In this paper we give explicit descriptions of the Brauer trees of cyclic $r$-blocks occurring in the finite classical groups $U_n(q), \quad SO_{2n+1}(q), \quad CSp_{2n}(q), \quad CSO_{2n}^e(q)$, where the prime $r$ for the modular representation theory is distinct from the prime $p$ dividing $q$ and $p$ is odd. The possible graphs occurring as Brauer trees of finite classical groups were described by Feit [3]. In this paper we complete his description by identifying the vertices with characters. An explicit description of Brauer trees for $GL_n(q)$ was given in [5]. We may suppose $r \neq 2$, since the trees have trivial structure for $r = 2$.

The Jordan decomposition of characters is compatible with blocks and induces graph isomorphisms of Brauer trees for cyclic blocks. This reduces the problem to one of constructing the projective indecomposable characters in a cyclic block $B$ where the non-exceptional characters of $B$ are unipotent characters in the sense of Deligne and Lusztig. The projective indecomposable characters in such unipotent cyclic blocks are most readily constructed by Frobenius induction from proper subgroups. In the context of classical groups, this effectively means Harish–Chandra induction from subparabolic subgroups. Despite this relative paucity of means, the tree of $B$ can be determined using combinatorial arguments on the partitions or symbols labeling the unipotent characters in $B$. In every case the tree has the form

$$
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \tau_1 \quad \tau_2 \quad \tau_3
\end{array}
$$

where the characters on one side of the exceptional vertex are distinguished from those on the other side by combinatorial properties of the partitions and symbols parametrizing them.

In Section 1 we define the combinatorial notation of abacus diagrams for partitions and symbols that will be required. Section 2 states the basic
classification of characters of a classical group $G$ into blocks. This classification is an analogue of the Nakayama theory of hooks and cores for blocks of the symmetric group. The basic properties of unipotent blocks $B$ with cyclic defect groups are described in Section 3. We also give combinatorial expressions for the ratios of the degrees of non-exceptional characters to the degrees of the exceptional characters in such a block. Section 4 contains the form of Harish–Chandra induction which will be used. Harish–Chandra induction will be used not only from subgroups of $G$ to $G$, but also from $G$ to larger classical groups containing $G$. Sections 5–8 constitute the main part of this paper. The proof falls naturally into two cases depending on whether centralizers of $r$-elements in $G$ contains a general linear group or a unitary group as a factor. The simpler case of a general linear factor is done in Section 5; the considerably more difficult case of a unitary factor is done in Sections 6–8. One cause for the greater difficulty in the unitary case is that the partitions or symbols parametrizing the unipotent characters in $B$ are not derived from the core of $B$ in a manner compatible with Harish–Chandra induction. Nevertheless, the configurations are sufficiently rigid to determine the tree. Section 9 contains the general case of an arbitrary cyclic block and a complete list of the trees. Finally, in the appendix we give the exact relation between two parametrizations of unipotent characters of $U_n(q)$ which appear in the work.

This research was done in several stages. The authors should like to acknowledge the support of the NSF, the DFG project at Essen, the ENS of Paris, and the University of Manchester-UMIST Group Representation Symposium.

1. PARTITIONS AND SYMBOLS

Let $\mathbb{N}$ be the set of non-negative integers, and let $n \in \mathbb{N}$. We consider sequences $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of integers from $\mathbb{N}$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$. Two such sequences $(\lambda_1', \lambda_2', \ldots, \lambda_r')$ are equivalent if their non-zero components are the same. An equivalence class of sequences is then a partition $\lambda$ of $n$. A sequence $(\lambda_1', \lambda_2', \ldots, \lambda_r')$ representing $\lambda$ determines a corresponding $\beta$-set, namely, $X = \{x_1, x_2, \ldots, x_r\}$, where $x_i = \lambda_i + (i - 1)$. A hook $v$ of $X$ is a pair $(y, x)$ of integers where $0 < y < x$, $y \notin X$, and $x \in X$. The length of $v$ is $e = x - y$, the leg length of $v$ is $|\{z \in X: y < z < x\}|$, and the residue of $v$ modulo $e$ is the residue class of $x - t + 1$ modulo $e$. We shall also say $v$ is an $e$-hook. These definitions correspond to the usual ones associated with the partition $\lambda$. Indeed, $v$ corresponds to the hook defined by the $(i, j)$-node in the Young diagram of $\lambda$, where

$$i = |z \in \mathbb{N}: z \in X, z \geq x| \quad \text{and} \quad j = |z \in \mathbb{N}: z \notin X, z \leq y|.$$
The equivalence relation on sequences induces an equivalence relation on $\beta$-sets. Namely, two $\beta$-sets $X = \{x_1, x_2, \ldots, x_t\}$, $X' = \{x'_1, x'_2, \ldots, x'_r\}$ are equivalent if, say $d = t' - t \geq 0$, and

$$\{x'_1, x'_2, \ldots, x'_r\} = \{0, 1, \ldots, d - 1, x_1 + d, \ldots, x_t + d\}.$$

We may write this as $X' = [0, d - 1] \cup (X + d)$. Then $(y, x) \mapsto (y + d, x + d)$ defines a bijection from hooks of $X$ to hooks of $X'$ preserving lengths, leg lengths, and residues.

Let $\lambda$ be a partition of $n$, $X$ a $\beta$-set for $\lambda$, and $e$ a positive integer. Following James [9], we arrange the non-negative integers in the array

$$
\begin{array}{cccccccc}
0 & 1 & 2 & \cdots & e - 1 \\
e & e + 1 & e + 2 & \cdots & 2e - 1 \\
2e & 2e + 1 & 2e + 2 & \cdots & 3e - 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{array}
$$

(1.1)

and consider the columns as runners of an abacus. The column containing $i$ for $0 \leq i \leq e - 1$ will be called the $i$th runner of the abacus. The integers $i, i + e, i + 2e, \ldots$ on the $i$th runner label positions $0, 1, 2\ldots$ on that runner. Placing a bead at the place $x$ of (1.1) for each $x \in X$ gives the abacus diagram of $X$.

If $v = (y, x)$ is an $e$-hook of $X$, then $X_1 = \{y\} \cup X \setminus \{x\}$ is a $\beta$-set for a partition $\lambda_1$ of $n - e$. We say that $\lambda_1$, $X_1$ are gotten from $\lambda$, $X$ by removing $v$, or equivalently, that $\lambda$, $X$ are gotten from $\lambda_1$, $X_1$ by adding $v$. The abacus diagram of $X_1$ is then gotten from that of $X$ by moving the bead at $x$ up one position on its runner. The $e$-core of $X$ is the unique $\beta$-set $X_\infty$ obtained from $X$ by successively removing $e$-hooks as often as possible. Thus there is a finite sequence $X = X_0, X_1, X_2, \ldots, X_h = X_\infty$ of $\beta$-sets such that each is obtained from the preceding one by removing an $e$-hook and such that $X_\infty$ has no $e$-hooks. The partition $\lambda_\infty$ represented by $X_\infty$ is also called the $e$-core of $\lambda$, and $\lambda_\infty$ is uniquely determined by $\lambda$.

A hook $v = (y, x)$ of length divisible by $e$ is said to be on the $i$th runner if $x$ is on the $i$th runner. In particular, hooks of length divisible by $e$ are on the same runner if and only if they have the same residue modulo $e$. Suppose hooks of length divisible by $e$ and residue $j$ modulo $e$ are on the $i$th runner. Let $Y_i = \{j: je + i \in X\}$, and let $\mu_i$ be the partition represented by the $\beta$-set $Y_i$. The $e$-quotient of $\lambda$ is the ordered sequence $(\mu_0, \mu_1, \ldots, \mu_{e-1})$. In particular, the $e$-quotient of $\lambda$ is $(\mu_0, \mu_1, \ldots, \mu_{e-1})$ if $|X| \equiv 1 \pmod e$.

A symbol $A = \{X, Y\}$ is an unordered pair of sets $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ of non-negative integers, called the rows of $A$. The symbol is a degenerate if $X = Y$, and two copies of each degenerate symbol
are introduced. Two symbols $A = \{X, Y\}$ and $A' = \{X', Y'\}$ are equivalent if

$$X' = [0, d - 1] \cup (X + d) \quad \text{and} \quad Y' = [0, d - 1] \cup (Y + d),$$

or

$$X = [0, d - 1] \cup (X' + d) \quad \text{and} \quad Y = [0, d - 1] \cup (Y' + d)$$

for some integer $d \geq 0$. The rank and defect of $A$, defined respectively by

$$\text{rk}(A) = \sum_{x \in X} x + \sum_{y \in Y} y - \left[ \frac{(a + b - 1)^2}{2} \right],$$

$$\text{def}(A) = |X| - |Y|,$$

are invariants of equivalence classes of symbols.

$\eta$ will be a fixed parameter taking values 1 or -1. A hook $v$ of a symbol $A = \{X, Y\}$ is a pair $(y, x)$ of integers where $0 \leq y < x$, and

$$y \not\in X, \; x \in X \quad \text{or} \quad y \not\in Y, \; x \in Y \quad \text{if} \quad \eta = 1,$$

$$y \not\in Y, \; x \in X \quad \text{or} \quad y \not\in X, \; x \in Y \quad \text{if} \quad \eta = -1.$$

We shall call $e = x - y$ the length of $v$ and say that $v$ is an $e$-hook. The symbol $A' = \{X', Y'\}$ gotten from $A$ by deleting $x$ from its row and inserting $y$ into the same or the other row according as $\eta = 1$ or $\eta = -1$, is said to be gotten by removing $v$. Equivalently, $A$ is said to be gotten from $A'$ by adding $v$. We shall also call hooks for the case $\eta = 1$ linear hooks, hooks for the case $\eta = -1$ cross hooks. If $A'$ is degenerate, both copies of $A'$ are considered as gotten from $A$ by removing $v$. Let $e$ be a fixed positive integer. The $e$-core $A_\infty$ of $A$ is the symbol gotten from $A$ by successively removing $e$-hooks as often as possible. Thus there is a sequence $A = A_0, A_1, \ldots, A_h = A_\infty$ of symbols such that each is obtained from the preceding one by removing an $e$-hook and such that $A_\infty$ has no $e$-hooks. If $A_\infty$ is degenerate and $A_0 \neq A_\infty$, both copies of $A_\infty$ are considered as the $e$-core of $A$.

The abacus diagram of a symbol $A = \{X, Y\}$ is obtained by associating the rows of $A$ with the columns of the array

$$\begin{array}{c|c}
0 & 0 \\
1 & 1 \\
2 & 2 \\
3 & 3 \\
\vdots & \vdots \\
\end{array}$$

(1.2)
say, $X$ with the first column and $Y$ with the second, and then placing a bead at the place $x$ of the first column for each $x \in X$, and a bead at the place $y$ of the second column for each $y \in Y$. Let $x$ be a position in the array. If $\eta = 1$. The $e$-track of $x$ is the set of positions $x'$ in the column of $x$ such that $(x' - x)/e$ is an integer. If $\eta = -1$, the $e$-track of $x$ is the zigzag set of positions $x'$ in the column of $x$ such that $(x' - x)/e$ is an even integer and positions $x''$ in the other column such that $(x'' - x)/e$ is an odd integer. So if $A'$ is gotten from $A$ by removing an e-hook $v = (y, x)$, then the abacus diagram of $A'$ is gotten from that of $A$ by moving the bead at $x$ up one position on its track. In particular, $A$ has no e-hooks if and only if beads come before spaces on all $e$-tracks.

2. Labeling of Characters

Let $V$ be a symplectic, orthogonal, or unitary space over a finite field $k$ of odd characteristic $p$, where $|k| = q$ if $V$ is a symplectic or orthogonal space, and $|k| = q^2$ if $V$ is a unitary space. Let $G$ be the isometry group $I_0(V)$ if $V$ is a unitary space or an orthogonal space of odd dimension, and let $G$ be the conformal isometry group $J_0(V)$ if $V$ is a symplectic or orthogonal space of even dimension. Thus $G$ is one of the groups $U(n(q), SO_{2n+1}(q), CSO_{2n+1}(q))$. In addition, let $V^*$ be a vector space over $k$ related to $V$ as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>$\dim V$</th>
<th>$V^*$</th>
<th>$\dim V^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>unitary</td>
<td>$n$</td>
<td>unitary</td>
<td>$n$</td>
</tr>
<tr>
<td>symplectic</td>
<td>$2n$</td>
<td>orthogonal</td>
<td>$2n + 1$</td>
</tr>
<tr>
<td>orthogonal</td>
<td>$2n + 1$</td>
<td>symplectic</td>
<td>$2n$</td>
</tr>
<tr>
<td>orthogonal</td>
<td>$2n$</td>
<td>orthogonal</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

In the first and last cases $V$ and $V^*$ are isometric spaces. If $V^*$ is a unitary or symplectic space, let $\pi: I_0(V^*) \rightarrow I_0(V^*)$ be the identity mapping, and $G^* = I_0(V^*)$. If $V^*$ is an orthogonal space, let $\pi: D_0(V^*) \rightarrow I_0(V^*)$ be the natural surjection of the special Clifford group $D_0(V^*)$ onto $I_0(V^*)$, and $G^* = D_0(V^*)$. We call $G^*$ a dual group of $G$.

Let $\mathbb{F}$ be an algebraic closure of $k$. We shall define a subset $\mathcal{F}$ of $k[X]$ to serve as elementary divisors of semisimple elements of $G$ and $G^*$, and integers $e_I$ for $I$ in $\mathcal{F}$ to serve as hook lengths in the block theory:

(1) Suppose $V$ is a unitary space. To each monic polynomial $A$ in $k[X]$ with non-zero roots $\omega$ in $\mathbb{F}$ corresponds a monic polynomial $\tilde{A}$ in
$k[X]$ with roots $\omega^{-q}$. Moreover, $\overline{\alpha} = \alpha$ if and only if $\alpha$ has odd degree $d$ and the roots $\omega$ of $\alpha$ have order dividing $q^d + 1$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{ \alpha: \alpha \text{ is monic, irreducible, } \alpha \neq X, \alpha = \overline{\alpha} \},$$

$$\mathcal{F}_2 = \{ \alpha \overline{\alpha}: \alpha \text{ is monic, irreducible, } \alpha \neq X, \alpha \neq \overline{\alpha} \}.$$

We call polynomials in $\mathcal{F}_1$ unitary, and polynomials in $\mathcal{F}_2$ linear. The degree $d_\Gamma$ of a polynomial $\Gamma$ in $\mathcal{F}$ is odd or even according as $\Gamma$ is unitary or linear. Let

$$\delta_\Gamma = \begin{cases} d_\Gamma & \text{if } \Gamma \in \mathcal{F}_1 \\ \frac{1}{2}d_\Gamma & \text{if } \Gamma \in \mathcal{F}_2 \end{cases}$$

and let $e_\Gamma$ be the smallest positive integer $i$ such that

$$|U(i, q^{2\delta_\Gamma})| \equiv 0 \pmod{r} \quad \text{if } \Gamma \in \mathcal{F}_1$$

$$|GL(i, q^{2\delta_\Gamma})| \equiv 0 \pmod{r} \quad \text{if } \Gamma \in \mathcal{F}_2.$$

In particular, $e_{X-1}$ is the order of $-q$ modulo $r$.

(2) Suppose $V$ is a symplectic or orthogonal space. To each monic polynomial $\alpha$ in $k[X]$ with non-zero roots $\omega$ in $F$ corresponds a monic polynomial $\alpha^*$ in $k[X]$ with roots $\omega^{-1}$. Let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_0 = \{ X-1, X+1 \},$$

$$\mathcal{F}_1 = \{ \alpha: \alpha \text{ is monic, irreducible, } \alpha \neq X, \alpha \neq X \pm 1, \text{ and } \alpha = \alpha^* \},$$

$$\mathcal{F}_2 = \{ \alpha \alpha^*: \alpha \text{ is monic, irreducible, } \alpha \neq X, \alpha \neq X \pm 1, \text{ and } \alpha \neq \alpha^* \}.$$

We call polynomials in $\mathcal{F}_1$ unitary, and polynomials in $\mathcal{F}_2$ linear. The degree $d_\Gamma$ of a polynomial $\Gamma$ in $\mathcal{F}$ is odd or even according as $\Gamma \in \mathcal{F}_0$ or $\Gamma \notin \mathcal{F}_0$. Let

$$\delta_\Gamma = \begin{cases} d_\Gamma & \text{if } \Gamma \in \mathcal{F}_0 \\ \frac{1}{2}d_\Gamma & \text{if } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \end{cases}$$

and let $e_\Gamma$ be the smallest positive integer $i$ such that

$$q^{2i} - 1 \equiv 0 \pmod{r} \quad \text{if } \Gamma \in \mathcal{F}_0$$

$$|U(i, q^{2\delta_\Gamma})| \equiv 0 \pmod{r} \quad \text{if } \Gamma \in \mathcal{F}_1$$

$$|GL(i, q^{2\delta_\Gamma})| \equiv 0 \pmod{r} \quad \text{if } \Gamma \in \mathcal{F}_2.$$

In particular, $e_{X-1}$, $e_{X+1}$ are the orders of $q^2$ modulo $r$. 
Given a semisimple element \( s \) of \( G^* \), let \( \pi(s) = \prod \pi \pi(s) \) be the primary decomposition of \( \pi(s) \), \( V^* = \sum V^* \) the corresponding orthogonal decomposition of \( V^* \), and \( m(s) \) the multiplicity of \( \Gamma \) as an elementary divisor of \( \pi(s) \). If \( V \) is a unitary space, let \( \Psi_1(s) = \{ \text{partitions } \lambda \text{ of } m(s) \} \).

If \( V \) is a symplectic or orthogonal space, let

\[
\Psi_2(s) = \{ \text{symbols } \lambda \text{ of rank } \left\lfloor \frac{1}{2} m(s) \right\rfloor \} \quad \text{for } \Gamma \in \mathcal{F}_0,
\]

\[
\Psi_3(s) = \{ \text{partitions } \lambda \text{ of } m(s) \} \quad \text{for } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2.
\]

Moreover, the following conditions are imposed on the symbols \( \lambda \) in \( \Psi_1(s) \) for \( \Gamma \in \mathcal{F}_0 \):

1. \( \text{def}(\lambda) \) is odd if \( V^*(s) \) is a symplectic or an odd-dimensional orthogonal space.
2. \( \text{def}(\lambda) \equiv 2 \pmod{4} \) if \( V^*(s) \) is an even-dimensional orthogonal space of type \(-1\).
3. \( \text{def}(\lambda) \equiv 0 \pmod{4} \) if \( V^*(s) \) is an even-dimensional orthogonal space of type \(1\), and both copies of degenerate symbols are in \( \Psi_3(s) \).

As usual, symbols may be replaced by equivalent symbols. Finally, we let

\[
\Psi(s) = \prod_{\Gamma \in \mathcal{F}} \Psi_\Gamma(s)
\]

be the set of sequences \( (\lambda_\Gamma) \), where \( \lambda_\Gamma \in \Psi_\Gamma(s) \).

The irreducible characters of \( G \) have been classified in \([10, 11]\) in the form \( \chi_{s, \lambda} \), where \( s \) runs over representatives for the semisimple conjugacy elements of \( G^* \) and \( \lambda \in \Psi(s) \). The main steps of this parametrization are as follows:

1. The irreducible characters of \( G \) are partitioned into disjoint geometric conjugacy classes \( \Pi(s) \) parametrized by the semisimple conjugacy classes \( (s) \) of \( G^* \). The geometric conjugacy class \( \delta(G, (s)) \) consists of the constituents of Deligne–Lusztig characters of the form \( R^{\hbar} \), where \( T^{\hbar} \) is a maximal torus of \( G^* \) containing \( s \).
2. A bijection \( \mathcal{L}_s : \delta(G, (s)) \to \delta(C_{G^*}(s)^*) \) is established, where \( C_{G^*}(s)^* \) is a dual group of \( C_{G^*}(s) \), such that

\[
\chi_{s, \lambda}(1) = \frac{|G|}{|C_{G^*}(s)|} \mathcal{L}_s(\chi_{s, \lambda})(1).
\]  

Moreover, there exists a sign \( \varepsilon_s \) depending only on \( G^* \) and \( s \) such that

\[
(\chi_{s, \lambda}, R^{\hbar}) = \varepsilon_s(\mathcal{L}_s(\chi_{s, \lambda}), R^{\hbar})
\]
for any \( \chi_{s,\lambda} \in \mathcal{G}(G, (s)) \) and any maximal torus \( T' \) of \( G^* \) containing \( s \). Here \( R_T^1 \) and \( R_T^1 \) are the Deligne–Lusztig characters of \( G \) and \( C_{G^*}(s)^* \) defined respectively by the pair \( (T', s) \) of \( G^* \) and the pair \( (T', 1) \) of \( C_{G^*}(s) \).

(3) A bijection is established between \( \mathcal{G}(C_{G^*}(s)^*, (1)) \) and \( \Psi(s) \).

We give formulas for the degrees of the unipotent characters of \( G \).

1. Suppose \( G \) is a unitary group. Let \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \) be a partition of \( n \), where \( \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_r \). Then

\[
\chi_{1,\lambda}(1) = D_{\lambda}(q) \prod_{i=1}^{n} (q^i - (-1)^i),
\]

(2.2)

where

\[
D_{\lambda}(q) = \frac{q^{d(\lambda)}}{\prod_{v} (q^{l(v)} - (-1)^{l(v)})},
\]

\( d(\lambda) = \sum_{j} (t - j) \lambda_j \), \( v \) runs over the hooks of \( \lambda \), and \( l(v) \) is the length of \( v \).

