < m$ since we eliminated at least one intersection with i. We also have $|j_1 \cap j| = |j_2 \cap j| = 0$ since we never intersected j in the construction. Notice that j_1 and j_2 are not oriented, and i is oriented.

Claim 2. *j_1 is not isotopic to j_2.*

Proof. Suppose that j_1 and j_2 are isotopic. Then, they are not linked, and there is a band B such that $\partial B \subseteq j_1 \cup j_2 \cup \partial_k$ and $\partial B \setminus (j_1 \cup j_2)$ is a disjoint union of two arcs $\varepsilon_1, \varepsilon_2$ on ∂_k, where each of ε_1 and ε_2 starts at an end point of j_1 and ends at an end point of j_2. By the construction, there is an end point of each of j_1 and j_2 near the end point of i on ∂_k, on the arc t shown in the figure, and there is not any point of j on t. Then, either $t = \varepsilon_1$ or $t = \varepsilon_2$. W.L.O.G. assume that $t = \varepsilon_1$. Then, $\varepsilon_1 \subseteq \partial B$ has the end point of i. Since $\varepsilon_1 \subseteq \partial_k$, it has only one side on S_c, and hence on B. Then, since $\varepsilon_1 \subseteq \partial B$ has the end point of i, by the construction we can see that the last intersection point along i of i and j has to lie in the band B. Then, since j does not intersect any of j_1 and j_2, and B has a point of j in the interior, j has to live in B. Then, since ε_1 does not contain any point of j, and j does not intersect any of j_1 and j_2, the end points of j has to lie on ε_2, which implies that j is an inessential arc on S_c. This gives a contradiction. Hence, j_1 and j_2 cannot be isotopic.

Claim 3. *Either j_1 or j_2 is of type* 1.1.

Proof. If j_1 and j_2 are linked, then a regular neighborhood of $j_1 \cup j_2 \cup \partial_k$ in S_c is a genus one surface with two boundary components by Lemma 4.4. In this case, both arcs have to be type 1.1 since representatives of encoding vertices for j_1, j_2 can be chosen in this surface and this surface has only one boundary component which is a boundary component of S_c. So, both arcs have to be type 1.1.

If j_1 and j_2 are unlinked, then a regular neighborhood, N, of $j_1 \cup j_2 \cup \partial_i$ is a sphere with four holes by Lemma 4.2. Encoding circles of j_1, j_2 can be chosen in N. Assume that j_1 is type 1.2 and has two encoding circles which are the two boundary components of the surface S_c. Then, j_2 would have to be type 1.1 since N can have at most two boundary components which are the boundary components of S_c and regular neighborhoods of $j_1 \cup \partial_k$ and $j_2 \cup \partial_k$ have only one common boundary component which is ∂_k. This proves Claim 3.

Let $r_1 \in \{j_1, j_2\}$ and r_1 be type 1.1. By the construction we get, $|i \cap r_1| < m$, $|j \cap r_1| = 0$. Now, using i and r_1 in place of i and j we can define a new type 1.1 arc r_2 with the properties,

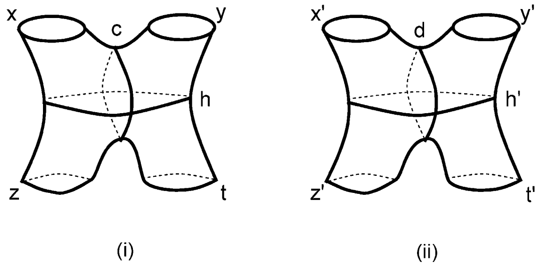

Fig. 14. Sphere with four holes.

$|i \cap r_2| < |i \cap r_1|, |r_1 \cap r_2| = 0$. By an inductive argument, we get a sequence of arcs such that every consecutive pair is disjoint, $i = r_{n+1} \to r_n \to r_{n-1} \to \cdots \to r_1 \to r_0 = j$. This gives us a path of type 1.1 arcs in $\mathscr{B}(S_c)$ between i and j. By using Lemmas 4.2 and 4.4, we can see a regular neighborhood of the union of each consecutive pair in the sequence and the boundary components of S_c, and encoding circles of these consecutive arcs. Then, by using the results of Lemmas 4.6 and 4.7, we can see that each pair of disjoint type 1.1 arcs give us the same boundary component. Hence, by using the sequence given above, we conclude that i and j give us the same boundary component. This proves Claim 1.

Let i_o be a properly embedded, type 1.1 arc on S_c connecting ∂_k to itself. Let $\partial^l = \partial^{l(i_o)}$. If i is a properly embedded, type 1.1 arc on S_c connecting ∂_k to itself, then by the arguments given above we have $\partial^l = \partial^{l(i)}$, and there exists a properly embedded arc j on S_d connecting ∂^l to itself such that $\lambda(\Delta_i) = \Delta_j$. Suppose that i is a properly embedded, type 1.2 arc on S_c connecting ∂_k to itself. Let Δ_i be the encoding simplex of $[i]$. Since i is type 1.2, Δ_i contains $[c]$ and only one other isotopy class of a nontrivial circle which is not isotopic to c in S. A pairwise disjoint representative set, P, of Δ_i corresponds to a nonembedded pair of pants on S. Let P' be a pairwise disjoint representative set of $\lambda([\Delta_i])$, containing d. By extending P to a pair of pants decomposition of S, and applying Lemma 3.7, we can see that P' corresponds to a unique nonembedded pair of pants of S which corresponds to a unique pair of pants, Q, in S_d which has two inessential boundary components containing ∂^l. Let j be a nontrivial properly embedded arc connecting ∂^l to itself in Q. Then, we have $\lambda(\Delta_i) = \Delta_j$. Hence, ∂^l is the boundary component that we want. □

We define a map $\sigma : \{\partial_1, \partial_2\} \to \{\partial^1, \partial^2\}$ using the correspondence which is given by Lemma 4.8.

