On Weak Monomorphisms in the Category of Convex Processes

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ABSTRACT

Convex processes have been introduced and studied by R. T. Rockafellar [3–5]. Some characterizations of monomorphisms and epimorphisms of the category of polyhedral convex processes have been given [6]. In the present paper we give a new class of morphisms in the category of all convex processes and we study the connections between this class and the class of monomorphisms.

Let X, Y be finite-dimensional real linear spaces. A multivalued mapping $T: X \to Y$ is said to be a convex (polyhedral convex) process if its graph

$$G(T) = \{(x, y) | y \in T(x)\} \subseteq X \times Y$$

is a convex cone (polyhedral convex cone) [4].

Convex processes have been introduced and studied by R. T. Rockafellar [3,4]. Polyhedral convex processes were investigated by him in [5].

In this paper we consider the category $\mathcal{C}$ (respectively $\mathcal{P}$), the objects of which are finite-dimensional, real linear spaces and the morphisms of which are convex processes (respectively polyhedral convex processes) [6] with the superposition $ST: X \to Z$ of processes $T: X \to Y$, $S: Y \to Z$ defined by

$$ST(x) = S(T(x)) = \bigcup_{y \in T(x)} S(y).$$

Let $\mathcal{K}$ be an arbitrary category. We shall say that a morphism $T \in \mathcal{K}$ is a *monomorphism* in the category $\mathcal{K}$ if for all morphisms $S_1, S_2 \in \mathcal{K}$ such that $TS_1 = TS_2$, one has $S_1 = S_2$ [2].

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Denote by $D(T)$ the effective domain for the convex process $T: X \to Y$, i.e.,

$$D(T) = \{x \mid T(x) \text{ is nonempty}\}.$$

DEFINITION 1. We shall say that $T \in \mathcal{C}$, $T: X \to Y$, is a weak monomorphism if

(i) $D(T) = X$,
(ii) $x \in X$, $T(x) \subseteq T(0)$ implies $x = 0$.

The purpose of this paper is to give a characterization of weak monomorphisms and to study connections between monomorphisms and weak monomorphisms in the category $\mathcal{C}$ (or $\mathcal{P}$). The class of weak monomorphisms in some cases is identical to the class of monomorphisms in the category $\mathcal{P}$. The question whether the class of weak monomorphisms in the category $\mathcal{P}$ is always equal to the class of monomorphisms in the category $\mathcal{P}$ still remains open. We shall show by example that the answer to the above question is negative in the category $\mathcal{C}$.

We have the following characterization of monomorphisms:

THEOREM 1. *Let $T \in \mathcal{C}$ ($T \in \mathcal{P}$), $T: X \to Y$. Then the following conditions are equivalent*:

(i) *T is a monomorphism in the category $\mathcal{C}$ ($\mathcal{P}$).*
(ii) *If $W, V \subseteq X$ are convex (polyhedral convex) sets such that $T(V) = T(W)$, then $W = V$.*

The proof of Theorem 1 for $T \in \mathcal{P}$ is in [6]; for $T \in \mathcal{C}$ this fact is easy to check.

Let us note that for a process T which is a function, a weak monomorphism is a linear monomorphism.

For a convex process $T: X \to Y$ we denote by $K(T)$ the following convex cone in the space X:

$$K(T) = \{x \mid 0 \in T(x)\}.$$

If $T: X \to Y$ is a monomorphism in the category $\mathcal{P}$ then $D(T) = X$, $K(T) = \{0\}$ [6]. It is not difficult to check that this remains true in the category $\mathcal{C}$.

In this paper for a subset A of the space X, we denote by $\operatorname{conv} A$ the convex hull [4] of A. For any $A \subseteq X$, $\operatorname{conv} A$ consist of all convex combinations of the elements of A.

PROPOSITION 1. *Every monomorphism in the category $\mathcal{C}$ (resp. $\mathcal{P}$) is a weak monomorphism.*

Proof. Let $T: X \to Y$ be a monomorphism in the category $\mathcal{C}$ ($\mathcal{P}$), $x_0 \in X$, and $T(x_0) \subseteq T(0)$; then $D(T) = X$ and

$$T(\text{conv}\{x_0, 0\}) = T(0).$$

Now applying Theorem 1 we obtain that

$$\text{conv}\{x_0, 0\} = \{0\}.$$

Hence $x_0 = 0$, so T is a weak monomorphism. ■

PROPOSITION 2. *If $T \in \mathcal{C}$ is a weak monomorphism, then $K(T) = \{0\}$.*