2. Suppose \( G \) is a symplectic or an orthogonal group. Let \( A \in \Psi(1) \), where \( A = \{X, Y\} \), \( X = \{x_1, x_2, ..., x_a\} \), and \( Y = \{y_1, y_2, ..., y_b\} \). Let

\[
D_{A}(q) = 2^{-c}q^{-f} \frac{A(X, q) A(Y, q)}{\theta(X, q^2) \theta(Y, q^2)} \prod_{(x, y) \in X \times Y} (q^x + q^y),
\]

(2.3)

where \( c = \lfloor (a + b - 1)/2 \rfloor \) if \( X \neq Y \), \( c = a = b \) if \( X = Y \), \( f = (a^2 + b^2 - 2) + \left( a + b - 4 \right) + ... \), and

\[
A(X, q) = \prod_{x, x' \in X, x' > x} (q^x - q^{x'})
\]

\[
\theta(X, q^2) = \prod_{x \in X} \left( \prod_{h=1}^{X} (q^{2h} - 1) \right)
\]

for any subset \( X \) of non-negative integers. By \([10, (2.8.1)]\)

\[
\chi_{1,A}(1) = D_{A}(q) |G : Z(G)|_{p'}.
\]

(2.4)

3. Blocks with Cyclic Defect Groups

By \([6, (12A), (13C)]\), the \( r \)-blocks \( B_{s,\mathcal{N}} \) of \( G \) are parametrized by pairs \((s, \mathcal{N})\), where \( s \) runs over representatives for the semisimple \( r' \)-conjugacy classes of \( G^* \) and \( \mathcal{N} \) is the \( r \)-core of an element in \( \Psi(s) \). The parametriza-
tion of blocks and characters of $G$ can be chosen so that an irreducible character $x_{i, \lambda}$ is in $B_{s, \mathcal{X}}$ if and only if

1. the $r'$-part of $t$ is conjugate to $s$ in $G^*$,
2. the $r$-part of $t$ is in a dual defect group of $B_{s, \mathcal{X}}$,
3. The $r$-core of $\lambda$ is a subset of $\mathcal{X}$.

We recall that the dual defect groups of $B_{s, \mathcal{X}}$ form a conjugacy class of $r$-subgroups of $G^*$ isomorphic to the defect groups of $B_{s, \mathcal{X}}$. The $r$-core of $\lambda = (\lambda_r)$ is

$$\{ \kappa = (\kappa_r); \kappa_r \text{ is an } e_r\text{-code of } \lambda_r \},$$

where the parameter $\eta$ in the definition of the $e_r$-core of a symbol $\lambda_r$ is determined by the congruence $q^{\delta_r \epsilon_r} \equiv \eta \pmod{r}$. In particular, the $r$-core of $\lambda$ is a set with 1, 2, or 4 elements.

Let $B = B_{s, \mathcal{X}}$ be a cyclic block of $G$, that is, a block with cyclic defect group, $(R, b)$ a maximal $B$-subpair, and $e_B = |N(R, b) : C(R)|$ the inertial index of $B$. By [1, Theorem 1, Part 1], $B$ has $e_B$ non-exceptional characters and $\frac{|R| - 1}{e_B}$ exceptional characters. If $|R| > r$, or if $|R| = r$ and $e_B < r - 1$, then the non-exceptional characters in $B$ are the $r$-rational irreducible characters in $B$. If $|R| = r$ and $e_B = r - 1$, then all irreducible characters in $B$ are $r$-rational, and any character in $B$ may be designated as the exceptional character of $B$. We may suppose the non-exceptional characters of $B$ are in $\mathfrak{S}(G, (s))$, the exceptional characters of $B$ are in $\Pi_y \mathfrak{S}(G, (sy))$, where $y$ runs over non-identity elements in a dual defect group centralizing $s$. This follows from the following result:

(3A) Let $T'$ be a maximal torus of $G^*$, $s$ an $r'$-element of $T'$, and $y$ an $r$-element of $T'$. Suppose $y^c \neq y$ in $C_{G^*}(s)$ for some $c$ relatively prime to $r$. Then

1. $R^{x}_{T'}$ is $r$-rational, and all irreducible constituents of $R^{x}_{T'}$ are $r$-rational.
2. $R^{y}_{T'}$ is not $r$-rational, and no irreducible constituent of $R^{y}_{T'}$ is $r$-rational.

Proof. Let $T$ be a maximal torus of $G$ such that $T' \simeq \text{Hom}(T, \mathbb{Q}_l^\times)$, and let $s, y$ correspond respectively to the characters $\zeta, \xi$ in $\text{Hom}(T, \mathbb{Q}_l^\times)$, so that $R^{x}_{T} = R^{\zeta}_{T}, R^{y}_{T} = R^{\xi}_{T}$. For any Galois automorphism $\sigma$ of $\mathbb{Q}_l$ fixing $r'$th roots of unity,

$$(R^{x}_{T})^\sigma = (R^{\zeta}_{T})^\sigma = R^{(c \sigma)}_{T} = R^{\zeta}_{T} = R^{x}_{T},$$

since the values of $\zeta$ are $r'$th roots of unity. Thus $R^{x}_{T'}$ is $r$-rational. Let $\chi$ be an irreducible constituent of $R^{x}_{T'}$. We claim $\langle \chi^\sigma, R^{x}_{T'} \rangle = \langle \chi, R^{x}_{T'} \rangle$ for all
$R^s_{T^*}$, where $t$ is a semisimple element of $G^*$ and $T^*$ is a maximal torus of $G^*$ containing $t$. Indeed, $\langle \chi^o, R^s_{T^*} \rangle = \langle \chi, R^s_{T^*} \rangle = 0$ if $t \not\sim s$ in $G^*$, since $\chi^o$ is a constituent of $(R^s_{T^*})^o = R^s_{T^*}$. On the other hand,

$$\langle \chi^o, R^s_{T^*} \rangle = \langle \chi^o, (R^s_{T^*})^o \rangle = \langle \chi, R^s_{T^*} \rangle = \langle \chi, R^s_{T^*} \rangle,$$

so the claim holds. But then $\chi^o = \chi$ by [11, (4.23)], so (1) holds. Now suppose $\sigma$ is a Galois automorphism of $\overline{\mathbb{Q}}$, fixing $r$'th roots of unity such that $\xi^o = \xi^c$. Then

$$(R^c_{T^*})^o = (R^c_{T^*})^c = R^{c,c} = R^{c,c}.$$ 

Since $sy$ and $sy^c$ are not conjugate in $G^*$, $(R^c_{T^*})^o \neq R^{c,c}_{T^*}$ and $R^{c,c}_{T^*}$ is not $r$-rational. If $\chi$ is any irreducible constituent of $R^c_{T^*}$, then $\chi^o \in \mathcal{S}(G, (sy^c))$ since $(\chi^o, R^{c,c}_{T^*}) = (\chi, R^{c,c}_{T^*})^o \neq 0$, so $\chi^o \neq \chi$ and (2) holds.

The elementary divisor of the cyclic block $B$ is the unique $\Gamma \in \mathcal{F}$ such that the following hold: If $\chi_{s, \lambda}$ is a non-exceptional character of $B$ and $\lambda = (\lambda_A)$, then $\lambda_A$ has an $e_A$-hook if and only if $d = I\lambda$. Moreover, $\lambda_{\Gamma}$ has a unique $e_{\Gamma}$-hook, and removing this $e_{\Gamma}$ hook gives the $e_{\Gamma}$-core of $\lambda_{\Gamma}$.

(3B) Let $B = B_{s, \kappa}$ be a cyclic block of $G$ with inertial index $e_B$ and elementary divisor $\Gamma$. Let $\kappa_{\Gamma}$ be the $\Gamma$-component of $\kappa$ in $\mathcal{X}$. If $V$ is a unitary space, then $e_B = e_{\Gamma}$. If $V$ is a symplectic or orthogonal space, then

$$e_B = \begin{cases} 
2e_{\Gamma} & \text{for } \Gamma \in \mathcal{F}_0 \text{ and a non-degenerate } \kappa_{\Gamma} \\
e B & \text{for } \Gamma \in \mathcal{F}_0 \text{ and a degenerate } \kappa_{\Gamma} \\
e \Gamma & \text{for } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2. 
\end{cases}$$

Proof: $|B \cap \mathcal{S}(G, (s))|$ is the number of $\lambda$ in $\mathcal{V}(s)$ such that $\lambda_{\Gamma}$ has $e_{\Gamma}$-core $\kappa_{\Gamma}$. If $V$ is a symplectic or orthogonal space and $\Gamma \in \mathcal{F}_0$, then there are $2e_{\Gamma}$ or $e_{\Gamma}$ possibilities for $\lambda_{\Gamma}$ according as $\kappa_{\Gamma}$ is non-degenerate or degenerate, so (3B) holds. We note $\lambda_{\Gamma}$ is never degenerate since $\lambda_{\Gamma}$ has a unique $e_{\Gamma}$-hook. In the remaining cases of (3B) there are $e_{\Gamma}$ possibilities for $\lambda_{\Gamma}$.

A block of $G$ of the form $B_{1, \kappa}$ is called unipotent. For the rest of this section, $B = B_{1, \kappa}$ is a cyclic unipotent block, $e = e_{X-1}$, and elements in $\mathcal{X}$ are identified with their $(X-1)$-components. Thus $\mathcal{X}$ consists of a partition, a non-degenerate symbol, or a pair of degenerate symbols.

Case 1. Suppose $V$ is a unitary space, so $-q$ has order $e$ modulo $r$. Let $\kappa$ be the partition in $\mathcal{X}$, and $X = \{x_1, x_2, \ldots, x_k\}$ a corresponding $\beta$-set. We may suppose each runner in the abacus diagram of $X$ contains beads by replacing $X$ by an equivalent $\beta$-set. Let $\rho_1, \rho_2, \ldots, \rho_e$ be the last beads on the $e$ runners of the diagram, arranged so that $\rho_1 > \rho_2 > \cdots > \rho_e$. In particular,
$|x \in X: \rho_i < x < \rho_i + e| = i - 1$. The partitions $\kappa_{\rho_i}$ gotten by adding the hooks $(\rho_i, \rho_i + e)$ to $\kappa$ label the non-exceptional characters in $B$. Let $\sigma_1, \sigma_2, \ldots, \sigma_s$ and $\tau_1, \tau_2, \ldots, \tau_t$ be the subsequences of $\rho$'s having one or the other parity. If $\rho$ is a $\sigma$ or a $\tau$, we write $\kappa_\sigma$ or $\kappa_\tau$ for $\kappa_{\rho_i}$. In addition, we write $\chi_\sigma$ for $\chi_{1, \kappa_\sigma}$, $\chi_\sigma$ for $\chi_{1, \kappa_\sigma}$, $\chi_\tau$ for $\chi_{1, \kappa_\tau}$, and $\chi_{\text{exc}}$ for an exceptional character in $B$.

(3C) Let $V$ be a unitary space, and let $\chi_{\rho_i}, \chi_{\text{exc}}$ be as above. Then

$$\frac{\chi_{\rho_i}(1)}{\chi_{\text{exc}}(1)} = \prod_{\rho_i < \rho_j} \frac{1}{q^{\rho_i - \rho_j} - (-1)^{\rho_i - \rho_j}} \prod_{\rho_i > \rho_j} \frac{q^{\rho_j - \rho_i}}{q^{\rho_i - \rho_j} - (-1)^{\rho_j - \rho_i}}.$$ 

Proof. (2.1) and (2.2) imply that

$$\frac{\chi_{\rho_i}(1)}{\chi_{\text{exc}}(1)} = (q^e - (-1)^e) \frac{D_{\kappa}(q)}{D_{\kappa}(q)},$$

since

$$\chi_{\rho_i}(1) = D_{\kappa}(1) \prod_{j=1}^n (q^j - (-1)^j),$$
$$\chi_{\text{exc}}(1) = \frac{D_{\kappa}(q)}{q^e - (-1)^e} \prod_{i=n-e+1}^n (q^i - (-1)^i) \prod_{i=1}^{n-e} (q^i - (-1)^i).$$

Let $Y = \{y_1, y_2, \ldots, y_k\}$ be the $\beta$-set corresponding to $\kappa_{\rho_i}$. If $\rho_i = x_\alpha$, then

$$y_j = \begin{cases} x_j & \text{for } j < \alpha \text{ and } j \geq \alpha + i \\ x_{j+1} & \text{for } \alpha \leq j < \alpha + i - 1 \\ x_\alpha + e & \text{for } j = \alpha + i - 1. \end{cases}$$

In particular,

$$d(\kappa_{\rho_i}) - d(\kappa) = \sum_{j=1}^k (k - j)(y_j - x_j)$$
$$= (k - (\alpha + i - 1))(x_\alpha + e - x_{\alpha + i - 1})$$
$$+ \sum_{j=\alpha}^{\alpha + i - 2} (k - j)(x_{j+1} - x_j)$$
$$= (k - (\alpha + i - 1))e + \sum_{j=1}^{i-1} (x_{\alpha + j} - x_\alpha)$$
$$= \sum_{j=1}^{i-1} (\rho_j - \rho_i),$$
where the last equality holds, since \( \rho_j - \rho_i = (x_{i+j'} - x_i) + (\rho_j - x_{i+j'}) \) for a unique element \( x_{i+j'} \) in the interval \((\rho_i, \rho_i + e)\) and \( k - (x + i - 1) = |x \in X: x > x_i + e| \). Now

\[
\frac{D_{k;\pi}(q)}{D_k(q)} = \frac{q^{d(k\pi) - d(\pi)}}{q^k - (-1)^k} \prod_{x < \rho_i, x \notin X} \frac{q^{\rho_i - x} - (-1)^{\rho_i - x}}{q^{\rho_i + e - x} - (-1)^{\rho_i + e - x}} \\
\times \prod_{\rho_i < x < \rho_i + e, x \notin X} \frac{1}{q^{\rho_i + e - x} - (-1)^{\rho_i + e - x} - (-1)^{\rho_i - x} - (-1)^{x - \rho_i} - (-1)^{\rho_i} - (-1)^{\rho_i - e}} \\
\times \prod_{x > \rho_i + e, x \notin X} \frac{1}{q^{x - \rho_i - e} - (-1)^{x - \rho_i} - (-1)^{\rho_i - e}}.
\]

(3C) now follows, since the factors indexed by \( x \in X \), grouped together by the runners of \( \rho_j \) for \( 1 \leq j \leq i - 1 \), have product

\[
\prod_{1 \leq j < i} \frac{q^{\rho_j - \rho_i}}{q^{\rho_j - \rho_i} - (-1)^{\rho_j - \rho_i}}
\]

while the factors indexed by \( x \notin X \), grouped together by the runners of \( \rho_j \) for \( j \geq i + 1 \), have product

\[
\prod_{j \geq i + 1} \frac{1}{q^{\rho_j - \rho_i} - (-1)^{\rho_j - \rho_i}}.
\]

(3D) Let \( V \) be a unitary space, and let \( \chi_{\sigma_1}, \chi_{\tau_1}, \chi_{\text{exc}} \) be as above. Then

\[
\frac{\chi_{\sigma_1}(1)}{\chi_{\text{exc}}(1)} = \frac{\chi_{\tau_1}(1)}{\chi_{\text{exc}}(1)} = \frac{1}{e} \quad \text{(mod } r)\text{).}
\]

\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\sigma_{i+1}}(1)} = \frac{\chi_{\tau_i}(1)}{\chi_{\tau_{i+1}}(1)} = -1
\]

**Proof:** Fix \( \rho_i \), and for \( \rho_j \neq \rho_i \), let \( z_j \) be the unique integer in the interval \((\rho_i, \rho_i + e)\) such that \( z_j \equiv \rho_j \pmod{e} \). If \( j \leq i - 1 \), then

\[
\frac{q^{\rho_j - \rho_i}}{q^{\rho_j - \rho_i} - (-1)^{\rho_j - \rho_i}} \equiv \frac{(-q)^{\rho_j - \rho_i}}{(-q)^{\rho_j - \rho_i} - 1} \equiv \frac{(-q)^{z_j - \rho_i}}{(-q)^{z_j - \rho_i} - 1} \equiv \frac{-1}{(-q)^{\rho_i - z_j} - 1} \quad \text{(mod } r)\text{).}
\]
If \( j \geq i + 1 \), then

\[
\frac{1}{q^{\rho_i - \rho_j} - (-1)^{\rho_i - \rho_j}} \equiv \frac{(-1)^{\rho_i - \rho_j}}{(-q)^{\rho_i - \rho_j} - 1} \pmod{r}.
\]

Since \( \rho_i - z_j \) runs over representatives of the non-zero residue classes modulo \( e \),

\[
\prod_{j \neq i} ((-q)^{\rho_i - z_j} - 1) \equiv \prod_{h=1}^{e-1} ((-q)^{h} - 1) \equiv (-1)^{e-1} \pmod{r},
\]

where the last congruence follows by setting \( X = 1 \) in the formal derivative of \( X^e - 1 \pmod{r} \). Hence by (3C)

\[
\frac{X_{\rho_i}(1)}{X_{\text{exc}}(1)} \equiv \left[ \prod_{j > i} (-1)^{\rho_i - \rho_j} \right] (-1)^{e-1} \frac{1}{e} \pmod{r}.
\]

In particular,

\[
\frac{X_{\rho_e}(1)}{X_{\text{exc}}(1)} \equiv \frac{1}{e} \pmod{r}.
\]

Suppose \( \rho_e \equiv \rho_{e-1} \equiv \cdots \equiv \rho_{e-u+1} \pmod{2} \), but \( \rho_e \not\equiv \rho_{e-u} \pmod{2} \). Then

\[
\frac{X_{\rho_{e-u}}(1)}{X_{\text{exc}}(1)} \equiv \left[ \prod_{j > e-u} (-1)^{\rho_{e-u} - \rho_j} \right] (-1)^u \frac{1}{e} \pmod{r}.
\]

The first part of (3D) now holds, since \( \{ \rho_e, \rho_{e-u} \} = \{ \sigma, \tau \} \). Suppose \( \rho_i, \rho_{i+v} \) are any two terms such that \( \rho_i \equiv \rho_{i+v} \pmod{2} \), \( \rho_i \not\equiv \rho_j \pmod{2} \) for \( i < j < i + v \). By (3.1)

\[
\frac{X_{\rho_i}(1)}{X_{\rho_{i+v}}(1)} \equiv \left[ \prod_{j > i} (-1)^{\rho_i - \rho_j} \prod_{j > i + v} (-1)^{\rho_{i+v} - \rho_j} \right] (-1)^v \pmod{r}.
\]

So the second part of (3D) also holds.
Case 2. Suppose \( V \) is a symplectic or orthogonal space, so \( q^2 \) has order \( e \) modulo \( r \). Let \( \eta = \pm 1 \) be defined by \( q^e \equiv \eta \) (mod \( r \)), and let \( A = \{ X, Y \} \) be a symbol in \( \mathcal{K} \). We may assume that \( d = |X| - |Y| \geq 0 \) and that each \( e \)-track of the abacus diagram of \( A \) contains beads. Let \( \sigma_1, \sigma_2, \ldots, \sigma_s \) be the beads in \( X \) which are last on their \( e \)-tracks, arranged so that \( \sigma_1 > \sigma_2 > \cdots > \sigma_s \). Similarly, let \( \tau_1, \tau_2, \ldots, \tau_t \) be the beads in \( Y \) which are last on their \( e \)-tracks, arranged so that \( \tau_1 > \tau_2 > \cdots > \tau_t \). The symbols \( A_{\sigma_i}, \ A_{\tau_j} \) gotten from \( A \) by adding the \( e \)-hooks \( (\sigma_i, \sigma_i + e), (\tau_j, \tau_j + e) \) for \( 1 \leq i \leq s \), for \( 1 \leq j \leq t \), respectively, label the non-exceptional characters in \( B \). We write \( \chi_{\sigma_i} \) for \( \chi_{1, A_{\sigma_i}}, \ \chi_{\tau_j} \) for \( \chi_{1, A_{\tau_j}} \), and \( \chi_{\text{exc}} \) for an exceptional character in \( B \).

(3E) Let \( V \) be a symplectic or an orthogonal space, and let \( A, s, t \) be defined as above.

1. If \( A \) is non-degenerate, then \( s = t = e \) for \( \eta = 1 \); \( s = e + d \), \( t = e - d \) for \( \eta = -1 \).

2. If \( A \) is degenerate, then \( s = t = e \).

Proof: Only the case \( A \) is non-degenerate and \( \eta = -1 \) requires proof. We proceed by induction on the rank of \( A \). The assumption \( |X| \geq |Y| \) implies \( s \geq 1 \). If \( t = 0 \), then the last bead on each \( e \)-track is in \( X \), so that \( d = e \), \( s = 2e \). So we may suppose \( t \geq 1 \). Suppose \( \sigma_i > \tau_j + e \) for some \( i, j \). Then the symbol \( A' = \{ X', Y' \} \) obtained from \( A \) by removing the linear hook \( (\tau_j + e, \sigma_i) \) on \( X \) is its own \( e \)-core, and \( |X'| - |Y'| = d \). Moreover, if \( s', t' \) are defined for \( A' \) as \( s, t \) were for \( A \), then \( s' = s, \ t' = t \), and (1) holds by induction. So we may suppose \( \sigma_i \leq \tau_j + e \), and likewise \( \tau_j \leq \sigma_i + e \), for all \( i, j \). Hence

\[
\tau_1 - e < \sigma_i < \tau_i + e, \quad \sigma_i - e < \tau_j < \sigma_i + e \quad (3.2)
\]

for all \( i, j \). But then \( \sigma_i = \sigma_{i-1} - 1 \), \( \tau_j = \tau_{j-1} - 1 \) for \( 1 < i \leq s \), \( 1 < j \leq t \), and (1) easily follows.

