Computers and Mathematics with Applications 57 (2009) 1645–1650

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the eigenvalues of a specially rank-r updated complex matrix[☆]

Zhiping Xiong, Bing Zheng [*]

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

ARTICLE INFO

Article history:
Received 28 March 2008
Received in revised form 5 February 2009
Accepted 27 February 2009

Keywords:
Rank-r updated matrix
Determinant
Characteristic polynomial
Eigenvalue
Leverrier's algorithm for m–D system

ABSTRACT

In this paper, an alternatively simpler proof to an eigenvalue theorem of a specially structured rank-r updated complex matrix is presented and also its characteristic polynomial is explicitly determined by Leverrier's algorithm for m–D system.

1. Introduction

Let A be an $m \times m$ complex matrix and $U, V \in C^{m\times r}$ be two full column rank matrices. The matrix $A + UV^*$ is called the rank-r updated matrix of A. The applications of rank-r updated matrices can be found in the fields of optimization theory, Markov chains, structured matrices, relative perturbation bounds for the eigenvalue problem etc.; see [1–7].

Recently, the eigenvalue problem of a specially structured rank-r updated matrix received much attention since it is tightly related to Google's or generalized Google's PageRank problems. Let S be an $m \times m$ row-stochastic matrix, i.e. a nonnegative matrix that satisfies $Se = e$, where the m-dimensional vector $e^* = (1, 1, \ldots, 1)$. The Google matrix $G(\alpha)$ is defined by $G(\alpha) = \alpha S + (1-\alpha)ev^*$, where $0 < \alpha < 1$, and v is an m-dimensional positive vector normalized by $e^*v = 1$ (i.e. v is a probability vector). The eigenvector of $G(\alpha)$ corresponding to the maximal eigenvalue 1 is called the PageRank. To rank the Web pages, we usually need to compute the pageRank of the Google matrix, which is the left eigenvector associated with the principle eigenvalue of the Google matrix $G(\alpha)$. Furthermore, a generalized Google matrix is defined as follows:

$$\tilde{G}(c, c_1, \ldots, c_r) = cA + \sum_{i=1}^{r}(c_i\lambda_i)x_i(v_i)^*, \quad c + \sum_{i=1}^{r} c_i = 1,$$

where $c, c_i \in C$, $1 \le i \le r$, $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues of the complex matrix A and $x_1, x_2, \ldots, x_r$ are linearly independent right eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, respectively. It is seen that the Google matrix $G(\alpha)$ is a special rank-1 updated matrix and generalized Google matrix $\tilde{G}(c, c_1, \ldots, c_r)$ is a special rank-r updated matrix. The eigenvalue problem of the (generalized) Google matrix has been investigated in [8–15].

☆ Supported by the start-up fund of Lanzhou University and the Natural Science Foundation of Gansu Province (3ZS051-A25-020), PR China.

* Corresponding author.
E-mail address: bzheng@lzu.edu.cn (B. Zheng).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.02.027

Motivated by the recent spectral analysis of the Google matrix in the computation of the PageRank for the Google Web search engine [8–10], Ding and Yao [3] considered the eigenvalue problem of a specially rank-r updated real matrix and then by the use of QR decomposition, Wu [12] extended Ding and Yao's results to the case when the matrix is complex. In this paper, we provide an alternative simpler approach to solve the eigenvalue problem for the specially structured rank-r updated complex matrix $A+UV^*$, where A is an $m\times m$ complex matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_m$ counting algebraic multiplicities, $U=(u_1,u_2,\ldots,u_r)$ is a rank-r matrix such that $u_1,u_2,\ldots,u_r$ are right eigenvectors of A corresponding to $\lambda_1,\lambda_2,\ldots,\lambda_r$, respectively, and $V=(v_1,v_2,\ldots,v_r)\in C^{m\times r}$ is an arbitrary matrix. By applying Leverrier's algorithm for m–D system [16], we can give the explicit expression of the characteristic polynomial of the specially structured rank-r updated matrix $A+UV^*$.

