Additive preservers of non-zero decomposable tensors

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Abstract

Let \( T \) be an additive mapping from a tensor product of vector spaces over a field into itself. We describe \( T \) for the following two cases: (i) \( T \) is surjective and sends non-zero decomposable elements to non-zero decomposable elements, and (ii) \( T(A) \) is a non-zero decomposable element if and only if \( A \) is a non-zero decomposable element.

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1. Introduction

The problem of characterizing linear mappings on spaces of matrices (operators) that preserve certain properties has attracted the attention of many mathematicians. The first result in this area is the classical theorem of Frobenius [9] that classifies linear mappings on the space of complex square matrices that preserve the determinant. Dieudonne [7] generalized the Frobenius theorem by describing all invertible linear mappings on the space of all \( n \times n \) matrices over an arbitrary field that send the set of all singular matrices into itself. A mapping from a space of matrices (operators) to another is called a rank-one preserver if it sends rank-one matrices (operators) to rank-one matrices (operators). Dieudonne’s result gives immediately the structure of invertible linear rank-one preservers on the space of all \( n \times n \) matrices over an arbitrary field. Marcus and Moyls [21] characterized linear rank-one preservers on rectangular matrices over algebraically closed fields of characteristic zero. This result leads naturally to the study of linear mappings on
tensor spaces that preserve non-zero decomposable elements. Westwick [25] obtained a general
decomposition theorem concerning linear mappings from one tensor product space to another
that send non-zero decomposable elements to non-zero decomposable elements. He deduced
from this general theorem that if (i) $T$ is a surjective linear mapping from the tensor product of
$n$ vector spaces $U_i$, $i = 1, \ldots, n$, of arbitrary dimension over any field onto itself that preserve
non-zero decomposable tensors or (ii) $T$ is a linear mapping from the tensor product of $n$ vector
spaces $U_i$, $i = 1, \ldots, n$, of finite dimension over an algebraically closed field into itself that
preserve non-zero decomposable tensors, then $T$ is induced by bijective linear mappings from
$U_i$ to $U_{\sigma(i)}$, $i = 1, \ldots, n$, for some permutation $\sigma$. Later, he [26,27] improved the result in [25] by
characterizing linear mappings from the tensor product of $m$ vector spaces to the tensor product
of $n$ vector spaces that send non-zero decomposable elements to decomposable elements. When
$m = n = 2$, this was proved by Lim in [15].

Recently many results on linear preservers have been extended to the additive analogue. The
first result concerning additive rank-one preservers was obtained by Omladic and Semrl [22].
They characterized surjective additive maps on the algebra $\mathcal{F}(X)$ of all bounded finite rank linear
operators on a real or complex Banach spaces $X$. A mapping from a space of matrices (operators)
to another is called rank-one non-increasing if it sends rank-one matrices (operators) to matrices
(operators) of rank less than or equal to one. Kurma [12] obtained a substantial generalization of
the result of Omladic and Semrl. He characterized rank-one non-increasing additive mappings on
$\mathcal{F}(X)$. A short alternative proof of his result as well as a slight extension to tensor spaces over
division rings was obtained in [20]. Bell and Sourour [1] classified surjective additive rank-one
preservers between block triangular matrices over arbitrary fields and also additive mappings
between those spaces preserving rank-one matrices in both directions when the underlying fields
have no isomorphic proper subfields. Chooi and Lim [6] extended some results of Bell and
Sourour. They obtained a general form of additive rank-one preservers from block triangular
matrix spaces to rectangular matrix spaces. Zhang and Sze [24] classified additive rank-one
preservers between rectangular matrix spaces over arbitrary fields. Additive rank-one preservers
on symmetric and Hermitian matrices were studied in [2,10,23] and some generalizations to rank-
one non-increasing additive mappings between the corresponding matrix spaces were obtained in
[13,18,19]. Very recently, a remarkable work of Huang and Semrl [11] completely characterizes
mappings $T$ from a space of $m \times m$ complex hermitian matrices to another space of all $n \times n$
complex hermitian matrices such that rank $(T(A) - T(B)) = 1$ whenever rank $(A - B) = 1$. An
additive mapping $f$ from a vector space to another vector space $V$ is said to be almost surjective
if $V$ is linearly spanned by $\text{Im} f$. In this note we study additive mappings from one tensor space
to another that send non-zero decomposable elements to non-zero decomposable elements. We
show that if (i) $T$ is a almost surjective additive mapping from the tensor product $W$ of $n$ finite
dimensional vector spaces $U_1, \ldots, U_n$ each of dimension at least 2 over a field into itself that
preserve non-zero decomposable elements or (ii) $T$ is an additive map on $W$ that preserves non-
zero decomposable elements in both directions, then $T$ is induced by quasilinear mappings from
$U_i$ to $U_{\sigma(i)}$, $i = 1, \ldots, n$, for some permutation $\sigma$.

2. Results

Throughout this paper $F$ denotes a field, $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ are vector spaces over $F$
with dimension at least 2. Let $\bigotimes_{i=1}^m U_i$ denote the tensor product of $U_1, \ldots, U_m$ over the field $F$. Let
$X = x_1 \otimes \cdots \otimes x_m$ and $Y = y_1 \otimes \cdots \otimes y_m$ be two decomposable elements in $\bigotimes_{i=1}^m U_i$. Then we
write $X \sim Y$ if $X - Y$ is decomposable. If $X$ and $Y$ are non-zero and $X \sim Y$, then we say that they are adjacent and in this case $x_i$ and $y_i$ are linearly independent for at most one $i$. If $X$ and $Y$ are non-zero and not adjacent, it is clear that one can always find a chain of non-zero decomposable elements $A_1, \ldots, A_{m+1}$ such that $X = A_1, A_i \sim A_{i+1}, i = 1, \ldots, m$, and $A_{m+1} = Y$. In this case, we call $A_1, \ldots, A_{m+1}$ a chain of non-zero decomposable elements joining $X$ and $Y$. An additive subgroup of $\otimes_{i=1}^m U_i$ is called a decomposable subgroup if it consists of decomposable elements. One could show that every decomposable subgroup of $\otimes_{i=1}^m U_i$ is of the form

$$u_1 \otimes \cdots \otimes u_{i-1} \otimes G \otimes u_{i+1} \otimes \cdots \otimes u_m$$

for some $i$, some non-zero vectors $u_j \in U_j$, $j \neq i$, and some subgroup $G$ of $U_i$. A decomposable subgroup is said to be of type-$i$ if it is of the form (1). If $G = U_i$, then it is called a maximal decomposable subgroup of type-$i$. Two maximal decomposable subgroups of type-$i$

$$u_1 \otimes \cdots \otimes u_{i-1} \otimes U_i \otimes u_{i+1} \otimes \cdots \otimes u_m$$

and

$$y_1 \otimes \cdots \otimes y_{i-1} \otimes U_i \otimes y_{i+1} \otimes \cdots \otimes y_m$$

are called adjacent if $u_j$ and $y_j$ are linearly independent for at most one $j$. Note that when $m = 2$, any two maximal decomposable subgroups of the same type are automatically adjacent. If two maximal decomposable subgroups $M_1$ and $M_2$ are adjacent, we write $M_1 \sim M_2$. A mapping from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$ is called decomposable if it sends non-zero decomposable elements to non-zero decomposable elements. A decomposable mapping is called regular if the image of any maximal decomposable subgroup contains at least two linearly independent vectors. Clearly every linear decomposable mapping is regular. We shall establish a general decomposition theorem for regular additive decomposable mappings from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$ where $m \geq 2$ and $n \geq 2$. Unless otherwise stated, we assume from now on $m \geq 2$ and $n \geq 2$.

