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On Some Numbers Related to Whitney Numbers of Dowling Lattices

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We study some polynomials arising from Whitney numbers of the second kind of Dowling lattices. Special cases of our results include well-known identities involving Stirling numbers of the second kind. The main technique used is essentially due to Rota. © 1997 Academic Press

1. INTRODUCTION

The present paper deals with the algebraic combinatorics of Dowling lattices and may be regarded as a continuation of [1]. We assume that the reader is familiar with Dowling lattices (see [4]), which are group-theoretic analogs of partition lattices. The rank-n Dowling lattice based on the group G is denoted $Q_n(G)$. Since the Whitney numbers of Dowling lattices based on groups of the same order are equal, we may denote the Whitney numbers of the second kind by $W_m(n, k)$ where m is the order of G. When G is trivial, the Whitney numbers $W_1(n, k)$ are the Stirling numbers $S(n + 1, k + 1)$. In the second section, we derive formulas for the generating functions for Dowling polynomials $D_m(n, x) = \sum_k W_m(n, k)x^k$ using the technique Rota used in [7] for Bell numbers $B_n = \sum_k S(n, k)$. These techniques are also applied to similar results. Section 3 introduces and examines Dowling lattices generalizations of the polynomials $F_n(x) = \sum_{k=1}^n k!\, S(n, k)x^k$ studied by Tanny in [8]. Two equivalent generalizations

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are studied here:

$$F_{m,1}(x) = \sum_k k!\, m^k W_m(n,k) x^k; \qquad F_{m,2}(x) = \sum_k k!\, W_m(n,k) x^k.$$

Dowling [4] has given combinatorial interpretations for the coefficients of both $F_{m,1}(x)$ and $F_{m,2}(x)$, which we present in the first part of Section 3. The rest of that section studies $F_{m,1}(x)$ "à la Riordan," i.e., explicit formulas, generating functions, etc., in the same vein as [1]. The results are essentially analogs of results in [8] for $F_n(x)$. In the last section, we prove that the sequence $k!\, W_m(n,k)$ is log-concave.

2. DOWLING POLYNOMIALS

In Rota's paper [7], formulas concerning Bell numbers are derived via a particularly elegant and simple technique from linear algebra. We prove here what we noted in [1]: it is possible to use techniques in the spirit of [7] to derive the formulas given in [1] for Dowling numbers, $D_m(n)$. We define Dowling polynomials by

$$D_m(n,x) = \sum_{k=0}^{n} W_m(n,k) x^k,$$

where $W_m(n,k)$ is the Whitney number of the second kind of the Dowling lattice $Q_n(G)$. The following generating function for $D_m(n,x)$ is known; the (new) proof illustrates applying the techniques of [7] to such identities.

PROPOSITION 1. *The exponential generating function for Dowling polynomials, $D_m(n,x)$, is given by*

$$\sum_{n\geq 0} D_m(n,x)\frac{z^n}{n!} = \exp\left\{z + x\,\frac{e^{mz}-1}{m}\right\}. \qquad (1)$$

Proof. Let V be the vector space of polynomials. Since any sequence of polynomials of degrees $0,1,2,\ldots,$ forms a basis of V, we may choose the following sequence as a basis:

$$\left(\frac{x-1}{m}\right)_k, \qquad k \geq 0.$$

where $(y)_k = y(y-1)(y-2)\cdots(y-k+1)$. Now, define the linear functional L_m on V as follows:

$$L_m\left(\left(\frac{x-1}{m}\right)_k\right) = \frac{x^k}{m^k}, \qquad k \geq 0.$$

Thus, L_m is defined since it is known on a basis. Recall that the Whitney numbers of the second kind, $W_m(n,k)$, satisfy (see [4, Corollary 6.1])

$$x^n = \sum_{k=0}^{n} \left(\frac{x-1}{m}\right)_k m^k W_m(n,k). \tag{2}$$

Applying L_m to (2) gives

$$\begin{aligned} L_m(x^n) &= L_m\left(\sum_{k=0}^{n} \left(\frac{x-1}{m}\right)_k m^k W_m(n,k)\right) \\ &= \sum_{k=0}^{n} W_m(n,k) x^k. \end{aligned}$$