(3F) Let \( V \) be a symplectic or an orthogonal space, and let \( \chi_{\sigma_i}, \ \chi_{\text{exc}} \) be as above. Then

\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} = \begin{cases} 
\prod_{\sigma_i < \sigma_j} \frac{1}{q^{\sigma_i - \sigma_j} - 1} \prod_{\sigma_j > \sigma_i} \frac{q^{\sigma_j - \sigma_i}}{q^{\sigma_j - \sigma_i} - 1} \\
n \prod_{\tau_j < \sigma_i} \frac{1}{q^{\sigma_i - \tau_j} + 1} \prod_{\tau_j > \sigma_i} \frac{q^{\tau_j - \sigma_i}}{q^{\tau_j - \sigma_i} + 1} \\
2 \prod_{\sigma_j < \sigma_i} \frac{1}{q^{\sigma_i - \sigma_j} - 1} \prod_{\sigma_j > \sigma_i} \frac{q^{\sigma_j - \sigma_i}}{q^{\sigma_j - \sigma_i} - 1} \\
n \prod_{\tau_j < \sigma_i} \frac{1}{q^{\sigma_i - \tau_j} + 1} \prod_{\tau_j > \sigma_i} \frac{q^{\tau_j - \sigma_i}}{q^{\tau_j - \sigma_i} + 1}
\end{cases}
\]
according as \( \mathcal{A} \) is non-degenerate or degenerate. A similar formula holds for \( \chi_{\tau_i}(1)/\chi_{\text{exc}}(1) \).

**Proof.** By (2.3) and (2.4)

\[
\chi_{\tau_i}(1) = |G:Z(G)|_{p'} D_{\tau_i}(1),
\]

\[
\chi_{\text{exc}}(1) = \frac{|G|_{p'}}{|C_{G^*}(y)^*|_{p'}} \frac{|C_{G^*}(y)^*|_{p'}}{|Z(C_{G^*}(y)^*)|_{p'}} D_\mathcal{A}(q),
\]

where \( \chi_{\text{exc}} \in \text{\mathcal{E}}(G, (y)) \). Thus

\[
\frac{\chi_{\tau_i}(1)}{\chi_{\text{exc}}(1)} = (q^e - \eta) \frac{D_{\tau_i}(q)}{D_\mathcal{A}(q)}.
\]

We may suppose \( \mathcal{A} \) is non-degenerate; the argument is similar when \( \mathcal{A} \) is degenerate except for the additional factor 2 in \( D_{\tau_i}(q)/D_\mathcal{A}(q) \). Then

\[
\frac{\chi_{\tau_i}(1)}{\chi_{\text{exc}}(1)} = (-1)^{\omega(\sigma_i)} (q^e - \eta) \prod_{x \in X, x \neq \sigma_i} \frac{q^{\sigma_i+e} - \eta q^e}{q^{\sigma_i} - q^e} \times \prod_{y \in Y} \frac{q^{\sigma_i+e} + \eta q^e}{q^{\sigma_i+q^e}} \prod_{h=1}^e \frac{1}{q^{2(\sigma_i+h)} - 1},
\]

where

\[
\omega(\sigma_i) = \begin{cases} 
|x \in X: x > \sigma_i| + |x \in X: x > \sigma_i + e| & \text{if } \eta = 1 \\
|x \in X: x > \sigma_i| + |y \in Y: y > \sigma_i + e| & \text{if } \eta = -1.
\end{cases}
\]

Let \( m_u, n_v \) be non-negative integers for \( 1 \leq u \leq s, 1 \leq v \leq t \) such that

\[
m_u e \leq \sigma_u < (m_u + 1)e, \quad n_v e \leq \tau_v < (n_v + 1)e.
\]

The product of factors indexed by \( x \in X, y \in Y \) over the \( e \)-track of \( \sigma_i \) is

\[
\prod_{k=1}^{m_i} \frac{q^{\sigma_i+e} - \eta^{k+1} q^e - \eta^{k} q^{\sigma_i-ke}}{q^{\sigma_i} - \eta^{k} q^{\sigma_i-ke}} = \frac{q^{(m_i + 1)e} - \eta^{m_i + 1}}{q^e - \eta},
\]

the product over the \( e \)-track of \( \sigma_u \) for \( u \neq i \) is

\[
\prod_{k=0}^{m_u} \frac{q^{\sigma_i+e} - \eta^{m_u+1} q^e - \eta^{m_u} q^{\sigma_i-m_u e}}{q^{\sigma_i} - q^{\sigma_u}} = q^{m_u e} \frac{q^{\sigma_i+e} - \eta^{m_u+1} q^{\sigma_u-m_u e}}{q^{\sigma_i} - q^{\sigma_u}}
\]

\[
= \begin{cases} 
\frac{q^{\sigma_i+e} - \eta^{m_u+1}}{q^{\sigma_i} - \eta^{m_u+1}} & \text{if } \sigma_u < \sigma_i \\
\frac{q^{\sigma_i+e} - \eta^{m_u+1}}{q^{\sigma_i} - \eta^{m_u+1}} & \text{if } \sigma_u > \sigma_i
\end{cases}
\]

\[
= \begin{cases} 
\frac{q^{\sigma_i+e} - \eta^{m_u+1}}{q^{\sigma_i} - \eta^{m_u+1}} & \text{if } \sigma_u < \sigma_i \\
\frac{q^{\sigma_i+e} - \eta^{m_u+1}}{q^{\sigma_i} - \eta^{m_u+1}} & \text{if } \sigma_u > \sigma_i
\end{cases}
\]
and the product over the e-track of $\tau_v$ is
\[
\prod_{k=0}^{n_v} \frac{q^{\sigma_{i+v}} + \eta^{k+1} q^{\tau_v - ke}}{q^{\sigma_i} + \eta^k q^{\tau_i - ke}}
\]
\[
= \begin{cases} 
q^{\sigma_i - \tau_i + (n_u + 1)e} + \eta^{n_u + 1} & \text{if } \tau_v \leq \sigma_i \\
-q^{\sigma_i - \tau_i + (n_u + 1)e} + \eta^{n_u + 1} & \text{if } \tau_v > \sigma_i
\end{cases}
\]

The numbers $(m_u + 1)e - \sigma_u$, $(n_v + 1)e - \tau_v$ for $1 \leq u \leq s$, $1 \leq v \leq t$, run over \{1, 2, ..., e\} twice. If $(m_u + 1)e - \sigma_u = (m_u + 1)e - \sigma_u'$ for distinct $u, u'$, then $\eta = -1$ and $m_u \not\equiv m_u'$ (mod 2). Similarly, if $(n_v + 1)e - \tau_v = (n_v + 1)e - \tau_v'$ for distinct $v, v'$, then $\eta = -1$ and $n_v \not\equiv n_v'$ (mod 2). On the other hand, if $(m_u + 1)e - \sigma_u = (n_v + 1)e - \tau_v$, then $m_u \equiv n_v$ (mod 2). Thus
\[
\prod_{u=1}^{s} \prod_{v=1}^{t} (q^{\sigma_i - \sigma_u + (m_u + 1)e} - \eta^{m_u + 1})(q^{\sigma_i - \tau_i + (n_v + 1)e} + \eta^{n_v + 1})
\]
\[
= \prod_{h=1}^{e} (q^{2(\sigma_i + h)} - 1).
\]
The two sides of the (3F) then differ at most by a sign, so they are the same.

(3G) Let $V$ be a symplectic or an orthogonal space, and let $\chi_{\sigma_i}$, $\chi_{\tau_i}$, $\chi_{\text{exc}}$ be as above. Then
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\sigma_{i+1}}(1)} = \frac{\chi_{\tau_{i}}(1)}{\chi_{\tau_{i+1}}(1)} \equiv -1 \pmod{r}
\]
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} = \frac{\chi_{\tau_{i}}(1)}{\chi_{\text{exc}}(1)} \equiv \frac{1}{2e} \text{ or } \frac{1}{e} \pmod{r}
\]
according as $A$ is non-degenerate or degenerate. In particular, $\chi_{\sigma_i}$ is linked to $\chi_{\sigma_i}$ in the Brauer tree only if $i \not\equiv i' \pmod{2}$, $\chi_{\tau_i}$ is linked to $\chi_{\tau_i'}$ only if $j \not\equiv j' \pmod{2}$, and $\chi_{\sigma_i}$ or $\chi_{\tau_i}$ is linked to the exceptional vertex only if $i \equiv s \pmod{2}$ or $j \equiv t \pmod{2}$.

Proof. We suppose $A$ is non-degenerate as before. In (3.3)
\[
\frac{q^{\sigma_i + e} - \eta q^x}{q^{\sigma_i} - q^x} = \eta, \quad \frac{q^{\sigma_i + e} + \eta q^x}{q^{\sigma_i} + q^x} = \eta \pmod{r}
\]
whenever $x, y$ are not on the $e$-track of $\sigma_i$. Moreover, as in the proof of (3E),
\[
\prod_{h=1, \sigma_i + h \neq (m_i + 1)e}^{e} (q^{2(\sigma_i + h)} - 1) \equiv (-1)^e e \quad \text{(mod } r) .
\]
Thus
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} \equiv (-1)^{\omega(\sigma_i)} + e - 1, \eta^{m_i + 1} \quad \text{(mod } r) , (3.4)
\]
\[
= \frac{1}{e} \frac{(-1)^{\omega(\sigma_i)} + e - 1}{e^{2m_i + 1e - 1}}.
\]
where $d$ is the defect of $\Lambda$. We claim $\omega(\sigma_i) \equiv 0 (\text{mod } 2)$. Indeed, if $q = 1$, then
\[
\omega(x_i) = \frac{i}{2} (\text{mod } 2).
\]
Similarly, $\omega(x_i+1) \equiv i (\text{mod } 2)$. So the claim follows. If $q = -1$, then
\[
\omega(x_i) = \frac{i}{2} (\text{mod } 2), \quad \omega(x_i+1) \equiv i (\text{mod } 2).
\]
Since the mapping $x \mapsto x + e$ is a bijection of $\{ x < x < x < \sigma_i \}$ onto
\[
\{ y < Y: \sigma_i + e < y < \sigma_i + 2e \},
\]
the claim follows. Thus by (3.4)
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} \equiv -1 \quad \text{(mod } r) .
\]
Now $\omega(\sigma_i) \equiv 0 (\text{mod } 2)$. Indeed, if $q = 1$, then $\omega(\sigma_i) = 0$; if $q = -1$, then
\[
\omega(x_i) = \frac{i}{2} (\text{mod } 2), \quad \omega(x_i+1) \equiv i (\text{mod } 2).
\]
Thus
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} \equiv \frac{1}{2e} (-1)^e \eta^d \quad \text{(mod } r) ,
\]
by (3.4). Since $s = e$ for $q = 1$ and $s = e + d$ for $q = -1$, in both cases
\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\text{exc}}(1)} \equiv \frac{1}{2e} \quad \text{(mod } r) ,
\]
follows. The statements on $\chi_{\sigma_i}(1)$ are similarly proved.
4. Harish-Chandra Induction

For the rest of this paper $\delta = 2$ or 1 according as $V$ is or is not a unitary space. Let $G = U_n(q)$, $SO_{2n+1}(q)$ or $CSp_{2n}(q)$, or $CSO_{2n}^c(q)$, and let $w, c$ be non-negative integers such that $n - \delta w = \frac{1}{2}c(c + 1)$, $c(c + 1)$, or $c^2$, respectively. In addition, let $d = c$, $d = 2c + 1$, $d = 2c$, respectively. We fix a flag

$$\mathfrak{F}: 0 \subset U_1 \subset U_2 \subset \cdots \subset U_w$$

of isotropic subspaces $U_i$ of dimension $i$, and a Levi subgroup $L_\mathfrak{F}$ of the stabilizer of $\mathfrak{F}$. The choice of $w$ implies $L_\mathfrak{F}$ has a unique unipotent cuspidal character $\zeta$. Let $\mathcal{E}_d(G, (1))$ be the Harish-Chandra family of unipotent characters of $G$ occurring in $R_{L_\mathfrak{F}}^G(\zeta)$. Since the centralizer algebra of $R_{L_\mathfrak{F}}^G(\zeta)$ is a Hecke algebra of type $C$, or $D_\ast$, or $E_\ast$, the characters in $\mathcal{E}_d(G, (1))$ are parametrized by the irreducible characters of a Weyl group $W$ of type $C$, or $D$, and type $D$ occurs if and only if $G = CSO_{2n}^c(q)$ and $c = 0$. More generally, for each subflag $\mathfrak{F}_0$ of $\mathfrak{F}$, let $L_{\mathfrak{F}_0}$ be a Levi subgroup of the stabilizer of $\mathfrak{F}_0$, and $\mathcal{E}_d(L_{\mathfrak{F}_0}, (1))$ the set of unipotent characters of $L_{\mathfrak{F}_0}$ occurring in $R_{L_{\mathfrak{F}_0}}^{L_\mathfrak{F}}(\zeta)$. The characters in $\mathcal{E}_d(L_{\mathfrak{F}_0}, (1))$ are parametrized by the irreducible characters of a corresponding parabolic subgroup $W_{\mathfrak{F}_0}$ of $W$. Now $W_{\mathfrak{F}_0} = W_{\mathfrak{F}_0,0} \times S_{t_1} \times S_{t_2} \times \cdots \times S_{t_r}$, where $W_{\mathfrak{F}_0,0}$ is a Weyl group of type $C_0$, or $D_0$, or $E_0$, $S_{t_j}$ is a symmetric group of degree $t_j$, and $w = w_0 + \sum_j t_j$. So irreducible characters $\phi$ of $W_{\mathfrak{F}_0}$ are labeled by sequences

$$([\alpha, \beta], \mu_1, \mu_2, \ldots, \mu_r)$$

consisting of a pair $[\alpha, \beta]$ of partitions with $|\alpha| + |\beta| = w_0$, and partitions $\mu_j$ of $t_j$ for $1 \leq j \leq r$. Here $[\alpha, \beta] = (\alpha, \beta)$ is ordered if $W_{\mathfrak{F}_0,0}$ is of type $C_{w_0}$, $[\alpha, \beta] - \{\alpha, \beta\}$ is unordered if $W_{\mathfrak{F}_0,0}$ is of type $D_{w_0}$. The sequences (4.1) will also label the characters $\chi$ in $\mathcal{E}_d(L_{\mathfrak{F}_0}, (1))$ corresponding to the $\phi$.

Let $\mathfrak{F}_0 \leq \mathfrak{F}_1$. We may suppose $L_{\mathfrak{F}_0} \leq L_{\mathfrak{F}_1}$ and $W_{\mathfrak{F}_0} \leq W_{\mathfrak{F}_1}$. If $\chi'$ in $\mathcal{E}_d(L_{\mathfrak{F}_0}, (1))$, $\chi$ in $\mathcal{E}_d(L_{\mathfrak{F}_1}, (1))$ are parametrized respectively by the characters $\phi'$ of $W_{\mathfrak{F}_0}$, $\phi$ of $W_{\mathfrak{F}_1}$, then

$$((\chi', R_{L_{\mathfrak{F}_0}}^{L_{\mathfrak{F}_1}}(\chi'))) = (\phi, \text{Ind}_{W_{\mathfrak{F}_0}}^{W_{\mathfrak{F}_1}}(\phi'))$$

(4.2)

by the Howlett-Lehrer comparison theorem [7, (5.9)]. The following two situations arise often in later sections.

(A) Suppose $\phi'$, $\phi$ have labels $([\alpha', \beta'], \mu'_1, \mu'_2, \ldots, \mu'_r)$, $([\alpha, \beta], \mu_1, \mu_2, \ldots, \mu_r)$, respectively, where all $\mu'_j$; $\mu_j$ are the trivial partition of 1. For $W$ of type $C_w$, $(\phi, \text{Ind}_{W_{\mathfrak{F}_0}}^{W_{\mathfrak{F}_1}}(\phi')) \neq 0$ if and only if we can add a sequence of 1-hooks to get $\alpha$ from $\alpha'$, $\beta$ from $\beta'$. For $W$ of type $D_w$ and $\alpha \neq \beta$, $(\phi, \text{Ind}_{W_{\mathfrak{F}_0}}^{W_{\mathfrak{F}_1}}(\phi')) \neq 0$ if and only if we can add a sequence of 1-hooks to get $\alpha$ from $\alpha'$, $\beta$ from $\beta'$, or alternatively, $\alpha$ from $\beta'$, $\beta$ from $\alpha'$. The claim for
type $C_w$ follows from the Littlewood–Richardson rule. To prove the claim for type $D_w$, we embed $W_{i_0}, W_{i_1}$ as subgroups of index 2 in Weyl groups $W_{i_0}, W_{i_1}$ of type $C_w, C_w$. Let $\phi$ be an irreducible character of $W_{i_0}$ covering $\phi$ so that $\phi$ has label $(\alpha, \beta)$ or $(\beta, \alpha)$. Then

$$(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) \neq 0$$

if and only if

$$(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) = (\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi'))) \neq 0.$$  

(B) Suppose $\phi', \phi$ have labels $([\alpha', \beta'], \mu_1), [\alpha, \beta]$, respectively, where $\mu_1'$ is a hook partition of $t_1$ of leg length $h$. For $W$ of type $C_w$, $(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) \neq 0$ if and only if we can add a hook $\sigma$ to $\alpha'$ to get $\alpha$, and a hook $\tau$ to $\beta'$ to get $\beta$. Here the sum of the lengths of $\sigma$ and $\tau$ is $t_1$, the sum of their leg lengths is $h$ or $h - 1$, and if one hook is empty, the other has leg length $h$. For $W$ of type $D_w$ and $\alpha \neq \beta$, $(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) \neq 0$ if and only if the preceding conditions hold, or the preceding conditions with $\alpha'$ and $\beta'$ interchanged hold. These claims follow from [8, 21.3] and the arguments used in the previous case.

The pair $[\alpha, \beta]$ in (A) and (B) can be encoded in a symbol $\mu_0 = \{X, Y\}$. For a symplectic or an orthogonal group, the symbol is the one given in [10, Section 3] by Lusztig. Namely $X, Y$ are $\beta$-sets for $\alpha, \beta$, respectively, such that $|X| - |Y| = d$, where $d = 2c + 1$ in a symplectic group, $d = 2c$ in an orthogonal group. For a unitary group, $X, Y$ are $\beta$-sets for $\alpha, \beta$, respectively, such that $Y$ contains the interval $[0, c - 1]$ and $|X| - |Y| = 1$. Similarly, the pair $[\alpha', \beta']$ can be encoded in a symbol $\mu'_0 = \{X', Y'\}$, and this can be done so that $|X| + |Y| = |X'| + |Y'|$. Then (A) and (B) can be expressed as follows:

(A') Suppose $\phi', \phi$ have labels $(\mu'_0, \mu'_1, ..., \mu'_c), (\mu_0, \mu_1, ..., \mu_c)$, where $\mu'_0 = \{X', Y'\}, \mu_0 = \{X, Y\}$ are symbols and the $\mu'_j, \mu_j$ are the trivial partition of 1 for $j' \geq 1, j \geq 1$. Suppose $|X'| + |Y'| = |X| + |Y|$, and $\mu_0$ is non-degenerate. Then $(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) \neq 0$ if and only if $\mu_0$ can be gotten from $\mu'_0$ by adding a sequence of linear $1$-hooks.

(B') Suppose $\phi', \phi$ have labels $(\mu_0, \mu_1), \mu_0$, where $\mu'_0 = \{X', Y'\}, \mu_0 = \{X, Y\}$ are symbols and $\mu'_1$ is a hook partition of $t_1$ of leg length $h$. Suppose $|X'| + |Y'| = |X| + |Y|$ and $\mu_0$ is non-degenerate. Then $(\phi, \text{Ind}_{W_{i_0}}^{W_{i_0}}(\phi')) \neq 0$ if and only if $\mu_0$ can be gotten from $\mu'_1$ by adding one linear hook to each row of $\mu'_1$. Here the sum of the lengths of the hooks is $t_1$, the sum of their leg lengths is $h$ or $h - 1$, and if one of the hooks is empty, the other is $\mu'_1$.

In the case of a unitary group, the symbol $\mu_0 = \{X, Y\}$ encoding $[\alpha, \beta]$ has additional significance. The ordered pair $(\alpha, \beta)$ parametrizes a uni-
potent character of the component $L_{\Theta,0}$ of $L_{\Theta}$ corresponding to $W_{\Theta,0}$. Since $L_{\Theta,0} = U_{\Theta}(q)$, where $n_0 = n - \sum_j 2t_j$, this character is also parametrized by a partition $\lambda_0$ of $n_0$. Let $Z_0$, $Z_1$ be the sets of integers defined by the conditions

\begin{align*}
X &= Z_0, \quad Y = [0, d-1] \cup (Z_1 + d) \quad \text{if } d \text{ is even} \\
X &= Z_1, \quad Y = [0, d-2] \cup (Z_0 + d-1) \quad \text{if } d \text{ is odd}.
\end{align*}

(4.3)

Then

\[ Z = \{2i: i \in Z_0\} \cup \{2i+1: i \in Z_1\} \quad (4.4) \]

is a $\beta$-set of odd cardinality for $\lambda_0$. This is proved in the appendix. In particular, if $\lambda_0$ is the analogous partition corresponding to $(\phi', \beta')$ in situation (A), then $(\phi, \text{Ind}_{W_{\Theta}}(\phi')) \neq 0$ if and only if $\lambda_0$ can be gotten from $\lambda_0'$ by adding a sequence of 2-hooks.

The language of symbols and abacus diagrams will be used in the next three sections of the paper for the symplectic, the orthogonal, and the unitary cases. When $G$ is the unitary group, we may also replace symbol $\mu_0$ by the symbol $A = \{Z_0, Z_1\}$. This replacement of (4.3) by (4.4) does not affect the descriptions in (A'), (B'), since Harish-Chandra induction involves only the characters in $\Pi_{\Theta} \delta_d(L_{\Theta},(1))$ for a fixed $d$. In using $A$ it will be convenient to use the numbers in $Z$ as names of the elements of $Z_0, Z_1$. We illustrate this by an example. Suppose $\sigma, \sigma'$ are even integers with $\sigma \in Z, \sigma' \notin Z, \sigma - \sigma' = 2k > 0$. So $\sigma$ and $\sigma'$ correspond to a bead and a space $k$ positions apart on a runner of the abacus diagram of $A$. We then speak of moving $\sigma$ up $k$ positions to $\sigma'$ to mean that the bead is moved up $k$ positions to the space.