**Lemma 2.1.** Let $T$ be a regular additive decomposable mapping from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$. If $M_1$ and $M_2$ are maximal decomposable subgroups of the same type in $\otimes_{i=1}^m U_i$, then $T(M_1)$ and $T(M_2)$ are decomposable subgroups of the same type in $\otimes_{i=1}^n V_i$.

**Proof.** Since there is a chain of maximal decomposable subgroups $N_1, \ldots, N_{m-2}$ such that $M_1 \sim N_1, \ldots, N_i \sim N_{i+1}, \ldots, N_{m-2} \sim M_2$, we may assume without loss of generality that $M_1 \sim M_2$. We may also assume that $M_1 = A \otimes U_m$ and $M_2 = B \otimes U_m$, where $A, B$ are linearly independent adjacent decomposable elements in $\otimes_{i=1}^{m-1} U_i$. Suppose that $T(M_1)$ and $T(M_2)$ are decomposable subgroups of different types. We may assume that

$$T(M_1) \subseteq v_1 \otimes V_2 \otimes C,$$

$$T(M_2) \subseteq V_1 \otimes v_2 \otimes D$$

for some non-zero vectors $v_1 \in V_1, v_2 \in V_2$, and some non-zero decomposable elements $C, D$ in $\otimes_{i=3}^n V_i$. Since $T$ is regular, we have $\dim \langle T(M_1) \rangle \geq 2, i = 1, 2$. Hence there exist vectors $u_1, u_2 \in U_m$ such that

$$T(A \otimes u_2) = v_1 \otimes w_2 \otimes C,$$

$$T(B \otimes u_1) = w_1 \otimes v_2 \otimes D,$$

where both $v_1, w_1$ and $v_2, w_2$ are linearly independent. Since $A \otimes u_1 \in M_1$ and $A \otimes u_1 \sim B \otimes u_1$, it follows that

$$T(A \otimes u_1) = cv_1 \otimes v_2 \otimes C$$
for some non-zero scalar \( c \) and \( C, D \) are linearly dependent. Similarly

\[
T(B \otimes u_2) = dv_1 \otimes v_2 \otimes D
\]

for some non-zero scalar \( d \). Hence \( T((A + B) \otimes (u_1 + u_2)) \) is not decomposable tensor since \( v_1 \otimes (w_2 + cv_2) \) and \( (w_1 + dv_1) \otimes v_2 \) are not adjacent. This is a contradiction since \( (A + B) \otimes (u_1 + u_2) \) is decomposable. This proves that \( T(M_1) \) and \( T(M_2) \) are decomposable subgroups of the same type. \( \Box \)

**Lemma 2.2.** Let \( U_1, \ldots, U_m \) and \( V \) be vector spaces over \( F \). Let \( f \) and \( g \) be additive mappings from \( \otimes_{i=1}^m U_i \) to \( V \) such that \( \langle f(A) \rangle = \langle g(A) \rangle \neq \{0\} \) for any non-zero decomposable element \( A \in \otimes_{i=1}^m U_i \). If \( f \) is regular, then \( f = \lambda g \) for some non-zero scalar \( \lambda \).

**Proof.** Let \( \Omega \) denote the set of all non-zero decomposable elements of \( \otimes_{i=1}^m U_i \). For any \( A \in \Omega \), we have \( f(A) = \sigma(A)g(A) \) for some non-zero scalar \( \sigma(A) \). Let \( B \) and \( C \) be two adjacent elements in \( \Omega \). Suppose that \( f(B) \) and \( f(C) \) are linearly independent.

Then \( f(B - C) = \sigma(B - C)g(B - C) \) and hence

\[
\sigma(B)g(B) - \sigma(C)g(C) = \sigma(B - C)(g(B) - g(C)).
\]

We thus obtain that \( \sigma(B) = \sigma(C) = \sigma(B - C) \). Suppose now that \( f(B) \) and \( f(C) \) are linearly dependent. Since \( f \) is regular, there exists \( D \in \Omega \) such that \( B, C, D \) are contained in a maximal decomposable subgroup and \( f(B), f(D) \) are linearly independent. Note that \( f(C) \) and \( f(D) \) are linearly independent. Hence from the previous case we have \( \sigma(B) = \sigma(D) = \sigma(C) \). Now for any two non-adjacent decomposable elements \( X \) and \( Y \) in \( \Omega \), there exists a chain of non-zero decomposable elements \( X_1, \ldots, X_{m+1} \) joining \( X \) and \( Y \). Hence \( \sigma(X) = \sigma(X_i) = \sigma(Y) \), \( i = 1, \ldots, m + 1 \). This shows that there exists a non-zero scalar \( c \) such that for any \( A \in \Omega \), \( f(A) = cg(A) \). Since \( f \) is additive, it follows that \( f = cg \). \( \Box \)

**Remark 2.3.** Lemma 2.2 is known for \( m = 1 \) (see [8, Lemma 6.3.4, p. 137]).

In order to describe some types of additive decomposable mappings, we need the following definitions and notation. For each positive integer \( s \), let \( \{s\} = \{1, \ldots, s\} \). Let \( \{J_1, \ldots, J_k\} \) be a partition of \( [m] \). Let \( \tau \) be a non-zero field endomorphism of \( F \). For each \( i = 1, \ldots, k \), let \( T_i : \otimes_{s \in J_i} U_s \to V_i \) be a \( \tau \)-quasilinear mapping. Then these \( k \) quasilinear mappings induce a \( \tau \)-quasilinear mapping \( T \) from \( \otimes_{i=1}^n U_i \) to \( \otimes_{i=1}^n V_i \) such that

\[
T(x_1 \otimes \cdots \otimes x_m) = T_1(\otimes_{s \in J_1} x_s) \otimes \cdots \otimes T_k(\otimes_{s \in J_k} x_s)
\]

for any decomposable element \( x_1 \otimes \cdots \otimes x_m \in \otimes_{i=1}^m U_i \) and we denote \( T \) by \( T_1 \otimes \cdots \otimes T_k \).

Note that \( T \) is a (almost surjective) decomposable mapping if and only if each \( T_i \) is a (almost surjective) decomposable mapping.

Let \( I \) be a non-empty proper subset of \( [n] \). For each non-zero decomposable element \( y := \otimes_{i \in I'} V_i \in \otimes_{i \in I} V_i \) where \( I' = [n] \setminus I \), let \( M_y \) denote the multiplication mapping from \( \otimes_{i \in I} V_i \) to \( \otimes_{i=1}^n V_i \) such that

\[
M_y(\otimes_{i \in I} x_i) = v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_n,
\]

where \( v_i = x_i \) if \( i \in I \) and \( v_i = y_i \) if \( i \notin I \). Clearly \( M_y \) is a linear decomposable mapping.
An additive decomposable mapping from a tensor space to another is said to be degenerate if its image consists of decomposable elements. For two positive integers \( k \leq n \), a mapping \( \phi \) from \( [k] \) to \( [n] \) is called order-preserving if \( \phi(i) < \phi(j) \) for \( i < j \).

