Reduction formulae of Littlewood–Richardson coefficients

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\section{0. Introduction}

\textit{Littlewood–Richardson coefficients} $c_{\lambda,\mu}^{\nu}$ are important in many fields of mathematics. They count the number of column strict (skew) tableaux on the shape $\nu/\lambda$ of content $\mu$ that satisfy a certain

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condition on a word derived from the tableau, which we call a Littlewood–Richardson tableau. They explain the rule of multiplication of two Schur functions: $s_{\lambda} \cdot s_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} s_{\nu}$, and the tensor product of two irreducible polynomial representations of $GL_n(\mathbb{C})$: $V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c_{\lambda, \mu}^{\nu} V(\nu)$. They also appear in the Schubert calculus of Grassmannians; $\sigma_{\lambda} \cdot \sigma_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} \sigma_{\nu}$, where $\sigma_{\lambda}$ is the Schubert class in the cohomology ring of a Grassmannian, indexed by the partition $\lambda$.

Two reduction formulae of Littlewood–Richardson coefficients, one is the conjugated formula of the other, had been proved by Griffiths–Harris and were importantly used in the proof of Pieri’s formula [9].

Recently, a factorization theorem of Littlewood–Richardson coefficients is proved by King, Tollu and Toumazet [12]; we call this KTT theorem for short in this article. Roughly speaking, the factorization theorem states that if $c_{\lambda, \mu}^{\nu} > 0$ and any one of Horn’s inequalities is an equality, then $c_{\lambda, \mu}^{\nu}$ can be written as a product of two Littlewood–Richardson coefficients indexed by certain subpartitions of $\lambda$, $\mu$, $\nu$. Our observation is that if the index sets in the KTT theorem have certain cardinalities then it gives a reduction formula of Littlewood–Richardson coefficients. Moreover, one of them is the first reduction formula by Griffiths–Harris.

We provide explicit statement of four reduction formulae as special cases of KTT theorem and extend them to more general and useful forms. We also prove conjugated formulae of those reductions that enable us to list eight useful reduction formulae including two classical reduction formulae by Griffiths–Harris.

In Section 1, we state (classical) reduction formulae I and II by Griffiths and Harris, and the factorization theorem by King, Tollu and Toumazet. Section 2 is devoted to four special cases of KTT theorem, which give reduction formulae of Littlewood–Richardson coefficients: We restate them by finding explicit conditions for $(I, J, K)$ to be an essential Horn triple, and provide (sketch of) bijective proofs for $r = 1$ and $r = 2$ cases in terms of tableaux. The main theorems, extensions and conjugations of special cases of KTT theorem, and their examples appear in Section 3. As an application of our main theorem, we prove that one of two reduction formulae is always applicable when the Littlewood–Richardson coefficient is 1 for distinct part partitions and make a conjecture that one of four reduction formulae is always applicable if the Littlewood–Richardson coefficient is 1 in Section 4. The proofs of the main theorems are given in Section 5 and open problems are stated in Section 6.

1. Preliminaries

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a nonincreasing sequence of nonnegative integers with finite positive numbers. The partition $(i_1, i_2, i_3, i_4, i_5, \ldots)$ with $m_j$ $i_j$’s is also denoted by $(i_1^{m_1}, i_2^{m_2}, \ldots)$. The size of $\lambda$ is $|\lambda| = \sum_i \lambda_i$ and the length of $\lambda$, $\ell(\lambda)$, is the number of positive numbers in $\lambda$. The Young diagram of a partition $\lambda$ is a left-justified array of boxes with $\lambda_i$ boxes in its $i$th row. For a partition $\lambda$, $\hat{\lambda}$ is the conjugate of $\lambda$, whose diagram is obtained by interchanging rows and columns of $\lambda$. For two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\nu = (\nu_1, \nu_2, \ldots)$, we write $\lambda \subseteq \nu$, if $\lambda_i \leq \nu_i$ for all $i$. For partitions $\lambda, \nu$ with $\lambda \subseteq \nu$, the skew diagram of shape $\nu/\lambda$ is the diagram consisting of boxes of $\nu$ which are not the boxes of $\lambda$, and the difference of $\nu$ and $\lambda$ is the sequence of nonnegative integers $\nu - \lambda = (\nu_1 - \lambda_1, \nu_2 - \lambda_2, \ldots)$. By $\nu \setminus \nu_K$ where $K$ is a (finite) subset of the indices, we mean the partition obtained by deleting all the $k$th part for $k \in K$ from $\nu$.

A skew tableau of shape $\nu/\lambda$ with content $\mu = (\mu_1, \mu_2, \ldots)$ is a filling of boxes of a skew diagram $\nu/\lambda$, with $\mu_i$ $i$’s, where entries are weakly increasing in rows and strictly increasing in columns. The reverse row word of a skew tableau $T$, denoted by $w(T)$, is the word obtained by reading the entries of $T$ from right to left and top to bottom. A word $w = x_1 \cdots x_r$ is called a lattice word if, for any $s \leq r$ and $i, x_1 \cdots x_s$ contains at least as many $i$’s as it contains $(i + 1)$’s. A skew tableau $T$ is a Littlewood–Richardson skew tableau (LR-tableau) if its reverse row word $w(T)$ is a lattice word.

**Definition 1.1.** For given partitions $\lambda$, $\mu$ and $\nu$, the Littlewood–Richardson coefficient (LR-coefficient) $c_{\lambda, \mu}^{\nu}$ is the number of LR-tableaux on the shape $\nu/\lambda$ of content $\mu$. 

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When \( n \geq \max(\ell(\lambda), \ell(\mu), \ell(\nu)) \) and \( m - n \geq \max(\nu_1, \mu_1, \nu_1) \), we let \( \nu^c = (m - n - \nu_1, m - n - \nu_2, \ldots, m - n - \nu_{n-1}, \ldots, m - n - \nu_1) \) be the complement of \( \nu \) with respect to the choice of \( n, m \). A well-known fact about LR-coefficients is that \( c_{\lambda, \mu}^\nu \) is equal to the universal Schubert coefficient \( s(\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_\nu) \) of three Schubert classes corresponding to the partitions \( \lambda, \mu \) and \( \nu \) inside the Grassmannian \( G(n, m) \). We always assume that \( \lambda, \mu \) and \( \nu \) are contained in the rectangular partition \((m - n)n\) for fixed \( m, n \). Note that if \( \lambda_1 > \nu_1 \) or \( \ell(\lambda) > \ell(\nu) \) then the corresponding LR-coefficient is not defined, and if \( \mu_1 > \nu_1 \) or \( \ell(\mu) > \ell(\nu) \) then the corresponding LR-coefficient is 0. Therefore the smallest possible choice of \( n, m - n \) is \( 1 \), \( \ell(\nu) \) respectively. We always assume that the three given partitions satisfy \( |\nu| = |\lambda| + |\mu| \), otherwise the corresponding LR-coefficient \( c_{\lambda, \mu}^\nu \) is 0.

We state the classical reduction formulae by Griffiths and Harris [9], whose combinatorial proofs in terms of LR-tableaux are given in [3,4] respectively. The second reduction formula may also be obtained by applying the first reduction formula to the conjugate of each partition and the following well-known fact:

**Lemma 1.2.** (See [7,10].) For partitions \( \lambda, \mu, \nu \), the following equality holds

\[
c_{\lambda, \mu}^\nu = c_{\lambda', \mu'}^{\nu'}.
\]

**Theorem 1.3 (Reduction formula I).** For any three indices \( 0 \leq i, j, k' \leq n \) with \( i + j + k' = 2n + 1 \), let \( k = n + 1 - k' \) then

\[
c_{\lambda, \mu}^\nu = \begin{cases} 
0 & \text{if } \lambda_i + \mu_j + \nu_{k'} > \nu_1, \\
\nu_{i\setminus|\nu|} & \text{if } \lambda_i + \mu_j + \nu_{k'} = \nu_1.
\end{cases}
\]

**Theorem 1.4 (Reduction formula II).** For any three coefficients \( \lambda_i, \mu_j \) and \( \nu_{k'} \) with \( \lambda_i + \mu_j + \nu_{k'} \geq 2\nu_1 + 1 \), let \( k = n + 1 - k' \) then

\[
c_{\lambda, \mu}^\nu = \begin{cases} 
0 & \text{if } i + j + k' > n, \\
c_{\lambda-(1^{k-1}), \mu-(1^{j})} & \text{if } i + j + k' = n.
\end{cases}
\]

Moreover, the formula can be applied when \( i = 0, j = 0 \) or \( k' = 0 \) by formally setting \( \lambda_0 = \mu_0 = \nu_0 = m - n \).

