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On the Loop-Free Decomposition of Stochastic Finite-State Systems

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Two generalizations of Bacon's theory of loop-free decomposition of probabilistic finite-state systems are proposed. The first of these consists of a modification of the structure of the decomposition which then allows for the decomposition of a larger class of systems. The second generalization subsumes the first: sufficient conditions for a stochastic finite-state system to be decomposable for a single initial distribution on its set of states are given.

1. INTRODUCTION AND RESULTS

Several years ago G. Bacon (1964) extended certain results obtained by J. Hartmanis (to be found in Hartmanis and Stearns (1966)) for the loop-free decomposition of deterministic sequential machines to probabilistic finite-state machines. Here, two generalizations of Bacon's theory will be presented. The first will consist of making a structural modification of Bacon's decomposition scheme allowing for the decomposition of a larger class of systems. The second generalization subsumes the first; it is shown that there exist stochastic finite-state systems which admit a loop-free decomposition for a proper subset of the set of all initial distributions on the set of states, while both Bacon's theory and our first generalization hold for the set of all initial distributions.

The results will be stated in this section; subsequent sections will contain proofs and pertinent examples.

A *stochastic finite-state system* (SFS) is the ordered triple $M = \{X, S, \{M(x)\}\}$ where X is a finite nonempty alphabet, S is a finite set of n states, and $\{M(x)\}$ is a set of $n \times n$ stochastic matrices consisting of a matrix $M(x)$ for each $x \in X$. The typical entry of $M(x)$, $m_{ij}(x)$, is interpreted as the probability

* This work was completed while the author was at the Polytechnic Institute of Brooklyn.

of transition of state i to state j with input x. A SFS may be viewed as representing the state behavior of a generalized probabilistic automaton (Turakainen, 1968), stochastic sequential machine (Carlyle, 1963), or simply as a controlled finite Markov chain (Kemeny and Snell, 1960).

From Kemeny and Snell (1960) we have the concept of "lumpability" of a finite Markov chain. Let π be a partition on S for some SFS M. We shall say that M is *lumpable* on π if for each pair of blocks A and B of π and for each state i and k in A and $x \in X$

$$\sum_{j \in B} m_{ij}(x) = \sum_{j \in B} m_{kj}(x)$$

holds. This is identical to stating that the stochastic process derived by lumping the states of the controlled Markov chain on π is itself a Markov chain. Lumpability is a restrictive property. The SFS $(\pi)M$ whose states are the blocks of π and whose transition matrices $\{(\pi)\, M(x)\}$ are obtained by lumping the $\{M(x)\}$ on π will be called the lumped system. Clearly, any SFS is lumpable on the trivial partitions on S.

Following Bacon (1964), we shall say that two partitions π_1 and π_2 are (*stochastically*) *independent* if

$$\sum_{j \in A_1 \cap A_2} m_{ij}(x) = \left[\sum_{j \in A_1} m_{ij}(x)\right]\left[\sum_{j \in A_2} m_{ij}(x)\right]$$

for all blocks A_1 of π_1 and A_2 of π_2 with $A_1 \cap A_2$ nonempty, all $i \in S$ and $x \in X$ of M. Stochastic independence of partitions is also a restrictive property not implying or implied by lumpability.

Bacon's (1964) quasi-series decomposition scheme is identical to that presented by Hartmanis and Stearns (1966) for deterministic sequential machines, insofar as it consists of constructing the decomposable SFS $M = \{X, S, \{M(x)\}\}$ as an interconnection of the "front" SFS; $(\pi)M$ and the "tail" SFS $D = \{\pi \times X, \tau, \{D(\delta)\}\}$ so that the states of $(\pi)M$ and the input to M are the input to D. To this effect, he gives the following necessary and sufficient conditions:

(i) there exists a nontrivial partition π on S such that M is lumpable on π;
(ii) there exists a nontrivial partition τ on S such that $\pi \cdot \tau = \Phi$;
(iii) π and τ are stochastically independent.

Bacon's decomposition scheme is identical to that in Hartmanis and Stearns (1966) and (i) and (ii) are a generalization of the conditions they give; however (iii) will be shown in Theorem 2 to be due to using their scheme of quasi-series decomposition of deterministic systems in the stochastic case as well and becomes superfluous if this scheme is modified.