**Theorem 2.4.** Let \( T \) be a non-degenerate regular additive decomposable mapping from \( \bigotimes_{i=1}^{m} U_i \) to \( \bigotimes_{i=1}^{n} V_i \). Then there exist a partition \( \{ J_1, \ldots, J_k \} \) of \( [m] \), an order-preserving mapping \( \phi \) from \( [k] \) to \( [n] \), \( \tau \)-quasilinear decomposable mappings \( T_i: \bigotimes_{s \in J_i} U_s \to V_{\phi(i)} \), \( i = 1, \ldots, k \), and a multiplication mapping \( M_y: \bigotimes_{i=1}^{k} V_{\phi(i)} \to \bigotimes_{i=1}^{n} V_i \) where \( y \) is a non-zero decomposable element in \( \bigotimes \{ V_j : j \in [n] \setminus \text{Im} \phi \} \) if \( n > k \), such that

\[
T = M_y \circ (T_1 \otimes \cdots \otimes T_k),
\]

where \( M_y \) is deleted if \( n = k \).

**Proof.** Let \( J \) be the subset of \( [n] \) consisting of all \( j \) such that \( T(M) \) is of type-\( j \) for some maximal decomposable subgroup \( M \) of \( \bigotimes_{i=1}^{m} U_i \). Write \( J = \{ j_1, \ldots, j_k \} \) where \( j_1 < j_2 < \cdots < j_k \).

Let \( \phi \) be the mapping from \( [k] \) to \( [n] \) such that \( \phi(i) = j_i, i = 1, \ldots, k \). Let \( J_i \) be the subset of \( [m] \) consisting of all positive integers \( s \) such that maximal decomposable subgroups of type-\( s \) in \( \bigotimes_{i=1}^{m} U_i \) are mapped to decomposable subgroups of type-\( j_i \) in \( \bigotimes_{i=1}^{n} V_i \). This is well-defined in view of Lemma 2.1, and \( \{ J_1, \ldots, J_k \} \) is a partition of \( [m] \). Since \( T \) is non-degenerate, we have \( k \geq 2 \). For convenience, we may assume that \( J = \{ 1, \ldots, k \} \) and for any \( s \in J_i \) and \( t \in J_{i+1}, i = 1, \ldots, k-1 \), we have \( s < t \). Thus \( J_k = \{ l, l+1, \ldots, m \} \) for some positive integer \( l \).

Suppose that \( k < n \). Let \( x_1 \otimes \cdots \otimes x_m \) be a non-zero decomposable element in \( \bigotimes_{i=1}^{m} U_i \) and \( T(x_1 \otimes \cdots \otimes x_m) = y_1 \otimes \cdots \otimes y_n \). Consider any non-zero decomposable element \( u_1 \otimes \cdots \otimes u_m \in \bigotimes_{i=1}^{m} U_i \). Since \( T(U_1 \otimes x_2 \otimes \cdots \otimes x_m) \) is of type \( \leq k \) and \( u_1 \otimes x_2 \otimes \cdots \otimes x_m \in U_1 \otimes x_2 \otimes \cdots \otimes x_m \), it follows that

\[
T(u_1 \otimes x_2 \otimes \cdots \otimes x_m) \in W \otimes y_{k+1} \otimes \cdots \otimes y_n,
\]

(2)

where \( W = \bigotimes_{i=1}^{k} V_i \). Similarly \( u_1 \otimes u_2 \otimes x_3 \otimes \cdots \otimes x_m \in u_1 \otimes U_2 \otimes x_3 \otimes \cdots \otimes x_m \) and \( T(u_1 \otimes U_2 \otimes x_3 \otimes \cdots \otimes x_m) \) is of type \( \leq k \), it follows from (2) that

\[
T(u_1 \otimes u_2 \otimes x_3 \otimes \cdots \otimes x_m) \in W \otimes y_{k+1} \otimes \cdots \otimes y_n.
\]

Continue the process, we see that

\[
T(u_1 \otimes \cdots \otimes u_m) \in W \otimes y_{k+1} \otimes \cdots \otimes y_n.
\]

We can now define an additive mapping \( S: \bigotimes_{i=1}^{m} U_i \to \bigotimes_{i=1}^{k} V_i \) as follows:

\[
S(u_1 \otimes \cdots \otimes u_m) = v \quad \text{if} \quad T(u_1 \otimes \cdots \otimes u_m) = v \otimes y_{k+1} \otimes \cdots \otimes y_n.
\]

Then clearly \( S \) is a non-degenerate regular decomposable mapping. Suppose that \( k = n \). We let \( S \) to be the same as \( T \).

**Claim.** For each non-zero decomposable element \( x \) in \( \bigotimes_{i=1}^{k-1} U_i \), there exists a non-zero decomposable element \( x' \) in \( \bigotimes_{i=1}^{k-1} V_i \) such that \( S(x \otimes Y) \subseteq x' \otimes V_k \) where \( Y = \bigotimes_{i=1}^{m} U_i \).

Suppose the contrary. Then there exist non-zero decomposable elements \( y_1, y_2 \) in \( Y \) such that

\[
S(x \otimes y_1) = z_1 \otimes v_1,
S(x \otimes y_2) = z_2 \otimes v_2
\]
for some linearly independent decomposable elements $z_1, z_2$ in $\otimes_{i=1}^{k-1} V_i$ and some non-zero elements $v_1, v_2 \in V_k$. Since there is a chain of adjacent decomposable elements joining $y_1, y_2$, we may assume that $y_1 \sim y_2$.

Let $H$ be a maximal decomposable subgroup containing $x \otimes y_1$ and $x \otimes y_2$. Then $S(H)$ is a decomposable subgroup of type-$k$ containing $z_1 \otimes v_1$ and $z_2 \otimes v_2$, a contradiction.

This shows that there exists an additive mapping $C_x : Y \to V_k$ such that

$$S(x \otimes f) = x' \otimes C_x(f)$$

for all $f$ in $Y$. Note that $C_x$ is regular.

Let $y, z$ be any pair of non-zero decomposable elements in $\otimes_{i=1}^{l-1} U_i$. Then we have

$$S(y \otimes f) = y' \otimes C_y(f),$$
$$S(z \otimes f) = z' \otimes C_z(f)$$

for some non-zero decomposable elements $y', z'$ in $\otimes_{i=1}^{k-1} V_i$ and for all $f$ in $Y$. Suppose that $f$ is a non-zero decomposable element. Assume that $y$ and $z$ are adjacent. Then $y \otimes f$ and $z \otimes f$ belong to a maximal decomposable subgroup $M$ of type-$s$ for some $s < l$. Hence $S(M)$ a decomposable subgroup of type-$t$ for some $t < k$. This shows that $C_y(f)$ and $C_z(f)$ are linearly dependent. Now, suppose that $y$ and $z$ are not adjacent. Then there exists a chain of decomposable elements joining $y$ and $z$. We conclude from the previous case that $C_y(f)$ and $C_z(f)$ must be linearly dependent. In view of Lemma 2.2, $C_y$ and $C_z$ are linearly dependent. This shows that there exist an additive mapping $C : Y \to V_k$ and an additive mapping $B : \otimes_{i=1}^{l-1} U_i \to \otimes_{i=1}^{k-1} V_i$ such that

$$S(x \otimes f) = B(x) \otimes C(f)$$

for all $f$ in $Y$ and $x$ in $\otimes_{i=1}^{l-1} U_i$.