Lemma 4.9. *σ is a bijection.*

Proof. Let x, y, z, t, h, c be essential circles on S such that x, y, z, t bound a 4-holed sphere and h, c be two circles which intersect geometrically twice and algebraically zero times in this subsurface as shown in Fig. 14(i). By using Lemma 4.6, we can see that there exist pairwise disjoint representatives x', y', z', t', h', d of $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([t]), \lambda([h]), \lambda([c])$, respectively, such that x', y', z', t' bound a sphere with four holes, and h', d intersect geometrically twice and algebraically zero times in this subsurface as shown in Fig. 14(ii).

The curve h corresponds to two arcs, say i_1, i_2 on S_c which start and end on different boundary components, ∂_k, ∂_l, of S_c, respectively. W.L.O.G. assume that the encoding circles of i_1 and i_2 are

$\{x,z,\partial_k\}$ and $\{y,t,\partial_l\}$ and they correspond to $\{x',z',\partial^m\}$ and $\{y',t',\partial^n\}$ for $m,n\in\{1,2\}$ which bound pairs of pants L and R, respectively. h' corresponds to two arcs, say j_1 and j_2 on L and R which start and end on different boundary components, ∂^m and ∂^n of S_d, respectively. By the definition of σ we have, $\sigma(\partial_k)=\partial^m$ and $\sigma(\partial_l)=\partial^n$ for $k,l,m,n\in\{1,2\}$, $k\neq l, m\neq n$. So, σ is onto. Hence, σ is a bijection. □

Lemma 4.10. *Let $[i]\in\mathscr{V}(S_c)$. If i connects ∂_k to ∂_l on S_c where $k,l\in\{1,2\}$, then there exists a unique $[j]\in\mathscr{V}(S_d)$ such that j connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i)=\Delta_j$.*

Proof. Let $[i]\in\mathscr{V}(S_c)$ and let i connect ∂_k to ∂_l on S_c where $k,l\in\{1,2\}$. Let Δ_i be the encoding simplex of i. Then, a pairwise disjoint representative set of $\lambda([\Delta_i])$ containing d corresponds to a unique pair of pants in S_d which has boundary components $\sigma(\partial_k)$ and $\sigma(\partial_l)$. By the classification of properly embedded arcs in pair of pants, there exists a unique isotopy class of nontrivial properly embedded arcs which start at $\sigma(\partial_k)$ and end at $\sigma(\partial_l)$, in this pair of pants. Let j be such an arc. We have $\lambda(\Delta_i)=\Delta_j$.

Let e be an essential properly embedded arc in S_d such that e connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i)=\Delta_e$. Then, we have $\Delta_e=\Delta_j=\lambda(\Delta_i)$. Let a,b,d be a pairwise disjoint representative set of $\lambda(\Delta_i)$ on S. Then there are properly embedded arcs j_1,e_1 isotopic to j,e, respectively, which are in pair of pants bounded by a,b,d. Since the genus of S is at least 3, there is at most one pair of pants which has a,b,d on its boundary. So, j_1,e_1 are in the same pair of pants. Since they both connect the same boundary components in this pair of pants, they are isotopic. So, $[j]=[e]$. Hence, $[j]$ is the unique isotopy class in S_d such that j connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i)=\Delta_j$. □

λ induces a unique map $\lambda_*:\mathscr{V}(S_c)\to\mathscr{V}(S_d)$ such that if $[i]\in\mathscr{V}(S_c)$ then $\lambda_*([i])$ is the unique isotopy class corresponding to $[i]$ where the correspondence is given by Lemma 4.10. Using the results of the following lemmas, we will prove that λ_* extends to an injective simplicial map on the whole complex, $\mathscr{B}(S_c)$.

Lemma 4.11. *$\lambda_*:\mathscr{V}(S_c)\to\mathscr{V}(S_d)$ extends to a simplicial map $\lambda_*:\mathscr{B}(S_c)\to\mathscr{B}(S_d)$.*

Proof. It is enough to prove that if two distinct isotopy classes of essential properly embedded arcs in S_c have disjoint representatives, then their images under λ_* have disjoint representatives. Let a,b be two disjoint representatives of two distinct classes in $\mathscr{V}(S_c)$. Let ∂_1,∂_2 be the two boundary components of S_c. We consider the following cases:

Case 1: Assume that a and b connect the two boundary components of S_c. W.L.O.G. assume that the end points of a map to the same point under the quotient map $q:S_c\to S$, and the end points of b map to the same point under q. By Lemma 4.1, we have, $(N,a,b)\cong(S_4^2,a_o,b_o)$ where N is a regular neighborhood of $a\cup b\cup\partial_1\cup\partial_2$. Then, we have $(q(N),q(a),q(b),q(\partial_1))\cong(M_o,m_1,m_2,m_3)$ where M_o,m_1,m_2,m_3 are as shown in Fig. 15.