We define an s-decomposition (abbreviation of strong decomposition) as that decomposition scheme in which the "tail" machine has as inputs the present and next state of the "front" machine (and the input) instead of just the present state. In the next section we shall prove

THEOREM 1. *For the* SFS $M = \{X, S, \{M(x)\}\}$, (i) *and* (ii) *are necessary and sufficient for the existence of an s-decomposition with "front"* SFS

$$(\pi)M = \{X, \pi, \{(\pi)\, M(x)\}\}$$

and "tail" SFS

$$T = \{\pi \times \pi \times X, \tau, \{T(\gamma)\}\}$$

where the blocks of π *are states of* $(\pi)M$ *and the blocks of* τ *are the states of* T. *Furthermore* $\gamma = (A, B, x) \in \pi \times \pi \times X$ (*present and next state of* $(\pi)M$ *and input, in that order*) *so that for* $C, D \in \tau$ *the typical entry of* $T(\gamma)$ *is*

$$t_{CD}(\gamma) = \frac{m_{ij}(x)}{(\pi)m_{AB}(x)}$$

where $i = A \cap C$ *and* $j = B \cap D$ *and* $(\pi)\, m_{AB}(x)$ *is an entry of* $(\pi)\, M(x)$, *if* $(\pi)\, m_{AB}(x) \neq 0$ *and* $A \cap C$ *and* $B \cap D$ *are nonempty. Otherwise* $t_{CD}(\gamma)$ *is unspecified ("don't-care").*

To relate Bacon's results to ours, we shall also prove

THEOREM 2. *For* $B \in \pi$ *let* $\gamma_1, \gamma_2 \in \pi \times \{B\} \times X$ *and suppose* (i) *and* (ii) *hold for the* SFS M, *where* B *is any block of* π. *Then* (iii) *is true if and only if*

$$T(\gamma_1) = T(\gamma_2)$$

for all $B \in \pi$.

This simply states that the s-decomposition reverts to Bacon's scheme if T is independent of the next state of $(\pi)M$; therefore the former subsumes the latter.

If in addition to (i) and (ii), M is also lumpable on τ, then T is independent of the present state of $(\pi)M$.

Suppose M is not lumpable on any partition on S; in the third section of this note it will be shown that a quasi-series decomposition may exist for the SFS started in a proper subset of the set of all distributions on S. For this we use the concept of "weak lumpability" of a Markov chain.

Following the presentation of Kemeny and Snell (1960), let α be a distribution on S (a n-dimensional stochastic row vector) and $A_1, A_2, ..., A_r$ a sequence of (not necessarily distinct) blocks of some partition π on S. For any α let α^i be the distribution obtained from α with knowledge that the SFS M is in a state of A_i.[1]

We shall call α^i the *restriction* of α to A_i. Introducing some more notation, $\alpha(k)$ will denote the projection of α along its kth coordinate and will be interpreted as the probability that the SFS is in $k \in S$ given α. Also we will use

$$\alpha(A_i) = \sum_{k \in A_i} \alpha(k).$$

It will be said that

(iv) M is *weakly lumpable* on π for α, if for every sequence $A_1, A_2, ..., A_r$ and all input words $x_1 x_2 \cdots x_{r-1} \in W(X)$[2]

$$(\pi)\, m^{\alpha}_{A_r A_S}(x) = [\cdots [[\alpha^1 M(x_1)]^2\, M(x_2)]^3 \cdots M(x_{r-1})]\, M(x)(A_S)$$

depends solely on A_r, A_S, α, and $x \in X$. Here we have used $[\alpha^1 M(x_1)]^2$ to denote the restriction of $\alpha^1 M(x_1)$ to A_2; furthermore (A_S) at the end of the expression for $(\pi)\, m^{\alpha}_{A_r A_s}(x)$ is as defined above for $\alpha(A_i)$. $(\pi)\, m^{\alpha}_{A_r A_s}(x)$ will be the state transition probability of the SFS $(\pi) M_\alpha$ with state set π and transition matrices $\{(\pi)\, M_\alpha(x)\}$ obtained as described above. The vector $(\pi)\alpha$ with entries $\alpha(A_i)$ for each block $A_i \in \pi$ is obtained from α so that when M is started in α, the lumped system $(\pi) M_\alpha$ is started in $(\pi)\alpha$. The following theorem, which will be proved in Section 3, generalizes Theorem 1 to the case where the SFS M admits a decomposition for the initial distribution α on S. Such a decomposition will be called a w-decomposition (abbreviation of weak decomposition).