Clearly $B$ is a regular decomposable mapping. If $k > 2$, by repeating the process we see that there are regular additive decomposable mappings $C_i : \otimes_{s \in J_i} U_s \to V_i$, $1 \leq i \leq k - 1$, such that

$$S(A_1 \otimes \cdots \otimes A_k) = C_1(A_1) \otimes \cdots \otimes C_k(A_k)$$

for all $A_i \in \otimes_{s \in J_i} U_s$, $i = 1, \ldots, k$ where $C = C_k$.

For any $\lambda \in F \setminus \{0\}$ and any non-zero decomposable element $A_i \in \otimes_{s \in J_i} U_s$, $i = 1, \ldots, k$,

$$S((\lambda A_1) \otimes A_2 \otimes \cdots \otimes A_k) = C_1(\lambda A_1) \otimes C_2(A_2) \otimes \cdots \otimes C_k(A_k)$$
$$= S(A_1 \otimes \lambda A_2 \otimes \cdots \otimes A_k)$$
$$= C_1(A_1) \otimes C_2(\lambda A_2) \otimes \cdots \otimes C_k(A_k).$$

This shows that $C_1(\lambda A_1) = \tau_{A_1}(\lambda) C_1(A_1)$ and $C_2(\lambda A_2) = \eta_{A_2}(\lambda) C_2(A_2)$ for some non-zero scalars $\tau_{A_1}(\lambda)$ and $\eta_{A_2}(\lambda)$. Clearly $\tau_{A_1}(\lambda) = \eta_{A_2}(\lambda)$. Hence there is a function $\tau$ on $F$ such that $\tau(\lambda) = \tau_{A_1}(\lambda) = \eta_{A_2}(\lambda)$ for any $\lambda \in F \setminus \{0\}$ and any non-zero decomposable element $A_i \in \otimes_{s \in J_i} U_s$, $i = 1, 2$. Let $\tau(0) = 0$. Clearly $\tau$ is additive on $F$.

Since

$$C_1((\lambda \mu) A_1) = \tau(\lambda \mu) C_1(A_1) = C_1(\lambda (\mu A_1))$$
$$= \tau(\lambda) C_1(\mu A_1) = \tau(\lambda) \tau(\mu) C_1(A_1),$$

it follows that $\tau$ is multiplicative on $F$. Thus $C_1$ is $\tau$-quasilinear. Similarly $C_i$ is $\tau$-quasilinear for $i \geq 2$. This completes our proof. \(\square\)
Remark 2.5. When $T$ is linear, Theorem 2.4 was obtained by Westwick [25] by first proving a combinatorial result concerning adjacency preserving mappings from one Cartesian product of finite number of sets to another. This combinatorial result is not applicable to additive decomposable mappings.

Example 2.6. Let $V = \otimes^3 \mathbb{R}^n$ and $W = \otimes^4 \mathbb{R}^n$ where $n$ is a positive integer $\geq 2$ and $\mathbb{R}$ is the real field. Note that $\mathbb{R}^n \otimes \mathbb{R}^n$ and $\mathbb{R}$ are isomorphic as vector spaces over the rational numbers and let $f : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ be an isomorphism. Let $v$ and $w$ be two fixed non-zero vectors in $\mathbb{R}^n$. Then the additive mapping $T : V \rightarrow V$ such that

$$T(x \otimes y \otimes z) = x \otimes v \otimes (f(y \otimes z))w$$

is a non-regular degenerate decomposable mapping and the additive mapping $S : W \rightarrow W$ such that

$$S(x_1 \otimes x_2 \otimes y \otimes z) = x_1 \otimes x_2 \otimes v \otimes (f(y \otimes z))w$$

is non-regular and non-degenerate.

Corollary 2.7. Let $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ be finite dimensional real vector spaces all with same dimension $k$. Let $T$ be a linear decomposable mapping from $\otimes_{i=1}^m U_i$ into $\otimes_{i=1}^n V_i$. If $k \neq \{2, 4, 8\}$, then $m \leq n$ and there exist a permutation $\sigma$ on $[m]$, an order-preserving mapping $\phi$ from $[m]$ to $[n]$, invertible linear mappings $T_i : U_{\sigma(i)} \rightarrow V_{\phi(i)}$, $i = 1, \ldots, m$, and a multiplication mapping $M_y : \otimes_{i=1}^m V_{\phi(i)} \rightarrow \otimes_{i=1}^n V_i$ where $y$ is a non-zero decomposable element in $\otimes\{V_j : j \in [n] \setminus \operatorname{Im} \phi\}$ if $n > m$, such that

$$T = M_y \circ (T_1 \otimes \cdots \otimes T_m),$$

where $M_y$ is deleted if $n = m$.

Proof. Since $T$ is linear, it follows that it is regular. Suppose that there is a linear decomposable mapping from $\otimes_{i=1}^m U_i \rightarrow V_s$ for some $J \subseteq [m], \sigma \in [n]$ with $|J| \geq 2$. This will in turn imply that there exists a linear decomposable mapping from $\mathbb{R}^k \otimes \mathbb{R}^k$ to $\mathbb{R}^k$. It is known that this occurs only if $k = 2, 4, 8$ (see [3]). We thus obtain a contradiction. It is now clear that our result follows from Theorem 2.4. □

Proposition 2.8. Let $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ be finite dimensional where $\dim U_i = s_i$, $i = 1, \ldots, m$ and $t = \max_i \dim V_i$. If $\sum_{i=1}^m s_i - m + 1 \leq t$, then there exists a linear degenerate decomposable mapping from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$. Conversely, if $F$ is an algebraically closed field and there exists a linear decomposable mapping from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$, then $\sum_{i=1}^m s_i - m + 1 \leq t$.

Proof. Necessity: We may assume that $U_i$ is the vector space of all polynomials of degree less than $s_i$ over $F$. Let $j$ be an integer such that $t = \dim V_j$. We may assume that $V_j$ is the vector space of all polynomials of degree less than $t$ over $F$. Then the multiplication of polynomials is a multilinear mapping from $\times_{i=1}^m U_i$ to $V_j$ since $\sum_{i=1}^m s_i - m + 1 \leq t$. This multilinear mapping induces a decomposable mapping from $\otimes_{i=1}^m U_i$ to $V_j$ and hence there exists a linear degenerate decomposable mapping from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$.

Sufficiency: Let $T$ be a degenerate linear decomposable map from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$. Let $D$ denote the set of all decomposable elements in $\otimes_{i=1}^m U_i$. Then it is known that $D$ is a homogeneous irreducible algebraic variety of dimension equal to $\sum_{i=1}^m s_i - m + 1$. Suppose that there exists a degenerate linear decomposable mapping $T$ from $\otimes_{i=1}^m U_i$ to $\otimes_{i=1}^n V_i$. Since $0 \notin T(D \setminus \{0\})$, it follows from [17, Lemma 1] that $T(D)$ is an irreducible algebraic variety of dimension equal to
posable mapping from \( x \) additive decomposable mappings. For each non-zero decomposable element \( \tau \otimes \) is a linear decomposable mapping from for some non-zero \( x_i \).