Thus, we obtain

$$L_m(x^n) = D_m(n,x). \tag{3}$$

Formula (3) is the analog of formula (4) of [7]. Now, we can evaluate (1):

$$\begin{aligned} \sum_{n\geq 0} D_m(n,x)\frac{z^n}{n!} &= \sum_{n\geq 0} L_m(x^n)\frac{z^n}{n!} \\ &= L_m\left(\sum_{n\geq 0}\frac{(xz)^n}{n!}\right) \\ &= L_m(e^{xz}). \end{aligned}$$

We write

$$e^{xz} = e^z(e^{mz})^{(x-1)/m} = e^z(1+v)^{(x-1)/m}, \qquad v = e^{mz}-1.$$

Consequently,

$$\begin{aligned} L_m(e^{xz}) &= e^z L_m\left((1+v)^{(x-1)/m}\right) \\ &= e^z L_m\left(\sum_k \left(\frac{x-1}{m}\right)_k \frac{v^k}{k!}\right), \end{aligned}$$

where, in the last equality we developed $(1+v)^l$ in a Taylor series. Using the definition of L_m we obtain

$$\begin{aligned} L_m(e^{xz}) &= e^z\left(\sum_{k\geq 0}\frac{(xv)^k}{m^k k!}\right) \\ &= e^z \exp\left(\frac{xv}{m}\right). \end{aligned}$$

Since $v = e^{mz}-1$, we obtain (1). ∎

Another example is the following result:

THEOREM 1. *The Dowling polynomials satisfy the recursion formula*

$$D_m(n+1, x) = (x+1) D_m(n, x) + x \sum_{i=0}^{n-1} \binom{n}{i} m^{n-i} D_m(i, x). \quad (4)$$

Proof. First, observe that we have

$$\left(\frac{x-1}{m}\right)_k = \frac{(x-1)}{m}\left(\frac{x-1}{m} - 1\right)_{k-1},$$

and then for any polynomial $P(x)$, we have

$$L_m((x-1)P(x-m)) = xL_m(P(x)). \quad (5)$$

Apply (5) with $P(x) = (x+m)^n$ we obtain

$$L_m(x^{n+1} - x^n) = xL_m\big((x+m)^n\big)$$

$$L_m(x^{n+1}) - L_m(x^n) = xL_m\left(\sum_k \binom{n}{k} m^{n-k} x^k\right)$$

$$L_m(x^{n+1}) = (1+x) L_m(x^n) + xL_m\left(\sum_{k=0}^{n-1} \binom{n}{k} m^{n-k} x^k\right)$$

$$L_m(x^{n+1}) = (1+x) L_m(x^n) + x \sum_{k=0}^{n-1} \binom{n}{k} m^{n-k} L_m(x^k),$$

which is what we wanted to prove. ∎

When $x = 1$, $D_m(n, 1) = D_m(n)$, the Dowling numbers, and Eq. (4) becomes

$$D_m(n+1) = 2D_m(n) + \sum_{i=0}^{n-1} \binom{n}{i} m^{n-i} D_m(i),$$

which appears in [1]. Also, we can derive this last formula by defining L_m on the basis as follows:

$$L_m\left(\left(\frac{x-1}{m}\right)_k\right) = \frac{1}{m^k}, \qquad k \geq 0.$$

For the completeness of the study, we give the representation of $D_m(n, x)$ as a series.