THEOREM 3. *Conditions* (ii) *and* (iv) *are sufficient for the existence of a w-decomposition of the* SFS *M with initial distribution α into* $(\pi)\, M_\alpha = \{X, \pi, \{(\pi)\, M_\alpha(x)\}\}$ *started in* $(\pi)\alpha$ *and* $T_\alpha = \{\pi \times \pi \times X, \tau, \{T_\alpha(\gamma)\}\}$ *started in* $(\tau)\alpha$, *where for blocks $A, B \in \pi$, $C, D \in \tau$, with $j = B \cap D$ and $j \in S$, $\gamma \in \pi \times \pi \times X$, each matrix $T_\alpha(A, B, x)$ satisfies*

$$\text{(v)} \qquad \sum_{A \in \pi} \sum_{C \in \tau} t_{CD}(A, B, x) \cdot \alpha(A) \cdot \alpha(C) = \frac{[\alpha \cdot M(x)](j)}{[(\pi)\, \alpha \cdot (\pi)\, M_\alpha(x)](B)}\,.$$

[1] E.g., for $\alpha = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $A_1 = \{1, 2\}$ $\alpha^1(\frac{2}{3}, \frac{1}{3}, 0)$.

[2] $W(X)$ is the set of all words over the alphabet X.

It is evident from (v) that $T(A, B, x)$ is not uniquely defined. It will be shown in Section 3, however, that a stochastic matrix $T(A, B, x)$ satisfying (v) may always be found.

Although a SFS may be easily tested for lumpability for each nontrivial partition on its set of states, this is not the case for weak lumpability where no procedure which will lead to an answer in a finite number of steps is known.

Theorem 3 may be reworded for the w-decomposition of a SFS into more than two subsystems; however, that is a straightforward extension. Extension of these results to systems with outputs may be carried out via arguments similar to Bacon's (1964) or by converting such systems (e.g., the stochastic sequential machine of Carlyle (1963)) to an equivalent "Moore type" system to which our arguments may be applied directly.

2. Proofs and Example: s-Decomposition

Let us start by showing sufficiency of (i) and (ii) in Theorem 1. Let $i \in S$; since $\pi \cdot \tau = \Phi$ there exist $A \in \pi$, $C \in \tau$ so that $i = A \cap C$. The existence of $(\pi)M$ follows from the definition of lumpability of M on π. For $B \in \pi$, $D \in \tau$, and $j = B \cap D$ we calculate

$$\text{Prob}[B, D \mid A, C, x] = \text{Prob}[D \mid A, B, C, x] \, \text{Prob}[B \mid A, C, x]$$

and since $(\pi)M$ is a SFS it follows that

$$m_{ij}(x) = \text{Prob}[D \mid C, A, B, x] \cdot (\pi) \, m_{AB}(x).$$

Since $\text{Prob}[D \mid C, A, B, x]$ defines the SFS T with inputs from $\pi \times \pi \times X$ and state set τ and we note that

$$t_{CD}(A, B, x) = \text{Prob}[D \mid C, A, B, x] = \frac{m_{ij}(x)}{(\pi) \, m_{AB}(x)}$$

and

$$\sum_{D \in \tau} t_{CD}(A, B, x) = \frac{\sum_{j \in B} m_{ij}(x)}{(\pi) \, m_{AB}(x)} = 1.$$

Defining a mapping $f : S \to \pi \times \tau$ with each $i \in S$ mapped into the corresponding (A, C), we define a subsystem of the SFS constructed from the connection of $(\pi)M$ and T which is isomorphic to M, since it has the same transition probabilities. This subsystem is persistent (nontransient) since at each step in its operation it remains in its state set with probability one.

To prove necessity, consider the interconnection of two SFS, $(\pi)M$ and T, where

$$(\pi)M = \{X, \pi, \{(\pi)\, M(x)\}\}$$

and

$$T = \{\pi \times \pi \times X, \tau, \{T(\gamma)\}\}$$

and construct the SFS

$$M = \{X, S, \{M(x)\}\}$$

where $S = \pi \times \tau$ and for each $i = (A, C) \in S, j = (B, D) \in S$, with $A, B \in \pi$ and $C, D \in \tau$, we define

$$m_{ij}(x) = (\pi)\, m_{AB}(x)\, t_{CD}(A, B, x)$$

where $(\pi)\, m_{AB}(x)$ and $t_{CD}(A, B, x)$ are entries of $(\pi)\, M(x)$ and $T(A, B, x)$, respectively. From this construction, M is isomorphic to the interconnection of $(\pi)M$ and T. Furthermore

$$\sum_{D \in \tau} m_{ij}(x) = (\pi)\, m_{AB}(x) = \sum_{D \in \tau} m_{kj}(x)$$

for any $k = (A, E)$ for any $E \in \tau$, since each $T(A, B, x)$ is stochastic. Therefore M is lumpable on π. That π and τ may be viewed as partitions on S is obvious; that $\pi \cdot \tau = \Phi$ follows from the definition of S. This completes the proof of Theorem 1.