When \( m = n = 2 \), Proposition 2.8 has been discussed in [4,14].

**Corollary 2.10.** Let \( U_1, \ldots, U_m \) and \( V_1, \ldots, V_m \) be finite dimensional where \( \dim U_i = s_i, i = 1, \ldots, m, s_i \leq s_{i+1}, i = 1, \ldots, m - 1 \) and \( F \) be algebraically closed. If \( T \) is a linear decomposable mapping from \( \otimes_{i=1}^{m} U_i \) to \( \otimes_{i=1}^{m} V_i \) and \( \max \{ \dim V_i : i = 1, \ldots, m \} < s_1 + s_2 - 1 \), then \( T \) is induced by \( m \) injective linear mappings.

**Proof.** Suppose that \( T \) is not induced by \( m \) injective linear mappings. Then by Theorem 2.4, there is a linear decomposable mapping from \( \otimes_{i \in J} U_i \rightarrow V_s \) for some \( J \subseteq [m], s \in [m] \) with \( |J| \geq 2 \). By Proposition 2.8, \( (\sum_{i \in J} \dim U_i) - |J| + 1 \leq \dim V_s \), a contradiction to the hypothesis that \( \dim V_s < s_1 + s_2 - 1 \). Hence \( T \) is induced by \( m \) injective linear mappings.

We shall now deduce from Theorem 2.4 a result concerning the structure of almost surjective additive decomposable mappings. For each non-zero decomposable element \( x_1 \otimes \cdots \otimes x_n \), we call each \( \langle x_i \rangle \) one of its factors.

**Theorem 2.11.** Let \( T \) be an almost surjective additive decomposable mapping from \( \otimes_{i=1}^{m} U_i \) to \( \otimes_{i=1}^{m} V_i \) where \( m \leq n \). Then \( m = n \) and there exist a permutation \( \sigma \) on \([m], \) injective, almost surjective \( \tau \)-quasilinear mappings \( T_i : U_{\sigma(i)} \rightarrow V_i, i = 1, \ldots, m, \) such that

\[
T = T_1 \otimes \cdots \otimes T_m.
\]

**Proof.** We shall show that \( T \) is regular. Suppose the contrary that there exists a maximal decomposable subgroup \( M \) of \( \otimes_{i=1}^{m} U_i \) such that \( T(M) \subseteq \langle y_1 \otimes \cdots \otimes y_n \rangle \) for some non-zero vectors \( y_i \in V_i, i = 1, \ldots, n \). Without loss of generality, we may assume that

\[
M = x_1 \otimes \cdots \otimes x_{m-1} \otimes U_m
\]

for some non-zero \( x_i \in U_i, i = 1, \ldots, m - 1 \). Let \( u_1 \otimes \cdots \otimes u_m \) be any non-zero decomposable element in \( \otimes_{i=1}^{m} U_i \). Then there is a chain of non-zero decomposable elements \( X_1, \ldots, X_m \) in \( \otimes_{i=1}^{m} U_i \) joining \( x_1 \otimes \cdots \otimes x_{m-1} \otimes u_m \) and \( u_1 \otimes \cdots \otimes u_m \). Since \( T \) preserves pairs of non-zero adjacent decomposable elements and \( m \leq n \), we see that \( \langle y_i \rangle \) is a factor of \( T(u_1 \otimes \cdots \otimes u_m) \) for some \( i \). For each \( i = 1, \ldots, n \), choose a vector \( v_i \notin \langle y_i \rangle \). Suppose that \( v_1 \otimes \cdots \otimes v_n \in (\text{Im} T) \).

Then

\[
v_1 \otimes \cdots \otimes v_n \in y_1 \otimes V_2 \otimes \cdots \otimes V_n + V_1 \otimes y_2 \otimes V_3 \otimes \cdots \otimes V_n + \cdots + V_1 \otimes \cdots \otimes V_{n-1} \otimes y_n.
\]

(3)

Let \( S_i \) be a linear mapping on \( V_i \) such that \( S_i(v_i) = v_i, S_i(y_i) = 0, i = 1, \ldots, n \). It follows from (3) that

\[
(S_1 \otimes \cdots \otimes S_n)(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n = 0,
\]

a contradiction. Hence \( T \) is regular. Since \( T \) is almost surjective, the result follows from Theorem 2.4.

**Corollary 2.12.** Let \( T \) be a surjective additive decomposable mapping from \( \otimes_{i=1}^{m} U_i \) onto itself where \( U_1, \ldots, U_m \) are finite dimensional. Then there is a permutation \( \sigma \) on \([m]\) and bijective semilinear mappings \( T_i : U_{\sigma(i)} \rightarrow U_i, i = 1, \ldots, m, \) such that

\[
T = T_1 \otimes \cdots \otimes T_m.
\]
Proof. By Theorem 2.11, there exist a permutation $\sigma$ on $[m]$ and injective, almost surjective $\tau$-quasilinear mappings $T_i : U_{\sigma(i)} \to U_i$, $i = 1, \ldots, m$, such that

$$T = T_1 \otimes \cdots \otimes T_m.$$ 

Hence $T$ is $\tau$-quasilinear. Since every surjective quasilinear mapping from a finite dimensional vector space onto itself is semilinear, it follows that $T$ is semilinear. This shows that $\tau$ is surjective and hence each $T_i$ is bijective $\tau$-semilinear. $\square$

Remark 2.13. Corollary 2.12 is known for $m = 2$, see [1, 22]. When $T$ is bijective and linear, Corollary 2.12 was proved by Westwick [25]. In this case, each $T_i$ in Corollary 2.12 will be linear.

Corollary 2.14. Let $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ be real vector spaces. Let $T$ be an almost surjective additive decomposable mapping from $\otimes_{i=1}^{m} U_i$ onto $\otimes_{i=1}^{n} V_i$ where $m \leq n$. Then $m = n$ and there is a permutation $\sigma$ on $[m]$ and bijective linear mappings $T_i : U_{\sigma(i)} \to V_i$, $i = 1, \ldots, m$, such that

$$T = T_1 \otimes \cdots \otimes T_m.$$ 

Proof. The result follows from Theorem 2.11 and the fact that identity mapping is the only non-zero endomorphism of the real field. $\square$

Example 2.15. Let $U$ denote the set of all complex numbers, the real quaternion and the Cayley numbers respectively. Then $U$ is a real vector space of dimension 2, 4, 8 respectively. Then there exists a linear mapping $S$ from $\otimes^4 U$ to $\otimes^2 U$ such that

$$S(x_1 \otimes \cdots \otimes x_4) = (x_1 \circ x_2) \otimes (x_3 \circ x_4)$$

for any $x_1, \ldots, x_4$ in $U$ where $\circ$ denotes the multiplication in $U$. Clearly $S$ is a surjective linear decomposable mapping.