5. UNIPOTENT BLOCKS, PART I

We assume in this section that one of the following holds:

1. $V$ is a unitary space and $e$ is even.
2. $V$ is a symplectic or orthogonal space and $r$ divides $q^e - 1$.

Let $B = B_{1,\kappa}$ be a cyclic unipotent block of $G$. We follow the notation of Section 3 for the labeling of the non-exceptional characters of $B$. In Case (1) we set $e = 2f$. Let $\kappa$ be the unique partition in $\kappa$. The partitions gotten by adding an $e$-hook to $\kappa$ are of 2 types: $\kappa_{\sigma_1}, \kappa_{\sigma_2}, \ldots, \kappa_{\sigma_l}$, where $\sigma_i$ is even, and $\kappa_{\tau_1}, \kappa_{\tau_2}, \ldots, \kappa_{\tau_l}$, where $\tau_i$ is odd. Let $Z$ be a $\beta$-set corresponding to $\kappa$ with $|Z|$ odd and $0, 1, \ldots, e - 1 \in Z$. If $A$ is the symbol $\{X, Y\}$ defined by

\[ X = \{i: 2i \in Z\}, \quad Y = \{i: 2i + 1 \in Z\}, \]
and if $A_{\sigma}, A_{\nu}$ are the analogous symbols derived from $\kappa_{\sigma}, \kappa_{\nu}$, then $A_{\sigma}$ and $A_{\nu}$ are gotten from $A$ by adding linear $f$-hooks to $X, Y$, respectively. In Case (2) we set $e = f$. The parameter $\eta = 1$, and $\mathcal{X}$ consists of a non-degenerate symbol or a pair of degenerate symbols. Let $A = \{X, Y\}$ be a symbol in $\mathcal{X}$ with $0, 1, \ldots, e-1$ in $X$ and $Y$. Let $A_{\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_r}$ be the symbols gotten from $A$ by adding a linear $f$-hook to $X$, and $A_{\tau_1}, A_{\tau_2}, \ldots, A_{\tau_s}$ the symbols gotten from $A$ by adding a linear $f$-hook to $Y$.

(5A) The Brauer tree of $B$ is

\[
\begin{array}{c}
\sigma_1 \sigma_2 \sigma_3 \ \cdots \ \cdots \ \cdots \\
\tau_1 \tau_2 \tau_3 \ \cdots \\
\end{array}
\]

if $A$ is non-degenerate

\[
\begin{array}{c}
\sigma_1 \ \cdots \\
\tau_1 \ \cdots \\
\end{array}
\]

if $A$ is degenerate.

**Proof.** We may suppose $f > 1$ since the case $f = 1$ is trivial by (3D) and (3G). We consider flags

\[ \mathcal{G}: 0 \subset U_1 \subset U_f, \quad \mathcal{G}': 0 \subset U_f, \quad \mathcal{G}'' : 0 \subset U_1, \]

where $U_1, U_f$ have dimensions $1, f$, respectively, and their corresponding Levi subgroups $L = L_{\mathcal{G}}, L' = L_{\mathcal{G}'}, L'' = L_{\mathcal{G}''}$. The unipotent character $\theta$ of $L$ labeled by $(\mu_1, \mu_2)$, where $\mu_1$ is the trivial partition of 1 and $\mu_2$ is the hook partition of $f-1$ of leg length $i-1$, is projective, so $R_{\mathcal{L}}^G(\theta)$ is a projective character of $G$. Since $R_{\mathcal{L}}^C(\theta) = R_{\mathcal{L}}^G(R_{\mathcal{L}}^C(\theta))$, it follows by the Littlewood-Richardson rule \[8, (21.3)]\] that (i) the irreducible constituents of $R_{\mathcal{L}}^C(\theta)$ are labeled by $(A, \lambda)$, where $\lambda$ is a hook partition of $f$ of leg length $i-1$, a hook partition of $f$ of leg length $i$, or a non-hook partition of $f$, (ii) the irreducible constituents of $B$ in $R_{\mathcal{L}}^G(\theta)$ are $\chi_{\sigma_1}$, $\chi_{\sigma_1+1}$, $\chi_{\tau_1}$, $\chi_{\tau_1+1}$. By (3D) and (3G) the projective indecomposable characters of $B$ in $R_{\mathcal{L}}^G(\theta)$ are either $\chi_{\sigma_1} + \chi_{\sigma_1+i}$ and $\chi_{\tau_1} + \chi_{\tau_1+i}$, or $\chi_{\sigma_1} + \chi_{\sigma_1+i}$ and $\chi_{\tau_1} + \chi_{\tau_1+i}$, and the latter case occurs only for non-degenerate $A$. Since

\[ R_{\mathcal{L}}^G(\theta) = R_{\mathcal{L}}^G(R_{\mathcal{L}}^C(\theta)), \]

the Littlewood-Richardson rule also implies the following: (i) The irreducible constituents $\chi''$ of $R_{\mathcal{L}}^C(\theta)$ are labeled by $(A'', \mu_1)$, where $A''$ is gotten from $A$ by adding a linear hook to each row of $A$. The sum of the hook lengths is $f-1$, the sum of the leg lengths is $i-1$ or $i-2$, and if one hook is empty, the other has leg length $i-1$. (ii) The irreducible constituents of $R_{\mathcal{L}}^G(\chi'')$ are labeled by symbols $\Xi$ gotten by adding a linear 1-hook to $A''$. In particular, $\Xi$ is $A_{\sigma_j}$ or $A_{\tau_j}$ only if $A''$ is gotten by adding a linear $(f-1)$-hook of leg length $i-1$ to a row of $A$, and $\Xi$ is then gotten from $A''$ by adding a linear 1-hook to the same row. Denote such a $A''$ leading to $A_{\sigma_j}$ or $A_{\tau_j}$ by $A''_{\sigma_j}$ or $A''_{\tau_j}$.
Suppose \( \Phi = \chi_{\sigma_i} + \chi_{\tau_{i+1}} \) is a projective indecomposable character of \( B \). Then there exists a projective indecomposable character \( \Phi'' \) of \( L'' \) such that

\[
\Phi \subseteq R_{L''}(\Phi''), \quad \Phi'' \subseteq R_{L''}(\theta')
\]

Moreover, \( \Phi'' \) contains irreducible characters labeled by \((A''_{\sigma_i}, \mu_i)\), \((A''_{\tau_{i+1}}, \mu_i)\). In particular, the \( f \)-cores of \( A''_{\sigma_i} \) and \( A''_{\tau_{i+1}} \) are the same or one of the two copies of a degenerate symbol. Let \( A''_{\sigma_i} = \{X', Y\} \), \( A''_{\tau_{i+1}} = \{X, Y''\} \). If \( A''_{\sigma_i} \) and \( A''_{\tau_{i+1}} \) have no linear \( f \)-hooks, then either \( X = X'' \), \( Y = Y'' \) or \( X = Y \), \( X'' = Y'' \), and both situations are impossible. If \( A''_{\sigma_i} \) and \( A''_{\tau_{i+1}} \) have linear \( f \)-hooks, these hooks occur in \( X'' \), \( Y'' \). Thus \( X \) is a row in the \( f \)-core of \( \{X', Y\} \), \( Y \) is a row in the \( f \)-core of \( \{X, Y''\} \). Since \( X \neq Y \), it follows \( X \) is the \( f \)-core of \( X'' \), \( Y \) is the \( f \)-core of \( Y'' \), which is impossible. Hence \( \chi_{\sigma_i} + \chi_{\tau_{i+1}} \) and \( \chi_{\tau_i} + \chi_{\tau_{i+1}} \) are projective indecomposable characters for \( 1 \leq i \leq f - 1 \).

It remains to link \( \sigma_f \) and \( \tau_f \) to the exceptional vertex. \( L' \) is a central product \( L_0 L' \), where \( L' \simeq GL_r(q') \). In particular, by [5, Theorem C] the principal block of \( L'_+ \) is cyclic, and \( \theta'_+ = \chi' + \sum \chi'_{exc} \), where \( \chi' \) is the Steinberg character and \( \chi'_{exc} \) runs over the exceptional characters in the principal block of \( L'_+ \), is a projective indecomposable character. Let \( \theta_0 \) be the projective unipotent character of \( L_0 \) labeled by \( \Lambda \), and let \( \theta' = \theta_0 \theta'_+ \). Then \( R_{L''}(\theta') \) is a projective character of \( G \), and \( \chi_{\sigma_f} \) and \( \chi_{\tau_f} \) are the only non-exceptional characters of \( B \) occurring in \( R_{L''}(\theta') \). Since \( \sigma_f \) and \( \tau_f \) are not linked by (3D) and (3G), \( \sigma_f \) and \( \tau_f \) are linked to the exceptional vertex. This completes the proof.

6. Unipotent Blocks, Part II

We assume in this section that one of the following holds:

1. \( V \) is a unitary space and \( e \) is odd.
2. \( V \) is a symplectic or orthogonal space and \( r \) divides \( q^r + 1 \).

Let \( B = B_{1,1} \) be a cyclic unipotent block of \( G \). We follow the notation of Section 3 for the labeling of the non-exceptional characters of \( B \). In Case (1) let \( \kappa \) be the unique partition in \( \mathcal{K} \). The partitions gotten by adding an \( e \)-hook to \( \kappa \) are of 2 types: \( \kappa_{\sigma_1}, \kappa_{\sigma_2}, ..., \kappa_{\sigma_r} \), where \( \sigma_i \) is even, and \( \kappa_{\tau_1}, \kappa_{\tau_2}, ..., \kappa_{\tau_r} \), where \( \tau_i \) is odd. Let \( Z \) be the \( \beta \)-set corresponding to \( \kappa \) with \( |Z| \) odd and 0, 1, ..., \( e - 1 \) in \( Z \). If \( A \) is the symbol \( \{X, Y\} \) defined by

\[
X = \{i: 2i \in Z\}, \quad Y = \{i: 2i + 1 \in Z\},
\]

and \( A_{\sigma_i}, A_{\tau_i} \) are the analogous symbols derived from \( \kappa_{\sigma_i}, \kappa_{\tau_i} \), then \( A_{\sigma_i} \) and \( A_{\tau_i} \) are gotten from \( A \) by adding cross \( e \)-hooks to \( X \), \( Y \), respectively. Since
A has odd defect, the characters $\chi_{\sigma}, \chi_{\tau}$ are in different Harish–Chandra families. Moreover, the argument of (3E) shows $\{s, t\} = \{ (e + 1)/2 + d, (e - 1)/2 - d \}$, where $\frac{1}{2}d(d+1)$ is the size of the 2-core $\kappa_\infty$ of $\kappa$. In Case (2) the parameter $\eta = -1$, and $\mathcal{H}$ consists of a non-degenerate symbol or a pair of degenerate symbols. Let $A = \{X, Y\}$ be a symbol in $\mathcal{H}$ with $0, 1, \ldots, e - 1$ in $X$ and $Y$. Let $A_{\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_t}$ be the symbols gotten from $A$ by adding a cross $e$-hook to $X$, and $A_{\tau_1}, A_{\tau_2}, \ldots, A_{\tau_t}$ the symbols gotten from $A$ by adding a cross $e$-hook to $Y$. By (3E) $s = e + d$, $t = e - d$, where $d = |X| - |Y| \geq 0$. The characters $\chi_{\sigma_i}, \chi_{\tau_i}$ are in the same Harish–Chandra family if and only if $d = 0$.

(6A) The Brauer tree of $B$ is

```
-...+.--..~-*...-
```

The proof of (6A) will be by induction. In this and the next section we establish the inductive step of the proof when $e = 3$. The remaining parts of the proof are in Section 8.

(6B) Suppose $e \geq 3$. Then the Brauer tree of $B$ contains the subgraphs

```
-...-
```

and

```
-...-n
```

Proof. The unipotent characters in $B$ are real-valued, so the tree of $B$ is equal to its real stem by [12, Theorem B; Theorem 9.2.1] In particular, no vertex in the tree is linked to 3 vertices. The following configuration is basic for our proof. Suppose there are characters of $B$ in the Harish–Chandra family associated with the flag $\mathfrak{F}: 0 \subset U_1 \subset U_2 \subset \cdots \subset U_w$ of isotropic subspaces $U_i$ of dimension $i$. We fix a subflag $\mathfrak{G}: 0 \subset U_1 \subset U_2 \subset \cdots \subset U_{w-h}$ of $\mathfrak{F}$, and a projective character $\theta$ of the Levi subgroup $L = L_{\mathfrak{G}}$. Then $R^G_L(\theta)$ is projective, and the component of $R^G_L(\theta)$ belonging to $B$ provides links in the tree.

If the characters $\chi_\sigma$ of $B$ are in the Harish–Chandra family associated to $\mathfrak{F}$, such configurations occur for $i = 1, 2, \ldots, s - 1$ with $h = \sigma_i - \sigma_{i+1}$ and $\theta$ the projective unipotent character of $L$ labeled by $(\Xi_{\sigma_i}, \mu_1, \mu_2, \ldots, \mu_{w-h})$, where

$$\Xi_{\sigma_i} = \{X \setminus \{\sigma_i\}, Y \cup \{\sigma_{i+1} + e\}\},$$

and $\mu_1, \mu_2, \ldots, \mu_{w-h}$ are the trivial partition of 1. In this case $A_{\sigma_i}, A_{\sigma_{i+1}}$ can be gotten by adding sequences of linear 1-hooks to $\Xi_{\sigma_i}$, but no other $A_{\sigma_z}$ can be so gotten, since

$$|x \in X \setminus \{\sigma_i\}: x \leq \sigma_i| < |x \in X \setminus \{\sigma_z\}: x \leq \sigma_z|,$$

$$|y \in Y \cup \{\sigma_{i+1} + e\}: y \leq \sigma_z + e| < |y \in Y \cup \{\sigma_z + e\}: y \leq \sigma_z + e|,$$
according as $x < i$, $x > i + 1$. We write $\Xi_{\sigma_i} \to A_{\sigma_i}$, $\Xi_{\sigma_i} \to A_{\sigma_{i+1}}$, $\Xi_{\sigma_i} \leftrightarrow A_{\sigma_i}$ for $x \neq i$, $i + 1$ as abbreviations for the preceding statements, and we say $\Xi_{\sigma_i}$ links $\sigma_i$ and $\sigma_{i+1}$ if $\chi_{\sigma_i} + \chi_{\sigma_{i+1}}$ is a projective indecomposable character in $R^G(\theta)$. Analogous definitions and statements hold for the characters $\chi_{r_j}$.

The following notation will also be used. For any symbol $\Sigma = \{S, T\}$ of positive defect with $|S| > |T|$, let $h(\Sigma) = S$, $l(\Sigma) = T$. For any finite subset $Z$ of $\mathbb{N}$, let
\[
I_k(Z) = |z \in Z : z \leq k|, \quad F_k(Z) = |z \in Z : z > k|.
\]

The reader may find it helpful to draw abacus diagrams for the arguments which follow.

(a) Suppose the $\chi_{\sigma_i}, \chi_{r_j}$ are in different Harish–Chandra families. Then $\Xi_{\sigma_i} \leftrightarrow A_{\sigma_i}$ for any $i$, since $\Xi_{\sigma_i}$ and $A_{\sigma_i}$ have different defects. Thus $\Xi_{\sigma_i}$ links $\sigma_i$ and $\sigma_{i+1}$ for $1 \leq i < s$. Similarly, $\Xi_{r_j}$ links $\tau_j$ and $\tau_{j+1}$ for $1 \leq j < t$, so (6B) holds.

(b) Suppose the $\chi_{\sigma_i}, \chi_{r_j}$ are in the same Harish–Chandra family. $V$ is then an orthogonal space, $d = 0$, $s = t = e$. If $A$ is degenerate, the argument in (a) shows $\sigma_i$ and $\sigma_{i+1}$ are linked for $1 \leq i < s$. So we may suppose $A$ is non-degenerate.

(c) There exists $k > 0$ such that $I_k(X) \neq I_k(Y)$ since $A$ has distinct rows. If $I_k(X) > I_k(Y)$, then $I_k(h(\Xi_{\sigma_i})) < I_k(h(A_{\tau_j}))$, so that $\Xi_{\sigma_i} \leftrightarrow A_{\tau_j}$ for any $j$. We shall abbreviate this argument by saying $\Xi_{\sigma_i} \leftrightarrow A_{\tau_j}$ for any $j$ by $I_k h$. Also $\Xi_{\tau_j} \leftrightarrow A_{\sigma_i}$ for any $i$ by $I_k l$. Thus
\[
\Xi_{\sigma_i} \text{ links } \sigma_i \text{ and } \sigma_{i+1} \text{ if } \sigma_{i+1} > k - e. \quad \Xi_{\tau_j} \text{ links } \tau_j \text{ and } \tau_{j+1} \text{ if } \tau_j \leq k.
\]

A similar consideration of $I_k h$ and $I_k l$ shows
\[
\Xi_{\sigma_i} \leftrightarrow A_{\tau_j} \text{ for } \tau_j \leq k - e \text{ and any } i, \quad \Xi_{\tau_j} \leftrightarrow A_{\sigma_i} \text{ for } \sigma_i > k \text{ and any } j.
\]

(d) We fix $k$ to be the least integer such that $I_k(X) \neq I_k(Y)$, say $I_k(X) > I_k(Y)$. Thus $k \in X$, $k \notin Y$, and either $i \in X \cap Y$ or $i \notin X \cup Y$ for $0 \leq i < k$. In particular, $k - e \in X \cap Y$, $k - e = \sigma_u$ for some $\sigma_u \in X$, and
\[
\sigma_{u-2} < \sigma_{u-1} = \tau_{u+1}, \quad \sigma_{u+2} = \tau_{u+2}, \ldots, \quad \sigma_s = \tau_e.
\]

Let $v$ be the integer such that $\tau_v \leq k < \tau_{v-1}$. Then
\[
\begin{array}{cccccc}
\sigma_1 & \sigma_2 & \sigma_{u-2} & \sigma_{u-1} & \tau_v & \tau_{v+1} & \tau_{v-1} & \tau_e \\
\end{array}
\]

are subgraphs of the tree by (c). Moreover, a consideration of $F_{k-1} l$ and $F_{k-1} h$ shows
\[
\Xi_{\sigma_i} \leftrightarrow A_{\tau_j} \quad \text{for } \sigma_i < k \text{ and } \tau_j > k, \quad \Xi_{\tau_j} \leftrightarrow A_{\sigma_i} \quad \text{for } \sigma_i < k - e \text{ and } \tau_{j+1} > k - e.
\]
(c) We claim \( \cdots \sigma_e \sigma_{e-1} \sigma_{e-2} \cdots \cdot \) is a subgraph of the tree. We may suppose \( u < e \), since the claim is trivial when \( u = e \). Suppose \( \mathcal{Z}_{\sigma_i} \) fails to link \( \sigma_i \) and \( \sigma_{i+1} \) for some \( \sigma_i < k \). Then \( \sigma_i \), \( \sigma_{i+1} \) are linked to \( \tau \)'s in the interval \( (k-e, k) \) by (c, d). Let the \( \tau \)'s in this interval be \( \tau_{u}, \tau_{u+1}, \ldots, \tau_{u+h} \). Here \( h \geq 1 \) since \( \sigma_i \), \( \sigma_{i+1} \) cannot be linked to the same \( \tau_j \) by (3G), and \( v + h < e \) since \( \tau_{u+1} = \sigma_{u+1} < k - e \). No vertex is linked to 3 other vertices. By (d) \( \tau_j \) is already linked to \( \tau_{j-1} \) and \( \tau_{j+1} \) for \( j = v + 1, \ldots, v + h \). Thus \( \sigma_{i+1}, \sigma_{i+2} \) are linked to \( \tau_v \), which is impossible, and the claim holds.

We claim \( \cdots \cdot \sigma_{e-1} \sigma_e \) is a subgraph of the tree if \( \sigma_u-1 < k \). Indeed, if \( u < e \), then \( \sigma_{e-1} \) and \( \sigma_e \) are linked by the preceding argument and the claim holds. Suppose \( u = e \). If \( \mathcal{Z}_{\sigma_{e-1}} \) fails to link \( \sigma_{e-1} \) and \( \sigma_e \), then \( \sigma_{e-1} \) and \( \sigma_e \) are linked to \( \tau \)'s in the interval \( (k-e, k) \) by (c, d). The \( \tau \)'s in this interval are \( \tau_{v}, \tau_{v+1}, \ldots, \tau_e \), where \( 1 < v < e \), so the links are \( \sigma_{e-1}, \sigma_e \) by (3G). Thus

\[
\begin{array}{cccccccc}
\sigma_1 & \sigma_2 & \sigma_{e-1} & \sigma_e & \tau_1 & \tau_{e-1} & \tau_e
\end{array}
\]

is a subgraph, and \( \mathcal{Z}_{\tau_{e-1}} \) cannot link \( \tau_{v-1} \) and \( \tau_v \). Hence \( \tau_{v-1}, \tau_v \) are linked to \( \sigma \)'s in the interval \( [k-e, k] \) by (c, d), so \( \sigma_{e-1}, \sigma_e \) is a link by (3G) and \( \sigma_1 < k \). Now \( \tau_e > \sigma_e \), so that \( \tau_1 > k, I_{\tau_{e-1}}(X) > I_{\tau_{e-1}}(Y) \). Thus

\[
\begin{array}{cccccccc}
\sigma_1 & \sigma_2 & \sigma_{e-1} & \sigma_e & \tau_1 & \tau_{e-1} & \tau_e
\end{array}
\]

is a subgraph by (c). This implies \( v = 2 \). But this is impossible, since \( \sigma_{e-1}, \sigma_e \) is not a link by (3G).

(f) Suppose \( \mathcal{Z}_{\tau_j} \) fails to link \( \tau_j \) to \( \tau_{j+1} \) for some \( \tau_j, \tau_{j+1} > k-e \). Then \( \tau_j, \tau_{j+1} \) are linked to \( \sigma \)'s in the interval \( [k-e, k] \) by (c, d). In particular, \( \sigma_{u-1} < k \) since \( \sigma_j, \sigma_{j+1} \) both cannot be linked to \( \sigma_u \) by (3G).

