NORTH-HOLLAND

Spaces of Matrices of Fixed Rank. II

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ABSTRACT

When $\min\{m, n\} = k + 1$, the exact value of $l(k, m, n)$, the maximum dimension of all possible linear spaces of rank k matrices of order $m \times n$, is known. The situation when $\min l(k, m, n) \geqslant k + 2$ is not clear. Partial results are obtained for $l(k, k + 2, k + 2)$ in this paper.

The maximum dimension $l(k, n, m)$ of a linear space $H \subseteq \mathscr{L}(\mathbb{C}^n, \mathbb{C}^m)$ each of whose nonzero members have rank k has been determined when $\min\{m, n\}$ is $k + 1$. By using Chern classes, an upper bound is obtained, and with examples these upper bounds are confirmed as best possible. The results are contained in [6], [7], and [8]. The situation when $\min\{m, n\} = k + 2$ is not as pleasant. The upper bound using the Chern classes is no longer necessarily the best possible.

In this paper we begin the second phase of determining $l(k, n, m)$ by considering $l(k, k + 2, k + 2)$. The general theory in [7] shows that the value of $l(k, k + 2, k + 2) = l$ is one of 3, 4, or 5. The following is a summary of the results obtained.

THEOREM. *Let $l = l(k, k + 2, k + 2)$. Then*

$$k \equiv 1 \pmod 3 \Rightarrow l = 3; \tag{1}$$

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$$k \equiv 0 \pmod 3) \Rightarrow l \geqslant 4$$

$$\textit{and we have equality when } k \not\equiv 0 \pmod 4); \tag{2}$$

$$k \equiv 2 \pmod 3) \textit{ does not determine } l,$$

$$\textit{since} \quad l(2,4,4) = 3 \textit{ and } l(8,10,10) = 4. \tag{3}$$

No examples have been found for which $l = 5$.

All examples of an H of dimension 4 apart from the case $k = 8$ come from the 4-dimensional $(k, k + 1, k + 2)$-spaces. Since $l(8, 9, 10) = 3$, the 4-dimensional (8, 10, 10)-space given is new.

For $l(k, k + 2, k + 2) = 5$ we must have $k \geqslant 32$.

In what follows the base space for the bundles is the complex projective space on H. The trivial line bundle is denoted by ξ, and the tautological line bundle by μ. The bundle map determined by H referred to below is found in [5].

Proof. The bundle map

$$\varphi : \mu \otimes \xi^{k+2} \to \xi^{k+2}$$

determined by H produces three elements

$$x = [\operatorname{Ker} \varphi],$$

$$y = [\operatorname{Rng} \varphi],$$

$$z = [\xi^{k+2}/\operatorname{Rng} \varphi]$$

in the λ-ring K, where K is the ring determined by the equivalence classes of vector bundles over $P(H)$. See [2] for details. It is well known that K is isomorphic to $Z[t]/(t - 1)^l$, and the isomorphism is induced by $s \to t$, where $s = [\mu]$ is the equivalence class of the tautological line bundle. See [1, p. 84]. These satisfy

$$x + y = (k + 2)s,$$

$$y + z = (k + 2).$$

Both x and z are equivalence classes of 2-bundles. We let $c_t : K \to Z[t]/t^l$ be the Chern class homomorphism from the additive group of K to the multiplicative group of units in $Z[t]/t^l$. Then x and z have images

$1 + a_1t + a_2t^2$ and $1 + c_1t + c_2t^2$ respectively, where a_1, a_2, c_1, c_2 are integers. We apply the Chern character ring homomorphism $\mathrm{ch} : K \to Q[t]/t^l$ on both sides of the equation $(k + 2)(s - 1) = x - z$ to get

$$\begin{aligned} k + 2 &= a_1 - c_1 \\ &= a_1^2 - c_1^2 - 2(a_2 - c_2) \\ &= a_1^3 - c_1^3 - 3(a_1a_2 - c_1c_2) \\ &= a_1^4 - c_1^4 - 4(a_1^2a_2 - c_1^2c_2) + 2(a_2^2 - c_2^2). \end{aligned}$$

These are obtained from the Hirzebruch polynomials, indicated in [2, pp. 18–19].

Let

$$\begin{aligned} \alpha &= a_1 - c_1, \\ \beta &= a_1 + c_1, \\ \gamma &= a_2 - c_2, \\ \delta &= a_2 + c_2. \end{aligned}$$

Then

$$k + 2 = \alpha \tag{4}$$

$$= \alpha\beta - 2\gamma \tag{5}$$

$$= \alpha\frac{\alpha^2 + 3\beta^2}{4} - \frac{3(\alpha\delta + \beta\gamma)}{2}. \tag{6}$$

If $l \geqslant 4$, then (4), (5) and (6) must have integer solutions. These imply

$$2\gamma = \alpha(\beta - 1), \tag{7}$$

$$6\delta = \alpha^2 + 3\beta - 4. \tag{8}$$

From (8) we have no integer solution if $\alpha \equiv 0 \pmod 3$. Therefore $k + 2 \equiv 0 \pmod 3 \Rightarrow l \leqslant 3$, and since $l(k, k+2, k+2) \geqslant 3$ for all k, we have proved (1).