We give an example [3, Example 4.9] showing usage of two reduction formulae to calculate an LR-coefficient.

**Example 1.5.** We use notations \( 1^{st} \) and \( 2^{nd} \) to indicate that the reduction formula I or the reduction formula II is used respectively.

\[
\begin{align*}
&c_{(9,6,6,6,5)}^{(4,4,3,2,0),(6,5,4,3,1)} \quad 1^{st} c_{(9,6,6,5)}^{(4,4,3,0),(6,5,3,1)} \\
&1^{st} c_{(9,6,5)}^{(4,4,0),(6,5,1)} \\
&2^{nd} c_{(8,5,4)}^{(3,3,0),(5,5,1)} \\
&2^{nd} c_{(7,4,4)}^{(3,3,0),(4,4,1)} \\
&2^{nd} c_{(6,3,3)}^{(3,3,0),(3,3,0)} = 1.
\end{align*}
\]
Remark 1.6. As Example 1.5 shows, reduction formulae I, II are useful in calculating LR-coefficients, even though they are not applicable in every occasion. Reduction formula I deletes one row of each partition and reduction formula II deletes one column of each partition.

When \( n \) is a positive integer, we let \([n] = \{1, \ldots, n\}\). Let \( r \leq n \) be a pair of positive integers, \( I = \{i_1, i_2, \ldots, i_r\} \), \( i_1 < i_2 < \cdots < i_r \) be an \( r \)-subset of \([n]\), and \( \lambda \) be a partition with \( c(\lambda) \leq n \). Then, we define \( \pi(I) = (i_r - r, i_{r-1} - (r - 1), \ldots, i_1 - 1) \) as a partition associated to \( I \) and \( \lambda_I = (\lambda_{i_1}, \ldots, \lambda_{i_r}) \) as a subpartition of \( \lambda \). Denoted by \( I^c \), we mean the complement of \( I \) in \([n]\), i.e. \( I^c = [n] - I \). For a pair of two positive integers \( r \leq n \), the set of essential Horn triples is

\[
R_r^n = \{ (I, J, K) \mid I, J, K \text{ are } r\text{-subsets of } [n], c_{\pi(I), \pi(J)} = 1 \}.
\]

Motivated by Horn’s inequalities giving an equivalent condition for an LR-coefficient to be nonzero, King, Tollu and Toumazet proved a theorem on the factorization of LR-coefficients (KTT theorem for short) enabling one to write an LR-coefficient as a product of two LR-coefficients of smaller partitions [12].

Theorem 1.7 (Horn’s inequalities). (See [13,14].) We let \( \lambda, \mu \) and \( \nu \) be partitions of lengths at most \( n \). Then \( c_{\lambda, \mu, \nu} > 0 \) if and only if \( |\nu| = |\lambda| + |\mu| \) and for all \( r \leq n \), \( \sum_{k \in K} v_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \) holds for all \( (I, J, K) \in R_r^n \).

Theorem 1.8 (KTT factorization theorem). (See [12].) We let \( \lambda, \mu \) and \( \nu \) be partitions of lengths at most \( n \) with \( c_{\lambda, \mu} > 0 \). Suppose that there exists \((I, J, K) \in R_r^n \) with \( r < n \), which satisfies the equality \( \sum_{k \in K} v_k = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \). Then, \( c_{\lambda, \mu, \nu} = c_{\lambda_{V_k, I_{K}}, \nu_{V_k}, \mu_{V_k}} \).

An immediate observation is if \( \lambda, \mu, \nu \) are partitions whose lengths are at most 2, then \( c_{\lambda, \mu, \nu} = 1 \) when it is nonzero. Hence, in Theorem 1.8, if \( r \leq 2 \) or \( r \geq n - 2 \), then \( c_{\lambda_I, \mu_J, \nu} = 1 \) or \( c_{\lambda_{I^c}, \mu_{J^c}, \nu} = 1 \), respectively. Therefore, some special cases of Theorem 1.8 give us reduction formulae of LR-coefficients, which we consider in detail in the next section.

2. Special cases of KTT theorem

It is not easy to check the condition \((I, J, K) \in R_r^n \), which is necessary for the use of KTT theorem. For special cases which give reduction formulae, we find explicit conditions for \((I, J, K) \) to be in \( R_r^n \). The aim of this section is to deduce reduction formulae of LR-coefficients from the KTT theorem in useful enough forms to be applied.

As we mentioned in the previous section, four special cases that always give reduction of LR-coefficients are when \( r = 1, 2, n - 1, n - 2 \). We therefore need to know which triples \((I, J, K) \) are in \( R_r^n \) when \( r = 1, 2, n - 1 \) or \( n - 2 \) in explicit ways.

One can obtain the following lemma directly from the definition of LR-coefficients.

Lemma 2.1. Let \( \lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \) be partitions with lengths at most two, and \( \nu = (\nu_1, \nu_2) \) be a partition with length two such that \( \lambda, \mu \subseteq \nu \). Then \( c_{\lambda, \mu} = 1 \) if and only if the following conditions are satisfied

\[
\begin{align*}
v_1 + v_2 &= \lambda_1 + \lambda_2 + \mu_1 + \mu_2, \\
v_1 - \lambda_1 &\leq \mu_1 \leq v_1 - \lambda_2, \\
v_2 - \lambda_2 &\leq \mu_2 \leq \min\{v_2 - \lambda_2, v_1 - \lambda_1\}.
\end{align*}
\]

In the following proposition, we describe the conditions for \((I, J, K) \) to be in \( R_r^n \) or \( R_r^2 \), while the proof can be easily done directly from the definition of \( R_r^n \) and Lemma 2.1. It is stated in [8] as well.
Definition 2.10. For a pair of two positive integers \( r \leq n \), define a set of triples of \( r \)-subsets of \([n]\) as follows

\[
C^n_r = \{(I, J, K) \mid I, J, K \text{ are } r\text{-subsets of } [n], \ C^{\sigma(K)}_{\sigma(I), \sigma(J)} = 1\}.
\]

Proposition 2.11. For a triple \((I, J, K)\) of \( r \)-subsets of \([n]\),

\( (I, J, K) \in C^n_r \) if and only if \((I^c, J^c, K^c) \in R^n_{n-r} \).

Proof. By Lemma 1.2, \((I, J, K) \in C^n_r \) if and only if \( \frac{C^{\sigma(K)}_{\sigma(I), \sigma(J)}}{C_r^I} = 1 \). For an \( r \)-subset \( I = \{i_1, i_2, \ldots, i_r\} \), \( i_1 < i_2 < \cdots < i_r \), of \([n]\),

\[
\tilde{\sigma}(I) = (r, \ldots, r, r-1, \ldots, r-1, \ldots, 1, 1, 0, \ldots, 0).
\]

Moreover, it is not hard to see that \( \pi(I^c) = \tilde{\sigma}(I) \). This completes the proof. \( \square \)

Due to Proposition 2.11, \( r = n-1, n-2 \) cases of Theorem 1.8 can be explicitly stated if we find conditions for a triple \((I, J, K)\) to be in \( C^n_r \) when \( r = 1, 2 \). The first part of the following proposition is immediate from the definition while the second part can be obtained by applying Lemma 2.1 to \( \sigma(I) \), \( \sigma(J) \) and \( \sigma(K) \).

Proposition 2.12. Let \( n \) be a positive integer. Then

1) \((\{i\}, \{j\}, \{k\}) \in C^n_1 \) if and only if \( i + j = k + n \).
2) \((\{i_1, i_2\}, \{j_1, j_2\}, \{k_1, k_2\}) \in C^n_2 \) if and only if the following conditions are satisfied

\[
i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 2n - 1, \quad \text{(2.13)}
\]

\[
k_2 + n \leq i_2 + j_2, \quad \text{(2.14)}
\]

\[
k_1 + n \leq i_2 + j_1, \quad \text{(2.15)}
\]

\[
k_1 + n \leq i_1 + j_2. \quad \text{(2.16)}
\]
We now can state KTT theorem for $r = 1, 2, n - 1, n - 2$ in combinatorial ways. One can check that KTT theorem for the case $r = n - 1$ is a part of reduction formula $I$.

**Theorem 2.17 (Special cases of KTT theorem).** (See [5].) We let $\lambda$, $\mu$, and $\nu$ be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $c_{\lambda, \mu}^\nu > 0$.

**(r = 1)** Suppose that there are $1 \leq i, j, k \leq n$ such that $i + j = k + 1$ and $v_k = \lambda_i + \mu_j$. Then $c_{\lambda, \mu}^\nu = c_{\lambda \setminus \lambda(i), \mu \setminus \mu(j)}^\nu$.