It is easy to see that $(M_o,m_1,m_2,m_3)\cong(N_o,n_o,m_o,z_o)$, where N_o,n_o,m_o,z_o are as given in Lemma 4.7. Then, there exists a homeomorphism $\varphi:(q(N),\ q(a),q(b),q(\partial_1))\to(N_o,n_o,m_o,z_o)$. Let $v=\varphi^{-1}(m_4)$, $y=\varphi^{-1}(m_5)$, $z=\varphi^{-1}(m_6)$. Suppose that $q(a)$ is isotopic to $q(b)$ on S. Since $q(a)$ is disjoint from $q(b)$, there exists an annulus having $q(a)\cup q(b)$ as its boundary. Since each of $q(a)$ and $q(b)$ intersects c transversely once, c cuts this annulus into a band B on S_c such that $\partial B\subseteq a\cup b\cup\partial S_c$.

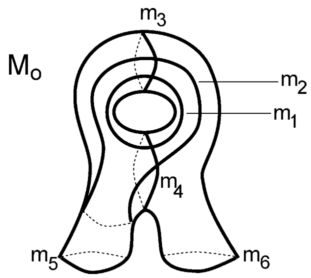

Fig. 15. Circles associated to type 2 arcs.

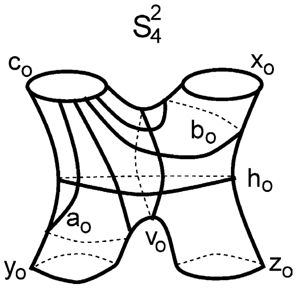

Fig. 16. Arcs and their encoding circles, I.

But this is not possible since a is not isotopic to b on S_c. Hence, $q(a)$ is not isotopic to $q(b)$ on S. Then, y and z are essential circles on S. Since $q(\partial_1)$ is equal to c on $S, q(\partial_1)$ is an essential circle on S. Since each of $q(a)$ and $q(b)$ intersects c transversely once, $q(a)$ and $q(b)$ are essential circles on S. Then, by Lemma 4.7, there exist $a' \in \lambda([q(a)])$, $b' \in \lambda([q(b)])$, $c' \in \lambda([q(\partial_1)]), N'$ and a homeomorphism $\chi : (N', a', b', c') \to (N, n_o, m_o, z_o)$. If we cut N' along c', we get two disjoint arcs, $\hat{a}, \hat{b}$ corresponding to a', b', respectively.

Since a connects ∂_1 and ∂_2, the encoding simplex of $[a]$ is $\{[c], \gamma\}$ where γ is the class of a 1-separating circle on S. There exists $x \in \gamma$ such that x bounds a genus 1 subsurface Q and $q(a)$ and c intersect transversely once on Q. Since $i([x],[c]) = 0, i([x],[q(a)]) = 0$, and λ is superinjective, we have $i(\lambda([x]),[d]) = 0, i(\lambda([x]), \lambda([q(a)])) = 0$. Then we choose a representative x' of $\lambda([x])$ which is disjoint from $a' \cup d$. By Lemma 3.7, x' is a 1-separating circle bounding a subsurface R containing $a' \cup d$. Then, it is easy to see that $\{[x'],[d]\}$ is the encoding simplex of $[\hat{a}]$. Then, since the encoding simplex of $[\hat{a}]$ is the image of the encoding simplex of $[a]$ under λ, and both a and $[\hat{a}]$ are type 2, by the definition of $\lambda_*, [\hat{a}]$ is the image of $[a]$ under λ_*. Similarly, $[\hat{b}]$ is the image of $[b]$ under λ_*. This shows that $\lambda_*([a])$ and $\lambda_*([b])$ have disjoint representatives $\hat{a}$ and $\hat{b}$, respectively.

Case 2: Assume that a, b are unlinked, connecting ∂_i to itself for some $i = 1, 2$. By Lemma 4.2, there is a homeomorphism ϕ such that $(S_4^2, a_o, b_o) \cong_\phi (N, a, b)$ where N is a regular neighborhood of $a \cup b \cup \partial_i$ and a_o, b_o are as shown in Fig. 16.

Since h_o, x_o and v_o, y_o are encoding circles for b_o, a_o, respectively, $\phi(h_o), \phi(x_o)$ and $\phi(v_o), \phi(y_o)$ are encoding circles for b and a, respectively. We have, $(S_4^2, c_o, x_o, y_o, z_o, h_o, v_o) \cong (N, \phi(c_o), \phi(x_o), \phi(y_o), \phi(z_o), \phi(h_o), \phi(v_o))$. Since a and b are essential arcs on S_c, $\phi(x_o)$ and $\phi(y_o)$ are essential circles on S.

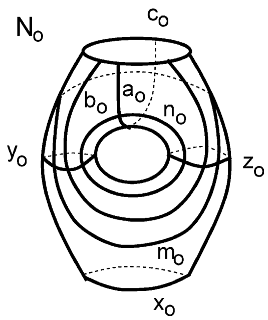

Fig. 17. Arcs and their encoding circles, II.

$\phi(c_o)$ is equal to c on S. So, $\phi(c_o)$ is an essential circle on S. Since a is not isotopic to b on $S_c, \phi(z_o)$ is an essential circle on S. So, we have that all of $\phi(c_o), \phi(x_o), \phi(y_o), \phi(z_o)$ are essential circles on S. Then, by Lemma 4.6, there exist $c' \in \lambda([\phi(c_o)])$, $x' \in \lambda([\phi(x_o)])$, $y' \in \lambda([\phi(y_o)])$, $z' \in \lambda([\phi(z_o)])$, $h' \in \lambda([\phi(h_o)])$, $v' \in \lambda([\phi(v_o)])$, N' and a homeomorphism χ s.t. $(S_4^2, c_o, x_o, y_o, z_o, h_o, v_o) \to_\chi (N', c', x', y', z', h', v')$. Then, h', x' are encoding circles for $\lambda_*([b])$ and v', y' are encoding circles for $\lambda_*([a])$, and $\chi(b_o), \chi(a_o)$ are disjoint representatives for $\lambda_*([b])$ and $\lambda_*([a])$, respectively.