THEOREM 2. *The Dowling polynomials are given by*

$$D_m(n,x) = e^{-x/m} \sum_{k\geq 0} \frac{x^k}{k!\,m^k}(mk+1)^n. \tag{6}$$

Proof. By (1) we have

$$\begin{aligned}\sum_{n\geq 0} D_m(n,x)\frac{z^n}{n!} &= \exp\left\{z + x\,\frac{e^{mz}-1}{m}\right\}\\ &= e^{-x/m}\left(\sum_{i\geq 0}\frac{z^i}{i!}\right)\left(\sum_{k\geq 0}\frac{x^k}{m^k k!}\,e^{mkz}\right)\\ &= e^{-x/m}\sum_{k\geq 0}\frac{x^k}{m^k k!}\sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{(mk)^{n-i}}{(n-i)!}\,\frac{1}{i!}\right)z^n.\end{aligned}$$

Equating the coefficients of z^n yields

$$D_m(n,x) = e^{-x/m}\sum_{k\geq 0}\frac{x^k}{m^k k!}(mk+1)^n,$$

the desired formula. ∎

For $m = 1$, Eq. (6) becomes

$$\begin{aligned}D_1(n,x) &= \sum_{k=0}^{n} S(n+1,k+1)x^k\\ &= e^{-x}\sum_{k\geq 0}\frac{x^k}{k!}(k+1)^n\\ &= e^{-x}\sum_{k\geq 0}\frac{x^k}{(k+1)!}(k+1)^{n+1}.\end{aligned}$$

Multiplying the last identity by x, we obtain

$$xD_1(n,x) = B_{n+1}(x) = e^{-x}\sum_{k\geq 1}\frac{k^{n+1}}{k!}x^k.$$

This is known, see [6]. If we put $m = 1$ in (6), we obtain formula (13) of [1]; also, $m = x = 1$ is Dobinsky's formula. Finally, we note that those formulas may be derived via the functional L_m.

3. THE GENERALIZATIONS OF $F_n(x)$ AND F_n

In this section, we give generalizations of $F_n(x)$ and F_n. We consider two quantities:

$$F_{m,1}(x) = \sum_k k!\, m^k W_m(n,k) x^k; \; F_{m,2}(x) = \sum_k K!\, W_m(n,k) x^k.$$

Note that $F_{m,1}(x/m) = F_{m,2}(x)$, so analytically both polynomials reduced to one. Thus, in this section we will work only with $F_{m,1}(x) = F_m(n, x)$. The results for $F_{m,2}(x)$ are easily derived by replacing x by x/m. Before beginning the study of these polynomials, we give Dowling's combinatorial interpretation of $k!\, m^k W_m(n,k)$ and $k!\, W_m(n,k)$. The discussion in Section 3.1 rests more heavily on the concepts of [4].

3.1. *Combinatorial Interpretation* (*Dowling*)

The following combinatorial interpretations of $k!\, W_m(n,k)$ and $k!\, m^k W_m(n,k)$ are due to T. A. Dowling [5]. I thank him for his permission to include them here.

For G a (multiplicative) group of order m, a partial G-partition is a set $\alpha = \{a_i\colon A_i \to G \mid i = 1, \dots, k\}$ of mappings $a_i\colon A_i \to G$ whose domains are non-empty, disjoint subsets of $\{1, 2, \dots, n\}$. The Dowling lattice $Q_n(G)$ consists of equivalence classes (under scalar multiplication by G) of partial G-partitions, where the class containing $\alpha = \{a_i\colon A_i \to G \mid i = 1, \dots, k\}$ is less than the class containing $\beta = \{b_j \to G \mid j = 1, \dots, h\}$ when each B_j is a union of some A_i and b_j restricts on those A_i to a multiple of a_i. The classes of partial G-partitions with k mappings have corank k in $Q_n(G)$; there are $W_m(n,k)$ of these. Suppose we add to the group G a zero element 0 satisfying $a \cdot 0 = 0 \cdot a = 0$ for all a in G, define addition (only for 0) by $a + 0 = 0 + a = a$, and call the resulting structure a "group." Then for any ordering of the k mappings, a partial G-partition α corresponds to a column-monomial $k \times n$ matrix over the "group", where "column−monomial" means that each column has at most one entry from G. When each row is replaced by a "linear combination" of a set of rows, with no row having a G-coefficient in more than one such linear combination, we get an element $\beta \geq \alpha$ in the lattice. Since there are $m^k k!$ representation of a given corank-k element of $Q_n(G)$ as a column-monomial matrix, $m^k k!\, W_m(n,k)$ is the number of $k \times n$ column-monomial matrices with nonzero entries from a group of order m. Consider the equivalence relation induced by scalar multiplication of rows by G, then $k!\, W_m(n,k)$ is the number of classes.