When $(k+2) \not\equiv 0 \pmod 3$, there are integer solutions, and so no more is available here.

If $l \geqslant 5$, then in addition to (4), (5), and (6) we have

$$k + 2 = \frac{\alpha\beta}{2}(\alpha^2 + \beta^2) - (\alpha^2 + \beta^2)\gamma - 2\alpha\beta\gamma + 2\gamma\delta. \tag{9}$$

These now imply

$$\alpha^2(2 - \beta) = 2 - \beta$$

and because $\alpha = k + 2 \neq 0$, we have

$$\beta = 2.$$

Then $\alpha = 2\gamma$ and $3\delta = 2\gamma^2 + 1$. Since $k = \alpha - 2$, it follows that $k \equiv 1 \pmod 2$ implies $l \leqslant 4$. Since δ is odd and $\delta + \gamma = 2a_2$ is even, we must have γ odd also. Then k is divisible by 4. Therefore

$$k \not\equiv 0 \pmod 4 \Rightarrow l \leqslant 4,$$

and (2) follows.

In order that $l \geqslant 5$ we must have m such that

$$k = 4m,$$

$$\alpha = 2(2m + 1),$$

$$\beta = 2,$$

$$\gamma = 2m + 1,$$

$$3\delta = 8m^2 + 8m + 3.$$

Then

$$a_1 = 2(m+1),$$

$$a_2 = \frac{(m+1)(4m+3)}{3},$$

$$c_1 = -2m,$$

$$c_2 = \frac{m(4m+1)}{3}.$$

For a_2 and c_2 to be integers we need $m(m+1) \equiv 0 \pmod 3$. When $m = 3n$ we have

$$a_1 = 2(3n+1),$$

$$a_2 = (3n+1)(4n+1),$$

$$c_1 = -6n,$$

$$c_2 = n(12n+1),$$

and when $m = 3n - 1$ we have

$$a_1 = 6n,$$

$$a_2 = n(12n-1),$$

$$c_1 = -6n+2,$$

$$c_2 = (3n-1)(4n-1).$$

We apply the Schwartzenberger conditions (see [5, p. 113] or [3, p. 166]).

When $m = 3n$, we have $k = 12n$ and then the s_4^2-condition implies

$$n(7n+1) \equiv 0 \pmod{12}.$$

When $m = 3n - 1$, $k = 12n - 4$. Then we have

$$n(5n+1) \equiv 0 \pmod{12}.$$

In the first case

$$n \equiv 0, 5, 8, 9 \pmod{12}.$$

and in the second case

$$n \equiv 0, 3, 4, 7 \pmod{12}.$$

The first value of k for which $l = 5$ is possible is $n = 3$, so that $k = 32$.

Finally, we must deal with the two special cases $k = 2$ and $k = 8$. The matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & C \\ 0 & 0 & 0 & 0 & 0 & -A & B & 0 & C & D \\ 0 & 0 & 0 & 0 & A & B & 0 & C & D & 0 \\ 0 & 0 & 0 & -A & 0 & 0 & C & -D & 0 & 0 \\ 0 & 0 & A & -B & 0 & 0 & D & 0 & 0 & 0 \\ 0 & -A & -B & 0 & -C & -D & 0 & 0 & 0 & 0 \\ -A & -B & 0 & -C & D & 0 & 0 & 0 & 0 & 0 \\ -B & 0 & -C & -D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & -D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is skew symmetric, has zero determinant and is of rank $\geqslant 8$ when any of A, B, C, or D is nonzero. It therefore represents a 4-dimensional (8, 10, 10)-space.

When $k = 2$, in the notation of Sylvester, there is an η satisfying (1), (2), and (3) (see [6, p. 2]), namely $\eta = 1 - 2\alpha + 2\alpha^2 \pmod{\alpha^4}$. Here we have $\eta^{-1} = 1 + 2\alpha + 2\alpha^2 \pmod{\alpha^4}$ and $(1 + \alpha)^4\eta = 1 + 2\alpha \pmod{\alpha^4}$. However, there does not exist a 4-dimensional (2, 4, 4)-space. This follows easily from Theorem 3A of [4], which states that such a space is either 2-decomposable or equivalent to a space of 3×3 matrices. In both cases the maximal dimension of the space is three. ■

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