Case 3: Assume that a connects one boundary component of S_c to itself and b connects the two boundary components of S_c to each other. The proof is similar to the proof of Case 2.

Case 4: Assume that a, b are linked, connecting ∂_i to itself for some $i = 1, 2$. By Lemma 4.4, there is a homeomorphism $\phi : (N_o, a_o, b_o) \to (N, a, b)$ where N is a regular neighborhood of $\partial_i \cup a \cup b$ and N_o, a_o, b_o are as in Fig. 17. y_o, z_o and m_o, n_o are encoding circles of a_0 and b_o, respectively. So, $\phi(y_o), \phi(z_o)$ and $\phi(m_o), \phi(n_o)$ are encoding circles for a and b, respectively. We have $(N, \phi(c_o), \phi(x_o), \phi(y_o), \phi(z_o), \phi(m_o), \phi(n_o)) \cong_{\phi^{-1}} (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$. $\phi(c_o)$ is equal to c on S. So, $\phi(c_o)$ is an essential circle on S. Since c is a non-separating circle on $S, \phi(x_o)$ is an essential circle on S. Since $\phi(y_o), \phi(z_o)$ and $\phi(m_o), \phi(n_o)$ are encoding circles for essential arcs a and b, respectively, $\phi(y_o), \phi(z_o), \phi(m_o)$ and $\phi(n_o)$ are essential circles on S. Then, by Lemma 4.7, there exist representatives $c'_o \in \lambda([\phi(c_o)]), x'_o \in \lambda([\phi(x_o)]), y'_o \in \lambda([\phi(y_o)]), z'_o \in \lambda([\phi(c_2)]), m'_o \in \lambda([\phi(m_o)]), n'_o \in \lambda([\phi(n_o)])$, N' and a homeomorphism $\chi : (N_o, c_o, x_o, y_o, z_o, m_o, n_o) \to_\chi (N', c'_o, x'_o, c'_o, z'_o, m'_o, n'_o)$. Since $\phi(y_o), \phi(z_o)$ and $\phi(m_o), \phi(n_o)$ are encoding circles for a and b, respectively, y'_o, z'_o and m'_o, n'_o are encoding circles for $\lambda_*([a]), \lambda_*([b])$, respectively. Existence of χ shows that $\lambda_*([a]), \lambda_*([b])$ have disjoint representatives. $\chi(a_o)$ and $\chi(b_o)$ are disjoint representatives for $\lambda_*([a]), \lambda_*([b])$, respectively.

We comment that the case where a connects one boundary component of S_c to itself and b connects the other boundary component of S_c to itself is easy to handle.

We have shown in all the cases that if two vertices have disjoint representatives, then λ_* sends them to two vertices which have disjoint representatives. Hence, λ_* extends to a simplicial map $\lambda_* : \mathscr{B}(S_c) \to \mathscr{B}(S_d)$. $\square$

Lemma 4.12. *Let $\lambda : \mathscr{C}(S) \to \mathscr{C}(S)$ be a superinjective simplicial map. Then, $\lambda_* : \mathscr{B}(S_c) \to \mathscr{B}(S_d)$ is injective.*

Proof. It is enough to prove that λ_* is injective on the vertex set, $\mathscr{V}(S_c)$. Let $[i], [j] \in \mathscr{V}(S_c)$ such that $\lambda_*([i]) = \lambda_*([j]) = [k]$. Then, by the definition of λ_*, the type of $[i]$ and $[j]$ are the same.

Assume they are both type 1.1. Let $\{[c],[x],[y]\}$ and $\{[c],[z],[t]\}$ be the encoding simplices for $[i]$ and $[j]$, respectively. Then, $\{\lambda([c]),\lambda([x]),\lambda([y])\}$ and $\{\lambda([c]),\lambda([z]),\lambda([t])\}$ are encoding simplices of $[k]$. So, $\{\lambda([x]),\lambda([y])\}=\{\lambda([z]),\lambda([t])\}$. Then, since λ is injective, we get $\{[x],[y]\}=\{[z],[t]\}$. This implies $[i]=[j]$. The cases where $[i],[j]$ are both type 1.2 or type 2 can be proven similarly to the first case by using the injectivity of λ. □

Lemma 4.13. *If an injection $\mu:\mathscr{B}(S_c)\to\mathscr{B}(S_d)$ agrees with $h_*:\mathscr{B}(S_c)\to\mathscr{B}(S_d)$ on a top dimensional simplex, where h_* is induced by a homeomorphism $h:S_c\to S_d$, then μ agrees with h_* on $\mathscr{B}(S_c)$.*