The key elements in this paper are the following lemmas:

Lemma 1.1. *Assume $A\in C^{m\times m}$ be a nonsingular matrix and $U,V\in C^{m\times r}$. Then*

$$\det(A+UV^*)=\det(A)\det(I+V^*A^{-1}U). \tag{1.1}$$

Proof. It is easy to see that the determinant identity (1.1) comes from the equality

$$\begin{pmatrix} I & O\\ V^* & I\end{pmatrix}\begin{pmatrix} I+A^{-1}UV^* & A^{-1}U\\ O & I\end{pmatrix}\begin{pmatrix} I & O\\ -V^* & I\end{pmatrix}=\begin{pmatrix} I & A^{-1}U\\ O & I+V^*A^{-1}U\end{pmatrix}. \quad \blacksquare$$

The next lemma is a special case of *Leverrier's algorithm* for m–D system given by G. Wang and B. Zheng [16], which provides an algorithm for computing the coefficients of the multi-variable characteristic polynomial (so called here) of an $m\times m$ matrix A.

Lemma 1.2. *Let $A=[a_{ij}]\in C^{m\times m}$ and z_i, $i=1,2,\ldots,m$, be complex variables. Then*

$$\begin{aligned} a(z_1,z_2,\ldots,z_m) &= \det\begin{pmatrix} z_1-a_{11} & -a_{12} & \cdots & -a_{1m}\\ -a_{21} & z_2-a_{22} & \cdots & -a_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{m1} & -a_{m2} & \cdots & z_m-a_{mm}\end{pmatrix}\\ &= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_m=0}^{1} a_{i_1,i_2,\ldots,i_m} z_1^{1-i_1}z_2^{1-i_2}\cdots z_m^{1-i_m}. \end{aligned} \tag{1.2}$$

The coefficients $a_{i_1,i_2,\ldots,i_m}$ can be determined sequentially by

$$\begin{aligned} & a_{0,0,\ldots,0}=1,\quad a_{1,0,\ldots,0}=-a_{11},\quad a_{0,1,0,\ldots,0}=-a_{22},\quad \ldots,\quad a_{0,0,\ldots,0,1}=-a_{mm},\\ & a_{i_1,i_2,\ldots,i_m}=-\frac{1}{i_1+i_2+\cdots+i_m}\mathrm{tr}(AB_{i_1,i_2,\ldots,i_m}),\\ & i_k=0,1,\ k=1,2,\ldots,m,\quad \sum_{k=1}^{m} i_k\neq 0, \end{aligned} \tag{1.3}$$

$$B_{0,\ldots,0}=\begin{pmatrix} 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{pmatrix},\quad B_{1,0,\ldots,0}=\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix},\quad \ldots,\quad B_{0,\ldots,0,1}=\begin{pmatrix} 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & 1\end{pmatrix},$$

$$\begin{aligned} B_{i_1,i_2,\ldots,i_m} &= B_{1,0,\ldots,0}(AB_{i_1-1,i_2,\ldots,i_m}+a_{i_1-1,i_2,\ldots,i_m}I)+\cdots+B_{0,\ldots,0,1}(AB_{i_1,i_2,\ldots,i_{m-1},i_m-1}+a_{i_1,i_2,\ldots,i_{m-1},i_m-1}I),\\ & i_k=0,1,\ k=1,2,\ldots,m, \end{aligned} \tag{1.4}$$

where $B_{i_1,i_2,\ldots,i_m}$ is a zero matrix and $a_{i_1,i_2,\ldots,i_m}$ is zero if one of $i_1,i_2,\ldots,i_m$ equals -1.

2. Main results

For an $m\times m$ complex matrix A, let $\sigma(A)\equiv\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ be the set of all eigenvalues of A, counting algebraic multiplicity. In this section, we will present the concrete expression of the characteristic polynomial of a specially rank-r updated matrix $A+UV^*$ of A. The eigenvalue problem of such special rank-r updated matrix has been studied by several authors, see [3,12]. But here we actually provide an alternatively more simple proof for their main results. The following is our main theorem.