Thus, $F_{m,1}(x)$ and $F_{m,2}(x)$ are the generating functions for such combinatorial objects. Most of the results we derive are the analogs of the formulas appearing in [8].

3.2. *Some Generating Functions*

Recall that $F_m(n, x) = F_{m,1}(x)$. Our first result is

THEOREM 3. *The generating function for $F_m(n, x)$ is given by*

$$\sum_{n \geq 0} F_m(n, x) \frac{z^n}{n!} = \frac{e^z}{1 - x(e^{mz} - 1)}. \tag{7}$$

Replacing x by x/m yields

$$\sum_{n \geq 0} F_{m,2}(x) \frac{z^n}{n!} = \frac{e^z}{1 - (x/m)(e^{mz} - 1)}.$$

Proof. We may use the explicit formula for $W_m(n, k)$ (see [1]), but the techniques of [7] allows us to avoid this formula. Define L_m on V as follows:

$$L_m\left(\left(\frac{x-1}{m}\right)_k\right) = k!\, x^k,\ k \geq 0.$$

Applying L_m to (2) yields

$$L_m(x^n) = F_m(n, x).$$

It follows that

$$\sum_{n \geq 0} F_m(n, x) \frac{z^n}{n!} = L_m(e^{xz}).$$

As in Section 1, we write

$$e^{xz} = e^z (e^{mz})^{(x-1)/m} = e^z (1 + v)^{(x-1)/m}.$$

This gives

$$\begin{aligned} L_m(e^{xz}) &= e^z L_m\left((1 + v)^{(x-1)/m}\right) \\ &= e^z \sum_k (xv)^k \\ &= \frac{e^z}{1 - xv} \\ &= \frac{e^z}{1 - x(e^{mz} - 1)}, \end{aligned}$$

which is (7). ∎

Tanny [8] gave a representation for $F_n(x)$ as an infinite series. The following is the analog for $F_m(n, x)$.

THEOREM 4. *The polynomials $F_m(n, x)$ (for all $m \geq 1$) satisfy*

$$F_m(n, x) = \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k (mk+1)^n. \tag{8}$$

Note that this series is convergent only for $x > -\frac{1}{2}$.

Proof. By (7), we have

$$\begin{aligned}
\sum_{n \geq 0} F_m(n, x) \frac{z^n}{n!} &= \frac{e^z}{1 - x(e^{mz} - 1)} \\
&= \frac{e^z}{(1 + x - x e^{mz})} \\
&= \frac{1}{x+1} \frac{e^z}{\left(1 - \frac{x}{x+1} e^{mz}\right)} \\
&= \frac{1}{x+1} e^z \sum_{k \geq 0} \left(\frac{x}{x+1} \right)^k e^{kmz} \\
&= \frac{1}{x+1} \left(\sum_{i \geq 0} \frac{z^i}{i!} \right) \sum_{k \geq 0} \left(\frac{x}{x+1} \right)^k \left(\sum_{l \geq 0} \frac{(kmz)^l}{l!} \right) \\
&= \frac{1}{x+1} \sum_{n \geq 0} \left(\sum_{k \geq 0} \left(\frac{x}{x+1} \right)^k \left(\sum_{i=0}^{n} \frac{(mk)^{n-i}}{(n-i)!} \frac{1}{i!} \right) \right) z^n.
\end{aligned}$$

Now, equate the coefficients of z^n, to obtain the desired formula. ∎

Putting $x = 1$ in (8) yields

$$F_m(n, 1) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} (mk+1)^n. \tag{9}$$

This is the analog of the representation of F_n given in [7].