Proof. Let $T \in \mathcal{C}$, $T: X \to Y$ be a weak monomorphism, and $x_0 \in K(T)$. Then $0 \in T(x_0)$, so $T(0) \subseteq T(x_0)$, and because of $D(T) = X$ we have

$$T(-x_0) \subseteq T(-x_0) + T(0) \subseteq T(-x_0) + T(x_0) \subseteq T(0).$$

Hence $x_0 = 0$, which proves the proposition. ■

LEMMA 1. *Let $T: X \to Y$ be a convex process with graph $G(T)$ closed, $x_1, x_2 \in D(T)$, and $T(x_1) \subseteq T(x_2)$. Then $x_1 - x_2 \in D(T)$ implies $T(x_1 - x_2) \subseteq T(0)$.*

Proof. Let $T: X \to Y$, $T \in \mathcal{C}$, be a process whose graph is a closed cone. Let $x_1, x_2, x_1 - x_2 \in D(T)$ and $T(x_1) \subseteq T(x_2)$. Take $y \in T(x_1 - x_2)$ and $y_1 \in T(x_1)$. Then by the assumption we have $y_1 \in T(x_2)$ and

$$y + y_1 \in T(x_1 - x_2) + T(x_2) \subseteq T(x_1).$$

Let us assume now that $y_1 + ky \in T(x_1)$ for $k = 1, 2, \ldots, n$. Then $y_1 + ny \in T(x_2)$ and

$$y_1 + (n+1)y = y + (y_1 + ny) \in T(x_1 - x_2) + T(x_2) \subseteq T(x_1).$$

Thus $y_1 + ny \in T(x_1)$ for $n = 1, 2, \ldots$. Hence

$$y + \frac{1}{n} y_1 = \frac{1}{n}(y_1 + ny) \in \frac{1}{n} T(x_1) = T\left(\frac{1}{n} x_1\right).$$

Since the cone $G(T)$ is closed, $y \in T(0)$. This means that $T(x_1 - x_2) \subseteq T(0)$. ■

By Lemma 1 we obtain immediately the following theorem:

THEOREM 2. *For $T \in \mathcal{P}$ the following conditions are equivalent:*

(i) *T is a weak monomorphism.*
(ii) *$D(T) = X$ and for every $x_1, x_2 \in X$, $T(x_1) \subseteq T(x_2)$ implies $x_1 = x_2$.*

We also use the classical definition of convergence of sets in a space X [1]. Let A_n, $n = 1, 2, \ldots$, be subsets of a space X. We shall say that $\lim A_n = A_0 \subseteq X$ if $\operatorname{Li} A_n = A_0 = \operatorname{Ls} A_n$, where $x \in \operatorname{Li} A_n$ if any neighborhood of x has common points with sets A_n for almost every n, and $x \in \operatorname{Ls} A_n$ if any neighborhood of x has common points with an infinite number of sets A_n.

For $T \in \mathcal{P}$ we have the following theorem:

THEOREM 3 [6]. *Let $T \in \mathcal{P}$, $T: X \to Y$, $x_n \in D(T)$, $n = 1, 2, \ldots$, and $\lim x_n = x_0$. Then $\lim T(x_n) = T(x_0)$.*

This is not true in the general case, when T is just a convex process, even if we assume that $G(T)$ is a closed cone.

PROPOSITION 3. *Let $T: X \to Y$ be a polyhedral convex process. Then for every $x \in D(T)$ there exists a polytope $W(x)$ in the space Y such that $T(x) = W(x) + T(0)$.*

Proof. Let $T: X \to Y$, $T \in \mathcal{P}$, $x \in D(T)$; then $T(x)$ is a nonempty polyhedral set, so there are a polyhedral cone $C(x)$ and a polytope $W(x)$ in the space Y [4] such that

$$T(x) = W(x) + C(x).$$

Since the set $W(x)$ is bounded, we have by Theorem 3

$$T(0) = \lim T\left(\frac{1}{n} x\right) = \lim \frac{1}{n} T(x) = \lim\left(\frac{1}{n} W(x) + \frac{1}{n} C(x)\right)$$
$$= \lim\left(\frac{1}{n} W(x) + C(x)\right) = C(x).$$

Therefore $T(x) = W(x) + T(0)$ for every $x \in D(T)$. ■

Applying Proposition 3 we can prove the following fact:

LEMMA 2. *Let $T \in \mathcal{P}$ be a weak monomorphism, $T: X \to Y$,* $\dim X = 1$, *and let $W \subseteq X$ be a convex set. Let $x_0 \in D(T)$ and $T(x_0) \subseteq T(W)$. Then $x_0 \in W$.*

The proof of this lemma is not difficult but long, so we omit it.
Applying Lemma 2 and Theorem 1, we get the next theorem:

THEOREM 4. *Let $T \in \mathcal{P}$, $T: X \to Y$,* $\dim X = 1$, *be weak monomorphism. Then T is a monomorphism in the category $\mathcal{P}$.*

Theorem 4 is very useful in constructing nonadditive monomorphisms in the category $\mathcal{P}$.