Theorem 2.1. *Let $A \in C^{m\times m}$, $U = (u_1, u_2, \dots, u_r) \in C^{m\times r}$ be a full column rank matrix such that $Au_i = \lambda_i u_i$, $i = 1, 2, \dots, r$ and $V = (v_1, v_2, \dots, v_r) \in C^{m\times r}$ be an arbitrary matrix. Then*

(1) *the characteristic polynomial of $A + UV^*$ is*

$$P_{A+UV^*}(\lambda) = \left(\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} s_{i_1,i_2,\dots,i_r}(\lambda-\lambda_1)^{1-i_1}(\lambda-\lambda_2)^{1-i_2}\cdots(\lambda-\lambda_r)^{1-i_r}\right) \times (\lambda-\lambda_{r+1})(\lambda-\lambda_{r+2})\cdots(\lambda-\lambda_m); \tag{2.1}$$

(2) *the eigenvalues of $A + UV^*$ are*

$$\{\delta_1, \delta_2, \dots, \delta_r, \lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_m\},$$

counting algebraic multiplicity, where δ_i, $1 \le i \le r$ are the solutions of the following equation:

$$\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} s_{i_1,i_2,\dots,i_r}(x-\lambda_1)^{1-i_1}(x-\lambda_2)^{1-i_2}\cdots(x-\lambda_r)^{1-i_r} = 0. \tag{2.2}$$

The coefficients $s_{i_1,i_2,\dots,i_r}$ in (2.1) or (2.2) can be determined sequentially by

$$s_{0,0,\dots,0} = 1, \quad s_{1,0,\dots,0} = -v_1{}^*u_1, \quad s_{0,1,0,\dots,0} = -v_2{}^*u_2, \quad \dots, \quad s_{0,0,\dots,0,1} = -v_r{}^*u_r,$$
$$s_{i_1,i_2,\dots,i_r} = -\frac{1}{i_1+i_2+\cdots+i_r}\operatorname{tr}(V^*UT_{i_1,i_2,\dots,i_r}),$$
$$i_k = 0, 1, k = 1, 2, \dots, r, \quad \sum_{k=1}^{r} i_k \ne 0,$$
$$T_{0,\dots,0} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad T_{1,0,\dots,0} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \dots, \quad T_{0,\dots,0,1} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$
$$T_{i_1,i_2,\dots,i_r} = T_{1,0,\dots,0}(V^*UT_{i_1-1,i_2,\dots,i_r} + s_{i_1-1,i_2,\dots,i_r}I) + \cdots + T_{0,\dots,0,1}(V^*UT_{i_1,i_2,\dots,i_{r-1},i_r-1} + s_{i_1,i_2,\dots,i_{r-1},i_r-1}I),$$
$$i_k = 0, 1, k = 1, 2, \dots, r,$$

where $T_{i_1,i_2,\dots,i_r}$ is a zero matrix and $s_{i_1,i_2,\dots,i_r}$ is zero if one of $i_1, i_2, \dots, i_r$ equals -1.

Proof. (1) Let $\lambda \notin \sigma(A)$ be any complex number. Then by applying Lemma 1.1, we have

$$\det(\lambda I - A - UV^*) = \det(I - V^*(\lambda I - A)^{-1}U)\det(\lambda I - A). \tag{2.3}$$

The conditions $Au_i = \lambda_i u_i$, $i = 1, 2, \dots, r$ imply that

$$(\lambda I - A)^{-1}u_i = \frac{1}{\lambda - \lambda_i}u_i, \quad i = 1, 2, \dots, r,$$

and hence the identity (2.3) becomes

$$\det(\lambda I - A - UV^*) = \det\left(I - V^*\left(\frac{u_1}{\lambda-\lambda_1}, \frac{u_2}{\lambda-\lambda_2}, \dots, \frac{u_r}{\lambda-\lambda_r}\right)\right)\det(\lambda I - A). \tag{2.4}$$

Note that

$$\det(\lambda I - A) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r)(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m) \tag{2.5}$$

and then combine (2.4) with (2.5), we have

$$\det(\lambda I - A - UV^*) = \det\left(I - V^*\left(\frac{u_1}{\lambda-\lambda_1}, \frac{u_2}{\lambda-\lambda_2}, \dots, \frac{u_r}{\lambda-\lambda_r}\right)\right)\det(\lambda I - A)$$
$$= \det\left(I - V^*\left(\frac{u_1}{\lambda-\lambda_1}, \frac{u_2}{\lambda-\lambda_2}, \dots, \frac{u_r}{\lambda-\lambda_r}\right)\right)(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r)\cdots(\lambda-\lambda_m)$$