**(r = 2)** Suppose that there are three subsets $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$, $K = \{k_1, k_2\}$ of $[n]$, which satisfy inequalities (2.6)–(2.9) and $v_{k_1} + v_{k_2} = \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2}$. Then $c_{\lambda, \mu}^\nu = c_{\lambda \setminus \lambda(i_1, i_2), \mu \setminus \mu(j_1, j_2)}^\nu$.

**(r = n − 1)** Suppose there are $1 \leq i, j, k \leq n$ such that $i + j = k + n$ and $\lambda_i + \mu_j = v_k$. Then, $c_{\lambda, \mu}^\nu = c_{\lambda \setminus \lambda(i, j), \mu \setminus \mu(\lambda)}^\nu$.

**(r = n − 2)** Suppose there are three subsets $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$, $K = \{k_1, k_2\}$ of $[n]$, which satisfy inequalities (2.13)–(2.16) and $v_{k_1} + v_{k_2} = \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2}$. Then, $c_{\lambda, \mu}^\nu = c_{\lambda \setminus \lambda(i_1, i_2), \mu \setminus \mu(j_1, j_2)}^\nu$.

KTT theorem has been proved by constructing a bijection between the corresponding sets of LR-tableaux. Bijective proofs in terms of LR-tableaux have been done for $r = n - 1$ case, the classical reduction formula $I$, in [3] and for $r = n - 2$ in [6].

We give (sketch of) proofs for $r = 1$ and $r = 2$ case of Theorem 2.17 by constructing bijections between two sets of LR-tableaux of corresponding shapes and contents in the following. The rigorous proofs require long and complicated work, and we only give main ideas and algorithms.

For a skew tableau $T$ with $n$ rows and integers $1 \leq h, \ell \leq n$, we let $n_T^h(\ell)$ be the number of $\ell$'s in the $h$th row of $T$.

2.1. $r = 1$

Throughout this subsection, we assume that $\lambda$, $\mu$, and $\nu$ are partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $c_{\lambda, \mu}^\nu > 0$, and $\lambda_i + \mu_j = v_k$ for integers $i, j, k$ such that $i + j = k + 1$.

We state a key lemma, that makes Algorithm 2.19 valid, without proof.

**Lemma 2.18.** For an LR-tableau $T$ of shape $\nu / \lambda$ with content $\mu$, if $j + 1 \leq \ell \leq k$ then

$$\sum_{h=\ell}^{k} n_T^h(\ell) = \sum_{h=\ell-1}^{k-1} n_T^h(\ell - 1),$$

$$\sum_{h=j}^{k} n_T^h(h) = \mu_j.$$

Let $T$ be an LR-tableau of shape $\nu / \lambda$ with content $\mu$. The following algorithm is to construct the corresponding LR-tableau $\Phi(T)$ of shape $(\nu \setminus v_k) / (\lambda \setminus \lambda(i))$, with content $\mu \setminus \mu(j)$ in a bijective way.

**Algorithm 2.19.** The reduced LR-tableau $\Phi(T)$ is obtained by applying the following algorithm:

**Step 1:** for $\ell = k$ downto $j + 1$ do

- Empty all boxes containing $\ell$ in $k$th row.
- Replace all $(\ell - 1)$'s in $(k - \ell + 1)$ consecutive rows from the $(\ell - 1)$st row to the $(k - 1)$st row with $\ell$'s.

end for

**Step 2:** Empty all boxes containing $j$ in the $k$th row.
Step 3: for $\ell = j + 1$ to $n$ do
   Replace all $\ell$'s with $(\ell - 1)$'s.
end for

Step 4: Slide each box at $(p, q)$-position, $1 \leq p \leq k - 1$, $\lambda_i + 1 \leq q$, one step down to remove all empty boxes.

Example 2.20. Let $\lambda = (5, 5, 3, 2, 2, 0)$, $\mu = (8, 7, 5, 5, 4, 1)$ and $\nu = (11, 9, 8, 8, 8, 3)$. We have $i = 3$, $j = 3$, $k = 5$, and $\lambda_3 + \mu_3 = \nu_5$.

\[
T = \begin{array}{cccccc}
\lambda & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

Steps 1, 2

\[
\begin{array}{cccccc}
\lambda & \mu \\
\vdots & \vdots \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

Step 3

\[
\begin{array}{cccccc}
\lambda & \mu \\
\vdots & \vdots \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

Step 4

\[
\begin{array}{cccccc}
\lambda & \mu \\
\vdots & \vdots \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

2.2. $r = 2$

Throughout this subsection, we assume that $\lambda$, $\mu$ and $\nu$ are partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $c_{\lambda, \mu}^{\nu} > 0$. We also assume that $i_1, i_2, j_1, j_2, k_1, k_2$ satisfy $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3$, $i_1 + j_1 \leq k_1 + 1$, $i_1 + j_2 \leq k_2 + 1$, $i_2 + j_1 \leq k_2 + 1$ and $\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = \nu_{k_1} + \nu_{k_2}$. The following is a key lemma that allows Algorithm 2.22 is valid.

Lemma 2.21. Let $T$ be an LR-tableau of shape $\nu/\lambda$ with content $\mu$.

(a) For $\ell = j_1 + 1, \ldots, k_1$, $\sum_{h=j+1}^{k_1} n_{i_1}^h (\ell) = \sum_{h=\ell+1}^{k_1} n_{i_1}^h (\ell - 1)$.
(b) For $\ell = j_1 + 1, \ldots, j_2 - 1$, $\sum_{h=k_2 - j_2 + 1}^{k_2} n_{i_1}^h (\ell) = \sum_{h=\ell+1}^{k_2} n_{i_1}^h (\ell - 1)$.
(c) For $\ell = j_2 + 1, \ldots, k_2$, $\sum_{h=k_2}^{k_2} n_{i_1}^h (\ell) = \sum_{h=\ell+1}^{k_2} n_{i_1}^h (\ell - 1)$.
(d) $\sum_{h=j_1}^{k_1} n_{i_1}^h (j_1) = \nu_{k_1} - \lambda_{i_1}$ and $\sum_{h=j_1}^{k_2} n_{i_1}^h (j_2) = \nu_{k_2} - \lambda_{i_2}$.
(e) $\sum_{h=j_1}^{k_1} n_{i_1}^h (j_1) + \sum_{h=k_2 - j_2 + 1}^{k_2} n_{i_1}^h (j_1) = \mu_{j_1}$ and $\sum_{h=j_2}^{k_2} n_{i_1}^h (j_2) = \mu_{j_2}$.

Let $T$ be an LR-tableau of shape $\nu/\lambda$ with content $\mu$. The following is an algorithm to construct the corresponding LR-tableau $\Phi(T)$ of shape $(\nu \setminus \nu_{[k_1, k_2]})/\lambda/\lambda_{[j_1, j_2]}$ with content $\mu \setminus \mu_{[j_1, j_2]}$ in a bijective way.

Algorithm 2.22. The reduced LR-tableau $\Phi(T)$ is obtained by applying the following algorithm:

Step 1: for $\ell = k_1$ downto $j_1 + 1$ do
   Empty all boxes containing $\ell$ in $k_1$th row.
   Replace all $(\ell - 1)$'s in $(k_1 - \ell + 1)$ consecutive rows from the $(\ell - 1)$st row to the $(k_1 - 1)$st row with $\ell$'s.
end for

Step 2: Empty all boxes containing $j_1$ in $k_1$th row.

Step 3: for $\ell = j_2 - 1$ downto $j_1 + 1$ do
Empty all boxes containing $\ell$ in $k_2$th row.
Replace all $(\ell - 1)$'s in $(j_2 - \ell)$ consecutive rows from the $(k_2 - j_2 + \ell)$th row to the $(k_2 - 1)$st row with $\ell$'s.

\textit{end for}

Step 4: Empty all boxes containing $j_1$ in $k_2$th row.

Step 5: \textbf{for} $\ell = k_2$ \textbf{downto} $j_2 + 1$ \textbf{do}
Empty all boxes containing $\ell$ in $k_2$th row.
Replace all $(\ell - 1)$'s in $(k_2 - \ell + 1)$ consecutive rows from the $(\ell - 1)$st row to the $(k_2 - 1)$st row with $\ell - 1$'s.
\textit{end for}

Step 6: Empty all boxes containing $j_2$ in $k_2$th row.

Step 7: \textbf{for} $\ell = j_1 + 1$ \textbf{to} $j_2$ \textbf{do}
Replace all $\ell$'s with $(\ell - 1)$'s.
\textit{end for}

Step 8: \textbf{for} $\ell = j_2 + 1$ \textbf{to} $n$ \textbf{do}
Replace all $\ell$'s with $(\ell - 2)$'s.
\textit{end for}

Step 9: Slide each box at $(p, q)$-position for $1 \leq p \leq k_1 - 1$, $\lambda_{i_1} + 1 \leq q$ or for $k_1 + 1 \leq p \leq k_2 - 1$, $\lambda_{i_2} + 1 \leq q$ down to remove all empty boxes.