Proof. Suppose that μ agrees with h_* on a top dimensional simplex, Δ. Then, $h_*^{-1}\circ\mu$ fixes Δ pointwise. Let w be a vertex in $\mathscr{B}(S_c)$. If $w\in\Delta$, then w is fixed by $h_*^{-1}\circ\mu$. Suppose that w is not in Δ. Take a top dimensional simplex containing w. Call it Δ'. There exists a chain $\Delta=\Delta_0,\Delta_1,\ldots,\Delta_m=\Delta'$ of top dimensional simplices in $\mathscr{B}(S_c)$, connecting Δ to Δ' such that any two consecutive simplices Δ_i,Δ_{i+1} have exactly one common face of codimension 1. This follows from the Connectivity Theorem for Elementary Moves of Mosher, [7], appropriately restated for surfaces with boundaries. Since $h_*^{-1}\circ\mu$ is injective, Δ_1 must be sent to a top dimensional simplex by $h_*^{-1}\circ\mu$. Let w_1 be the vertex of Δ_1 which is not in Δ. Since Δ is fixed by $h_*^{-1}\circ\mu$, the common face of Δ and Δ_1 is fixed. Δ_1 is the unique top dimensional simplex containing the common codimension 1 face of Δ and Δ_1 other than Δ. Since $h_*^{-1}\circ\mu(\Delta_1)$ is a top dimensional simplex having this common face which is different from Δ,Δ_1 must be sent onto itself by $h_*^{-1}\circ\mu$. This implies that w_1 is fixed. By an inductive argument, we can prove that all the top dimensional simplices in the chain are fixed. This shows that w is fixed. Hence, $h_*^{-1}\circ\mu$ is the identity on $\mathscr{B}(S_c)$ and μ agrees with h_* on $\mathscr{B}(S_c)$. □

We have proven that λ is an injective simplicial map which preserves the geometric intersection 0 and 1 properties. Using these properties and following N.V. Ivanov's proof of his Theorem 1.1 [3], it can be seen that λ_* agrees with a map, h_*, induced by a homeomorphism $h:S_c\to S_d$ on a top dimensional simplex in $\mathscr{B}(S_c)$. Then, by Lemma 4.13, it agrees with h_* on $\mathscr{B}(S_c)$. Then, again by Ivanov's proof, λ agrees with a homeomorphism on $\mathscr{C}(S)$. This proves Theorem 1.4.

Remark. Note that at the end of the proof of Theorem 1.4 we appealed to Ivanov's proof. In his proof of Theorem 1.1 [3], by using Lemma 1 in his paper and using some homotopy theoretic results about the complex of curves on surfaces with boundary, he shows that an automorphism of the complex of curves preserves the geometric intersection 1 property if the surface has genus at least 2. He uses this property to induce automorphisms on the complex of arcs using automorphisms of the complex of curves and gets an element of the extended mapping class group which agrees with the automorphism of the complex of curves. In this paper, for closed surfaces of genus at least 3, we prove that a superinjective simplicial map λ preserves the geometric intersection 1 property by using Ivanov's Lemma 1 and using some elementary surface topology arguments avoiding any homotopy theoretic results about the complex of curves. These elementary surface topology arguments can be used to replace the usage of the homotopy theoretic results about the complex of curves in Ivanov's proof to show that an automorphism of the complex of curves preserves the geometric intersection 1 property. Then we follow Ivanov's ideas and use this property to get an injective simplicial map of the complex of arcs of S_c (c is nonseparating) using a superinjective simplicial map of the complex of curves of S and follow Ivanov to get an element of the extended mapping class group that we want.

5. Injective homomorphisms of subgroups of mapping class groups

In this section we assume that $\Gamma' = ker(\varphi)$, where $\varphi : Mod_S^* \to Aut(H_1(S, \mathbb{Z}_3))$ is the homomorphism defined by the action of homeomorphisms on the homology.

For a group G and for subsets $A, H \subseteq G$, where $A \subseteq H$, the centralizer of A in $H, C_H(A)$, and center of $G, C(G)$, are defined as follows:

$$C_H(A) = \{h \in H : ha = ah \ \forall a \in A\}, \quad C(G) = \{z \in G : zg = gz \ \forall g \in G\}.$$

Lemma 5.1. *Let H be a subgroup of a group G and let $A \subseteq H$. Then*

$$H \cap C(C_G(A)) \subseteq C(C_H(A)).$$

Proof. Let $h \in H \cap C(C_G(A))$. Then, $h \in H \cap C_G(A)$. It is clear that $H \cap C_G(A) = C_H(A)$. So, $h \in C_H(A)$. Let $k \in C_H(A)$. Then, $k \in C_G(A) \cap H \subseteq C_G(A)$. Then, since $h \in C(C_G(A)), h$ commutes with k. Therefore, $h \in C(C_H(A))$. □

Lemma 5.2. *Let K be a finite index subgroup of Mod_S^*. Let $f : K \to Mod_S^*$ be an injective homomorphism. Let $\Gamma = f^{-1}(\Gamma') \cap \Gamma'$. Let G be a free abelian subgroup of Γ of rank $3g-3$. If $a \in G$, then*

$$\operatorname{rank} C(C_{\Gamma'}(f(a))) \leqslant \operatorname{rank} C(C_\Gamma(a)).$$

Proof. Let $A = f(G) \cap C(C_{\Gamma'}(f(a)))$ and $B = \langle f(G), C(C_{\Gamma'}(f(a)))\rangle$ be the group generated by $f(G)$ and $C(C_{\Gamma'}(f(a)))$. Since $a \in G$ and G is abelian, $f(a) \in f(G)$ and $f(G)$ is abelian. Then, $f(G) \subseteq C_{\Gamma'}(f(a))$ since $f(G) \subseteq \Gamma'$. Then, B is an abelian group. Since rank $G = 3g-3$ and f is injective on G, rank $f(G) = 3g-3$. The maximal rank of an abelian subgroup of Mod_S^* is $3g-3$, [1]. So, since $f(G) \subseteq B$ and B is an abelian group and rank $f(G) = 3g-3$ we have rank $B = 3g-3$.