As an example consider the SFS M with X consisting of only one letter, $S = \{1, 2, 3, 4\}$ and

$$M(x) = \begin{bmatrix} 0.3 & 0.2 & 0.5 & 0 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.5 & 0.1 & 0.2 & 0.2 \end{bmatrix}.$$

M is lumpable on $\pi = \{\overline{1, 2}; \overline{3, 4}\}$ forming $(\pi)M$ with

$$(\pi)\, M(x) = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix},$$

and for $\tau = \{\overline{1, 3}; \overline{2, 4}\}$ we have $\pi \cdot \tau = \Phi$ so that (i) and (ii) are satisfied. However M does not satisfy (iii):

$$m_{11}(x) = 0.3 \neq [m_{11}(x) + m_{12}(x)][m_{11}(x) + m_{13}(x)] = 0.4$$

so that it does not admit Bacon's quasi-series decomposition. However, it does admit an s-decomposition with $T = \{\pi \times \pi \times X, \tau\{T(\gamma)\}\}$ and calling $\mathrm{I} = \{1, 2\}$, $\mathrm{II} = \{3, 4\}$

$$T(\mathrm{I}, \mathrm{I}, x) = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix}, \qquad T(\mathrm{I}, \mathrm{II}, x) = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix},$$

$$T(\mathrm{II}, \mathrm{I}, x) = \begin{bmatrix} 1 & 0 \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix}, \qquad T(\mathrm{II}, \mathrm{II}, x) = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

where the first row and column of each $T(\gamma)$ corresponds to block $\{1, 3\}$ of τ.

Theorem 1 may be readily generalized to the general loop-free decomposition (similar to Bacon's (1964) Theorem 3); the extension is straightforward and is omitted here.

To prove Theorem 2, first assume that (iii) is true. Continuing with the same notation used in the proof of Theorem 1,

$$m_{ij}(x) = (\pi)\, m_{AB}(x) \cdot \sum_{j \in D} m_{ij}(x)$$

so

$$t_{CD}(A, B, x) = \sum_{j \in D} m_{ij}(x)$$

for $i = A \cap C$. But then, if $E \in \pi$,

$$t_{CD}(A, E, x) = \sum_{j \in D} m_{ij}(x),$$

so that $T(A, E, x) = T(A, B, x)$ and the $\{T(\gamma)\}$ are independent of the next state of $(\pi)M$. Conversely, assume that $T(\gamma_1) = T(\gamma_2)$ for all $B \in \pi$ such that $\gamma_1, \gamma_2 \in \pi \times \{B\} \times X$. Then for any $E \in \pi$ and $k = E \cap D$

$$\frac{m_{ij}(x)}{(\pi)\, m_{AB}(x)} = \frac{m_{ik}(x)}{(\pi)\, m_{AE}(x)}$$

But

$$\sum_{j \in D} m_{ij}(x) = \sum_{E \in \pi} m_{ik}(x) = \sum_{E \in \pi} m_{ij}(x) \frac{(\pi)\, m_{AE}(x)}{(\pi)\, m_{AB}(x)},$$

and since $(\pi)\, M(x)$ is a stochastic matrix

$$\sum_{j\in D} m_{ij}(x) = \frac{m_{ij}(x)}{(\pi)\, m_{AB}(x)}$$

which is (iii). Therefore the proof of Theorem 2 is complete.

3. Proof and Example: w-Decomposition

Let us first prove Theorem 3 using the notation employed in its statement. The probability of transition in the interconnection of $(\pi)M_\alpha$ and T_α started in $(\pi)\alpha$ and $(\tau)\alpha$, respectively, is

$$\mathrm{Prob}[B, D \mid (\pi)\alpha, (\tau)\alpha, x] = \mathrm{Prob}[D \mid (\pi)\alpha, B, (\tau)\alpha, x]\, \mathrm{Prob}[B \mid (\pi)\alpha, x]$$

and introducing

$$t_{CD}((\pi)\, \alpha, B, x) = \sum_{A\in\pi} t_{CD}(A, B, x) \cdot \alpha(A)$$

we obtain that

$$\mathrm{Prob}[D \mid (\pi)\, \alpha, (\tau)\, \alpha, x] = \sum_{C\in\tau} t_{CD}((\pi)\, \alpha, B, x) \cdot \alpha(C),$$

and since

$$\mathrm{Prob}[B \mid (\pi)\alpha, x] = [(\pi)\alpha \cdot (\pi)\, M_\alpha(x)](B)$$

we conclude that

$$\mathrm{Prob}[B, D \mid (\pi)\alpha, (\tau)\alpha, x] = [\alpha \cdot M(x)](j)$$