Example 2.16. Let $K$ be any field. Let $F := K(x)$ be the field of fractions of the ring of polynomials over $K$. Let $n$ be a positive integer $\geq 2$. Let $\tau : F \to F$ be the mapping defined by

$$\tau : \frac{p(x)}{q(x)} \to \frac{p(x^n)}{q(x^n)}.$$ 

Then $\tau$ is a field homomorphism of $F$ and it can shown that $1, x, \ldots, x^{n-1}$ is a basis of $F$ over $\tau(F)$ (see [8, Proposition 6.3.9, p. 139]). Let $A$ be a non-empty set. Let $U$ be a vector space over $F$ with a basis $\{u_\alpha : \alpha \in A\}$ $\cup \{v_\alpha : \alpha \in A\}$ and $W$ be a vector space over $F$ with a basis $\{w_\alpha : \alpha \in A\}$. Let $f : U \to W$ be the $\tau$-quasilinear mapping such that

$$f(u_\alpha) = w_\alpha, \quad f(v_\alpha) = x w_\alpha, \quad \alpha \in A.$$ 

Let $n \geq 3$ and $T = \otimes^n f$. Then $T$ is a decomposable mapping from $\otimes^n U$ to $\otimes^n W$. Note that $T$ is not injective although $f$ is. It is easily checked that $T$ is surjective although $f$ is not.

Our next result characterizes additive mappings $T$ from $\otimes_{i=1}^{m} U_i$ to $\otimes_{i=1}^{n} V_i$ that preserve non-zero decomposable elements in both directions, i.e., for any $A \in \otimes_{i=1}^{m} U_i$, $A$ is non-zero decomposable if and only if $T(A)$ is non-zero decomposable. Let $k$ be a positive integer. A non-zero element in $\otimes_{i=1}^{m} U_i$ is said to have rank $k$ if it is the sum of $k$, but not less than $k$, non-zero decomposable elements.
Theorem 2.17. Let $T$ be an additive mapping from $\otimes_{i=1}^{m} U_i$ to $\otimes_{i=1}^{n} V_i$ that preserve non-zero decomposable elements in both directions. Then $m \leq n$ and there exist a permutation $\sigma$ on $[m]$, an order-preserving $\phi$ from $[m]$ to $[n]$, injective $\tau$-quasilinear mapping $T_i : U_{\sigma(i)} \rightarrow V_{\phi(i)}$, $i = 1, \ldots, m$, and a multiplication mapping $M_y : \otimes_{i=1}^{m} V_{\phi(i)} \rightarrow \otimes_{i=1}^{n} V_i$ where $y$ is a non-zero decomposable element in $\otimes\{V_j : j \in [n]\setminus\text{Im} \phi\}$ if $n > m$, such that

$$T = M_y \circ (T_1 \otimes \cdots \otimes T_m),$$

where $M_y$ is deleted if $n = m$.

Proof. We shall show that $T$ is regular. We first show that $T$ sends rank-2 elements to rank-2 elements. Let $J$ be a rank-2 tensor of $\otimes_{i=1}^{m} U_i$. Suppose that $T(J) = 0$. Without loss of generality, we may assume that

$$J = x_1 \otimes \cdots \otimes x_m + y_1 \otimes \cdots \otimes y_m,$$

where $x_1 \otimes x_2 + y_1 \otimes y_2$ is of rank 2. Let

$$K = y_1 \otimes x_2 \otimes \cdots \otimes x_m.$$

Then $J + K$ is of rank 2. However $T(J + K) = T(K)$ is of rank one, a contradiction. Hence $T(J)$ is of rank 2. This shows that $T$ preserves rank-2 tensors. Hence $T$ is non-degenerate.

Let $M := U_1 \otimes A$ be a maximal type-1 decomposable subgroup of $\otimes_{i=1}^{m} U_i$ where $A$ is a non-zero decomposable element of $\otimes_{i=2}^{m} U_i$. Suppose that the linear span of $T(M)$ is 1-dimensional. Let $B \sim A$, $B$ are linearly independent. Let $x$, $y$ be linearly independent vectors in $U_1$. Let $T(x \otimes A) = C$, $T(y \otimes A) = D$ and $T(y \otimes B) = E$. Then $C$, $D$ are linearly dependent and $D \sim E$. Hence $C \sim E$. Since $T(x \otimes A + y \otimes B) = C + E$ and $x \otimes A + y \otimes B$ is of rank 2, it follows that $C + E$ is of rank 2, a contradiction to the fact that $C \sim E$. Hence $T(M)$ has at least 2 linearly independent vectors. Similarly the image of any maximal decomposable subgroup of $\otimes_{i=1}^{m} U_i$ has at least 2 linearly independent vectors. This proves that $T$ is regular. By Theorem 2.18, there exist a partition $\{J_1, \ldots, J_k\}$ of $[m]$, an order-preserving mapping $\phi$ from $[k]$ to $[n]$, $\tau$-quasilinear decomposable mappings $T_i : \otimes_{s \in J_i} U_s \rightarrow V_{\phi(i)}$, $i = 1, \ldots, k$, and a multiplication mapping $M_y : \otimes_{i=1}^{k} V_{\phi(i)} \rightarrow \otimes_{i=1}^{n} V_i$ where $y$ is a non-zero decomposable element in $\otimes\{V_j : j \in [n]\setminus\text{Im} \phi\}$ if $n > k$, such that

$$T = M_y \circ (T_1 \otimes \cdots \otimes T_k),$$

where $M_y$ is deleted if $n = k$. Since $T$ preserves rank-2 tensors, it is clear that each of $J_1, \ldots, J_k$ has only one element. Hence $k = m \leq n$ and our result follows.

Theorem 2.18. Let $T$ be a surjective additive mapping from $\otimes_{i=1}^{m} U_i$ to $\otimes_{i=1}^{n} V_i$ that preserve non-zero decomposable elements in both directions. Then $m = n$ and there is a permutation $\sigma$ on $[m]$ and bijective semilinear mappings $T_i : U_{\sigma(i)} \rightarrow V_i$, $i = 1, \ldots, m$, such that

$$T = T_1 \otimes \cdots \otimes T_m.$$
\( c \in F \), there exists a non-zero decomposable element \( B \) in \( \otimes_{i=1}^{m} U_i \) such that \( T(B) = cT(A) \). Since \( cT(A) \sim T(A) \), it follows that \( B \sim A \). Without loss of generality, we may assume that \( B = x_1 \otimes \cdots \otimes x_{m-1} \otimes u \) for some non-zero \( u \in U_m \). Let \( C = x \otimes x_2 \otimes \cdots \otimes x_m \) where \( x \) and \( x_1 \) are linearly independent. Since \( A \sim C \), it follows that \( T(A) \sim T(C) \) and hence \( T(B) \sim T(C) \).

We have \( B \sim C \) and this shows that \( u = \lambda x_m \) for some non-zero scalar \( \lambda \in F \). Hence

\[
T(B) = T(\lambda A) = \tau(\lambda)T(A) = cT(A).
\]

Thus \( \tau(\lambda) = c \). This shows that \( \tau \) is surjective. Hence each \( T_i \) is semilinear, \( i = 1, \ldots, m \).

Clearly each \( T_i \) is bijective and the proof is complete. \( \square \)

**Remark 2.19.** When \( m = n = 2 \), Theorem 2.18 was studied in [22].

In [22] Omladic and Semrl constructed an example of a bijective additive mapping on the algebra of bounded finite rank operators on infinite dimensional complex Banach space which preserves rank one operators but does not preserve them in both directions. Using their idea one can construct similar examples on \( U \otimes V \) where \( U \) is an \( n \)-dimensional vector space and \( V \) is an infinite dimensional vector space over \( F \) when \( F \) has an endomorphism \( \tau \) such that \( [F : \tau(F)] = n \geq 2 \).