(g) Suppose \( \cdots \cdot \sigma_{e-1} \sigma_e \) is a subgraph of the tree. If

\[
\begin{array}{cccccccc}
\tau_1 & \tau_2 & \tau_{e-1} & \tau_e
\end{array}
\]

is not a subgraph, then \( \mathcal{Z}_{\tau_j} \) fails to link \( \tau_j \) and \( \tau_{j+1} \) for some \( j \), so \( \tau_j, \tau_{j+1} \) are linked to \( \sigma \)'s with \( \sigma \leq k \) by (c). These \( \sigma \)'s are necessarily \( \sigma_1 \) and \( \sigma_e \). In particular, \( \sigma_1 \leq k \). Since \( \tau_1 > k \) by (d), it follows that \( I_{\tau_1-1}(X) > I_{\tau_1-1}(Y) \). So the tree contains

\[
\begin{array}{cccccccc}
\tau_2 & \tau_3 & \tau_{e-1} & \tau_e
\end{array}
\]

by (c), and \( j = 1 \). By (3G) the tree is then

\[
\begin{array}{cccccccc}
\tau_1 & \sigma_e & \sigma_{e-1} & \sigma_2 & \sigma_1 & \tau_2 & \tau_3 & \tau_{e-1} & \tau_e
\end{array}
\]

(6.1)
Suppose \( \tau_2 < k - e \). Then \( \sigma_2 = \tau_2 < k - e \), so that \( u = 1 \), \( \sigma_1 = k - e \), and \( \sigma_i = \tau_i \) for all \( i \geq 2 \). Let \( A' = \{ X', Y' \} \), where \( X' = X \), and \( Y' \) is obtained from \( Y \) be removing \( \tau_i \) and inserting \( \sigma_i + e \), that is, \( Y' \) is gotten from \( Y \) by moving \( \tau_i \) up to \( \sigma_i + e \). Then \( A' \) has defect 0 and no \( e \)-hooks. We define \( \sigma_i' \), \( \tau_i' \) for \( A' \) in the way \( \sigma_i, \tau_i \) were defined for \( A \). So \( \sigma_i' = \tau_i' = \tau_i \) for \( i \geq 2 \), \( \sigma_1' = \tau_1' = \tau_1 - e \), and \( \tau_1' = \sigma_1 + e \). We also define the symbols \( A'_{\tau_i}, A'_{\sigma_i} \) in the obvious way. Then \( A'_{\tau_1}, A'_{\tau_2} \rightarrow A_{\tau_2} , A'_{\tau_1} \rightarrow A_{\sigma_2} , A'_{\sigma_2} \rightarrow A_{\sigma_1} \) by \( \sigma, e + e \). Let \( \chi_{\sigma}, \chi_{\tau} \) be in the Harish-Chandra family associated with \( \mathcal{F} : 0 < U_1 < U_2 < \cdots < U_w \), and let \( \mathcal{G} : 0 < U_1 < U_2 < \cdots < U_w - h \) with \( h = \tau_1 - (\sigma_1 + e) \). By induction \( L = L_{\mathcal{G}} \) has a projective indecomposable character \( \theta \) of the form \( \chi_{\tau_1}' + \chi_{\tau_2}' \). Here \( \chi_{\tau_i}' \) is the unipotent character of \( L \) labeled by \( (A_{\tau_i}', \mu_1, \mu_2, \ldots, \mu_h) \), where \( \mu_1, \mu_2, \ldots, \mu_h \) are the trivial partition of 1. This is impossible, since \( R^G_{L}(\theta) \) contains \( \chi_{\tau_1}' \), but not \( \chi_{\sigma_1}' \). The preceding induction argument will be used repeatedly with the appropriate notation \( A', \sigma_i', \tau_i' \), \( L, \theta \). In such cases it will be understood that \( A' \) has defect 0 and no \( e \)-hooks, \( \sigma_i', \tau_i' \), \( L \) have their obvious meanings, and \( \theta \) is a projective character of \( L \).

Thus \( \tau_2 > k - e \). Then \( k - e \leq \sigma_i \leq \sigma_1 \leq k \) by (f), and \( \sigma_e = k - e \). The interval \( [k - e, k] \) \( \subseteq X \), and all but one integer in this interval is a \( \sigma \). Since \( \sigma_e < \tau_e \), we may suppose without loss of generality that \( k = e \). We claim that each integer in the interval \( [1, e - 1] \) on \( Y \) is a \( \tau \). Indeed, if some such integer is not a \( \tau \), then its \( e \)-track contains a \( \tau > 2e \). In particular, \( \tau_1 > 2e \). Also, there exists \( \tau_j \) with \( e < \tau_j < \tau_1 \). We choose \( \tau_j \) minimal with respect to this property, and then choose \( \sigma_i \) maximal so that \( \sigma_i + e < \tau_j \). Such a \( \sigma_i \) exists since \( \sigma_e + e < \tau_j \). Let \( A' = \{ X', Y' \} \), where \( X' = X \), and \( Y' \) is gotten from \( Y \) by moving \( \tau_j \) up to \( \sigma_i + e \). Then \( \tau_1 = \tau_2 = \tau_2 \) or \( \sigma_i + e \), and \( A'_{\tau_1} \rightarrow A_{\tau_1} \). On the other hand, \( A'_{\tau_1} \rightarrow A_{\tau_2} \), since \( \tau_1 + e \) and \( \tau_1 \) are the largest elements in \( h(A_{\tau_2}) \) and \( h(A_{\tau_1}) \), respectively. Similarly, \( A'_{\sigma_1} \rightarrow A_{\sigma_1} \), since \( \tau_1 \) and \( \tau_1 - e \) are the largest elements in \( l(A'_{\tau_2}) \) and \( l(A_{\sigma_1}) \), respectively. Let \( \theta = \chi_{\tau_1}' + \chi_{\tau_2}' \). Then \( R^G_{L}(\theta) \) contains \( \chi_{\tau_1} \), but not \( \chi_{\sigma_1} \). This contradicts (6.1), so the claim holds and

\[
\{ \sigma_1, \sigma_2, \ldots, \sigma_e \} \subseteq \{ 0, 1, 2, \ldots, e \}, \quad \{ \tau_2, \tau_3, \ldots, \tau_e \} = \{ 1, 2, \ldots, e - 1 \}. \tag{6.2}
\]

Since \( \tau_1 > \sigma_i \) for all \( i \), it follows by (3F) that

\[
\frac{\chi_{\sigma_i}(1)}{\chi_{\tau_i}(1)} = q^{\tau_i} \prod_{i=1}^{e-1} \frac{q^i(q^{\tau_i+1})}{q^i + 1} \prod_{i=1}^{e-1} \frac{q^{\sigma_i(q^{\tau_i+1})+1}}{q^{\sigma_i+1} + 1} > 1.
\]

This contradicts (6.1), so (6B) holds in case (g).

(h) Suppose \( \bullet \) is not an edge in the tree. Then \( \sigma_{u-1} \geq k \) by (e), and \( \sigma_{u-1}, \sigma_u \) are linked by \( \mathcal{E}_{\sigma_{u-1}} \) to \( \tau \)'s with \( \tau > k - e \) by (c). Since \( e \geq 3 \), \( \bullet \) or \( \bullet \) is an edge in the tree. So it is not the
case that \(\sigma_{u-1}, \sigma_u\) are both linked to additional \(\tau\)'s with \(\tau \leq k - e\). We shall use this fact to rule out three configurations.

Suppose there exist \(i \leq u - 1\) and \(j\) such that \(\tau_j > \sigma_i + e\). We may choose \(i\) so that \(\tau_j < \sigma_{i-1} + e\). Let \(A' = \{X', Y'\}\), where \(X' = X\), and \(Y'\) is gotten from \(Y\) by moving \(\tau_i\) up to \(\sigma_i + e\). Then \(\sigma'_i = \sigma_i\) for \(i \neq \alpha\), \(\sigma'_i = \tau_j - e\), \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\), \(\Lambda'_{\sigma'_i-1} \rightarrow \Lambda_{\sigma_i-1}\). On the other hand, \(\Lambda'_{\sigma'_i+e} \rightarrow \Lambda_{\sigma_i+e}\) for \(\alpha > \beta\). Similarly, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\) for \(\alpha < \beta\). Finally, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\tau_j}\) for \(\tau_j > k - e\) by \(I_{k-e}\). Let \(\theta = \chi_{\sigma_{u-1}}^{+} + \chi_{\sigma_u}^{-}\). The only characters of \(B\) in \(R^{'\theta}(\theta)\) are \(\chi_{\sigma_{u-1}}, \chi_{\sigma_u}\), and \(\chi_{\tau_j}\)'s with \(\tau_j \leq k - e\). This is impossible by the preceding paragraph.

Suppose there exist \(j < u\) and \(i\) such that \(\tau_j > \sigma_i + e\). We may choose \(i\) so that \(\sigma_{i+1} < \tau_j + e\). Let \(A' = \{X', Y'\}\), where \(X'\) is gotten from \(X\) by moving \(\sigma_i\) up to \(\tau_j + e\), and \(Y' = Y\). Then \(\sigma'_i = \sigma_i\) for \(i \neq \alpha\), \(\sigma'_i = \tau_j + e\), \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\), \(\Lambda'_{\sigma'_i-1} \rightarrow \Lambda_{\sigma_i-1}\). On the other hand, \(\Lambda'_{\sigma'_i+e} \rightarrow \Lambda_{\sigma_i+e}\) for \(\alpha > \beta\) by \(I_{\sigma_i+e}\). Similarly, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\) for \(\alpha < \beta\) by \(I_{\sigma_i}\). Finally, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\tau_j}\) for \(\tau_j > k - e\) by \(I_{k-e}\). As before, this is impossible.

Suppose \(u < e\) and \(\sigma_{u+1} + e > \sigma_u\). Then \(k - e < \tau_{u+1} + e < k\) since \(\sigma_{u+1} = \tau_{u+1}\). Let \(A' = \{X', Y'\}\), where \(X'\) is gotten from \(X\) by moving \(\sigma_{u-1}\) up to \(\tau_{u+1} + e\), and \(Y' = Y\). Then \(\sigma'_i = \sigma_i\) for \(i \neq u - 1\), \(\sigma'_{u-1} = \tau_{u+1} + e\), \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\), \(\Lambda'_{\sigma'_{u-1}} \rightarrow \Lambda_{\sigma_{u-1}}\). On the other hand, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\sigma_i}\) for \(\alpha > \beta\) by \(I_{\sigma_i}\). Similarly, \(\Lambda'_{\sigma'_{u-1}} \rightarrow \Lambda_{\sigma_{u-1}}\) for \(\alpha < \beta\) by \(I_{\sigma_i}\). Finally, \(\Lambda'_{\sigma'_i} \rightarrow \Lambda_{\tau_j}\) for \(\tau_j > k - e\) by \(I_{k-e}\). As before, this is impossible.

(i) Suppose \(\sigma_{u-1} \geq k\). Suppose \(u < e\), so \(\sigma_{u+1} + e < \sigma_u\) by (h). Since \(\sigma_{u+1} = \tau_{u+1}\), it follows \(\tau_{u+1} + e < k - e\). Let \(A' = \{X', Y'\}\), where \(X' = X\), and \(Y'\) is gotten from \(Y\) by moving \(\tau_{u+1}\) down to \(\sigma_{u+1} + e\). Then \(A'\) is a symbol of defect 0 for an overgroup \(G'\) of \(G\), and \(A'\) has no \(e\)-hooks. With obvious notation \(\sigma' = \sigma\) for \(\alpha \neq u + 1\), \(\sigma'_u = \tau_{u+1}\), \(\tau'_j = \tau_j\) for \(j \neq u\), \(\tau_{u+1} + e\) is a subgraph of the tree of \(B'\) labeled by \(A'\). Then

\[
\tau_1 \ldots \tau_u = \tau_{u+1} (6.3)
\]

is a subgraph of the tree of \(B'\) by (f). Indeed, \(\sigma'_{u+1} = k' - e, \sigma_u > k'\), so that \(\tau'_j\) and \(\tau'_{j+1}\) are linked by \(\Xi_{\tau'_j}\) for \(1 \leq j \leq u\).
Now \( A_{\sigma} \to A'_{\sigma} \). But \( A_{\sigma} \to A'_{\sigma} \) for \( \alpha > \beta \) by \( I_{k+e} h \), \( A_{\sigma} \to A'_{\sigma} \) for \( \alpha < \beta \) by \( I_{k-e} l \), \( A_{\sigma} \to A'_{\sigma} \) for \( \tau_i > k-e \) by \( I_{k-e} l \), and \( A_{\sigma} \to A'_{\sigma} \) for \( \tau_i \leq k-e \) by \( I_{k+e} h \). Also \( A_{\sigma_{u-1}} \to A'_{\sigma_{u-1}} \). But \( A_{\sigma_{u-1}} \to A'_{\sigma_{u-1}} \) by \( I_k h \), \( A_{\sigma_{u-1}} \to A'_{\sigma_{u}} \) for \( \alpha > u \) by \( I_{k+e} h \), \( A_{\sigma_{u-1}} \to A'_{\sigma_{u}} \) for any \( j \) by \( I_k h \).

Let \( \tau_i > k-e \). Then \( A_{\tau_i} \to A'_{\tau_i} \). But \( A_{\tau_i} \to A'_{\tau_i} \) for \( \tau_{i+1} > \tau_i \) by \( I_{k} h \). Also \( A_{\tau_i \tau_i} \to A'_{\tau_i \tau_i} \) for any \( \sigma_i \). Indeed, \( A_{\tau_i} \to A'_{\tau_i \tau_i} \) by \( I_{k+e} h \), \( A_{\tau_i} \to A'_{\tau_i} \) for \( \tau_i \leq u \) by \( I_{k} h \), \( A_{\tau_i} \to A'_{\tau_i \tau_i} \) by \( I_k h \) if \( \sigma_{u-1} > k \). The final cases, \( A_{\tau_i} \to A'_{\tau_i \tau_i} \) if \( \sigma_{u-1} = k \), and \( A_{\tau_i} \to A'_{\tau_i} \) require a closer look at the rows of the symbols. In the first case, we must get \( h(A_{\tau_i \tau_i-1}) \) from \( h(A_{\tau_i}) \) by adding at most \( e \) linear 1-hooks. In the second case, we must get \( h(A'_{\tau_i}) \) from \( h(A_{\tau_i \tau_i-1}) \) by adding altogether \( e \) linear 1-hooks. Both cases are impossible.

We now consider the projective character \( R^\theta_{G'}(\theta) \), where \( L \) is an appropriate Levi subgroup of \( G' \) with \( G \) as one component and 1-dimensional tori as the remaining components, \( \theta = \chi_{\sigma_{u}} + \chi_{r_{\sigma}} \) or \( \chi_{\sigma_{u-1}} + \chi_{r_{\sigma}} \). Here \( \chi \neq \beta \), \( \tau_{\alpha} \) and \( \tau_{\beta} > k-e \), \( \alpha \) and \( \beta < u+1 \), and the \( \chi \)'s denote the extensions of the corresponding characters of \( G \) to \( L \) by trivial extensions to the 1-dimensional tori. Then \( R^\theta_{G'}(\theta) \) contains projective indecomposable characters of \( B' \) of the form \( \chi'_{\sigma_{u}} + \chi'_{r_{\sigma}} \) and \( \chi'_{\sigma_{u-1}} + \chi'_{r_{\sigma}} \) for some \( i \leq u-1 \). But \( \tau_{r_{\sigma}} \) or \( \tau_{r_{\beta}} \) is already linked to two distinct \( \tau \)'s by \((6.3)\). This is impossible, and thus \( u = e \).

(j) Suppose \( \sigma_{u-1} \) is not an edge of the tree. By \((6, i)\)

\[
 u = e, \quad \tau_j = \sigma_{e-1} + \tau_{e} \quad \text{for all } j, \quad \sigma_i < \tau_{e} + \tau_{e} \quad \text{for all } i. \quad \text{(6.4)}
\]

We may assume without loss of generality that \( k = e \), so that \( 0 = \sigma_{e} < \tau_{e} \), and \( \tau_j > 0 \) for all \( j \). We claim \( \bullet \cdots \bullet \cdots \bullet \cdots \) is a subgraph of the tree. If \( \sigma_{e-1} > e \), this holds by \((f)\). So \( \sigma_{e-1} = e \) by \((e)\). Thus \( X = [0, 2e-2], \ Y = [0, 2e-1] \setminus \{e\} \) by \((6.4)\), and

\[
 \begin{align*}
 \sigma_1 &= 2e-2, & \sigma_2 &= 2e-3, \ldots, & \sigma_{e-1} &= e, & \sigma_e &= 0, \\
 \tau_1 &= 2e-1, & \tau_2 &= 2e-2, \ldots, & \tau_{e-1} &= e+1, & \tau_e &= e-1.
\end{align*}
\]

In particular, \( I_{\tau_2}(X) > I_{\tau_2}(Y) \), so that

\[
 \bullet \cdots \bullet \cdots \bullet \cdots
\]

is a subgraph of the tree by \((c)\). Let \( A' = \{X', Y'\} \), where \( X' = X \), and \( Y' \) is gotten from \( Y \) by moving \( \tau_{e-1} \) one space to \( e \). Then \( \sigma'_{\alpha} = \sigma_{\alpha} \) for \( \alpha \neq e \), \( \sigma'_{e} = 1 \), \( \tau'_{\beta} = \tau_{\beta} \) for \( \beta = e-1 \), \( \tau'_{e-1} = e \). Now \( A'_{\sigma_{u}} \to A_{\sigma_{u}} \), \( A'_{\sigma_{u-1}} \to A_{\sigma_{u-1}} \). On the other hand, \( A_{\sigma_{u}} \to A_{\sigma_{u}} \) for \( i < e \) by \( I_{k} l \), \( A_{\sigma_{u}} \to A_{\sigma_{u}} \) for any \( j \) by \( I_{k} l \).

Similarly, \( A'_{\sigma_{u-1}} \to A_{\sigma_{u}} \) for \( i < e-1 \) by \( I_{k} l \), \( A'_{\sigma_{u-1}} \to A_{\sigma_{u}} \) for \( j = 1, 2 \) since \( A_{\tau_1} \), \( A_{\tau_2} \) cannot be gotten by adding a single linear 1-hook to \( A'_{\sigma_{u-1}} \). Let
Then $R_\lambda^G(\theta)$ contains projective indecomposable characters of the form $\chi_{\sigma_{e-1}} + \chi_{\tau}$ and $\chi_{\sigma_e} + \chi_{\sigma}$, where $\alpha, \beta \geq 3$. If $\sigma_{t_1}$ fails to link $\tau_1$ and $\tau_2$, then $\tau_1, \tau_2$ are linked to $\sigma_{e-1}$ or $\sigma_e$ by (f). This is impossible since $\sigma_{e-1}$ and $\sigma_{e-2}$ are also linked. This proves the claim. By (3G) the tree of $B$ is

$$\sigma_1 \cdots \sigma_{e-2} \sigma_{e-1} \tau_1 \tau_2 \sigma_e$$

(6.5)

We show (6.4) and (6.5) lead to a contradiction. Now $t_e = \sigma_{e-1}$ or $t_e = \sigma_{e-1} - 1$. Indeed, the interval $(\sigma_{e-1}, \sigma_e + e)$ contains $\sigma_1, \sigma_2, \ldots, \sigma_{e-2}$ so that $t_e \geq \sigma_{e-1} - 1$, while the interval $(\tau_e, \sigma_{e-1} + e)$ contains $\tau_1, \tau_2, \ldots, \tau_{e-1}$ so that $t_e \leq \sigma_{e-1}$.

Suppose $t_e = \sigma_{e-1}$, so that $t_e > e$. Let $A = \{X', Y'\}$, where $X' = X$ and $Y'$ is gotten from $Y$ by moving $t_e$ up to $e$. Then $t_e = e$ and $A_{t_e} \rightarrow A_{t_e}$. On the other hand, $A_{t_e} \rightarrow A_{t_j}$ for $j < e$ by $I_{t_e}l$, and $A_{t_e} \rightarrow A_{t_e}$ by $I_{t_e}l$. Let $\theta = \chi_{t_e} + \sum \chi_{\text{exc}}$, where $\chi_{\text{exc}}$ runs over the exceptional characters of the cyclic block containing $\chi_{t_e}$. By induction $\theta$ is projective. Since $R_\lambda^G(\theta)$ contains $\chi_{t_e}$, but not $\chi_{t_e-1}$, nor $\chi_{t_{e-1}}$, this contradicts (6.1).

Suppose $t_e = \sigma_{e-1} - 1$. The preceding argument applies if $t_e > e$. Hence we may suppose $t_e < e$, and then $\sigma_{e-1} = e$, $\sigma_e = e - 1$, $\sigma_{e-j} = e + j$ for $j \geq 1$ by (e), (6.4). Let $A' = \{X', Y'\}$, where $X' = X$ and $Y'$ is gotten from $Y$ by moving $t_{e-1}$ up to $e$. Then $t_{e-1} = e - 1$ and $A_{t_{e-1}} \rightarrow A_{t_{e-1}}$. On the other hand, $A_{t_{e-1}} \rightarrow A_{t_j}$ for $j < e$ by $I_{t_{e-1}}l$, and $A_{t_{e-1}} \rightarrow A_{t_{e-1}}$ by $I_{t_{e-1}}l$. This contradicts (6.1) as before, and completes the proof of (6B).

7. Unipotent Blocks, Part III

We continue with the hypothesis and notation of Section 6.

(7A) Suppose $e \geq 3$ and $\chi_{\sigma_1}, \chi_{\tau}$ are in different Harish-Chandra families. Then the Brauer tree of $B$ contains the subgraph $\bullet \longrightarrow \bullet$.