for $j = B \cap D$. We observe that we may choose

$$t_{CD}(A, B, x) = t_{CD}((\pi)\alpha, B, x)$$

so that to exhibit a matrix $T_\alpha(A, B, x)$, it suffices to exhibit a stochastic matrix $T_\alpha((\pi)\alpha, B, x)$. We note that (v) may be rewritten as

$$(\tau)\alpha \cdot T_\alpha((\pi)\alpha, B, x) = \xi_B$$

where ξ_B is the stochastic row vector each of whose entries are the right hand side of (v) for each $j \in B$; since there may be more $D \in \tau$ than $j \in B$, those entries of ξ_B corresponding to those $D \in \tau$ for which $D \cap B = \Phi$ are zero. Since, given any two stochastic vectors, it is always possible to find a stochastic matrix which transforms one into the other, provided the vectors are of the same dimension, the matrix $T_\alpha((\pi)\alpha, B, x)$, and therefore the matrix $T_\alpha(A, B, x)$ for each A, are determined, yielding the SFS T_α, although not uniquely.

From the argument above, we may also conclude that M started in α has the same transition probabilities as that subsystem of the interconnection of $(\pi)M_\alpha$ and T_α defined by the mapping $f: S \to \pi \times \tau$ such that $f(j) = (B, D)$ if $j = B \cap D$ (and f is well defined since by (ii) there exists a unique pair B, D for each $j \in S$), when $(\pi)M_\alpha$ and T_α are started in $(\pi)\alpha$, $(\tau)\alpha$, respectively. This subsystem is persistent since at each step of its operation it remains in its state set with probability one. This establishes sufficiency of (ii) and (iv) as stated in Theorem 3.

As an example consider the three state SFS M whose transition matrix

$$M(x) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{7}{8} & \frac{1}{8} & 0 \end{bmatrix}$$

we take from Kemeny and Snell (1960). For any vector $\alpha = (1 - 3a, a, 2a)$ we have

$$\alpha M(x) = \left(\frac{1}{4} + a, \frac{1}{4} - \frac{a}{3}, \frac{1}{2} - \frac{2a}{3}\right) = (1 - 3b, b, 2b)$$

for $b = \frac{1}{4} - a/3$. For the partition $\pi = \{\overline{1}; \overline{2, 3}\}$ on S with $A = \{1\}$ and $B = \{2, 3\}$ we calculate

$$\alpha^A = (1, 0, 0), \qquad \alpha^B = (0, \tfrac{1}{3}, \tfrac{2}{3}).$$

Since α^A is α with $a = 0$ and α^B is α with $a = \frac{1}{3}$ it follows that

$$[[\cdots[\alpha^1 M(x)]^2 \cdots M(x)]^{r-1} M(x)]^A = \alpha^A$$

and

$$[[\cdots[\alpha^1 M(x)]^2 \cdots M(x)]^{r-1} M(x)]^B = \alpha^B$$

for any sequence $A_1, A_2, \ldots, A_{r-1}$ of blocks of π. Therefore M is weakly lumpable on π for α (choosing α to be a stochastic vector) forming $(\pi)\, M_\alpha = \{X, \pi, \{(\pi)\, M_\alpha(x)\}\}$ with

$$(\pi)\, M_\alpha(x) = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{7}{12} & \frac{5}{12} \end{bmatrix}.$$

Letting $\tau = \{\overline{1, 2}; \overline{3}\}$ and $C = \{1, 2\}$, $D = \{3\}$, it follows that $\pi \cdot \tau = \Phi$ and with $(\pi)\alpha = (1 - 3a, 3a)$, $(\pi)\alpha \cdot (\pi)\, M_\alpha(x) = (\frac{1}{4} + a, \frac{3}{4} - a)$, it follows that

$$\sum_{E \in \tau} t_{EC}((\pi)\, \alpha, A, x) \cdot \alpha(E) = 1.$$

Therefore

$$\sum_{E \in \tau} t_{ED}((\pi)\, \alpha, A, x) \cdot \alpha(E) = 0$$

and

$$(1 - 2a, 2a)\, T_\alpha((\pi)\alpha, A, x) = (1, 0).$$

Hence

$$T_\alpha((\pi)\, \alpha, A, x) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = T_\alpha(A, A, x) = T_\alpha(B, A, x).$$

Similarly

$$(1 - 2a, 2a)\, T_\alpha((\pi)\alpha, B, x) = (\tfrac{1}{3}, \tfrac{2}{3})$$

which is satisfied by

$$T_\alpha((\pi)\, \alpha, B, x) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = T_\alpha(A, B, x) = T_\alpha(B, B, x)$$

so that we have exhibited the w-decomposition of M for any α of the specified form.

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