DEFINITION 2. A convex process $T: X \to Y$ is said to be an additive process if for any $x_1, x_2 \in D(T)$,

$$T(x_1 + x_2) = T(x_1) + T(x_2).$$

EXAMPLE. Let us define $T \in \mathcal{P}$, $T: R \to R^2$, as follows:

$$T(x) = \{(y, z) \mid y \leqslant x \leqslant z, y \leqslant 2z\} \qquad \text{for} \quad x \in R.$$

Hence

$$T(x) = \begin{cases} (x, x) + T(0), & x \geqslant 0, \\ \operatorname{conv}\{(x, x/2), (2x, x)\} + T(0), & x < 0. \end{cases}$$

Of course $DT = R$, and it is easy to check that $T(x_0) \subseteq T(0)$ implies $x_0 = 0$, so T is a weak monomorphism. We note that $T(-1) + T(1) \neq T(0)$, so T is a nonadditive process, but applying Theorem 4, we get that T is a monomorphism in the category $\mathcal{P}$.

THEOREM 5. *Let $T \in \mathcal{C}$ ($T \in \mathcal{P}$), $T: X \to Y$, be an additive weak monomorphism. Then T is a monomorphism in the category $\mathcal{C}$ ($\mathcal{P}$).*

Proof. Let $T \in \mathcal{C}$ ($T \in \mathcal{P}$), $T: X \to Y$ be an additive weak monomorphism, and let W, V be convex (polyhedral convex) sets such that $T(W) = T(V)$.

Then for any $w \in W$ we have $T(w) \subseteq T(V)$, so

$$0 \in T(0) = T(w-w) = T(w) + T(-w)$$
$$\subseteq T(V) + T(-w) \subseteq T(V-w).$$

Hence there is an element $v \in V$ such that $0 \in T(v-w)$. This means that $v - w \in K(T)$. Now using the fact that $K(T) = \{0\}$, we get $W \subseteq V$. Analogously we have $V \subseteq W$, so $W = V$. Now, applying Theorem 1, we obtain that T is a monomorphism in the category $\mathcal{C}(\mathcal{P})$. ■

In the category $\mathcal{C}$ neither Theorem 2, Lemma 2, Lemma 1, Theorem 4, nor Proposition 3 is true. Indeed, let us define $T: R \to R^2$ as follows:

$$T(0) = \{(y,z) \mid -z < y < z\} \cup \{(0,0)\},$$

$$T(x) = \begin{cases} \{(y,y) \mid 0 < y < x\} + T(0) & \text{for} \quad x > 0, \\ (x,-x) + T(0) & \text{for} \quad x < 0. \end{cases}$$

It is not difficult to check that T is a convex process and $T(x) \subseteq T(0)$ implies $x = 0$. Hence, because $D(T) = X$, the T so defined is a weak monomorphism; but:

(i) $T(\frac{1}{2}) \subseteq T(1)$,
(ii) $T(0)$ does not contain $T(\frac{1}{2} - 1) = T(-\frac{1}{2})$,
(iii) $T(1) = T(\{x \in R \mid 0 < x \leqslant 1\})$;

hence T is not a monomorphism in the category $\mathcal{C}$.

Now let us show that the analogue of Proposition 3 is not true in the category $\mathcal{C}$. Indeed, let $T: R \to R$ be a process such that

$$G(T) = \{(x,y) \mid y > 0\} \cup \{(0,0)\};$$

then

$$T(0) = \{y \mid y \geqslant 0\}, \qquad T(x) = \{y \mid y > 0\} \quad \text{for } x \neq 0,$$

and obviously there does not exist a bounded set $W(x)$ such that $T(x) = W(x) + T(0)$ for $x \neq 0$.

Let us note that in this example T is such that the closure of $G(T)$ is a polyhedral convex cone. Even in the case when $G(T)$ is a closed cone, Proposition 3 need not be true. Indeed, let us define $T: R \to R^2$ as follows:

$$G(T) = \{(x, y, z) \mid z \geq 0, x^2 + y^2 - z^2 \leq 0\}.$$

Then

$$T(0) = \{(y, z) \mid z \geq 0, y^2 - z^2 \leq 0\},$$

$$T(1) = \{(y, z) \mid z \geq 0, y^2 - z^2 \leq -1\},$$

and obviously there does not exist a bounded set W such that

$$T(1) = W + T(0).$$

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