We have the following exact sequence:

$$1 \to A \to f(G) \oplus C(C_{\Gamma'}(f(a))) \to B \to 1.$$

This gives us

$$\operatorname{rank} f(G) + \operatorname{rank} C(C_{\Gamma'}(f(a))) = \operatorname{rank} A + \operatorname{rank} B.$$

Since rank $f(G) = \operatorname{rank} B$, we get

$$\operatorname{rank} C(C_{\Gamma'}(f(a))) = \operatorname{rank} A. \tag{1}$$

Then, by using Lemma 5.1, we have $A = f(G) \cap C(C_{\Gamma'}(f(a))) \subseteq f(\Gamma) \cap C(C_{\Gamma'}(f(a))) \subseteq C(C_{f(\Gamma)}(f(a)))$. Since f is injective on $K, C(C_{f(\Gamma)}(f(a)))$ is isomorphic to $C(C_\Gamma(a))$. So, we have rank $A \leqslant \operatorname{rank} C(C_\Gamma(a))$. Then, by Eq. (1) we have

$$\operatorname{rank} C(C_{\Gamma'}(f(a))) \leqslant \operatorname{rank} C(C_\Gamma(a)). \qquad \square$$

Lemma 5.3. *Let Γ be a finite index subgroup of Mod_S^* and $\Gamma \subseteq \Gamma'$. Let $N \in \mathbb{Z}^*$ such that $t_\alpha^N \in \Gamma$ $\forall \alpha \in \mathscr{A}$. Then, $C(C_\Gamma(t_\alpha^N))$ is an infinite cyclic subgroup of $\langle t_\alpha \rangle$ $\forall \alpha \in \mathscr{A}$.*

Proof. Let $\beta \in \mathscr{A}$ such that $i(\alpha, \beta) = 0$. Let $f \in C(C_\Gamma(t_\alpha^N))$. Then, $f(\alpha) = \alpha$. We want to show that $f(\beta) = \beta$. Since $i(\alpha, \beta) = 0$, t_α commutes with t_β. Then, t_α^N commutes with t_β^N. Then, since $t_\beta^N \in \Gamma, t_\beta^N \in C_\Gamma(t_\alpha^N)$. Since $f \in C(C_\Gamma(t_\alpha^N)), f$ commutes with t_β^N. Then, $f(\beta) = \beta$. Since f fixes the isotopy class of every circle which has geometric intersection zero with α, the reduction, $\hat{f}$, of f along α [1] fixes the isotopy class of every circle on S_a where $a \in [\alpha]$. Since $\Gamma \subseteq \Gamma', f \in \Gamma'$. Hence, by Lemma 1.6 [2], $\hat{f}$ restricts to each component Q of S_a and the restriction of $\hat{f}$ to Q is either trivial or infinite order. By Lemma 5.1 [4] and Lemma 5.2 [4], this restriction is finite order. Hence, it is trivial. Then, $f = t_\alpha^r$ for some $r \in \mathbb{Z}$, [1]. So, $C(C_\Gamma(t_\alpha^N)) \subseteq \langle t_\alpha \rangle$. Since $t_\alpha^N \in C(C_\Gamma(t_\alpha^N))$, $C(C_\Gamma(t_\alpha^N))$ is a nontrivial subgroup of an infinite cyclic group. Hence, it is infinite cyclic. □

Lemma 5.4. *Let K be a finite index subgroup of Mod_S^*. Let $N \in \mathbb{Z}^*$ such that $t_\alpha^N \in K$ $\forall \alpha \in \mathscr{A}$. Let $f : K \to Mod_S^*$ be an injective honmomorphism. Then $f(t_\alpha^N)$ is a reducible element of infinite order $\forall \alpha \in \mathscr{A}$.*

Proof. Since f is an injective homomorphism on K and t_α^N is an infinite order element of K, $f(t_\alpha^N)$ is an infinite order element of Mod_S^*. So, $f(t_\alpha^N)$ is either a reducible element or p-Anosov. Suppose it is p-Anosov. Let J be a maximal system of circles containing a where $[a] = \alpha$ and J' be the set of isotopy classes of these circles. Let $T_{J'}$ be the subgroup of Mod_S^* generated by t_β^N for all β in J'. $T_{J'}$ is a free abelian subgroup of K and it has rank $3g - 3$. Since f is an injection, $f(T_{J'})$ is also a free abelian subgroup of rank $3g - 3$. It contains $f(t_\alpha^N)$. By [6], $C_{Mod_S^*}(f(t_\alpha^N))$ is a virtually infinite cyclic group and $f(T_{J'}) \subseteq C_{Mod_S^*}(f(t_\alpha^N))$. Then $3g - 3 \leqslant 1$. This gives us $3g \leqslant 4$. But this is a contradiction to the assumption that $g \geqslant 3$. Hence, $f(t_\alpha^N)$ is a reducible element of infinite order.