$$= \det \begin{pmatrix} 1-\dfrac{v_1{}^*u_1}{\lambda-\lambda_1} & -\dfrac{v_1{}^*u_2}{\lambda-\lambda_2} & \cdots & -\dfrac{v_1{}^*u_r}{\lambda-\lambda_r} \\ -\dfrac{v_2{}^*u_1}{\lambda-\lambda_1} & 1-\dfrac{v_2{}^*u_2}{\lambda-\lambda_2} & \cdots & -\dfrac{v_2{}^*u_r}{\lambda-\lambda_r} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{v_r{}^*u_1}{\lambda-\lambda_1} & -\dfrac{v_r{}^*u_2}{\lambda-\lambda_2} & \cdots & 1-\dfrac{v_r{}^*u_r}{\lambda-\lambda_r} \end{pmatrix} (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_m)$$

$$= \det \begin{pmatrix} \lambda-\lambda_1-v_1{}^*u_1 & -v_1{}^*u_2 & \cdots & -v_1{}^*u_r \\ -v_2{}^*u_1 & \lambda-\lambda_2-v_2{}^*u_2 & \cdots & -v_2{}^*u_r \\ \vdots & \vdots & \ddots & \vdots \\ -v_r{}^*u_1 & -v_r{}^*u_2 & \cdots & \lambda-\lambda_r-v_r{}^*u_r \end{pmatrix} (\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m). \tag{2.6}$$

On the other hand, by Lemma 1.2 we have

$$\det \begin{pmatrix} \lambda-\lambda_1-v_1{}^*u_1 & -v_1{}^*u_2 & \cdots & -v_1{}^*u_r \\ -v_2{}^*u_1 & \lambda-\lambda_2-v_2{}^*u_2 & \cdots & -v_2{}^*u_r \\ \vdots & \vdots & \ddots & \vdots \\ -v_r{}^*u_1 & -v_r{}^*u_2 & \cdots & \lambda-\lambda_r-v_r{}^*u_r \end{pmatrix}$$

$$= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} s_{i_1,i_2,\ldots,i_r}(\lambda-\lambda_1)^{1-i_1}(\lambda-\lambda_2)^{1-i_2}\cdots(\lambda-\lambda_r)^{1-i_r}. \tag{2.7}$$

Substituting (2.7) into (2.6) yields

$$\det(\lambda I - A - UV^*) = \left(\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} s_{i_1,i_2,\ldots,i_r}(\lambda-\lambda_1)^{1-i_1}(\lambda-\lambda_2)^{1-i_2}\cdots(\lambda-\lambda_r)^{1-i_r}\right) \times (\lambda-\lambda_{r+1})(\lambda-\lambda_{r+2})\cdots(\lambda-\lambda_m).$$

Since the above equality is true for all $\lambda \notin \sigma(A)$, we now assert that the characteristic polynomial of $A + UV^*$ is

$$P_{A+UV^*}(\lambda) = \left(\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} s_{i_1,i_2,\ldots,i_r}(\lambda-\lambda_1)^{1-i_1}(\lambda-\lambda_2)^{1-i_2}\cdots(\lambda-\lambda_r)^{1-i_r}\right) \times (\lambda-\lambda_{r+1})(\lambda-\lambda_{r+2})\cdots(\lambda-\lambda_m).$$

The Conclusion (2) is obvious. ∎

Remark. As a main result in [12], G.Wu get the expression (2.6) of the characteristic polynomial of the rank-r updated matrix $A + UV^*$ by using the QR decomposition and it is also the statement of Theorem 3.1 in [3] when the case of real matrices. So here we present a much more simple proof for his main result. In particular, the concrete polynomial form can be determined by Leverrier's algorithm for m–D system.

Corollary 2.1. *Let A be an m by m complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ counting algebraic multiplicity, $U = (u_1, u_2, \ldots, u_r) \in C^{m\times r}$ and $V = (v_1, v_2, \ldots, v_r) \in C^{m\times r}$ be full column rank matrices. If $Au_i = \lambda_i u_i$, $v_i{}^*A = \lambda_i v_i{}^*$ and $\lambda_i \neq \lambda_j$ for $i \neq j$, then*

(1) *the characteristic polynomial of $A + UV^*$ is*

$$P_{A+UV^*}(\lambda) = (\lambda-\lambda_1-v_1{}^*u_1)(\lambda-\lambda_2-v_2{}^*u_2)\cdots(\lambda-\lambda_r-v_r{}^*u_r)(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m);$$

(2) *the eigenvalues of $A + UV^*$ are*

$$\{\lambda_1+v_1{}^*u_1, \lambda_2+v_2{}^*u_2, \ldots, \lambda_r+v_r{}^*u_r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_m\},$$

counting algebraic multiplicity.