**Example 2.20.** Let \( F \) be a field having a non-surjective endomorphism \( \tau \) such that \( [F : \tau(F)] = n \) and \( 1, x, \ldots, x^{n-1} \) is a basis of \( F \) over \( \tau(F) \). Let \( U \) be an \( n \)-dimensional vector space over \( F \) with a basis \( u_1, \ldots, u_n \) and \( V \) be an infinite dimensional vector space over \( F \). It is possible to choose two bases \( \{v_i, \alpha : \alpha \in A, i \in [n]\} \) and \( \{w_\alpha : \alpha \in A\} \) of \( V \) over \( F \). Let \( f : U \to U \) be the \( \tau \)-quasilinear mapping such that \( f(u_i) = u_i + \alpha, i = 1, \ldots, n \). Let \( g : V \to V \) be the \( \tau \)-quasilinear mapping such that \( g(v_i, \alpha) = x^{i-1}w_\alpha, \alpha \in A, i \in [n] \). Then both \( f \) and \( g \) are injective. Let \( c \in F \). Then \( c = \sum_{i=1}^{n} \tau(c_i)x^{i-1} \) for some \( c_i \in F \) and we have \( g(\sum_{i=1}^{n} c_iv_i, \alpha) = c\omega_\alpha \) for \( \alpha \in A \). This shows that \( g \) is surjective. Let \( T = f \otimes g \). Then \( T \) is a \( \tau \)-quasilinear decomposable mapping on \( U \otimes V \). Since \( T \) sends rank-2 tensor \( u_1 \otimes v_1, \alpha + u_2 \otimes v_2, \alpha \) to rank-1 tensor \( (u_1 + xu_2) \otimes w_\alpha \), it follows that \( T \) preserves non-zero decomposable elements in only one direction. Since \( T(\sum_{i=1}^{n} u_i \otimes y_i) = \sum_{i=1}^{n} u_i \otimes g(y_i) \) for \( y_i \in V \) and \( g \) is bijective, we see that \( T \) is bijective.

A pair \((A, B)\) of elements of a tensor space is called rank-additive if \( \rho(A + B) = \rho(A) + \rho(B) \) where \( \rho(A) \) denotes the rank of \( A \). A mapping \( T \) on a tensor space is said to preserve rank-additivity if \((T(A), T(B))\) is a rank-additive pair whenever \((A, B)\) is a rank-additive pair.

**Corollary 2.21.** Let \( T \) be a non-zero additive mapping from \( \otimes_{i=1}^{m} U_i \) into itself where \( \dim U_1 \geq \dim U_2 \geq \cdots \geq \dim U_m \) and \( U_i \) is of finite dimension for \( i \geq 2 \) with \( \dim U_2 = 1 \). If \( T \) preserves rank-additivity, then there exist a permutation \( \sigma \) on \([m]\) and quasilinear mappings \( T_i : U_{\sigma(i)} \to U_i, i = 1, \ldots, m \), such that

\[
T = T_1 \otimes \cdots \otimes T_m,
\]

where \( T_{\sigma^{-1}(i)} \) sends linearly independent sets to linearly independent sets for \( i \geq 2 \) and \( T_{\sigma^{-1}(1)} \) sends any \( t \) linearly independent vectors to \( t \) linearly independent vectors.

**Proof.** Let \( A, B \) be non-zero decomposable elements of \( \otimes_{i=1}^{m} U_i \). We shall show that \( \rho(T(A)) = \rho(T(B)) \). As there is a chain of decomposable elements joining \( A, B \), it suffices to consider only the case that \( A, B \) are adjacent. Without loss of generality, we may assume that
\[ A = x_1 \otimes y_1 \otimes C \quad \text{and} \quad B = x_1 \otimes y_2 \otimes C \]

for some \( x_1 \in U_1, y_1, y_2 \in U_2 \) and some non-zero decomposable element \( C \in \otimes_{i=3}^{m} U_i \). If \( y_1, y_2 \) are linearly dependent, we can choose a decomposable tensor \( D = x_1 \otimes y_3 \otimes C \) adjacent to \( A \) and \( B \) such that \( y_1, y_3 \) are linearly independent. Hence we can further assume that \( y_1, y_2 \) are linearly independent. Let \( x_2 \in U_1 \) be linearly independent to \( x_1 \). Let \( G = (x_1 + x_2) \otimes (y_1 + y_2) \otimes C \) and \( H = x_2 \otimes (y_1 + y_2) \otimes C \). Then \( G + (-A) = H + B, G + (-B) = H + A \) and \( (G, -A), (H, B), (G, -B), (H, A) \) are rank-additive pairs, it follows that

\[ \rho(T (G)) + \rho(T (-A)) = \rho(T (H)) + \rho(T (B)) \]

and

\[ \rho(T (G)) + \rho(T (-B)) = \rho(T (H)) + \rho(T (A)). \]

This implies that

\[ \rho(T (A)) = \rho(T (B)). \]

Hence \( \rho(T (A)) \neq 0 \) since \( T \) is non-zero.

Let \( \{D_j : j \in J\} \) be a basis of \( \otimes_{i=1}^{m} U_i \) consisting of decomposable tensors. Then every element of \( \otimes_{i=1}^{m} U_i \) is of the form \( \sum_{j \in J} u_j \otimes D_j \) where \( u_j \in U_1 \) and hence the maximal rank of \( \otimes_{i=1}^{m} U_i \) is finite, say, \( s \). Let \( M \) be an element of rank \( s \) in \( \otimes_{i=1}^{m} U_i \). Then \( M = \sum_{i=1}^{s} M_i \) for some decomposable elements \( M_i, i = 1, \ldots, s \). We have \( \rho(T (M)) = \sum_{i=1}^{s} \rho(T (M_i)) \) and hence \( \rho(T (M_i)) = 1, i = 1, \ldots, s \). This implies that \( T \) preserves non-zero decomposable elements.

It is clear that \( T \) also preserves rank \( k \) tensors for \( k \leq s \). Hence \( T \) preserves non-zero decomposable elements in both directions. By Theorem 2.17, there exist a permutation \( \sigma \) on \([m]\) and quasilinear mappings \( T_i : U_{\sigma(i)} \rightarrow U_i, i = 1, \ldots, m \), such that

\[ T = T_1 \otimes \cdots \otimes T_m. \]

Let \( e_1, \ldots, e_t \) be \( t \) linearly independent vectors in \( U_1 \), \( f_1, \ldots, f_t \) be \( t \) linearly independent vectors in \( U_2 \) and \( E \) be a non-zero decomposable element in \( \otimes_{i=3}^{m} U_i \). Then it is known that \( L := \sum_{i=1}^{t} (e_i \otimes f_i \otimes E) \) is of rank \( t \) (see [16]). Let \( T_{\sigma^{-1}(1)}(e_i) = w_i \) and \( T_{\sigma^{-1}(2)}(f_i) = z_i \), \( i = 1, \ldots, t \). Let \( P_\sigma : \otimes_{i=1}^{m} U_i \rightarrow \otimes_{i=1}^{m} U_i \) be the linear mapping such that

\[ P_\sigma (x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}. \]