\textbf{Proof.} (a) Suppose there exist pairs $(\sigma_i, \tau_j)$ such that $\sigma_i > \tau_j + e$. Choose one with $i$ maximal, and among pairs with this $i$, choose one with $j$ minimal. Let $A = \{X', Y'\}$, where $X'$ is gotten from $X$ by moving $\sigma_i$ up to $\tau_j + e$, and $Y' = Y$. Then $\sigma_i = \sigma_s < \tau_j + e$ if $i < s$, and $\sigma_i = \tau_j + e$ if $i = s$. In either case, $A_{\sigma_i} \rightarrow A_{\sigma_i}$, but $A_{\tau_j} \rightarrow A_{\tau_j}$ for $a < s$ by $I_{\sigma_i}l$. Let $\theta = \chi_{\sigma_i} + \sum \chi_{\text{exc}}$, where $\chi_{\text{exc}}$ runs over the exceptional characters in the cyclic block of $B$ containing $\chi_{\sigma_i}$. Since $\chi_{\sigma_i}$ is the only non-exceptional character of $B$ contained in $R_\lambda^G(\theta)$, $\sigma_i$ and $\sigma$ are linked.

(b) Suppose there exist pairs $(\sigma_i, \tau_j)$ such that $\tau_j > \sigma_i + e$. Choose one with $j$ maximal and among pairs with this $j$, choose one with $i$ minimal. Let $A' = \{X', Y'\}$, where $X' = X$ and $Y'$ is gotten from $Y$ by moving $\tau_j$ up to $\sigma_i + e$.
\( \sigma_i + e \). Then \( \sigma_i' = \sigma_i \) if \( i < s \), and \( \sigma_i' = \tau_j - e < \sigma_{i-1} = \sigma_{i-1} \) if \( i = s \). In either case \( A_{\sigma_i}' \to A_{\sigma_i} \). But \( A_{\sigma_i}' \not\to A_{\sigma_i} \) for \( \sigma < s \) by \( I_{\sigma, l} \). This implies as before that \( \sigma_s \) and \( \tilde{\sigma} \) are linked.

(c) The roles of \( \sigma \) and \( \tau \) can be interchanged in (a, b), so we may assume that \( \sigma_1 < \tau_i + e, \tau_i < \sigma_s + e \) in proving (7A). Thus \( \tau_i + e - \sigma_s \geq s \delta \) since \( \sigma_s < \tau_i + e \) for all \( i \), and similarly \( \sigma_s + e - \tau_i \geq t \delta \) since \( \tau_i < \sigma_s + e \) for all \( j \). These inequalities and the relation \( 2e = (s + t) \delta \) imply

\[
\begin{align*}
\delta \sigma_i &= 2e - t \delta \geq \tau_i + e - \sigma_s \geq s \delta, \\
\tau_j &= 2e - s \delta \geq \sigma_s + e - \tau_j \geq t \delta.
\end{align*}
\]

Thus the inequalities are equalities, and

\[
\begin{align*}
\sigma_i &= \sigma_1 - (i - 1) \delta & \text{for } 1 \leq i \leq s, \\
\tau_j &= \tau_1 - (j - 1) \delta & \text{for } 1 \leq j \leq t.
\end{align*}
\]

(7.1)

In particular, beads come before spaces on each runner of the abacus diagram of \( \Lambda \).

(d) Suppose \( V \) is a unitary space. Then \( \kappa \) is the triangular partition \( \{0, 1, ..., d\} \) represented by the \( \beta \)-set \( Z = [0, e - 1] \cup \{e + 2i - 1: 1 \leq i \leq d\} \), \( s = \frac{1}{2}(e + 1) + d \), \( t = \frac{1}{2}(e - 1) - d \), and

\[
\begin{align*}
\sigma_{s-1} &= 2i, \\
\tau_{t-1} &= 2d + (2j + 1)
\end{align*}
\]

(7.2)

for \( 0 \leq i \leq s - 1 \), \( 0 \leq j \leq t - 1 \). The following inequalities hold:

\[
\begin{align*}
\chi_{\sigma_s}(1) &> q \chi_{\sigma_{s-1}}(1) \quad (7.3) \\
\chi_{\tau_t}(1) &> q \chi_{\tau_{t-1}}(1) \quad \text{if } t \geq 2 \quad (7.4) \\
\chi_{\tau_t}(1) &> \chi_{\sigma_t}(1) \quad \text{if } t \geq 1 \quad (7.5) \\
\chi_{\tau_{t-1}}(1) &> \chi_{\sigma_t}(1) \quad \text{if } t \geq 2 \quad (7.6) \\
\chi_{\tau_{t-1}}(1) &> \chi_{\sigma_t}(1) \quad \text{if } t \geq 1. \quad (7.7)
\end{align*}
\]

Indeed, (7.2) and (3C) imply

\[
\begin{align*}
\frac{\chi_{\sigma_s}(1)}{\chi_{\sigma_{s-1}}(1)} &= q^{2e} - 2 \frac{(q^{2d} - 1) + (q^2 - 1)}{(q^{2d} - 1) + (q^2 - 1)}, \\
\frac{\chi_{\tau_t}(1)}{\chi_{\tau_{t-1}}(1)} &= q^{2e} - 2d - 5 \frac{(q^{2d+3} + 1)(q^2 - 1)}{(q^2 - 1) + (q^{2d+3} - 1)}.
\end{align*}
\]
Inequality (7.3) is equivalent to each of the following inequalities:

\[ q^{2e-3}(q^{2d+1} - 1)(q^2 - 1) > (q^{e+2d-1} - 1)(q^{e-2} + 1), \]
\[ q^{2e-3}(q^{2d+1} - 1)(q^2 - 1) > q^{2e+2d-3} + q^{e+2d-1} - q^{e-2} - 1, \]
\[ q^{2e+2d-4}(q^2 - 1) + q^{e-2}(q^{e+1} - q^{e-1} + 1) > q^{e+2d-1} - 1, \]

and the last inequality holds since \( 2e + 2d - 4 \geq e + 2d - 1 \). Inequality (7.4) is equivalent to each of the following inequalities:

\[ q^{2e-2d-6}(q^{2d+3} + 1)(q^2 - 1) > (q^{e-2} + 1)(q^{e-2d-3} - 1), \]
\[ q^{2e-2d-6}(q^{2d+3} + 1)(q^2 - 1) > q^{2e-2d-6} + q^{e-2d-5} - q^{e-2d-6} - 1, \]
\[ q^{2e-3}(q^2 - 1) + q^{2e-2d-6}(q^2 - q - 1) > q^{e-2d-3} - q^{e-2} - 1, \]

and the last inequality holds since \( q^{2e-3}(q^2 - 1) > q^{e-2d-3} \). Equations (7.2) and (3C) imply

\[
\frac{\chi_{t1}(1)}{\chi_{01}(1)} = \frac{q^M}{q^{2d+1} + 1} \prod_{i = (e - 2d - 1)/2}^{(e + 2d - 1)/2} (q^{2i} - 1) \prod_{i = 0}^{d} \frac{1}{(q^{2i+1} + 1)^2},
\]
\[
\frac{\chi_{t,-1}(1)}{\chi_{01}(1)} = \frac{q^M - 2e + 2d + 5(q^{e-2} + 1)}{(q^2 - 1)(q^{2d+1} + 1)(q^{2d+3} + 1)}
\times \prod_{i = (e - 2d - 3)/2}^{(e + 2d - 1)/2} (q^{2i} - 1) \prod_{i = 0}^{d-1} \frac{1}{(q^{2i+1} + 1)^2},
\]

where \( M = \frac{1}{2}(e - 2d - 3)(e - 2d - 1) + \frac{1}{2}(e - 1)^2 \). Inequalities (7.5) and (7.6) are clear if \( d = 0 \). We may suppose \( d \geq 1 \). Since \( e = 2t + 2d + 1 \), it follows that \( e \geq 2d + 3 \) if \( t \geq 1 \), and \( e \geq 2d + 5 \) if \( t \geq 2 \). We rewrite

\[
\frac{\chi_{t1}(1)}{\chi_{01}(1)} = q^M \frac{(q^{e+1} - 1)(q^{e+3} - 1) \ldots (q^{e+2d-1} - 1)}{(q + 1)^2 (q^3 + 1)^2 \ldots (q^{2d-1} + 1)^2}
\times \frac{(q^{e-1} - 1)(q^{e-3} - 1) \ldots (q^{e-2d-3} - 1)}{q^{2d+1} + 1}.
\]

Here \( q^{e+i} - 1 > (q^i + 1)^2 \) for \( i = 1, 3, \ldots, 2d - 1 \), since

\[ q^{e+i} - q^{2i} - 2q^i = q^i(q^e - q^i - 2) \geq q^i(q^{i+4} - q^i - 2). \]

Also \( q^{e-1} - 1 \geq q^{2d+1} + 1 \), as \( q^{e-1} - q^{2d+1} = q^{2d+1}(q^{e-2d-2} - 1) \geq 2 \). Thus \( \chi_{t1}(1)/\chi_{01}(1) \geq q^M \geq 1 \) and (7.5) holds. Similarly, we rewrite

\[
\frac{\chi_{t,-1}(1)}{\chi_{01}(1)} = q^M - 2e + 2d + 5 \frac{(q^{e+1} - 1)(q^{e+3} - 1) \ldots (q^{e+2d-1} - 1)}{(q + 1)^2 (q^3 + 1)^2 \ldots (q^{2d-1} + 1)^2}
\times \frac{(q^{e-1} - 1)(q^{e-3} - 1) \ldots (q^{e-2d-3} - 1)}{(q^{2d+3} + 1)(q^2 - 1)} \frac{q^{e-2} + 1}{q^{2d+1} + 1}.
\]
Here \( e \geq 2d + 5 \) so that \( q^{e-2d-3} - 1 \geq q^2 - 1, q^{e-2} + 1 > q^{2d+1} + 1 \). Also, \( q^{2d+3}(q^{e-2d-4} - 1) > 2 \), so that \( q^{e-1} - 1 > q^{2d+3} + 1 \). We have already noted \( q^{e+i} - 1 > (q^i + 1)^2 \) for \( i = 1, 3, \ldots, 2d - 1 \). But for \( i = 1 \), the stronger inequality \( q^{e+1} - 1 > q^{e-3}(q + 1)^2 \) holds, since \( q^{e+1} - 1 > q^{e+1} - q^{e-3} = q^{e-3}(q^4 - 1) > q^{e-3}(q + 1)^2 \). Thus

\[
\frac{\chi_{\tau_{i-1}}(1)}{\chi_{\sigma_i}(1)} > q^{M - e + 2d + 2(q^{e-3} - 1)(q^{e-5} - 1) \cdots (q^{e-2d-1} - 1)}.
\]

But \( M \geq 1, q^{e-3} - 1 > q^{e-4} \), so that

\[
\frac{\chi_{\tau_{i-1}}(1)}{\chi_{\sigma_i}(1)} > q^{1 - e + 2d + 2q^{e-4}} = q^{2d - 1} \geq 1,
\]

and (7.6) holds. Finally, (7.2) and (3C) imply

\[
\frac{\chi_{\tau_i}(1)}{\chi_{\text{exc}}(1)} = \prod_{i=0}^{d} \frac{1}{q^{2i+1} + 1} \prod_{i=1}^{(e-2d-3)/2} \left( \frac{q^{2i-1}}{q^{2i-1} + 1}, \frac{q^{2i}}{q^{2i} - 1} \right) \times \prod_{i=(e-2d-3)/2}^{(e-3)/2} \frac{1}{q^{2i+1} + 1},
\]

so (7.7) holds, since \( q^{2i-1}q^{2i} < (q^{2i-1} + 1)(q^{2i} - 1) \).

(3D) implies that \( \sigma_i \) and \( \tau_i \) are not linked. If \( \circ \) separates the vertices labeled by \( \sigma_i \)'s and \( \tau_i \)'s, or if \( t = 0 \), then (7A) holds by (7.3) and (7.4). So we may suppose \( t > 0 \) and \( \circ \) is one end of the tree. If the other end is \( \tau_t \), then \( t = 1 \) by (7.4). This is impossible, since \( \tau_t \) is not linked to \( \sigma_t \) by (7.5). If the other end is \( \tau_{t_1} \), then \( t \geq 2 \) and \( \bullet \cdots \bullet \) is a subgraph of the tree. This is impossible since

\[
\chi_{\tau_t}(1) > q\chi_{\tau_{t-1}}(1) > \chi_{\tau_{t-1}}(1) + \chi_{\sigma_t}(1)
\]

by (7.4) and (7.6). So the other end is necessarily \( \sigma_1 \) by (7.3), and \( \tau_t \) is linked to \( \circ \). This too is impossible by (7.7). Thus (7A) holds when \( V \) is a unitary space.

(e) Suppose \( V \) is a symplectic or an orthogonal space. By (7.1) we may suppose that \( A = \{ X, Y \} \), where \( X = [0, e + d - 1], Y = [0, e - 1] \), and \( d \geq 1 \). In particular,

\[
\{ \sigma_y, \sigma_{y-1}, \ldots, \sigma_1 \} = \{ 0, 1, 2, \ldots, e + d - 1 \},
\]

\[
\{ \tau_t, \tau_{t-1}, \ldots, \tau_1 \} = \{ d, d + 1, \ldots, e - 1 \},
\]
s = e + d, \ t = e - d. The following inequalities hold:

\[ \chi_{\sigma_1}(1) > \chi_{\sigma_{t-1}}(1) \]  
\[ \chi_{\tau_1}(1) > q \chi_{\tau_{t-1}}(1) \quad \text{if} \ t \geq 2 \]  
\[ \chi_{\text{exc}}(1) > \chi_{\tau_1}(1) \quad \text{if} \ t \geq 1 \]  
\[ \chi_{\tau_{t-1}}(1) > \chi_{\sigma_1}(1) \quad \text{if} \ t \geq 2. \]  

Indeed, (3F) implies

\[ \frac{\chi_{\sigma_1}(1)}{\chi_{\sigma_{t-1}}(1)} = q^{2e-1} \frac{(q^{d-1} + 1)(q - 1)}{(q^{e+d-1} - 1)(q^{e-1} + 1)} \]
\[ \frac{\chi_{\tau_1}(1)}{\chi_{\tau_{t-1}}(1)} = q^{2e-d-2} \frac{(q^{d+1} + 1)(q - 1)}{(q^{e-1} + 1)(q^{e-d-1} - 1)}. \]

Now (7.8) is equivalent to each of the following inequalities:

\[ q^{2e-1}(q^{d-1} + 1)(q - 1) > (q^{e+d-1} - 1)(q^{e-1} + 1), \]
\[ q^{2e-1}(q^{d-1} + 1 + q - 1) > q^{2e+d-2} + q^{e+d-1} - q^{e-1} - 1, \]
\[ q^{2e+d-2}(q - 2) + q^{2e} - q^{2e-1} - q^{e+d-1} > -q^{e-1} - 1. \]

Since \( e - d - t \geq 0 \) so that \( q^{2e} - q^{2e-1} - q^{e+d-1} = q^{2e-1}(q - 1 - q^{d-e}) \geq 0 \), the last inequality holds. Inequality (7.9) is equivalent to each of the following inequalities:

\[ q^{2e-d-3}(q^{d+1} + 1)(q - 1) > (q^{e+1} + 1)(q^{e-d-1} - 1), \]
\[ q^{2e-d-3}(q^{d+2} - q^{d+1} + q - 1) > q^{2e-d-2} + q^{e-d-1} - q^{e-1} - 1, \]
\[ q^{2e-d-3}(q^{d+2} - q^{d+1} - 1) > -(q^{e-1} - q^{e-d-1}) - 1. \]

The last inequality holds, since the right-hand side is negative. (3F) implies

\[ \frac{\chi_{\tau_1}(1)}{\chi_{\text{exc}}(1)} = \prod_{i=1}^{e-d-1} \frac{q^i}{q^i - 1} \prod_{i=1}^{d} \frac{1}{q^i + 1} \prod_{i=0}^{e-1} \frac{q^i}{q^i + 1} \]
\[ = \prod_{i=1}^{e-d-1} \frac{q^i}{q^i - 1} \frac{q^i - 1}{q^i + 1}. \]

Thus (7.10) holds, since \( e - d - 1 < e - 1 \) and

\[ \frac{q^i}{q^i - 1} \frac{q^i - 1}{q^i + 1} = \frac{q^{2i-1}}{q^{2i-1} + q^i - q^{i-1} - 1} < 1 \]
for $1 \leq i \leq e - d - 1$. Finally, (3F) implies

$$\frac{\chi_{s_r - 1}}{\chi_{s_1}(1)} = \frac{1}{q - 1} \prod_{i=1}^{d+1} \frac{q^i - 1}{q^i + 1} \prod_{i=d}^{e-2} \frac{q^i}{q^i + 1} \prod_{i=d}^{e-1} (q^i - 1)$$

$$\times \prod_{i=d}^{e-2} \frac{q^i}{q^i + 1} \prod_{i=d}^{e-1} (q^i - 1) \prod_{i=d}^{e-1} \frac{q^i}{q^i + 1} \prod_{i=d}^{e-1} (q^i - 1)$$

$$> \frac{q^{e-d-1} + 1}{q - 1} \prod_{i=d+1}^{e} \frac{q^i}{q^i + 1} \prod_{i=d+1}^{e-1} (q^i - 1) \prod_{i=d}^{e-1} \frac{q^i}{q^i + 1} \prod_{i=d}^{e-1} (q^i - 1)$$

since $q^{e-d-1} + 1 > q^{d+1}$. Now $q^{e-d-1} + 1 > q^i + 1$ for $1 \leq i \leq d + 1$ since $e - d \geq 2$. Thus

$$\prod_{i=d+1}^{e} \frac{q^i}{q^i + 1} \prod_{i=d+1}^{e-1} (q^i - 1) \prod_{i=d}^{e-1} \frac{q^i}{q^i + 1} \prod_{i=d}^{e-1} (q^i - 1)$$

and (7.11) holds.

Now $s \equiv t \pmod{2}$ since $s + t = 2e$. If $s$ and $t$ are odd, then

$$\frac{\chi_{s_1}}{\chi_{s_1}(1)} = \frac{\chi_{s_1}}{\chi_{s_1}(1)} = 1 \pmod{r}$$

by (3G). Thus $\circ$ separates the vertices labeled by $\sigma$'s and $\tau$'s, and (7A) follows from (7.8), (7.9). If $s$ and $t$ are even, then

$$\frac{\chi_{\sigma_1}}{\chi_{\sigma_1}(1)} = \frac{\chi_{\tau_1}}{\chi_{\tau_1}(1)} = -1 \pmod{r}.$$
Suppose $e \geq 3$ and $\chi_{\sigma}, \chi_{\tau}$ are in the same Harish-Chandra family. Then the Brauer tree of $B$ contains the subgraph $\sigma_e \leftrightsquigarrow \tau_e$.

**Proof.** By hypothesis, $V$ is a symplectic or an orthogonal space, $d = 0$, so that $\delta = 1$, $s = t = e$. We may suppose $0 = \sigma_e \leq \tau_e$.

(a) We first show that $\sigma_e$ and $\circ$ are linked. Suppose there exists $\tau_j$ with $\tau_j > \sigma_{e-1} + e$. Let $A' = \{X', Y'\}$, where $X' = X$, and $Y'$ is gotten from $Y$ by moving $\tau_j$ up to $\sigma_{e-1} + e$. Then $\sigma_e = \sigma_e$, and $A'_{\sigma_e} \leftrightarrow A_{\sigma_e}$. But $A'_{\sigma_e} \leftrightarrow A_{\sigma_e}$ for $i < e$ by $I_0l, A'_{\sigma_e} \leftrightarrow A_{\tau_j}$ by $I_0l$. Let $\theta = \chi'_{\sigma_e} + \sum \chi'_{\text{exc}}$, where $\chi'_{\text{exc}}$ runs over the exceptional characters in the cyclic block of $L$ containing $\chi'_{\sigma_e}$.

(b) Suppose there exists $\sigma_i$ with $\sigma_i > \sigma_{e+1} + e$. Let $A' = \{X', Y'\}$, where $X'$ is gotten from $X$ by moving $\sigma_i$ up to $\tau_{e+1} + e$, and $Y' = Y$. The preceding argument implies $\sigma_{e}$ is linked to $\circ$. So we may suppose $\sigma_i \leq \tau_{e+1} + e$ for all $i$.

(c) From (a) it follows that $\sigma_{e-1} + e - \tau_e \geq e$ since $\tau_1, \tau_2, ..., \tau_e$ are in $[\tau_e, \sigma_{e-1} + e)$. From (b) it follows $\tau_{e+1} + e - \sigma_{e+1} \geq e - 1$ since $\sigma_1, \sigma_2, ..., \sigma_{e-1}$ are in $[\sigma_{e-1}, \tau_e + e)$. Thus $1 \geq \sigma_{e-1} - \tau_{e+1} \geq 0$, and either $\tau_e = \sigma_{e-1} - 1$ or $\tau_e = \sigma_{e-1} - 1$.

Suppose $\tau_e = \sigma_{e-1} - 1$. Then

\[
\begin{align*}
\{\tau_e, \tau_{e-1}, ..., \tau_1\} &= \{a, a+1, a+2, ..., a+e-1\}, \\
\{\sigma_e, \sigma_{e-1}, ..., \sigma_1\} &= \{0, a, a+1, a+2, ..., a+e-1\} \setminus \{a+u\},
\end{align*}
\]

where $1 \leq u \leq e - 1$. Moreover, $a > e$ since $e \notin Y$. By (3F)

\[
\frac{\chi_{\sigma_e}(1)}{\chi_{\sigma_{e-1}}(1)} = q^a \prod_{i=1, i \neq u}^{e-1} \frac{q^a(q^i - 1)^{e-1-i} q^a(q^i + 1)}{q^{a+i-1}} > q^{e+1} \prod_{i=1, i \neq u}^{e-1} \frac{q^a(q^i - 1)}{q^{a+i-1}} = q^3,
\]

since $q[q^a(q^i - 1)/(q^{a+i-1})] > 1$ for $i \geq 1$. Also by (3F)

\[
\frac{\chi_{\tau_1}(1)}{\chi_{\tau_{e-1}}(1)} = \frac{q^a(q^{a+e-1} + 1)^{e-1-i} q^{a+i}(q^i - 1)}{q^a + 1} \prod_{i=1}^{e-1} \frac{q^{a+i}(q^{a+e-1-i} + 1)}{q^{a+i-1}} > q,
\]

since

\[
\frac{q^{a+e-1} + 1}{q^a + 1} > q, \quad \frac{q^a q^{a+1}(q - 1)}{q^{a+1} + 1} > 1.
\]
Equations (7.13) and (7.14) imply that $\sigma_{e}$ is not an end of the tree, and that

$$\tau_e \neq \tau_{e-1} \neq \tau_1$$

is not a subgraph of the tree. Since $\sigma_{e}$ and $\tau_{e}$ are not linked, it follows $\sigma_{e}$ and $\bigcirc$ are linked.