Proof. (1) Let $\lambda \notin \sigma(A)$ be any complex number. Then from the proof of Theorem 2.1, we have

$$\det(\lambda I - A - UV^*) = \det\begin{pmatrix} 1-\frac{v_1{}^*u_1}{\lambda-\lambda_1} & -\frac{v_1{}^*u_2}{\lambda-\lambda_2} & \cdots & -\frac{v_1{}^*u_r}{\lambda-\lambda_r} \\ -\frac{v_2{}^*u_1}{\lambda-\lambda_1} & 1-\frac{v_2{}^*u_2}{\lambda-\lambda_2} & \cdots & -\frac{v_2{}^*u_r}{\lambda-\lambda_r} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{v_r{}^*u_1}{\lambda-\lambda_1} & -\frac{v_r{}^*u_2}{\lambda-\lambda_2} & \cdots & 1-\frac{v_r{}^*u_r}{\lambda-\lambda_r} \end{pmatrix}(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_m).$$

Since

$$Au_i = \lambda_i u_i, \qquad v_i{}^*A = \lambda_i v_i{}^* \quad \text{and} \quad \lambda_i \neq \lambda_j \quad \text{for } i \neq j,$$

we can easily get $v_i{}^*u_j = 0$, where $i \neq j$, $1 \le i, j \le r$. Thus,

$$\begin{aligned}\det(\lambda I - A - UV^*) &= \det\begin{pmatrix} \lambda-\lambda_1-v_1{}^*u_1 & 0 & \cdots & 0 \\ 0 & \lambda-\lambda_2-v_2{}^*u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-\lambda_r-v_r{}^*u_r \end{pmatrix}(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m) \\ &= (\lambda-\lambda_1-v_1{}^*u_1)(\lambda-\lambda_2-v_2{}^*u_2)\cdots(\lambda-\lambda_r-v_r{}^*u_r)(\lambda-\lambda_{r+1})(\lambda-\lambda_{r+2})\cdots(\lambda-\lambda_m).\end{aligned}$$

The above equality is true for all $\lambda \notin \sigma(A)$. Therefore we have

$$P_{A+UV^*}(\lambda) = (\lambda-\lambda_1-v_1{}^*u_1)(\lambda-\lambda_2-v_2{}^*u_2)\cdots(\lambda-\lambda_r-v_r{}^*u_r)(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m).$$

The conclusion (2) is obvious. ■

Corollary 2.2. *Let A be an m by m complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ counting algebraic multiplicity, $U = (u_1, u_2, \ldots, u_r) \in C^{m\times r}$ be full column rank matrices. If $Au_i = \lambda_i u_i$ and $U^*U = \mathrm{diag}(\|u_1\|_2^2, \|u_2\|_2^2, \ldots, \|u_r\|_2^2)$, then*

(1) *the characteristic polynomial of $A + UU^*$ is*

$$P_{A+UU^*}(\lambda) = (\lambda-\lambda_1-\|u_1\|_2^2)(\lambda-\lambda_2-\|u_2\|_2^2)\cdots(\lambda-\lambda_r-\|u_r\|_2^2)(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_m);$$

(2) *the eigenvalues of $A + UU^*$ are*

$$\{\lambda_1+\|u_1\|_2^2, \lambda_2+\|u_2\|_2^2, \ldots, \lambda_r+\|u_r\|_2^2, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_m\},$$

counting algebraic multiplicity.

3. Example

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 4 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are $\sigma(A) = \{\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -1, \lambda_4 = 0\}$. We let $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$, where $Au_i = \lambda_i u_i$, $i = 1, 2, 3$ and