\textbf{Example 2.23.} Let $\lambda = (8, 7, 6, 5, 4, 2, 2, 2, 1, 0)$, $\mu = (11, 6, 6, 5, 3, 3, 3, 3, 3, 2)$ and $\nu = (13, 11, 10, 10, 6, 6, 6, 6, 4)$. We have $i_1 = 4$, $i_2 = 6$, $j_1 = 2$, $j_2 = 5$, $k_1 = 5$, $k_2 = 9$ and $\lambda_4 + \lambda_6 + \mu_2 + \mu_5 = \nu_5 + \nu_9$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example223.png}
\caption{Example 2.23}
\end{figure}
3. Extension and conjugation of Theorem 2.17

Each statement of Theorem 2.17 is restrictive to apply for the computation of LR-coefficients, compared to the classical reduction formulae (Theorems 1.3 and 1.4) by Griffiths–Harris. For example, it is not an easy work to check the positivity of the LR-coefficient. We, however, can argue that each statement of Theorem 2.17 is still valid without the condition \( c_{\lambda, \mu} > 0 \).

The proof of KTT theorem by King et al. [12] gives a bijection between a set of LR-hives and a set of pairs of smaller LR-hives of corresponding partitions. That means, \( c_{\lambda, \mu} \neq 0 \) along a product of two LR-coefficients in KTT theorem is nonzero. Therefore, we just have to add conditions for the LR-coefficients of one (or two) part partitions to be nonzero in each case of Theorem 2.17. Note that no extra condition is necessary for \( r = 1, n - 1 \) case since the condition \( \lambda_i + \mu_j = v_k \) guarantees \( c_{(\lambda_i), (\mu_j)} \neq 0 \), meanwhile we need to add conditions \( v_{k_1} - \lambda_{i_1} \leq \mu_{j_1} \leq v_{k_1} - \lambda_{i_2} \) and \( v_{k_2} - \lambda_{i_1} \leq \mu_{j_2} \leq \min\{v_{k_2} - \lambda_{i_2}, v_{k_1} - \lambda_{i_1}\} \) as in Lemma 2.1 when \( r = 2, n - 2 \).

In the following theorem we state four reduction formulae derived from Theorem 2.17 in their most general forms so that they also give conditions for \( c_{\lambda, \mu} = 0 \) as in the classical reduction formulae. We also relax the necessary conditions by replacing the equality with an inequality in (*) of each case.

**Theorem 3.1 (Extension of Theorem 2.17).** Let \( \lambda, \mu \) and \( \nu \) be partitions with \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq n \) and \( |\nu| = |\lambda| + |\mu| \).

**Case (r = 1)** Suppose that there are \( 1 \leq i, j, k \leq n \) satisfying the following condition;

\[ i + j \leq k + 1. \]

\((*)\)

(a) If \( \lambda_i + \mu_j < v_k \), then \( c_{\lambda, \mu}^{(v_k)} = 0 \).

(b) If \( \lambda_i + \mu_j = v_k \), then \( c_{\lambda, \mu}^{(v_k)} = c_{\lambda, (\mu), (\nu)}^{(v_k)} \).

**Case (r = 2)** Suppose that there are \( 1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq n \) such that \( i_1 < i_2, j_1 < j_2, k_1 < k_2 \) satisfying the following conditions;

\[ i_1 + i_2 + j_1 + j_2 \leq k_1 + k_2 + 3. \]

\[ i_1 + j_1 \leq k_1 + 1. \]

\[ i_1 + j_2 \leq k_2 + 1. \]

\[ i_2 + j_1 \leq k_2 + 1. \]

\((*)\)

(a) If \( \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} < v_{k_1} + v_{k_2} \), then \( c_{\lambda, \mu}^{(v_{k_1} + v_{k_2})} = 0 \).

(b) If \( \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = v_{k_1} + v_{k_2} \), then
\( c^v_{\lambda, \mu} = \begin{cases} 
\nu^{(v \setminus \{1\})} & \text{if } v_{k_1} - \lambda_i \leq \mu_j \leq v_{k_1} - \lambda_i, \text{ and } \\
\nu^{(\lambda \setminus \{i\}, \mu \setminus \{j\})} & \text{if } v_{k_1} - \lambda_i \leq v_{k_2} - \lambda_i \leq \min\{v_{k_2} - \lambda_i, v_{k_1} - \lambda_i\}, \\
0 & \text{otherwise.} 
\end{cases} \)

\((r = n - 1)\) Suppose there are \(1 \leq i, j, k \leq n\) satisfying the following condition;

\[ i + j \geq k + n. \quad (\ast) \]

(a) If \(\lambda_i + \mu_j > v_k\), then \(c^v_{\lambda, \mu} = 0\).

(b) If \(\lambda_i + \mu_j = v_k\), then \(c^v_{\lambda, \mu} = c^{v \setminus \{1\}}_{\lambda \setminus \{i\}, \mu \setminus \{j\}}\).

\((r = n - 2)\) Suppose that there are \(1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq n\) such that \(i_1 < i_2, j_1 < j_2, k_1 < k_2\) satisfying the following conditions;

\[ i_1 + i_2 + j_1 + j_2 \geq k_1 + k_2 + 2n - 1, \]
\[ k_2 + n \leq i_2 + j_2, \]
\[ k_1 + n \leq i_2 + j_1, \]
\[ k_1 + n \leq i_1 + j_2. \quad (\ast) \]

(a) If \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} > v_{k_1} + v_{k_2}\), then \(c^v_{\lambda, \mu} = 0\).

(b) If \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = v_{k_1} + v_{k_2}\), then

\[ c^v_{\lambda, \mu} = \begin{cases} 
\nu^{(v \setminus \{1\})} & \text{if } v_{k_1} - \lambda_i \leq \mu_j \leq v_{k_1} - \lambda_i, \text{ and } \\
\nu^{(\lambda \setminus \{i\}, \mu \setminus \{j\})} & \text{if } v_{k_2} - \lambda_i \leq \mu_j \leq \min\{v_{k_2} - \lambda_i, v_{k_1} - \lambda_i\}, \\
0 & \text{otherwise.} 
\end{cases} \]

As we mentioned, reduction formula II (Theorem 1.4) is obtained by applying the reduction formula I (Theorem 1.3) to the LR-coefficient with respect to the conjugate partitions. In the following, we state the conjugated versions of reduction formulae in Theorem 3.1 not in terms of \(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}\) but in terms of \(\lambda, \mu, \nu\).

**Theorem 3.2** (Conjugation of Theorem 3.1). Let \(\lambda, \mu, \nu\) be partitions with \(\ell(\lambda), \ell(\mu), \ell(\nu) \leq n\) and \(|\nu| = |\lambda| + |\mu|\).

\((\widetilde{r} = 1)\) Suppose that there are \(\lambda, \mu, \nu\) such that \(\lambda_i > \lambda_{i+1}, \mu_j > \mu_{j+1}\) and \(v_{k-1} > v_k\) satisfying the following condition;

\[ \lambda_{i+1} + \mu_{j+1} + 1 \leq v_k. \quad (\ast) \]

(a) If \(i + j < k\), then \(c^v_{\lambda, \mu} = 0\).

(b) If \(i + j = k\), then \(c^v_{\lambda, \mu} = c^{v \setminus \{1\}}_{\lambda \setminus \{i\}, \mu \setminus \{j\}}\).