Suppose $\tau_{e} = \sigma_{e-1} - 1 = a$. Then

$$\{\sigma_{e}, \sigma_{e-1}, ..., \sigma_1\} = \{0, a + 1, a + 2, ..., a + e - 1\}$$

$$\{\tau_{e}, \tau_{e-1}, ..., \tau_1\} = \{a, a + 1, a + 2, ..., a + e\} \setminus \{a + w\},$$

where $1 \leq w \leq e$. If $a = 0$, then $w = e$ and $\Lambda$ is degenerate. In this case

$$\frac{\chi_{\sigma_{e}}(1)}{\chi_{\sigma_{e}-1}(1)} = 2 \prod_{i=1}^{e-1} \frac{1}{q^{i-1}} \prod_{i=0}^{e-1} \frac{1}{q^{i} + 1} < 1,$$  \hspace{1cm} (7.16)

so (7B) holds. We may suppose $a > 0$. By (3F)

$$\frac{\chi_{\sigma_{e}}(1)}{\chi_{\sigma_{e}-1}(1)} = \frac{q^a + 1}{q^a} \prod_{i=1}^{e-1} \frac{q^{a+1}(q^{i-1} + 1)}{q^{a+i+1} + 1} > q^a + 1 \prod_{i=1}^{e-2} \frac{q^{a+1}(q^{i-1} + 1)}{q^{a+i+1} + 1} > q \prod_{i=2}^{e-1} \frac{(q^a + (i+1))^2}{(q^a + (i+1))^2},$$

since

$$\frac{q^a + 1}{q^a} > \frac{q^{a+1}(q^{w-1} + 1)}{q^{a+w-1}} \quad \text{and} \quad \frac{2q^{3a+2}(q-1)}{(q^{a+2} + 1)(q^{a+1} + 1)} > 1.$$

But the following inequalities are equivalent for $i \geq 2$:

$$q^{2a+2}(q^i - 1)(q^{i-1} + 1) > (q^{a+i+1} - 1)(q^{a+i} + 1),$$

$$q^{2a+2i+1} + q^{2a+i+2} - q^{a+i+1} - q^{2a+2} > q^{2a+2i+1} + q^{a+i+1} - q^{a+i} - 1,$$

and the last inequality holds, since

$$q^{2a+i+1} \leq \frac{1}{2} q^{2a+i+2}, \quad q^{2a+2} \leq \frac{1}{4} q^{2a+i+2}, \quad q^{a+i+1} \leq \frac{1}{4} q^{2a+i+2}.$$

Thus the first of the following inequalities hold.

$$\chi_{\sigma_{e}}(1) > q \chi_{\sigma_{e}-1}(1) \quad \text{and} \quad \chi_{\sigma_{e}}(1) > q \chi_{\tau_{1}}(1).$$

(7.17)  \hspace{1cm} (7.18)
The second follows from the relations

\[
\frac{\chi_{\sigma_e}(1)}{\chi_{\tau_1}(1)} = q^{a+e} \prod_{i=1}^{e-1} \frac{q^{a+i}(q^{e-i}+1)}{q^{a+i}+1}
\]

if \( w \neq e \)

\[
\frac{\chi_{\sigma_e}(1)}{\chi_{\tau_1}(1)} = q^{a+e} \prod_{i=1}^{e-1} \frac{q^{a+i}(q^{e-i}+1)}{q^{a+i}+1} \prod_{i=0, i \neq w}^{e-2} \frac{q^{a+i}(q^{e-i}+1)}{q^{a+i}+1}
\]

if \( w = e \)

which hold by (3F). As before (7.17), (7.18) imply that \( \sigma_e \) and \( \varnothing \) are linked.

(d) Finally, we show \( \tau_e \) and \( \varnothing \) are linked. We may suppose \( \tau_e > 0 \); otherwise the roles of \( \sigma_e \) and \( \tau_e \) can be interchanged in the preceding argument. In particular, \( e \in X \) and \( \tau_1 > \sigma_e + e = e \). Among all pairs \( (\sigma_i, \tau_i) \) with \( \tau_j > \sigma_j + e \), choose one with \( i \) minimal, and among pairs with this \( i \), choose one with \( j \) maximal. Let \( A' = \{ X', Y' \} \), where \( X' = X \) and \( Y' \) is gotten from \( Y \) by moving \( \tau_j \) up to \( \sigma_j + e \). Then \( \tau_e = \tau_1 \) if \( j < e \), \( \tau_e = \tau_1 + e \) if \( j = e \), and \( A'_{\tau_e} \rightarrow A_{\tau_1} \). But \( A'_{\tau_e} \nleftrightarrow A_{\tau_1} \) by \( I_{\tau_e} \). Suppose \( \sigma_1 > e \). Then \( A'_{\tau_e} \nleftrightarrow A_{\sigma_1} \) by \( I_{\sigma_1} \).

Let \( \theta = \chi_{\tau_e} + \sum \chi_{\text{exc}}' \), where \( \chi_{\text{exc}}' \) runs over the exceptional characters in the cyclic block of \( L \) containing \( \chi_{\tau_e}' \). Since \( \chi_{\tau_e} \) occurs in \( R_L^G(\theta) \), \( \tau_e \) and \( \varnothing \) are linked. Thus we may suppose \( \sigma_1 \leq e \), so that \( \sigma_1 = e - 1 \) or \( e \). If \( \tau_e < \sigma_1 \), then \( A'_{\tau_e} \nleftrightarrow A_{\sigma_1} \) by \( I_{\sigma_1} \), and we argue as before. Thus \( \tau_e \geq \sigma_1 \). If \( \tau_e < e \), then \( \tau_e = \sigma_1 = e - 1 \), \( (\sigma_1, \tau_e) = (\sigma_1, \tau_{e-1}) \). So in particular, \( I_{2e-2}(l(A'_{\tau_e})) = e - 1 \), \( I_{2e-2}(l(A_{\sigma_1})) = 2e - 2 \). If \( \tau_e > e \), then \( I_{2e-2}(l(A'_{\tau_e})) \leq e + 1 \), \( I_{2e-2}(l(A_{\sigma_1})) = 2e - 1 \). In either case \( A'_{\tau_e} \nleftrightarrow A_{\sigma_1} \), since \( e - 1 < 2e - 2 \) and \( e + 1 < 2e - 1 \), and we argue as before. This completes the proof of (7B).

8. Unipotent Blocks, Conclusion

We first complete the proof of (6A) for the case \( e \geq 3 \). Suppose \( G \) is one of the groups

\[ U_e(q), \quad SO_{2e+1}(q), \quad CSP_{2e}(q), \quad CSO_{2e}^+(q), \quad CSO_{2e+2}^+(q), \]

and \( B \) is the principal block of \( G \). If \( G = U_e(q), SO_{2e+1}(q), \) or \( CSP_{2e}(q) \), then \( \chi_\sigma, \chi_\lambda \) are in different Harish-Chandra families, and hence are correctly linked by part (a) of the proof of (6B). Moreover, \( \kappa \) is the empty partition, \( d = 0 \) when \( G = U_e(q); \ A = \{ X, Y \} \) with \( X = \{ 0, 1, 2, \ldots, e \} \), \( Y = \{ 0, 1, 2, \ldots, e - 1 \} \), \( d = 1 \) when \( G = SO_{2e+1}(q) \) or \( CSP_{2e}(q) \). The inequalities (7.3)–(7.11) remain valid, and together with (3D), imply \( \varnothing \) is correctly linked to the non-exceptional vertices. If \( G = CSO_{2e}^+(q) \), then \( A \) is the degenerate symbol (7.15) with \( a = 0, w = e \), so the \( \sigma \)'s are correctly linked by parts (a, b) of the proof of (6B). Moreover, the inequality (7.16) holds, so \( \varnothing \) is correctly linked. Finally, if \( G = CSO_{2e+2}^+(q) \), then \( A \) is the
symbol \((7.15)\) with \(a = 1, w = e - 1\), so the inequalities \((7.17), (7.18)\) hold. The smallest \(k\) such that \(I_k(X) \neq I_k(Y)\) is \(k = e\), and \(I_e(X) > I_e(Y)\). Thus

\[
\cdots \sigma_2 \sigma_1 \cdots \quad \text{and} \quad \cdots \tau_2 \tau_1 \cdots
\]

are subgraphs of the tree by parts (c, e) of the proof of (6B). If \(\tau_1\) and \(\tau_2\) are not linked, then \(\Sigma_{\tau_1}\) necessarily links \(\tau_1\) and \(\sigma_e\), \(\tau_2\) and \(\sigma_1\) by (3G). Since \(\chi_{\tau_1}(1) < \chi_{\text{exc}}(1)\) by (3F), the tree has the form (6.1), which is impossible by (7.18). Thus the \(\tau\)'s are correctly linked. We then argue as in parts (c, d) of (7B) to show \(\sigma\) is correctly linked. Hence (6A) holds when \(e \geq 3\).

We now prove (6A) when \(e = 2\). Then \(V\) is a symplectic or an orthogonal space. Let \(A = \{X, Y\}\) be in \(\mathcal{X}\). If \(\chi_{\sigma}, \chi_{\tau}\) are in different Harish-Chandra families, then \(d = 1\) or \(2, s = 4\) or \(3, t = 0\) or \(1, \) and the \(\chi_{\sigma}\) and the \(\chi_{\tau}\) are correctly linked by part (a) of the proof of (6B). If \(s = 4\), then (6A) holds by (3G). If \(s = 3\), then \(\sigma\) separates the \(\chi_{\sigma}\) and \(\chi_{\tau}\) by (3G). We need to rule out the case

\[
\sigma_3 \sigma_2 \sigma_1 \tau_1
\]

We may suppose

\[
\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 0, \quad \tau_1 = 1
\]

by induction and parts (a–c) of the proof of (7A). This configuration is also the minimal configuration for this situation. Then \(\chi_{\sigma_1}(1) > \chi_{\sigma_2}(1)\) by (3F), and (6A) holds.

Suppose then the \(\chi_{\sigma}, \chi_{\tau}\) are in the same Harish-Chandra family so that \(d = 0\). We may suppose \(A\) is non-degenerate. Let \(k\) be the least integer such that \(I_k(X) \neq I_k(Y)\). If \(I_k(X) > I_k(Y)\) and \(\sigma_1 < k\), then \(\sigma_1\) and \(\sigma_2\) are linked. Otherwise \(\Sigma_{\sigma_1}\) would link \(\sigma_1, \sigma_2\) to \(\tau\)'s in the interval \((k - 2, k)\) by (c, d) of the proof of (6B). Since the interval contains at most one \(\tau\), this is impossible by (3G).

Case 1. Suppose \(\sigma_2 = \tau_2 = 0\), so that \(\sigma_1\) and \(\tau_1\) are necessarily odd. We may suppose \(\sigma_1 < \tau_1\). Then \(k = \sigma_1 + 2, I_k(X) > I_k(Y)\), and \(\sigma_1\) and \(\sigma_2\) are linked. Now

\[
\frac{\chi_{\sigma_1}(1)}{\chi_{\text{exc}}(1)} = \frac{1}{2} \frac{q^{\sigma_1}}{q^{\sigma_1} - 1} \cdot \frac{q^{\tau_1}}{q^{\tau_1} + 1} < 1,
\]

\[
\frac{\chi_{\tau_2}(1)}{\chi_{\text{exc}}(1)} = \frac{1}{2} \frac{q^{\tau_1}}{q^{\tau_1} - 1} \cdot \frac{q^{\sigma_1}}{q^{\sigma_1} + 1} < 1.
\]

\(\sigma\) can only be linked to \(\sigma_2\) or \(\tau_2\) by (3G), so \(\sigma\) is not an end-point of the tree by the preceding inequalities. It follows (6A) must hold.
Case 2. Suppose $\sigma_2 = 0$ and $\tau_2 > 0$. Then $k = 2$, $I_2(X) > I_2(Y)$. Now
\[
\begin{align*}
\chi_{\sigma_2}(1) &= q^{\sigma_1 \tau_2} \frac{q^{\tau_1} - 1}{q^{\tau_1} + 1} \frac{q^{\tau_2} - 1}{q^{\tau_2} + 1} \\
\chi_{\text{exc}}(1) &= q^{\sigma_1 \tau_2} \frac{q^{\tau_1} - 1}{q^{\tau_1} + 1} \frac{q^{\tau_2} - 1}{q^{\tau_2} + 1} \\
\chi_{\sigma_1}(1) &= \frac{1}{q^{\tau_1} - 1} q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} \\
\chi_{\text{exc}}(1) &= \frac{1}{q^{\tau_1} - 1} q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} \\
\chi_{\sigma_2}(1) &= \frac{q^{\tau_1 - \tau_2}}{q^{\tau_2} + 1} q^{\epsilon_3 \alpha} \\
\chi_{\sigma_1}(1) &= \frac{1}{q^{\tau_1} - 1} q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} \\
\chi_{\sigma_2}(1) &= \frac{1}{q^{\tau_1} - 1} q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} \\
\chi_{\text{exc}}(1) &= \frac{1}{q^{\tau_1} - 1} q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta}
\end{align*}
\]
where $\alpha = |\sigma_1 - \tau_2|$, $\beta = |\sigma_1 - \tau_1|$, and the $\epsilon_i$ are 0 or 1. We claim $\circ$ is not an end-point of the tree. For suppose it were. If $\circ$ and $\tau_2$ are linked, then $\chi_{\text{exc}}(1) < \chi_{\tau_2}(1)$, so that
\[q^{\tau_1 - \tau_2} - 1 q^{\epsilon_3 \alpha} > q^{\epsilon_4 \beta} + 1 \geq 3.\]
But $q^{\tau_1 - \tau_2}/(q^{\tau_1 - \tau_2} - 1) \leq 2$ and $q^{\epsilon_3 \alpha}/(q^{\epsilon_4 \beta} + 1) \leq 1$, which is impossible. If $\circ$ and $\sigma_2$ are linked, then $\chi_{\text{exc}}(1) < \chi_{\sigma_2}(1)$, so that
\[q^{\sigma_1 \tau_2} - 1 q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} > 1.\]
If $\sigma_1 > \tau_2$, then
\[q^{\sigma_1 \tau_2} - 1 q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} = q^{\sigma_1 \tau_2} - 1 q^{\epsilon_3 \alpha} q^{\epsilon_4 \beta} < 1,\]
which is impossible. So $\sigma_1 \leq \tau_2$. Suppose $\tau_2 > 2$. Let $A' = \{X', Y'\}$, where $X' = X$ and $Y'$ is gotten from $Y$ by moving $\tau_2$ up to 2. Then $\{\sigma_2, \sigma'_2\} = \{\sigma_1, \tau_2 - 2\}$, $\{\tau_2, \tau'_2\} = \{2, \tau_1\}$, and $A''_{\tau_2} \rightarrow A_{\tau_2}$. But $A''_{\tau_2} \rightarrow A_{\tau_1}$ by $I_{\tau_2} l$ and $A''_{\tau_2} \rightarrow A_{\sigma_i}$ for any $i$ by $I_{\tau_i - 1} l$. Let $\theta = \chi_{\tau_2} + \sum \chi_{\text{exc}}$, where $\chi_{\text{exc}}$ runs over the exceptional characters in the cyclic block of $L$ containing $\chi_{\tau_2}$. Since $\chi_{\tau_2}$ is the only non-exceptional character of $B$ occurring in $R^G_L(\theta)$, $\circ$ and $\tau_2$ are also linked, which is impossible. Thus $\tau_2 = 1$, $\sigma_1 = 1$, $\sigma_1$ and $\sigma_2$ are linked, and the tree is
\[
\begin{tikzpicture}
  \node [label=left:$\tau_1$] (1) at (0,0) {};
  \node [label=right:$\tau_2$] (2) at (1,0) {};
  \node [label=above:$\sigma_1$] (3) at (2,0) {};
  \node [label=above:$\sigma_2$] (4) at (3,0) {};
  \node [label=right:$\circ$] (5) at (4,0) {};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
\end{tikzpicture}
\]
(8.1)

Let $A' = \{X', Y'\}$, where $X'$ is gotten from $X$ by moving $\sigma_1$, down to position 3. Then $A'$ is a symbol of defect 0 for an overgroup $G'$ of $G$, and $A'$
has no $e$-hooks. With obvious notation, $\sigma' = 0$, $\sigma'_1 = 3$, $\tau'_2 = -1$, and $\tau'_1 = \tau_1$. Here we may suppose $A'$ has additional entries $-1$ in each row. Let $B'$ be the cyclic block of $G'$ labeled by $A'$. Then $\chi'_{\text{exc}}(1) > \chi'_{\tau'_2}(1)$, $\chi'_{\text{exc}}(1) > \chi'_{\sigma'_2}(1)$, so the exceptional vertex of the tree of $B'$ must be linked to $\sigma'_2$ and to $\tau'_2$. Now $A_{\sigma_1} \rightarrow A'_{\sigma_1}$, but $A_{\sigma_1} \not\rightarrow A'_{\sigma_2}$ by $I_2 h$. Also, $A_{\tau_2} \rightarrow A'_{\tau_2}$, but $A_{\tau_2} \not\rightarrow A'_{\sigma_1}$ by $I_3 l$. We now consider the projective character $R_L^G(\theta)$, where $L$ is an appropriate Levi subgroup of $G'$ containing $G$ and 1-dimensional tori, and $\theta = \chi_{\sigma_1} + \chi_{\tau_2}$. Here $\chi_{\sigma_1}$, $\chi_{\tau_2}$ denote the extensions of these characters of $G$ to $L$ by letting $\chi_{\sigma_1}$, $\chi_{\tau_2}$ be trivial on the 1-dimensional tori. Since the non-exceptional characters of $B'$ in $R_L^G(\theta)$ include $\chi'_{\sigma_1}$ and $\chi'_{\tau_2}$, but not $\chi'_{\sigma_2}$, the tree of $B'$ is

Now $A_{\tau_1} \rightarrow A'_{\tau_1}$, but $A_{\tau_1} \not\rightarrow A'_{\sigma_2}$ by $I_4 h$. We apply the preceding considerations to the character $\theta = \chi_{\tau_1} + \chi_{\tau_2}$ of $L$, which is projective by (8.1). Since the non-exceptional characters of $B'$ in $R_L^G(\theta)$ include $\chi'_{\tau_1}$, but not $\chi'_{\sigma_2}$, this is a contradiction. Thus we have proved the claim $\circ$ is not an end-point of the tree.

Finally, suppose the tree is

\[ \tau_1 \quad \sigma_2 \quad \tau_2 \quad \sigma_1 \quad \circ \quad \bullet \]

(8.2)

In particular, $\chi_{\sigma_1}(1) < \chi_{\tau_2}(1)$. If $\tau_2 > \sigma_1$, then

\[
\frac{1}{q^{\sigma_1} - 1} \cdot \frac{q^{\tau_2 - \sigma_1}}{q^{\tau_2 - \sigma_1} + 1} \cdot \frac{q^{\tau_1 - \sigma_1}}{q^{\tau_1 - \sigma_1} + 1} < \frac{q^{\tau_2 - \tau_2}}{q^{\tau_2 - \tau_2} - 1} \cdot \frac{1}{q^{\tau_2 - 1}} \cdot \frac{1}{q^{\tau_2 - \sigma_1} + 1}.
\]

Now $\tau_1 > \tau_2 > \sigma_1$ and $q \geq 2$ imply

\[
\frac{q^{\tau_1 - \sigma_1}}{q^{\tau_1 - \sigma_1} + 1} \geq \frac{4}{5}, \quad \frac{q^{\tau_1 - \tau_2}}{q^{\tau_1 - \tau_2} - 1} \leq 2.
\]

So

\[
\frac{4 q^{\tau_2 - \sigma_1}}{5 q^{\sigma_1 - 1}} \leq \frac{2}{q^{\tau_2 + 1}}, \quad \text{or} \quad q^{\tau_2 - \sigma_1} \frac{q^{\tau_2 + 1}}{q^{\sigma_1 - 1}} \leq \frac{5}{2}.
\]

This is impossible since

\[
q^{\tau_2 - \sigma_1} \frac{q^{\tau_2 + 1}}{q^{\sigma_1 - 1}} > q^{2(\tau_2 - \sigma_1)} \geq 4.
\]
So we may suppose \( r_2 \leq \sigma_1 \). If \( \sigma_1 > 2 \), then \( Z_{\tau_1} \not\subseteq A_{\sigma_1} \) by part (c) of the proof of (6B). Thus \( \tau_1 \) and \( \tau_2 \) are linked by \( Z_{\tau_1} \). This contradicts (8.2), so we may suppose \( \sigma_2 = 2 \), whence \( \tau_2 = 1 \). Let \( A' = \{ X', Y' \} \), where \( X' = X \) and \( Y' \) is gotten from \( Y \) by moving \( \tau_2 \) down to position 2. With the previous notation, we have \( \sigma'_1 = -1 \), \( \sigma'_1 = 2 \), \( \tau'_1 = 2 \), \( \tau'_1 = \tau_1 \). Now \( \tau'_1 \) and \( \tau'_2 \) are linked by part (f) of the proof of (6B). Thus \( \bullet \rightarrow \bullet \rightarrow \circ \) and \( \circ \rightarrow \circ \) are edges in the tree of \( B' \). Here \( \sigma'_1 \) and \( \sigma'_2 \) are linked, since \( \chi'_{\tau'_2}(1) < \chi'_{\text{exc}}(1) \) rules out

\[
\begin{array}{c}
\sigma_1 \quad \sigma_2 \\
\tau_1 \quad \tau_2 \\
\circ
\end{array}
\]

Now \( A_{\sigma_2} \to A'_{\sigma'_2} \), but \( A_{\sigma_2} \not\subseteq A'_{\sigma'_1} \) by \( I_0 \). Also \( A_{\tau_1} \to A'_{\tau'_1} \), but \( A_{\tau_1} \not\subseteq A'_{\sigma'_1} \), since the largest integer in \( h(A_{\tau_1}) \) is greater than the largest integer in \( h(A'_{\sigma'_1}) \). The character \( \theta = \chi_{\tau_1} + \chi_{\sigma_2} \) of \( L \) is projective by (8.2). Since \( R^G_L(\theta) \) contains \( \chi'_{\sigma'_2} \), but not \( \chi'_{\sigma'_1} \), this is impossible. This final contradiction rules out (8.2), and (6A) holds when \( e = 2 \).