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then from Theorem 2.1, we have

$$P_{A+UV^*}(\lambda) = \left(\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} s_{i_1,i_2,i_3}(\lambda-\lambda_1)^{1-i_1}(\lambda-\lambda_2)^{1-i_2}(\lambda-\lambda_3)^{1-i_3}\right)(\lambda-\lambda_4)$$

and

$$s_{0,0,0} = 1, \qquad s_{1,0,0} = -v_1{}^*u_1 = -1, \qquad s_{0,1,0} = -v_2{}^*u_2 = 0, \qquad s_{0,0,1} = -v_3{}^*u_3 = 1$$

and

$$T_{0,0,0}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad T_{1,0,0}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad T_{0,1,0}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad T_{0,0,1}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$T_{1,0,1}=\begin{pmatrix}1&0&-4\\0&0&0\\1&0&-1\end{pmatrix},\qquad s_{1,0,1}=3,\qquad T_{1,1,0}=\begin{pmatrix}0&0&0\\-1&-1&0\\0&0&0\end{pmatrix},\qquad s_{1,1,0}=0,$$

$$T_{0,1,1}=\begin{pmatrix}0&0&0\\0&1&0\\0&3&0\end{pmatrix},\qquad s_{0,1,1}=0,\qquad T_{1,1,1}=\begin{pmatrix}0&-12&0\\-1&3&4\\-3&-3&0\end{pmatrix},\qquad s_{1,1,1}=12.$$

Thus

$$P_{A+UV^*}(\lambda)=(\lambda^3-3\lambda^2-12)(\lambda-\lambda_4)=(\lambda-\rho_1)(\lambda-\rho_2)(\lambda-\rho_3)(\lambda-\lambda_4)$$

and

$$\sigma(A+UV^*)=\{\rho_1,\rho_2,\rho_3,\lambda_4\},$$

where

$$\rho_1=\sqrt[3]{7+4\sqrt{3}}+\sqrt[3]{7-4\sqrt{3}}+1,$$
$$\rho_2=W\sqrt[3]{7+4\sqrt{3}}+\bar{W}\sqrt[3]{7-4\sqrt{3}}+1,$$
$$\rho_3=\bar{W}\sqrt[3]{7+4\sqrt{3}}+W\sqrt[3]{7-4\sqrt{3}}+1,$$
$$W=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\qquad \bar{W}=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}.$$

References

[1] A. Brauer, Limits for the characteristic roots of a matrix IV: Applications to stochastic matrices, Duke Math. J. 19 (1952) 75–91.
[2] J. Ding, A. Zhou, Eigenvalues of rank-one updated matrices with some applications, Appl. Math. Lett. 20 (2007) 1223–1226.
[3] J. Ding, G. Yao, The eigenvalue problem of a specifically updated matrix, Appl. Math. Comput. 185 (2007) 415–420.
[4] C.D. Meyer, Generalized inversion of modified matrices, SIAM J. Appl. Math. 24 (1973) 315–323.
[5] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Science Press, Beijing, 2004.
[6] Y. Wei, X. Li, F. Bu, F. Zhang, Relative perturbation bounds for the eigenvalues of diagonalizable and singular matrices—Application of perturbation theory for simple invariant subspaces, Linear Algebra Appl. 419 (2006) 765–771.
[7] Y. Yuan, R. Byrd, Non-quasi-Newton updates for unconstrained optimization, J. Comput. Math. 13 (1995) 95–107.
[8] A.N. Langville, C.D. Meyer, Deeper inside Pagerank, Internet Math. 1 (2004) 335–380.
[9] A.N. Langville, C.D. Meyer, A survey of eigenvector methods for web information retrieval, SIAM Rev. 47 (2005) 135–161.
[10] L. Eldén, The eigenvalues of the Google matrix, Technical Report LITH-MAT-R-2004-01.
[11] G. Wu, Y. Wei, A power-Arnoldi algorithm for computing PageRank, Numer. Linear Algebra Appl. 14 (2007) 521–546.
[12] G. Wu, Eigenvalues and Jordan canonical form of a successively rank-1 updated matrix with applications to Google's PageRank problem, J. Comput. Appl. Math. 216 (2008) 364–370.
[13] T.H. Haveliwala, S.D. Kamvar, The second eigenvalue of the Google matrix, Stanford University Technical Report, 2003.
[14] S. Serra Capizzano, Google PageRank problem: The model and the analysis, in: Proceedings of the Dagstuhl Conference in Web Retrieval and Numerical Linear Algebra, 2007.
[15] L. Page, S. Brin, R. Motwami, T. Winograd, The PageRank citation ranking: Bring order to the web, Technical Report, Computer Science Department, Stanford University, 1998.
[16] G.R. Wang, B. Zheng, Leverrier's algorithm for *m–D* system, Numer. Math. J.Chinese Univ. (English Ser.) 13 (1) (2004) 10–23.