In [1], we gave a relation between Stirling numbers of the second kind and the $W_m(n, k)$. In the next result we give the analog between $F_m(n, x)$ and $F_n(x)$.

Recall the generating function for the $F_n(x)$, see [8].

$$\sum_{n\geq 0} F_n(x)\,\frac{z^n}{n!} = \frac{1}{1-x(e^z-1)}. \tag{10}$$

Using this we prove

THEOREM 5. *The sequence $F_m(n,x)$ satisfies*

$$F_m(n,x) = \sum_{i=0}^{n} \binom{n}{i} m^i F_i(x). \tag{11}$$

Proof. Let $t = mz$ in (7), we obtain

$$\begin{aligned}\sum_{n\geq 0} F_m(n,x)\frac{t^n}{m^n n!} &= \frac{e^{t/m}}{1-x(e^t-1)}\\ &= \left(\sum_{i\geq 0}\frac{t^i}{m^i i!}\right)\left(\sum_{l\geq 0}F_l(x)\frac{t^l}{l!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{F_i(x)}{m^{n-i}(n-i)!\,i!}\right)t^n.\end{aligned}$$

The identification of the coefficients gives (11). ∎

4. LOG-CONCAVITY OF $k!\,W_m(n,k)$

It is known that the polynomials $F_n(x)$ have only real zeros. In fact we have the identity

$$F_n(x) = \sum_{k=1}^{n} A(n,k)\,x^{n-k+1}(x+1)^{k-1}, \tag{12}$$

where $A(n,k)$ is the Eulerian number. It is known and easy to establish (see [3]), that Eulerian polynomials have only real zeros, and so do the polynomials $F_n(x)$.

We are convinced that it is possible to find the analog of this relation for $F_m(n,x)$, and hence to find new numbers generalizing the Eulerian ones, although we were unable to do this. But all is not lost. In the following we prove that all the zeros of $F_{m,2}(x)$ (and then $F_{m,1}(x)$) are real and negative. The same result also holds for $D_m(n,x)$, see [2].

THEOREM 6. *For all $m \geq 1$ and $n \geq 1$, the polynomials $F_{m,2}(x)$ have only real negative zeros. Consequently the sequence $k!\,W_m(n,k)$ is strictly log−concave.*

Proof. The case $m = 1$ is known, and reduces to (12), see [3, 8]. So, suppose that $m > 1$. For small n we can compute $F_{m,2}(x) = F_{m,2}(n, x)$. Indeed, we have

$$F_{m,2}(1, x) = 1 + x,\ F_{m,2}(2, x) = 1 + (m + 2)x + 2x^2, \dots$$

Thus, in these cases the result holds. Now, using the following recursion formula between the $W_m(n, k)$, see [4],

$$W_m(n, k) = W_m(n - 1, k - 1) + (1 + mk)W_m(n - 1, k),$$

we obtain

$$F_{m,2}(n, x) = (x + 1)F_{m,2}(n - 1, x) + (x^2 + mx)F'_{m,2}(n - 1, x),$$

where F' is the derivative of F. Assume that the result holds for $n - 1$. Define the polynomial $H(x)$ as follows:

$$H(x) = x(x + m)^{m-1}F^m_{m,2}(n - 1, x),$$

where F^m is the m-th power of F. By the induction hypothesis, $H(x)$ has $mn - m + m - 1 + 1 = mn$ real zeros, and by Rolle's Theorem, $H'(x)$ has at least $mn - 1$ real zeros, but

$$H'(x) = m(x + m)^{m-2}F^{m-1}_{m,2}(n - 1, x)F_{m,2}(n, x).$$

The degree of $H'(x)$ is $m - 2 + mn - m - n + 1 + n = mn - 1$, so $F_{m,2}(n, x)$ has n real zeros. This completes the induction. ∎

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