Lemma 5.5. *Let K be a finite index subgroup of Mod_S^* and $f : K \to Mod_S^*$ be an injective homomorphism. Let $\alpha \in \mathscr{A}$. Then there exists $N \in \mathbb{Z}^*$ such that*

$$\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) = 1.$$

Proof. Let $\Gamma = f^{-1}(\Gamma') \cap \Gamma'$. Let $\alpha \in \mathscr{A}, N \in \mathbb{Z}^*$ such that $t_\alpha^N \in \Gamma$. Then, since $\langle f(t_\alpha^N) \rangle \subseteq C(C_{\Gamma'}(f(t_\alpha^N)))$, and $\langle f(t_\alpha^N) \rangle$ is an infinite cyclic group we have

$$\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) \geqslant 1. \tag{2}$$

By Lemma 5.3, $\operatorname{rank} C(C_\Gamma(t_\alpha^N)) = 1$. Since t_α^N is in a free abelian subgroup of Γ of rank $3g - 3$, by Lemma 5.2, $\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) \leqslant \operatorname{rank} C(C_\Gamma(t_\alpha^N))$. So, we get

$$\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) \leqslant 1. \tag{3}$$

(2) and (3) imply that $\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) = 1$. □

Lemma 5.6. *Let $\alpha, \beta \in \mathscr{A}$ and i, j be nonzero integers. Then, $t_\alpha^i = t_\beta^j \Leftrightarrow \alpha = \beta$ and $i = j$.*

Proof. If $\alpha=\beta$ and $i=j$, then obviously $t_\alpha^i=t_\beta^j$. To see the other implication, we consider the reduction systems. Since β is the canonical reduction system for t_β^j and α is the canonical reduction system for t_α^i and $t_\alpha^i=t_\beta^j$, their canonical reduction systems must be equal. So, $\alpha=\beta$. Then we have $t_\alpha^i=t_\alpha^j$. Then, $i=j$ since t_α is an infinite order element in Mod_S^*. □

Lemma 5.7. *Let K be a finite index subgroup of Mod_S^*, and $f: K \to Mod_S^*$ be an injective homomorphism. Then $\forall \alpha \in \mathscr{A}$, $f(t_\alpha^N)=t_{\beta(\alpha)}^M$ for some $M,N\in\mathbb{Z}^*, \beta(\alpha)\in\mathscr{A}$.*

Proof. Let $\Gamma=f^{-1}(\Gamma')\cap\Gamma'$. Since Γ is a finite index subgroup we can choose $N\in Z^*$ such that $t_\alpha^N\in\Gamma$, for all α in $\mathscr{A}$. By Lemma 5.4, $f(t_\alpha^N)$ is a reducible element of infinite order in Mod_S^*. Let C be a realization of the canonical reduction system of $f(t_\alpha^N)$. Let c be the number of components of C and p be the number of p-Anosov components of $f(t_\alpha^N)$. Since $t_\alpha^N\in\Gamma, f(t_\alpha^N)\in\Gamma'$. By Theorem 5.9 [4], $C(C_{\Gamma'}(f(t_\alpha^N)))$ is a free abelian group of rank $c+p$. By Lemma 5.5, $c+p=1$. Then, either $c=1$, $p=0$ or $c=0, p=1$. Since there is at least one curve in the canonical reduction system we have $c=1, p=0$. Hence, $f(t_\alpha^{N\cdot k})=t_{\beta(\alpha)}^M$ for some $M,k\in\mathbb{Z}^*,\beta(\alpha)\in\mathscr{A}$. □

Remark. Suppose that $f(t_\alpha^M)=t_\beta^P$ for some $\beta\in\mathscr{A}$ and $M,P\in\mathbb{Z}^*$ and $f(t_\alpha^N)=t_\gamma^Q$ for some $\gamma\in\mathscr{A}$ and $N,Q\in Z^*$. Since $f(t_\alpha^{M\cdot N})=f(t_\alpha^{N\cdot M}), t_\beta^{PN}=t_\gamma^{QM}, P,Q,M,N\in\mathbb{Z}^*$. Then, $\beta=\gamma$ by Lemma 5.6. Therefore, by Lemma 5.7, f gives a correspondence between isotopy classes of circles and f induces a map, $f_*:\mathscr{A}\to\mathscr{A}$, where $f_*(\alpha)=\beta(\alpha)$.

In the following lemma, we use a well known fact that $ft_\alpha f^{-1}=t_{f(\alpha)}^{\varepsilon(f)}$ for all α in $\mathscr{A}, f\in Mod_S^*$, where $\varepsilon(f)=1$ if f has an orientation preserving representative and $\varepsilon(f)=-1$ if f has an orientation reversing representative.

Lemma 5.8. *Let K be a finite index subgroup of Mod_S^*. Let $f:K\to Mod_S^*$ be an injective homomorphism. Assume that there exists $N\in\mathbb{Z}^*$ such that $\forall\alpha\in\mathscr{A},\exists Q\in\mathbb{Z}^*$ such that $f(t_\alpha^N)=t_\alpha^Q$. Then, f is the identity on K.*

Proof. We use Ivanov's trick to see that $f(kt_\alpha^Nk^{-1})=f(t_{k(\alpha)}^{\varepsilon(k)\cdot N})=t_{k(\alpha)}^{Q\cdot\varepsilon(k)}$ and $f(kt_\alpha^Nk^{-1})=f(k)$ $f(t_\alpha^N)f(k)^{-1}=f(k)t_\alpha^Qf(k)^{-1}=t_{f(k)(\alpha)}^{\varepsilon(f(k))\cdot Q}$ $\forall\alpha\in\mathscr{A}$, $\forall k\in K$. Then, we have $t_{k(\alpha)}^{Q\cdot\varepsilon(k)}=t_{f(k)(\alpha)}^{\varepsilon(f(k))\cdot Q}\Rightarrow k(\alpha)=f(k)(\alpha)$ $\forall\alpha\in\mathscr{A}$, $\forall k\in K$ by Lemma 5.6. Then, $k^{-1}f(k)(\alpha)=\alpha$ $\forall\alpha\in\mathscr{A}$, $\forall k\in K$. Then, $k^{-1}f(k)$ commutes with t_α $\forall\alpha\in\mathscr{A}$, $\forall k\in K$. Since Mod_S^* is generated by Dehn twists, $k^{-1}f(k)\in C(Mod_S^*)$ $\forall k\in K$. Since the genus of S is at least 3, $C(Mod_S^*)$ is trivial. So, $k=f(k)$ $\forall k\in K$. Hence, $f=id_K$. □