9. THE GENERAL CASE

(9A) Let \( B = B_{s,x} \) be a cyclic block of \( G \), let \( H = C_{G^*}(s)^* \), and let \( \mathcal{L}_s; \mathcal{E}(G, (s)) \to \mathcal{E}(H, (1)) \) be the Jordan decomposition of characters of \( \mathcal{E}(G, (s)) \). Then there exists a cyclic unipotent block \( B' \) of \( H \) such that \( \mathcal{L}_s \) induces a bijection

\[
\mathcal{E}(G, (s)) \cap B \to \mathcal{E}(H, (1)) \cap B'
\]

between the non-exceptional characters of \( B \) and the non-exceptional characters of \( B' \).

Proof: \( H \) has a central product decomposition \( H_0(\prod_{\Delta \neq \pm 1, \Delta \in \mathcal{F}} H_\Delta) \) of the following type:

1. If \( V \) is a unitary space, then
   \[
   H_\Delta \cong \begin{cases} 
   GL_{m_\Delta}(s)(q^{2\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_2 \\
   U_{m_\Delta}(s)(q^{\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_1.
   \end{cases}
   \]

The product \( H_0(\prod_{\Delta} H_\Delta) \) is a direct product.

2. If \( V \) is an orthogonal space of odd dimension, then
   \[
   H_\Delta \cong \begin{cases} 
   GL_{m_\Delta}(s)(q^{\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_2 \\
   U_{m_\Delta}(s)(q^{\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_1,
   \end{cases}
   \]

\[
H_0 \cong I_0(U_{X-1}) \times I_0(U_{X+1}),
\]
where $U_{X-1}$, $U_{X+1}$ are orthogonal spaces of dimensions $m_{X-1}(s) + 1$, $m_{X+1} + 1$. The product $H_0(\prod \Delta \bar{H}_\Delta)$ is a direct product.

(3) If $V$ is a symplectic space or an orthogonal space of even dimension, then

$$H_\Delta \simeq \begin{cases} GL_{m_{\Delta}(s)}(q^{\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_2 \\ U_{m_{\Delta}(s)}(q^{\delta_\Delta}) & \text{if } \Delta \in \mathcal{F}_1, \end{cases}$$

$$H_0 \simeq \langle \tau, I_0(U_{X-1}) \times I_0(U_{X+1}) \rangle.$$

Here $U_{X-1}$ and $U_{X+1}$ are respectively symplectic and orthogonal spaces of dimensions $m_{X-1}(s) - 1$ and $m_{X+1}(s)$ if $V$ is a symplectic space, and $U_{X-1}$ and $U_{X+1}$ are orthogonal spaces of dimensions $m_{X-1}(s)$ and $m_{X+1}(s)$ if $V$ is an orthogonal space. The product $H_+ = \prod_{\Delta \notin \mathcal{F}_0} H_\Delta$ is a direct product, $[\tau, H_+] = 1$, and

$$H_0 Z(H_+)/Z(H_+) \simeq \langle \tau_{X-1} \tau_{X+1}, I_0(U_{X-1}) \times I_0(U_{X+1}) \rangle,$$

where $\langle \tau_{X-1}, I_0(U_{X-1}) \rangle = J_0(U_{X-1})$, $\langle \tau_{X+1}, I_0(U_{X+1}) \rangle = J_0(U_{X+1})$. It will be convenient to set $H_{X-1} = I_0(U_{X-1})$, $H_{X+1} = I_0(U_{X+1})$ in case (2), and $H_{X-1} = J_0(U_{X-1})$, $H_{X+1} = J_0(U_{X+1})$ in case (3). Let $\kappa \in \mathfrak{F}$, $\kappa = (\kappa_\Delta)$. Each $\kappa_\Delta$ labels a unipotent block $B'_\Delta$ of $H_\Delta$, where $B'_\Delta$ is cyclic for the elementary divisor $\Gamma$ of $B$, and $B'_\Delta$ has trivial defect group for $\Delta \neq \Gamma$. The characters in $B'_X \times B'_{X+1}$ restrict irreducibly to $H_0$. In cases (1) and (2) this is clear; in case (3) this follows from the fact that characters in $\delta(J_0(U_{X-1})), (s_{X-1}))$ and $\delta(J_0(U_{X+1})), (s_{X+1}))$ restrict irreducibly to $I_0(U_{X-1})$ and $I_0(U_{X+1})$ since neither $\pi(s_{X-1})$ nor $\pi(s_{X+1})$ have both 1 and $-1$ as eigenvalues. These restrictions then form a cyclic block $B'_0$ of $H_0$ with nontrivial defect group if $\Gamma = X \pm 1$, and trivial defect group if $\Gamma \neq X \pm 1$. Now $H$ is a quotient of the direct product $H_0 \times (\prod_{\Delta \notin \mathcal{F}, \Delta \neq X \pm 1} H_\Delta)$ by a central subgroup, so the block $B'_0 \times (\prod_{\Delta \notin \mathcal{F}, \Delta \neq X \pm 1} B'_\Delta)$ of the direct product determines a unique block $B'$ of $H$, which then has the desired properties.

We extend $\mathcal{L}_s$ to a bijection, denoted also by $\mathcal{L}_s$, between the vertices of the tree $\mathcal{T}$ of $B$ and the tree $\mathcal{T}'$ of $B'$ by having the exceptional vertices in $\mathcal{T}$ and $\mathcal{T}'$ correspond.

(9B) $\mathcal{L}_s$ induces a graph isomorphism of $\mathcal{T}$ onto $\mathcal{T}'$.

Proof. Let $\Gamma$ be the elementary divisor of $B$. The proof of (9A) shows $\mathcal{T}'$ is isomorphic to the tree $\mathcal{T}'_\Gamma$ of $B'_\Gamma$. In the arguments of Sections 5–8 establishing the links in $\mathcal{T}'_\Gamma$, we used (1) inequalities and congruences on the degrees of the characters in $B'_\Gamma$, and (2) Harish–Chandra induction. We
claim that analogous arguments can be given for the corresponding links in $\mathcal{F}$. Indeed,

$$\chi_{s,\lambda}(1) = \frac{|G|_{p'}}{|H|_{p'}} \chi_{1,\lambda}(1),$$

$$\chi_{\text{exc}}(1) = \frac{|G|_{p'}}{|H|_{p'}} \chi_{\text{exc}}(1),$$

where $\mathcal{L}_s(\chi_{s,\lambda}) = \chi_{s,\lambda}$, and $\chi_{\text{exc}}, \chi_{\text{exc}}'$ are exceptional characters in $B, B'$, respectively. The second equation holds since $\chi_{\text{exc}} \in \mathcal{E}(G, (sy))$, $\chi_{\text{exc}}' \in \mathcal{E}(H, (y))$ for some $r$-element $y$ in $C_{\alpha}(s)$. Moreover, $\chi_{1,\lambda}'(1) = c \chi_{1,\lambda,1}'(1)$, $\chi_{\text{exc}}(1) = c \chi_{\text{exc}}'(1)$ for some constant $c$, where $\chi_{1,\lambda,1}', \chi_{\text{exc}}'$ are the characters of $B'_{\text{r}}$ corresponding to $\chi_{1,\lambda}', \chi_{\text{exc}}'$. Thus the inequalities and congruences carry over to the degrees of the characters in $B$.

In Harish-Chandra induction, we considered a parabolic subgroup $M_{\text{r}}$ of $H_{\text{r}}$, a projective indecomposable character $\theta_{\text{r}}$ of $M_{\text{r}}$ in a cyclic unipotent block, and the constituents of $R_{M_{\text{r}}}^{H_{\text{r}}}(\theta_{\text{r}})$ in $B'_{\text{r}}$. Let $M$ be the subparabolic subgroup of $H$ defined by replacing the factor $H_{\text{r}}$ in $H_0(\prod_{\mathcal{A} \neq x \pm 1} H_{\mathcal{A}})$ by $M_{\text{r}}$. This replacement is to mean the following when $\Gamma = X + 1$ and $H_0 = \langle \tau, I_0(U_{X-1}) \times I_0(U_{X+1}) \rangle$: Suppose $I = X - 1$. Then

$$H_{X-1} = \langle \tau_{X-1}, I_0(U_{X-1}) \rangle, \quad M_{X-1} = \langle \tau_{X-1}, I_0(U_{X-1,0}) \times M_{X-1,+} \rangle,$$

where $U_{X-1,0}$ is a non-singular subspace of $U_{X-1}$ and $M_{X-1,+} \cong \prod_j GL_{n_j}(q)$. Then the factor $H_0$ is replaced by

$$M_0 = \langle \tau, I_0(U_{X-1,0}) \times M_{X-1,+} \times I_0(U_{X+1}) \rangle.$$

A similar interpretation holds when $\Gamma = X + 1$. Thus we may write $M = M_0(\prod_{\mathcal{A} \neq x \pm 1} M_{\mathcal{A}})$, where essentially only one factor in this product differs from the corresponding factor in $H = H_0(\prod_{\mathcal{A} \neq x \pm 1} H_{\mathcal{A}})$. The projective irreducible character in the blocks $B'_{\mathcal{A}}$ or $B'_d$ corresponding to the other factors, together with $\theta_{\mathcal{A}}', \theta_d'$, define a projective indecomposable character $\theta'$ in a cyclic block $b'$ of $M$. The irreducible constituents of $R_{\mathcal{A}}^H(\theta')$ in $B'$ are then in bijection with the irreducible constituents of $R_{M_{\mathcal{A}}}^{H_{\mathcal{A}}}(\theta_{\mathcal{A}}')$ in $B'_{\mathcal{A}}$, so the arguments by Harish-Chandra induction establishing links in $\mathcal{F}'$ carry over to the corresponding links in $\mathcal{F}$ when we replace $\mathcal{F}_r, H_r, M_r, \theta_r$ respectively by $\mathcal{F}', H, M, \theta'$.

Let $L$ be a subparabolic subgroup of $G$ such that $s \in L^*$ and $M = C_{L^*}(s)^*$, and let $\mathcal{L}_{L^*}: \mathcal{E}(L, (s)) \to \mathcal{E}(M, (1))$ be the Jordan decomposition of characters in $\mathcal{E}(L, (s))$. The block $b'$ of $M$ determines a cyclic block $b$ of $L$ given by the relation of (9A), and $\theta'$ determines a projective indecomposable character $\theta$ in $b$. Indeed, if $\theta' = \psi_1' + \psi_2'$ is a sum of non-exceptional characters $\psi_1', \psi_2'$ in $b'$, then $\theta = \psi_1 + \psi_2$, where $\psi_1 = \mathcal{L}_{L^*}(\psi_1)$,
\( \psi' = \mathcal{L}_{L'}(\psi) \). If \( \theta' = \psi' + \sum \psi'_{\text{exc}} \) is the sum of a non-exceptional character \( \psi' \) and exceptional characters \( \psi'_{\text{exc}} \) in \( B' \), then \( \theta = \psi + \sum \psi_{\text{exc}} \), where \( \psi' = \mathcal{L}_{L'}(\psi) \) and \( \psi_{\text{exc}} \) runs over the exceptional characters in \( b \). By [11, Chap. 8]

\[
(\chi, R^G_G(\psi)) = (\chi', R^H_H(\psi')),
\]
for \( \chi \in \mathcal{E}(G, (s)) \), \( \psi \in \mathcal{E}(L, (s)) \), \( \chi' = \mathcal{L}_S(\chi) \), \( \psi' = \mathcal{L}_{L'}(\psi) \). Thus any statement derived from \( R^H_H(\theta') \) by Harish–Chandra induction on links between non-exceptional characters \( \chi_{1, \lambda} \), \( \chi'_{1, \mu} \) in \( B' \) holds for the corresponding non-exceptional characters \( \chi_{s, \lambda} \), \( \chi'_{s, \mu} \) in \( B \). Also, any statement derived from \( R^H_H(\theta') \) by Harish–Chandra induction on links between the exceptional characters in \( B' \) and a non-exceptional character \( \psi' \) in \( B' \) holds for the exceptional characters in \( B \) and the corresponding non-exceptional character \( \chi'_{s, \lambda} \) in \( B \), since the exceptional characters in \( B' \) and \( B \) are those characters in \( B' \) and \( B \) not in \( \mathcal{E}(H, (1)) \) and \( \mathcal{E}(G, (s)) \), respectively.

We summarize the trees \( \mathcal{T} \) for a cyclic block \( B = B_{\lambda, \chi} \) of \( G \) with elementary divisor \( \Gamma \). It suffices to give the trees \( \mathcal{T}' \). Let \( \kappa \in \mathcal{H} \), and let \( \kappa_\Gamma \) be the \( \Gamma \)-component of \( \kappa \).

1. If \( \Gamma \) is a linear elementary divisor, then \( \mathcal{T}' \) is

\[
\begin{array}{c}
\circ \\
\sigma_1 \sigma_2 \sigma_{\Gamma}
\end{array}
\]

2. If \( \Gamma \) is a unitary elementary divisor, then \( \mathcal{T}' \) is

\[
\begin{array}{c}
\circ \\
\sigma_1 \sigma_2 \sigma_3 \\
\tau_1 \tau_2 \tau_1
\end{array}
\]

Here \( s = t = e_\Gamma \) when \( e_\Gamma \) is even; \( s = e_\Gamma + d \), \( t = e_\Gamma - d \), and \( \frac{1}{2}d(d+1) \) is the size of the 2-core of \( \kappa_\Gamma \) when \( e_\Gamma \) is odd.

3. If \( \Gamma \in \mathcal{F}_0 \) and \( \kappa_\Gamma \) is a non-degenerate symbol, then \( \mathcal{T}' \) is

\[
\begin{array}{c}
\circ \\
\sigma_1 \sigma_2 \sigma_3 \\
\tau_1 \tau_2 \tau_1
\end{array}
\]

Here \( s = t = e_\Gamma \) when \( q^{e_\Gamma} \equiv 1 \pmod{r} \); \( s = e_\Gamma + d \), \( t = e_\Gamma - d \), and \( d \) is the defect of \( \kappa_\Gamma \) when \( q^{e_\Gamma} \equiv -1 \pmod{r} \).

4. If \( \Gamma \in \mathcal{F}_0 \) and \( \kappa_\Gamma \) is a degenerate symbol, then \( \mathcal{T}' \) is

\[
\begin{array}{c}
\circ \\
\sigma_1 \sigma_2 \sigma_{\Gamma}
\end{array}
\]

Cases (2–4) follow from the descriptions in Sections 5–8 of trees in unipotent blocks. In case (1) \( B_\Gamma \) is a block of \( GL_{mr}(q^{\delta_\Gamma}) \) labeled by the partition \( \kappa_\Gamma \), and \( \mathcal{T}_\Gamma \) has the desired form by [5, Theorem C].
APPENDIX: THE TWO PARAMETRIZATIONS OF UNIPOTENT CHARACTERS OF $U_n(q)$

The unipotent characters of $G = U_n(q)$ have two parametrizations:

1. By partitions $\lambda$ of $n$. Here a unipotent character $\chi_{1,\lambda}$ has label $\lambda$ if

$$\chi_{1,\lambda} = \pm \frac{1}{n!} \sum_{w \in W} \phi_\lambda(w w_0) R_{T_n}^G(1),$$

where $\phi_\lambda$ is the irreducible character of the symmetric group $S_n$ labeled by $\lambda$, $w_0$ is the element in $S_n$ of longest length, and $T_n$ is a maximal torus of $G$ corresponding to $w$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$, then

$$\chi_{1,\lambda}(1) = q^{d(\lambda)} \prod_{i=1}^r \frac{(q^i - (-1)^i)}{\prod_v (q^{l(v)} - (-1)^{l(v)})},$$

where $d(\lambda) = \sum_j (t - j) \lambda_j$, $v$ runs over the hooks of $\lambda$, and $l(v)$ is the length of $v$.

2. By triples $(d, \alpha, \beta)$. Here $d \geq 0$ and $(\alpha, \beta)$ is an ordered pair of partitions such that $|\alpha| + |\beta| = \frac{1}{2}[n - \frac{1}{2}(d+1)]$. Such $(\alpha, \beta)$ label the irreducible characters of the centralizer algebra associated with $\mathcal{E}_d(G, 1)$, and thus $(d, \alpha, \beta)$ labels the irreducible characters $\chi_{d,\alpha,\beta}$ in $\mathcal{E}_d(G, 1)$. The degree of $\chi_{d,\alpha,\beta}$ has been given by Lusztig. Namely, let $\{X, Y\}$ be a symbol associated with $(\alpha, \beta)$ as described in [10, (4.6)], where $|X| = |Y| + 1$. Then

$$\chi_{d,\alpha,\beta}(1) = D_{X,Y}(q^2, q^{2d+1}) |G : Z(G)|_{p'},$$

where

$$D_{X,Y}(q^2, q^{2d+1}) = \frac{A(X, q^2) A(Y, q^2)}{(q^{2x} + 2d - 1 + q^{2y})} \times \frac{1}{\prod_{x \in X} \prod_{h=1}^x (q^{2h} - 1)(q^{2h+2d-1} + 1)} \times (1 + q^{2d+1})^b q \left( \exp \left[ (2d-1) \binom{b}{2} + 2f \right] \right).$$

Here $A(S, q^2) = \prod_{s,x \in X, s > x} (q^{2s} - q^{2x})$, $b = |Y|$, and $f = \binom{2b-1}{2} + \binom{2b-3}{2} + \cdots$.

The following refines Lusztig's characterization [10, (9.6)] of the set of partitions labeling the characters in $\mathcal{E}_d(G, 1)$.
PROPOSITION. Let \( \lambda \) be a partition of \( n \) with 2-core \( \lambda_\infty \) and 2-quotient \( (\lambda_0, \lambda_1) \), and let \( |\lambda_\infty| = \frac{1}{2}d(d+1) \). Then \( \chi_{1,\lambda} = \chi_{d,\alpha,\beta} \), where \( \lambda_0 = \alpha, \lambda_1 = \beta \) for even \( d \), and \( \lambda_0 = \beta, \lambda_1 = \alpha \) for odd \( d \).

Proof. We proceed by induction. If \( \lambda \) has no 2-hooks, then \( \chi_{1,\lambda} \) is the unique unipotent cuspidal character of \( G \), and \( \chi_{1,\lambda} = \chi_{d,\emptyset,\emptyset} \) by [10, (9.4)]. So we may suppose \( \lambda \) has a 2-hook \( v \), say at position \((i, j)\) of the Young diagram. Let \( \lambda' \) be the partition of \( n - 2 \) gotten from \( \lambda \) by removing \( v \). Then

\[
\left( \frac{\chi_{1,\lambda}(1)}{\chi_{1,\lambda'}(1)} \right)_p = \begin{cases} 
q^{2i-2} & \text{if } v \text{ has leg length 0} \\
q^{2i-1} & \text{if } v \text{ has leg length 1}
\end{cases}
\tag{A.3}
\]

by (A.1). In particular, this implies that the partitions \( \lambda \) of \( n \) gotten by adding a 2-hook to a fixed \( \lambda' \) parametrize unipotent characters \( \chi_{1,\lambda} \) of \( G \) of distinct degrees. Moreover, by [4, (1C)] these \( \chi_{1,\lambda} \) are precisely the irreducible constituents of \( R^G_L(\xi) \), where

\[
L = U_{n-2}(q) \times GL_1(q^2), \quad \xi = \chi'_{1,\lambda'} \times 1,
\]

\( \chi_{1,\lambda'} \) is the unipotent character of \( U_{n-2}(q) \) labeled by \( \lambda' \), and 1 is the trivial character of \( GL_1(q^2) \). Let \( \lambda'_\infty \) be the 2-quotient and \( (\lambda'_0, \lambda'_1) \) the 2-core of \( \lambda' \). In particular, \( |\lambda'_\infty| = \frac{1}{2}d(d+1) \). By induction \( \chi_{1,\lambda'} = \chi_{d,\alpha',\beta'} \), where \( \alpha' = \lambda'_0, \beta' = \lambda'_1 \) for even \( d \), and \( \alpha' = \lambda'_1, \beta' = \lambda'_0 \) for odd \( d \). The Howlett–Lehrer comparison theorem [7, (5.9)] implies that \( \chi_{d,\alpha,\beta} \) is a constituent of \( R^G_L(\xi) \). Thus to complete the proof, it suffices to show

\[
\left( \frac{\chi_{1,\lambda}(1)}{\chi_{1,\lambda'}(1)} \right)_p = \left( \frac{\chi_{d,\alpha,\beta}(1)}{\chi_{d,\alpha',\beta'}(1)} \right)_p.
\tag{A.4}
\]

This is a straightforward calculation using (A.1) and (A.2).

REFERENCES

12. H. F. TUAN, Groups whose orders contain a prime to the first power, Ann. of Math. 45 (1944), 110–140.