\((\widetilde{r} = 2)\) Suppose that there are \(\lambda_{i_1} > \lambda_{i_2}, \mu_{j_1} > \mu_{j_2}, v_{k_1} > v_{k_2}\) such that \(\lambda_{i_1} > \lambda_{i_1+1}, \lambda_{i_2} > \lambda_{i_2+1}, \mu_{j_1} > \mu_{j_1+1}, \mu_{j_2} > \mu_{j_2+1}\) and \(v_{k_1} > v_{k_1+1}, v_{k_2} > v_{k_2+1}\) satisfying the following conditions:

\[ \lambda_{i_1+1} + \lambda_{i_2+1} + \mu_{j_1+1} + \mu_{j_2+1} + 1 \leq v_{k_1} + v_{k_2}, \]
\[ \lambda_{i_2+1} + \mu_{j_2+1} + 1 \leq v_{k_2}, \]
\(\lambda_{j+1} + \mu_{j+1} + 1 \leq v_k,\)
\(\lambda_{j+1} + \mu_{j+1} + 1 \leq v_k.\)

\((*)\)

(a) If \(i_1 + i_2 + j_1 + j_2 < k_1 + k_2,\) then \(c_{\lambda, \mu} = 0.\)
(b) If \(i_1 + i_2 + j_1 + j_2 = k_1 + k_2,\) then

\[c_{\lambda, \mu}^\nu = \begin{cases} 
  c_{\lambda_i (2k_1, 1k_2 - k_1)} - (2j_1, 1j_2 - j_1) & \text{if } k_2 - i_2 \leq j_2 \leq k_2 - i_1, \text{ and } k_1 - i_2 \leq j_1 \leq \min(k_1 - i_1, k_2 - i_2), \\
  0 & \text{otherwise.}
\end{cases}\]

\(\text{(r = n - 1)}\) Suppose that there are \(\lambda, \mu, v_k\) such that \(\lambda_i > \lambda_{i+1},\) \(\mu_j > \mu_{j+1}\) and \(v_k > v_{k+1}\) satisfying the following condition;

\(\lambda_i + \mu_j \geq v_1 + v_{k+1} + 1.\)

\((*)\)

(a) If \(i + j > k,\) then \(c_{\lambda, \mu}^\nu = 0.\)
(b) If \(i + j = k,\) then \(c_{\lambda, \mu}^\nu = c_{\lambda}^{(1^k)} - (1^k), \mu^{-(1^k)}.\)

\(\text{(r = n - 2)}\) Suppose that there are \(\lambda_{i+1} > \lambda_{i+2}, \mu_{j+1} > \mu_{j+2}, v_{k+1} > v_{k+2}\) such that \(\lambda_{i+1} > \lambda_{i+1}, \lambda_{i+2} > \lambda_{i+2}, \mu_{j+1} > \\
\mu_{j+1}, \mu_{j+2} > \mu_{j+2} \text{ and } v_{k+1} = v_{k+1}, v_{k+2} > v_{k+2}\) satisfying the following conditions;

\(\lambda_{i+1} + \lambda_{i+2} + \mu_{j+1} + \mu_{j+2} \geq v_{k+1} + v_{k+2} + 2v_1 + 1,\)
\(\lambda_{i+1} + \mu_{j+1} \geq v_1 + v_{k+1} + 1,\)
\(\lambda_{i+1} + \mu_{j+1} \geq v_1 + v_{k+1} + 1,\)
\(\lambda_{i+2} + \mu_{j+1} \geq v_1 + v_{k+2} + 1.\)

\((*)\)

(a) If \(i_1 + i_2 + j_1 + j_2 > k_1 + k_2,\) then \(c_{\lambda, \mu} = 0.\)
(b) If \(i_1 + i_2 + j_1 + j_2 = k_1 + k_2,\) then

\[c_{\lambda, \mu}^\nu = \begin{cases} 
  c_{\lambda_i (2k_1, 1k_2 - k_1)} - (2j_1, 1j_2 - j_1) & \text{if } k_2 - i_2 \leq j_2 \leq k_2 - i_1, \text{ and } k_1 - i_2 \leq j_1 \leq \min(k_1 - i_1, k_2 - i_2), \\
  0 & \text{otherwise.}
\end{cases}\]

Moreover, the formulae can be applied when we formally set \(\lambda_0 = \mu_0 = v_1\) and \(v_{n+1} = 0\) in each case.

**Remark 3.3.** To obtain a reduction deleting one column, the strictness conditions, \(\lambda_i > \lambda_{i+1}\) for example, in each case of Theorem 3.2 are essential; if \(\lambda_i = \lambda_{i+1}\) for example in \(r = 1\) case, then we can use \(i - 1\) instead of \(i\) and obtain 0 for LR-coefficient.

**Remark 3.4.** The necessary conditions for \(r = 1, 2\) cases of Theorem 3.2 are different from the ones for \(r = n - 1, n - 2\) cases. For example, a relation among \(\lambda_{i+1}, \mu_{j+1}, v_k\) is necessary for \(r = 1\) case, while we need a relation among \(\lambda_i, \mu_j, v_{k+1}\) for \(r = n - 1\) case. These are because directions of inequalities in the necessary conditions for \(r = 1\) and \(r = n - 1\) cases of Theorem 3.1 are opposite, and considering the conjugated version of those conditions makes the difference.

We state a direct implication of \(r = n - 1\) cases of Theorems 3.1 and 3.2 (reduction formulae I and II by Griffiths and Harris). The consequence is also direct from the definition of LR-coefficients. The following can be effectively used for the calculation of LR coefficients.
Corollary 3.5. Let $\lambda$, $\mu$ and $\nu$ be partitions with $\ell(\lambda)$, $\ell(\mu)$, $\ell(\nu) \leq n$ and $|\nu| = |\lambda| + |\mu|$. Then either we can show that $c_{\lambda,\mu}^{\nu} = 0$ or can find partitions $\lambda'$, $\mu'$, $\nu'$ such that $c_{\lambda,\mu}^{\nu'} = c_{\lambda',\mu'}^{\nu'}$, where $\nu' \neq \lambda'$, $\nu' \neq \mu'$ and $\lambda_n' = \mu_n' = 0$.

Proof. It is well known that $c_{\lambda,\mu}^{\nu} = c_{\mu,\lambda}^{\nu}$ and we argue with $\lambda$ only.

Suppose that $\lambda_n = 1$ then we can apply $r = n-1$ case of Theorem 3.1 with $i = k = 1$ and $j = n$ to show that if $\mu_n > 0$ then $c_{\lambda,\mu}^{\nu} = 0$ and if $\mu_n = 0$ then $c_{\lambda,\mu}^{\nu} = c_{\lambda,\mu|n]}^{\nu|n]}$.

Suppose that $\lambda_n \neq 0$, then $r = n-1$ case of Theorem 3.2 can be applied with $i = n$, $j = 0$ and $k = n$ to show that $c_{\lambda,\mu}^{\nu} = c_{\lambda,(\nu)}^{\nu}$. By successively using one of above two arguments, we can obtain $\lambda'$, $\mu'$, $\nu'$ with desired properties. $\Box$

We give examples of Theorems 3.1 and 3.2 in calculating LR-coefficients.

Notations $=,$, $\equiv$, $\equiv$ (or $=,$, $\equiv$, $\equiv$) respectively are used to indicate that $r = 1,2,n-1$, $r = n - 2$ case of Theorem 3.1 (Theorem 3.2, respectively) is used. Notations $(n-1)^*$ and $\tilde{n-1}^*$ are used to indicate that we use $r = n-1$ case of Theorems 3.1 and 3.2 several times to obtain corresponding $\lambda'$, $\mu'$, $\nu'$ in Corollary 3.5.

Example 3.6.

\[
\begin{align*}
C_{(15,14,14,13,13,13,13,11,11,10,10,9,9,9,9,9,9,7,5,4,3,1,0)}^{\lambda} & = C_{(13,12,11,11,11,11,9,7,5,4,3,1,0)}^{\lambda}, (9,8,7,6,5,5,5,5,5,4,2,2,0,0,0,0,0) \\
\tilde{n-1}^* & = 2 (15,14,14,13,13,13,13,11,11,10,10,10,9,8,8,8,6,6,4,2,1,0) \\
\tilde{n-2} & = (12,11,11,10,10,10,10,10,8,8,7,7) \\
n-2 & = (10,10,9,8,8,8,6,6,4,2,1,0) \\
\tilde{n-2} & = C_{(9,9,9,9,9,9,7,7,5,4,2,1,0)}^{\lambda}, (6,5,5,5,4,4,2,2,2,0,0,0) \\
(n-1)^* & = C_{(9,9,9,9,9,7,7,5,4,2,1,0)}^{\lambda}, (6,5,5,5,4,4,2,2,2,0,0,0) \\
\tilde{n-2} & = (7,7,7,7,5,5,4,4,4) \\
n-2 & = C_{(6,5,5,5,4,2,1,0)}^{\lambda}, (4,3,3,2,2,2,2,2,2,0,0,0) \\
\tilde{n-2} & = (7,7,7,5,5,5,5,4,4) \\
\tilde{n-2} & = C_{(6,5,5,5,4,2,1,0)}^{\lambda}, (4,3,3,2,2,2,2,2,2,0,0,0) \\
\tilde{n-2} & = (7,7,7,5,5,5,5,4) \\
n-2 & = C_{(6,5,5,5,4,2,1,0)}^{\lambda}, (4,3,3,2,2,2,2,2,2,0,0,0) \\
n-1 & = (5,5,5,5,4,4,3) \\
n-2 & = C_{(4,4,3,2,0)}^{\lambda}, (3,2,1,1,1,1) \\
n-2 & = C_{(4,4,3,2,0)}^{\lambda}, (3,2,1,1,1,1) \\
n-2 & = C_{(3,2,0)}^{\lambda}, (3,2,1,1,1,1) \\
= 1.
\end{align*}
\]