Then

\[ T(L) = P_\sigma \left( \sum_{i=1}^{t} w_i \otimes z_i \otimes J \right) \]

for some non-zero decomposable tensor \( J \) in \( \otimes_{i=3}^{m} U_i \) and it is of rank \( t \). This implies that \( \sum_{i=1}^{t} w_i \otimes z_i \) is of rank \( t \) and hence \( w_1, \ldots, w_t \) as well as \( z_1, \ldots, z_t \) are linearly independent. Hence both \( T_{\sigma^{-1}(1)} \) and \( T_{\sigma^{-1}(2)} \) send \( t \) linearly independent vectors to \( t \) linearly independent vectors. Similarly we can show that \( T_{\sigma^{-1}(i)} \) sends linearly independent sets to linearly independent sets for \( i \geq 3 \). This completes the proof. \( \square \)

**Remark 2.22.** When \( m = 2 \), Corollary 2.21 was obtained in [5].

The following example shows that for each positive integer \( t \), there exists a quasilinear mapping on a certain infinite dimensional vector space which preserves \( t \) linearly independent vectors but not \( t + 1 \) linearly independent vectors.
**Example 2.23** [8, Theorem 7.2.12, p. 169]. Let $C$ denote the field of complex numbers and $C^{(N)}$ the vector space of all infinite sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of complex numbers $x_i$ such that the set $\{i : x_i \neq 0\}$ is finite. Let $B$ be any transcendental basis of $C$ over the field of rational numbers. Let $\alpha_1, \ldots, \alpha_t$ be $t$ distinct elements in $B$. Then there exists a monomorphism $\tau : C \to C$ such that $\alpha_1, \ldots, \alpha_t$ are algebraically independent over $\tau(C)$. The mapping $f : C^{(N)} \to C'$ defined by

$$f(x_1, x_2, \ldots, x_n, \ldots) = \left( \sum_i \tau(x_i)\alpha_1^i, \ldots, \sum_i \tau(x_i)\alpha_t^i \right)$$

is $\tau$-quasilinear and sends $t$ linearly independent vectors to $t$ linearly independent vectors.

It is known that every non-degenerate additive mapping from $U_1 \otimes U_2$ to $V_1 \otimes V_2$ that sends decomposable elements to decomposable elements is induced by two quasilinear mappings. This was essentially proved by Kurma [12] (see also [5,20]). We apply this result to give an improvement of Theorem 2.4 when $m = 2$ as follows:

**Theorem 2.24.** Let $T$ be a non-degenerate additive mapping from $\otimes_{i=1}^2 U_i$ to $\otimes_{i=1}^n V_i$ that sends decomposable elements to decomposable elements where $n > 2$. Then there exist distinct $s, t$ in $[n]$, $\tau$-quasilinear mappings $T_1 : U_1 \to V_s$, $T_2 : U_2 \to V_t$, and a multiplication mapping $M_y : V_s \otimes V_t \to \otimes_{i=1}^n V_i$ such that

$$T = M_y \circ (T_1 \otimes T_2),$$

where $y$ is a non-zero decomposable element in $\otimes_{j \neq s,t} V_j$.

**Proof.** Suppose that for any two decomposable elements $A, B \in \otimes_{i=1}^2 U_i$, we have $T(A) \sim T(B)$. Then it is easily seen that the image of the set of all decomposable elements under $T$ is contained in a decomposable subgroup of $\otimes_{i=1}^n V_i$ and hence $T$ is degenerate, a contradiction to the hypothesis. This shows that there exist $x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2$ such that $T(x_1 \otimes y_1) - T(x_2 \otimes y_2)$ is not decomposable. Let $T(x_1 \otimes y_1) = C$ and $T(x_2 \otimes y_2) = D$. Note that $T(x_1 \otimes y_2)$ and $T(x_2 \otimes y_1)$ cannot be both zero, otherwise $T((x_1 + x_2) \otimes (y_1 - y_2))$ is not decomposable, a contradiction. Hence we may assume without loss of generality that $T(x_1 \otimes y_2) \neq 0$. Since $x_1 \otimes y_2 \sim x_1 \otimes y_1$ and $x_1 \otimes y_2 \sim x_2 \otimes y_2$, it follows that $T(x_1 \otimes y_2) \sim C$ and $T(x_1 \otimes y_2) \sim D$. Hence $T(x_1 \otimes y_2)$ and $C$ have at least $m - 1$ factors in common. Similarly $T(x_1 \otimes y_2)$ and $D$ have at least $m - 1$ factors in common. Hence $C, D$ have $m - 2$ factors in common and we may assume that

$$C = w_1 \otimes z_1 \otimes E \quad \text{and} \quad D = w_2 \otimes z_2 \otimes E$$

for some linearly independent vectors $w_1, w_2 \in V_1$, some linearly independent vectors $z_1, z_2 \in V_2$ and a non-zero decomposable element $E \in \otimes_{i=3}^n V_i$. Thus $T(x_1 \otimes y_2) = \lambda w_1 \otimes z_1 \otimes E$ or $T(x_1 \otimes y_2) = \lambda w_1 \otimes z_2 \otimes E$ for some non-zero $\lambda$ in $F$. We shall show that $\text{Im} T \subseteq V_1 \otimes V_2 \otimes E$.

**Case 1.** $T(x_1 \otimes y_2) = \lambda w_2 \otimes z_1 \otimes E$ for some non-zero $\lambda$ in $F$. Since $T(x_2 \otimes y_1) \sim C$ and $T(x_2 \otimes y_1) \sim D$, we have $T(x_2 \otimes y_1) = \mu w_1 \otimes z_2 \otimes E$ or $\eta w_2 \otimes z_1 \otimes E$ for some $\mu, \eta \in F$. The latter case is not possible since

$$T((x_1 + x_2) \otimes (y_1 + y_2)) = w_1 \otimes z_1 \otimes E + w_2 \otimes (z_2 + (\lambda + \eta)z_1) \otimes E$$

is not a decomposable element. This shows that $T(x_2 \otimes y_1) = \mu w_1 \otimes z_2 \otimes E$ and hence $T((x_1 + x_2) \otimes (y_1 + y_2)) = w_1 \otimes (z_1 + \mu z_2) \otimes E + w_2 \otimes (\lambda z_1 + z_2) \otimes E$. Since this image is decomposable, we have $\mu \neq 0$. Now let $x \otimes y \in U_1 \otimes U_2$ such that $T(x \otimes y) = w \otimes z \otimes J \neq 0$ where
$J$ is a non-zero decomposable element in $\otimes_{i=3}^n V_i$. Clearly there exist $i, j$ such that $T(x_i \otimes y_j) = w \otimes u \otimes z \otimes v \otimes E$ where $w, u$ as well as $z, v$ are linearly independent. By the previous argument, we see that $J$ and $E$ are linearly dependent. Hence $T(x \otimes y) \in V_1 \otimes V_2 \otimes E$. This proves that $\text{Im } T \subseteq V_1 \otimes V_2 \otimes E$.

**Case 2.** $T(x_1 \otimes y_2) = \lambda w_1 \otimes z_2 \otimes E$ for some non-zero $\lambda$ in $F$. Using the same arguments as in Case 1, we can show that $\text{Im } T \subseteq V_1 \otimes V_2 \otimes E$.

Theorem 2.24 now follows from the result of Kurma. □

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**References**