Corollary 5.9. *Let $g: Mod_S^*\to Mod_S^*$ be an isomorphism and $h: Mod_S^*\to Mod_S^*$ be an injective homomorphism. Assume that there exists $N\in\mathbb{Z}^*$ such that $\forall\alpha\in\mathscr{A},\exists Q\in\mathbb{Z}^*$ such that $h(t_\alpha^N)=g(t_\alpha^Q)$. Then $g=h$.*

Proof. Apply Lemma 5.8 to $g^{-1}h$ with $K=Mod_S^*$. Since for all α in $\mathscr{A}, g^{-1}h(t_\alpha^N)=t_\alpha^Q$, we have $g^{-1}h=id_K$. Hence, $g=h$.

Lemma 5.10. *Let α, β be distinct elements in $\mathscr{A}$. Let i, j be two nonzero integers. Then, $t_\alpha^i t_\beta^j = t_\beta^j t_\alpha^i \Leftrightarrow i(\alpha, \beta) = 0$.*

Proof. $t_\alpha^i t_\beta^j = t_\beta^j t_\alpha^i \Leftrightarrow t_\alpha^i t_\beta^j t_\alpha^{-i} = t_\beta^j \Leftrightarrow t_{t_\alpha^i(\beta)}^j = t_\beta^j \Leftrightarrow t_\alpha^i(\beta) = \beta$ (by Lemma 5.6) $\Leftrightarrow i(\alpha, \beta) = 0$ ("$\Leftarrow$" : Clear. "$\Rightarrow$": $i(\alpha, \beta) \neq 0 \Rightarrow t_\alpha^m(\beta) \neq \beta$ for all $m \in \mathscr{Z}^*$ since α is an essential reduction class for t_α.). $\square$

By the remark after Lemma 5.7, we have that $f : K \to Mod_S^*$ induces a map $f_* : \mathscr{A} \to \mathscr{A}$. We prove the following lemma which shows that f_* is a superinjective simplicial map on $\mathscr{C}(S)$.

Lemma 5.11. *Let $f : K \to Mod_S^*$ be an injection. Let $\alpha, \beta \in \mathscr{A}$. Then,*

$$i(\alpha, \beta) = 0 \quad \Leftrightarrow \quad i(f_*(\alpha), f_*(\beta)) = 0.$$

Proof. There exists $N \in \mathbb{Z}^*$ such that $t_\alpha^N \in K$ and $t_\beta^N \in K$. Then we have the following: $i(\alpha, \beta) = 0 \Leftrightarrow t_\alpha^N t_\beta^N = t_\beta^N t_\alpha^N$ (by Lemma 5.10) $\Leftrightarrow f(t_\alpha^N) f(t_\beta^N) = f(t_\beta^N) f(t_\alpha^N)$ (since f is injective on K) $\Leftrightarrow t_{f_*(\alpha)}^P t_{f_*(\beta)}^Q = t_{f_*(\beta)}^Q t_{f_*(\alpha)}^P$ where $P = M(\alpha_1, N)$, $Q = M(\alpha_2, N) \in \mathbb{Z}^*$ (by Lemma 5.7) $\Leftrightarrow i(f_*(\alpha), f_*(\beta)) = 0$ (by Lemma 5.10). $\square$

Now, we prove the second main theorem of the paper.

Theorem 5.12. *Let f be an injective homomorphism, $f : K \to Mod_S^*$, then f is induced by a homeomorphism of the surface S and f has a unique extension to an automorphism of Mod_S^*.*

Proof. By Lemma 5.11, f_* is a superinjective simplicial map on $\mathscr{C}(S)$. Then, by Theorem 1.4, f_* is induced by a homeomorphism $h : S \to S$, i.e. $f_*(\alpha) = H(\alpha)$ for all α in $\mathscr{A}$, where $H = [h]$. Let $\chi^H : Mod_S^* \to Mod_S^*$ be the isomorphism defined by the rule $\chi^H(K) = HKH$ for all K in Mod_S^*. Then for all α in $\mathscr{A}$, we have the following:

$$\chi^{H^{-1}} \circ f(t_\alpha^N) = \chi^{H^{-1}}(t_{f_*(\alpha)}^M) = \chi^{H^{-1}}(t_{H(\alpha)}^M) = H^{-1} t_{H(\alpha)}^M H = t_{H^{-1}(H(\alpha))}^{M \cdot \varepsilon(H^{-1})} = t_\alpha^{M \cdot \varepsilon(H^{-1})}.$$

Then, since $\chi^{H^{-1}} \circ f$ is injective, $\chi^{H^{-1}} \circ f = id_K$ by Lemma 5.8. So, $\chi^H|_K = f$. Hence, f is the restriction of an isomorphism which is conjugation by H, (i.e. f is induced by h).

Suppose that there exists an automorphism $\tau : Mod_S^* \to Mod_S^*$ such that $\tau|_K = f$. Let $N \in Z^*$ such that $t_\alpha^N \in K$ for all α in $\mathscr{A}$. Since $\chi^H|_K = f = \tau|_K$ and $t_\alpha^N \in K$, $\tau(t_\alpha^N) = \chi^H(t_\alpha^N)$ for all α in $\mathscr{A}$. Then, by Corollary 5.9, $\tau = \chi^H$. Hence, the extension of f is unique.

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