Example 3.7.

\[
\begin{align*}
C_{(8,7,5,3,3,1,1,1)}^{(8,7,5,3,3,1,1,1)} & = C_{(7,4,3,0,0,0,0,0)}^{(7,4,3,0,0,0,0,0)}, (7,6,1,1,0,0,0,0,0) \\
\tilde{n-2} & = C_{(6,5,4,2,2,1,1)}^{(6,5,4,2,2,1,1)}, (5,5,1,0,0,0,0,0) \\
\tilde{n-2} & = 0.
\end{align*}
\]
Remark 3.8. There must be triples $\lambda$, $\mu$, and $\nu$ where we cannot apply any of the reduction formulae from Theorem 3.1 or Theorem 3.2. Otherwise every LR-coefficient will be either 0 or 1 as long as the triple satisfy $|\nu| = |\lambda| + |\mu|$. For example, when $\lambda = (6, 5, 4, 3, 2, 1, 0)$, $\mu = (7, 3, 2, 1, 1, 0)$ and $\nu = (9, 7, 6, 6, 3, 3, 2)$, no reduction is applicable. This also is related to the fact that four sets $R_n^r$, $r = 1, 2, n - 1, n - 2$, do not exhaust all the essential triples which play essential roles to determine the positivity of the LR-coefficients.

Remark 3.9. Rank 2 reductions, $r = 2$ and $r = n - 2$ reductions, are not just compositions of rank 1 reductions: When $\lambda = (5, 5, 3, 2, 2, 0, 0, 0)$, $\mu = (4, 3, 3, 3, 1, 0, 0, 0)$ and $\nu = (6, 6, 5, 5, 4, 4, 2, 2, 2)$, none of $r = 1$, $r = n - 1$ reductions or their conjugated reductions is applicable while $r = n - 2$ reduction is applicable.

4. Applications

In this section, we consider the cases when the LR-coefficient is 1.

The following theorem has been conjectured by Fulton [8] and was proved by Knutson, Tao and Woodward [14]. For a partition $\alpha = (\alpha_1, \ldots, \alpha_t)$ and a positive integer $N$, we let $N\alpha$ denote the partition $(N\alpha_1, \ldots, N\alpha_t)$.

Theorem 4.1. For any positive integer $N$, we have $c_{N\alpha, \mu}^\nu = 1$ if and only if $c_{N\lambda, N\mu}^\nu = 1$.

An immediate conclusion, due to Buch [1], of the previous theorem and the work by Knutson, Tao and Woodward [14] on essential inequalities defining the facets of Littlewood–Richardson cone is the following.

Corollary 4.2. For $n \geq 3$, if each of partitions $\lambda$, $\mu$, $\nu$ consists of $n$ distinct integers, and if each of the inequalities in Theorem 1.7 holds strictly, then $c_{\lambda, \mu}^\nu$ must be at least 2.

We now state a theorem as an application of reduction formulae in Theorem 3.1.

Theorem 4.3. If each of the partitions in $\lambda$, $\mu$, $\nu$ consists of $n$ distinct integers and $c_{\lambda, \mu}^\nu = 1$, then we can repeatedly apply either $(r = 1)$ or $(r = 2)$ reduction of Theorem 3.1 to obtain triples of empty partitions, whose LR-coefficient is 1.

Proof. We use an inductive proof on the length $n$ of the partition $\nu$. Obviously $(r = 1)$ reduction is applicable when $n = 1$, and $(r = 2)$ reduction is applicable if $n = 2$.

Let $n \geq 3$ and suppose that the theorem holds for all positive integers less than $n$. Then, because of Corollary 4.2, one of the inequalities in Theorem 1.7 must be an equality. We therefore have $(I, J, K) \in R_n^r$ for some $0 < r < n$ such that

$$c_{\lambda, \mu}^\nu = c_{\lambda I, \mu J}^\nu c_{\lambda K, \mu F}^\nu$$

by the factorization theorem (Theorem 1.8). This implies $c_{\lambda I, \mu J}^\nu = 1$ and $c_{\lambda K, \mu F}^\nu = 1$ and the theorem holds for two triples $(\lambda I, \mu J, \nu K)$, $(\lambda K, \mu F, \nu K)$ by induction hypothesis since they are triples of partitions of distinct parts also. We will show that $(r = 1)$, $(r = 2)$ reductions for $(\lambda I, \mu J, \nu K)$ imply the $(r = 1)$, $(r = 2)$ reductions for $(\lambda, \mu, \nu)$ respectively.

Let $I = \{i_1, i_2, \ldots, i_r\}$, $J = \{j_1, j_2, \ldots, j_r\}$ and $K = \{k_1, k_2, \ldots, k_r\}$. Suppose that $(r = 1)$ reduction is applicable to $(\lambda I, \mu J, \nu K)$. Then, there are $1 \leq a, b, c \leq r$ such that $a + b \leq c + 1$ and $\lambda_{i_a} + \mu_{j_b} = \nu_{k_c}$.

Moreover, since $(I, J, K) \in R_n^r$, we have $c_{\pi(I), \pi(J)}^\pi(K) = 1$. Remember that the $(r + 1 - s)$th element of $\pi(I)$ is $s_3 - s$ for $1 \leq s \leq r$. By the first part of $(r = n - 1)$ reduction in Theorem 3.1 for the triple $(\pi(I), \pi(J), \pi(K))$, we can deduce the following: Whenever $(r + 1 - s) + (r + 1 - t) \geq (r + 1 - u) + r$
or equivalently $s + t \leq u + 1$, we have $(i_s - s) + (j_t - t) \leq k_u - u$ and hence $i_s + j_t - k_u \leq s + t - u \leq 1$ that is $i_s + j_t \leq k_u + 1$. We now have $i_s + j_b \leq k_s + 1$ since $a + b \leq c + 1$. This implies that $(r = 1)$ reduction is applicable to the triple $(\lambda, \mu, \nu)$ with three indices $i_s, j_b$ and $k_c$.

As the second case, let us suppose that $(r = 2)$ reduction is applicable to $(\lambda_1, \mu_j, \nu_k)$. Then there are $a_1, a_2, b_1, b_2$ and $c_1, c_2$ such that

$$a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2 + 3,$$

$$a_1 + b_1 \leq c_1,$$

$$a_1 + b_2 \leq c_2 + 1,$$

$$a_2 + b_1 \leq c_2 + 1$$

and

$$v_{k_1} + v_{k_2} = \lambda_{ia_1} + \lambda_{ia_2} + \mu_{jb_1} + \mu_{jb_2}.$$

Moreover $c_{(\lambda_{ia_1}, \lambda_{ia_2}), (\mu_{jb_1}, \mu_{jb_2})} = 1$. We show that $i_{a_1}, i_{a_2}, j_{b_1}, j_{b_2}$ and $k_{c_1}, k_{c_2}$ satisfy the $(r = 2)$ reduction condition, which completes the proof. If we apply the $(r = n - 2)$ of Theorem 3.1 to $(\pi(I), \pi(J), \pi(K))$ then we obtain the following: Whenever

$$s_1 + s_2 + t_1 + t_2 \leq u_1 + u_2 + 3,$$

$$s_1 + t_1 \leq u_1 + 1,$$

$$s_1 + t_2 \leq u_2 + 1,$$

$$s_2 + t_1 \leq u_2 + 1,$$ \hfill (4.4)

we have

$$(i_{s_1} - s_1) + (i_{s_2} - s_2) + (j_{t_1} - t_1) + (j_{t_2} - t_2) \leq (k_{u_1} - u_1) + (k_{u_2} - u_2).$$

Note that $a_1, a_2, b_1, b_2$ and $c_1, c_2$ satisfy the conditions (4.4) and we have the inequality

$$(i_{a_1} - a_1) + (i_{a_2} - a_2) + (j_{b_1} - b_1) + (j_{b_2} - b_2) \leq (k_{c_1} - c_1) + (k_{c_2} - c_2)$$

and this implies that

$$i_{a_1} + i_{a_2} + j_{b_1} + j_{b_2} \leq a_1 + a_2 + b_1 + b_2 - c_1 - c_2 + k_{c_1} + k_{c_2} \leq 3 + k_{c_1} + k_{c_2}. \quad (4.5)$$

If we apply the $(r = n - 1)$ of Theorem 3.1 to $(\pi(I), \pi(J), \pi(K))$ then we obtain the following: Whenever $s + t \leq u + 1$, we have $i_s + j_t \leq k_u$. Note that triples $a_1, b_1, c_1, a_1, b_2, c_2$ and $a_2, b_1, c_2$ satisfy the necessary conditions for the following identities:

$$i_{a_1} + j_{b_1} \leq k_{c_1}, \quad i_{a_1} + j_{b_2} \leq k_{c_2}, \quad i_{a_2} + j_{b_1} \leq k_{c_2}. \quad (4.6)$$

Because of (4.5) and (4.6) $(r = 2)$ reduction is application to $(\lambda, \mu, \nu)$ with indices $i_{a_1}, i_{a_2}, j_{b_1}, j_{b_2}$ and $k_{c_1}, k_{c_2}$. □
Remark 4.7. Theorem 4.3 certainly gives a polynomial time algorithm in \( n \). To check if \( (r = 1) \) or \( (r = 2) \) reduction is applicable requires polynomial time (quadratic or quintic, respectively) complexity and at most \( n \) reductions are enough. Hence the worst time complexity of the algorithm from Theorem 4.3 to decide the LR-coefficient is 1 is \( O(n^6) \). In [2], Bürgisser and Ikenmeyer also suggested a polynomial time algorithm (in \( n \) and \( \log |\nu| \)) to decide multiplicity freeness of the LR-coefficients when the given triples have distinct parts. Their algorithm checks if certain cycles exist in a network constructed from given three partitions. They, however, did not give the degree of the complexity.

Remark 4.8. Theorem 4.3 does not hold if a partition among \( \lambda, \mu, \nu \) has repeated parts. A simple example is \( \lambda = (3, 0, 0), \mu = (4, 2, 0) \) and \( \nu = (5, 3, 1) \). It is easy to check that \( c_{\lambda, \mu}^\nu = 1 \), but neither \( (r = 1) \) nor \( (r = 2) \) reduction is applicable.

Our extensive experiment with maple programming and some rigorous arguments on small cases lead us to believe that rank 1 reductions \( (r = 1), (r = n - 1), (r = 1), (r = n - 1) \) are enough to reduce a triple of partitions to that of empty partitions if the corresponding LR-coefficient is 1.

Conjecture 4.9. If \( c_{\lambda, \mu}^\nu = 1 \), then one of \( (r = 1), (r = n - 1), (r = 1), (r = n - 1) \) reductions is always applicable.

Example 4.10. To the triple \( \lambda = (3, 0, 0), \mu = (4, 2, 0), \nu = (5, 3, 1) \) in Remark 4.8, both \( (r = 1) \) and \( (r = n - 1) \) reductions are applicable.

Example 4.11. The following shows an example of Conjecture 4.9

\[
\begin{align*}
\{10, 10, 6.5, 4.3, 3.3, 1.1, 1\} & \quad \{6.5, 3.1, 1.0, 0.0, 0.0, 0.0\} \\
\{7, 6.5, 3.0, 0.0, 0.0, 0.0\} \quad \{6.5, 5.5, 3.0, 0.0, 0.0, 0.0\} \quad \{5.4, 2.0, 0.0, 0.0\}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{c_{\{10, 10, 6.5, 4.3, 3.3, 1.1\}}^{\{7, 6.5, 3.1, 1.0, 0.0, 0.0\}, \{6.5, 5.5, 3.0, 0.0, 0.0, 0.0\}}^{\{5.4, 2.0, 0.0, 0.0\}}} & = \frac{1}{c_{\{8, 8.3, 2.2, 2\}^{\{5.4, 3.2, 0.0, 0.0\}, \{5.4, 4.2, 0.0, 0.0\}}}^{\{5.4, 2.0, 0.0, 0.0\}}} = \frac{1}{c_{\{7.1, 4.0, 4.0\}}}^{\{3.3\}} = 1.
\end{align*}
\]

5. Proofs

5.1. Proof of Theorem 3.1

We first assume that the inequality in each (\*) is an equality. Then the conditions for \( i, j, k \) are the ones for the triple \( (i, j, k) \) to be an element of \( R^\nu_1 \) when \( r = 1 \), and \( (i^c, j^c, k^c) \) to be an element of \( R^\nu_{n-1} \) when \( r = n - 1 \). The conditions for \( i_1, j_1, j_2, k_1, k_2 \) are the ones for the triple \( (I, J, K) \), where \( I = [i_1, i_2], J = [j_1, j_2], K = [k_1, k_2] \), to be an element of \( R^\nu_2 \) when \( r = 2 \), and \( (F^c, J^c, K^c) \) to be an element of \( R^\nu_{n-2} \) when \( r = n - 2 \).

Now one can check that the condition for the first case (case (a)) in each case coincides with \( \sum_{k \in K} v_k > \sum_{j \in J} \lambda_j + \sum_{i \in I} j_j \) with corresponding essential Horn triple \( (I, J, K) \). This proves the first part of each case due to Theorem 1.7. The second part of each case is Lemma 2.17 without the nonzero condition for \( c_{\lambda, \mu}^\nu \). Note here that two conditions \( v_{k_2} - \lambda_{i_1} \leq \mu_{j_2} \leq \min\{v_{k_1} - \lambda_{i_1}, v_{k_1} - \lambda_{i_1}\} \)

\( v_{k_1} - \lambda_{i_1} \leq \mu_{j_1} \leq v_{k_1} - \lambda_{i_2} \) are equivalent to the condition \( c_{\lambda_1, \lambda_2}^{(v_{k_1}, v_{k_2})} \neq 0 \) by Theorem 2.1, with which reduction occurs and without which the LR-coefficient vanishes.
Assume that the inequality of \((*)\) in each case of Theorem 3.1 is strict. We consider only \(r = 2\) case here; other cases can be shown in similar ways. We, hence, will show the following:

Suppose there are \(1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq n\) such that \(i_1 < i_2, j_1 < j_2, k_1 < k_2\) satisfying the following conditions

\[
i_1 + i_2 + j_1 + j_2 < k_1 + k_2 + 3, \\
i_1 + j_1 \leq k_1 + 1, \\
i_1 + j_2 \leq k_2 + 1, \\
i_2 + j_1 \leq k_2 + 1. \quad (***)
\]

(i) If \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} < v_{k_1} + v_{k_2}\), then \(c^v_{\lambda, \mu} = 0\).

(ii) If \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = v_{k_1} + v_{k_2}\), then

\[
c^v_{\lambda, \mu} = \begin{cases} 
\frac{v_i v_{k_1} v_{k_2}}{c_{\lambda \lambda(i_1, i_2) \mu \mu(j_1, j_2)}} & \text{if } v_{k_1} - \lambda_{i_1} \leq \mu_{j_1} \leq v_{k_1} - \lambda_{i_2}, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

We first show that we can make the condition (*** an equality by changing \(k_1\) and \(k_2\) without altering other conditions.

**Lemma 5.1.** We can find \(\bar{k}_1 \leq k_1\) and \(\bar{k}_2 \leq k_2\) so that

\[
i_1 + i_2 + j_1 + j_2 = \bar{k}_1 + \bar{k}_2 + 3, \\
i_1 + j_1 \leq \bar{k}_1 + 1, \\
i_1 + j_2 \leq \bar{k}_2 + 1, \\
i_2 + j_1 \leq \bar{k}_2 + 1. \quad (5.2)
\]

**Proof.** Let \(x = k_1 + 1 - (i_1 + j_1) \geq 0\) and \(y = k_2 + 2 - (i_2 + j_2)\). Then

\[
x + y = k_1 + k_2 + 3 - (i_1 + i_2 + j_1 + j_2) > 0.
\]

If \(y < 0\), then let \(\bar{k}_1 = k_1 - x - y > 0\) and \(\bar{k}_2 = k_2\). If \(y \geq 0\), then let \(\bar{k}_1 = k_1 - x > 0\) and \(\bar{k}_2 = k_2 - y = i_2 + j_2 - 2 > 0\). One can check that \(\bar{k}_1, \bar{k}_2\) satisfy the desired properties. \(\square\)

Note that \(v_{\bar{k}_1} \geq v_{k_1}\) and \(v_{\bar{k}_2} \geq v_{k_2}\).

Assuming \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} < v_{k_1} + v_{k_2}\), we can see that \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} < v_{k_1} + v_{\bar{k}_2}\) and we can apply the current theorem (when we have an equality in \((*)\)) with \(i_1, i_2, j_1, j_2, \bar{k}_1, \bar{k}_2\) to conclude that \(c^v_{\lambda, \mu} = 0\). Therefore, the case (i) is proved.

When \(\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} = v_{k_1} + v_{k_2}\), there are two cases to be considered: The first case is when \(v_{\bar{k}_1} = v_{k_1}\) and \(v_{\bar{k}_2} = v_{k_2}\), and the second case is when either \(v_{\bar{k}_1} > v_{k_1}\) or \(v_{\bar{k}_2} > v_{k_2}\).

For the first case, i.e., when \(v_{\bar{k}_1} = v_{k_1}\) and \(v_{\bar{k}_2} = v_{k_2}\), by applying the current theorem (when we have equality in \((*)\)) with \(i_1, i_2, j_1, j_2, \bar{k}_1, \bar{k}_2\), we can show that

\[
c^v_{\lambda, \mu} = c_{\lambda \lambda(i_1, i_2) \mu \mu(j_1, j_2)} = c_{\lambda \lambda(i_1, i_2) \mu \mu(j_1, j_2)},
\]
when \( c^{(\nu_1, \nu_2)}_{(\lambda_1, \lambda_2), (\mu_j, \mu_{j_2})} \neq 0 \). If \( c^{(\nu_1, \nu_2)}_{(\lambda_1, \lambda_2), (\mu_j, \mu_{j_2})} = 0 \) then \( c^{(*)}_{(\lambda_1, \lambda_2), (\mu_j, \mu_{j_2})} = 0 \), and we have \( c^{(*)}_{\lambda, \mu} = 0 \) by applying the current theorem when we have an equality in \((*)\) with \( i_1, i_2, j_1, j_2, k_1, k_2 \).

For the second case, that is when either \( \nu^*_{k_1} > v_{k_1} \) or \( \nu^*_{k_2} > v_{k_2} \) is true, since \( \nu^*_i + \nu_{i} = k \), we can see that \( c^{(*)}_{\lambda, \mu} = 0 \). On the other hand, by letting \( \lambda' = \lambda \setminus \lambda_{\{i, j\}}, \mu' = \mu \setminus \mu_{\{j_1, j_2\}}, \nu' = \nu \setminus \nu_{\{k_1, k_2\}} \) and considering

\[
\lambda'_{i_1} + \lambda'_{j_2} + \mu'_{j_1} + \mu'_{j_2} = \lambda_{i_1} + \lambda_{j_2} + \mu_{j_1} + \mu_{j_2} + 1 \\
\leq \lambda_{i_1} + \lambda_{j_2} + \mu_{j_1} + \mu_{j_2} \\
= \nu_1 + \nu_2 \\
< \nu_{k_1} + \nu_{k_2},
\]

we can show that \( c^{(*)}_{\lambda', \mu'} = 0 \). This proves the case (ii). \( \square \)

5.2. Proof of Theorem 3.2 when \( r = n - 2 \)

Theorem 3.2 is a conjugated version of Theorem 3.1. More precisely, we can obtain Theorem 3.2 by applying Theorem 3.1 to the conjugated triple \( \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \) as reduction formula II has been obtained from reduction formula I by Griffiths and Harris [9].

A proof for the conjugated \( r = n - 1 \) case, i.e. reduction formula II, is in [3] and we follow the same line of that proof for \( r = n - 2 \) case. Other cases can be proved similarly and we omit the proof for \( r = 1, 2 \).

We first state a useful lemma which can be easily proved by looking at Young diagrams of a partition and its conjugate.

**Lemma 5.6.** For a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), let \( \tilde{\lambda} \) be its conjugate. Then, obviously from the definition of the conjugate partition, \( \tilde{\lambda}_{\beta} \) is the number of \( i \)'s such that \( \lambda_i \geq \beta \). Moreover, if \( \lambda_i = \beta \) the followings are satisfied:

(i) \( \tilde{\lambda}_{i \beta} \geq i \), and \( \tilde{\lambda}_{i \beta} = i \) if and only if \( \lambda_i > \lambda_{i+1} \).

(ii) If \( \lambda_{i-1} > \lambda_i \), then \( \tilde{\lambda}_{i+1} = i - 1 \).

Suppose that there are \( \lambda_{i_1} > \lambda_{i_2}, \mu_{j_1} > \mu_{j_2}, v_{k_1} > v_{k_2} \) such that \( \lambda_{i_1} > \lambda_{i_1+1}, \lambda_{i_2} > \lambda_{i_2+1}, \mu_{j_1} > \mu_{j_1+1}, \mu_{j_2} > \mu_{j_2+1} \) and \( v_{k_1} > v_{k_1+1}, v_{k_2} > v_{k_2+1} \) satisfying the following conditions;

\[
\lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2} \geq v_{k_1+1} + v_{k_2+1} + 2v_1 + 1, \\
\lambda_{i_1} + \mu_{j_1} \geq v_1 + v_{k_1+1} + 1, \\
\lambda_{i_2} + \mu_{j_2} \geq v_1 + v_{k_2+1} + 1, \\
\lambda_{i_2} + \mu_{j_1} \geq v_1 + v_{k_2+1} + 1. \quad (*)
\]

If we let

\[
\lambda_{i_1} = a_1, \quad \lambda_{i_2} = a_2, \\
\mu_{j_1} = b_1, \quad \mu_{j_2} = b_2, \\
v_{k_1+1} = c_1 - 1, \quad v_{k_2+1} = c_2 - 1,
\]

then \( a_1 > a_2, b_1 > b_2 \) and \( c_1 > c_2 \). Moreover, by Lemma 5.6, we have
\[ x_{a_1} = i_1, \quad x_{a_2} = i_2, \]
\[ \tilde{\mu}_b = j_1, \quad \mu_{b_2} = j_2. \]
\[ v_{\tilde{c}_1} = k_1, \quad v_{\tilde{c}_2} = k_2. \]

Therefore the given conditions can be written as follows

\[
a_1 + a_2 + b_1 + b_2 \geq (c_1 - 1) + (c_2 - 1) + 2v_1 + 1 \tag{5.7}
\]
\[
= c_1 + c_2 + 2v_1 - 1, \tag{5.8}
\]
\[
a_1 + b_1 \geq v_1 + c_1, \tag{5.9}
\]
\[
a_1 + b_2 \geq v_1 + c_2, \tag{5.10}
\]
\[
a_2 + b_1 \geq v_1 + c_2. \tag{5.11}
\]

One can check that Eqs. (5.8)–(5.11) give exactly the same conditions for the \( r = n - 2 \) case of Theorem 3.1 with \( v_1 \) instead of \( n \). Therefore we can apply Theorem 3.1 to the triple \( \tilde{\lambda}, \tilde{\mu}, \tilde{v} \) to obtain

(i) if \( \tilde{\lambda}_a + \tilde{\lambda}_b + \tilde{\mu}_b > \tilde{v}_{\tilde{c}_1} + \tilde{v}_{\tilde{c}_2} \), then \( c_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{v}} = 0 \),
(ii) if \( \tilde{\lambda}_a + \tilde{\lambda}_b + \tilde{\mu}_b = \tilde{v}_{\tilde{c}_1} + \tilde{v}_{\tilde{c}_2} \), then

\[
c_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{v}} = \begin{cases} 
\tilde{c}_{\tilde{\lambda}, \tilde{\mu}, \tilde{c}_1}^{\tilde{v}} & \text{if } \tilde{v}_{\tilde{c}_1} - \tilde{\lambda}_a \leq \tilde{\mu}_b \leq \tilde{v}_{\tilde{c}_2} - \tilde{\lambda}_a, \\
0 & \text{otherwise},
\end{cases}
\]

which also can be written as, by Lemma 1.2,

(a) if \( i_1 + i_2 + j_1 + j_2 > k_1 + k_2 \), then \( c_{\lambda, \mu}^{v} = 0 \),
(b) if \( i_1 + i_2 + j_1 + j_2 = k_1 + k_2 \), then

\[
c_{\lambda, \mu}^{v} = \begin{cases} 
c_{\lambda, \mu, (i_1, i_2, j_1, j_2)}^{v} & \text{if } k_2 - i_2 \leq j_2 \leq k_2 - i_1, \text{ and } k_1 - i_2 \leq j_1 \leq \min\{k_1 - i_1, k_2 - i_2\}, \\
0 & \text{otherwise.}
\end{cases}
\]

This finishes the proof of Theorem 3.2 when \( r = n - 2 \). \( \square \)

**Remark 5.12.** The proof of Theorem 3.2 works particularly easily when \( i_1 = 0, j_1 = 0 \) or \( k_2 = n \) with \( \lambda_0 = \mu_0 = v_1 \) and \( v_{n+1} = 0 \).

6. **Open problems**

1. Characterize primitive triples. A triple of partitions \((\lambda, \mu, \nu)\) is a primitive if \( c_{\lambda, \mu}^{\nu} > 0 \) but no reduction is applicable.
2. Give a proof of Conjecture 4.9.

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References