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Transitive ovoids of the Hermitian surface of $PG(3, q^2)$, q even

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Abstract

As it is well known, the transitive ovoids of $PG(3, q)$ are the non-degenerate quadrics and the Suzuki–Tits ovoids (see in: A. Blokhuis, J.W.P. Hirschfeld, D. Jugnickel, J.A. Thas (Eds.), Finite Geometries, Proceedings of the Fourth Isle of Thorns Conference, Developments in Mathematics, Kluwer, Boston, 2001, pp. 121–131). Kleidman (J. Algebra 117 (1988) 117) classified the 2-transitive ovoids of finite classical polar spaces. Kleidman's result was partially improved by Gunawardena (J. Combin. Theory Ser. A 89 (2000) 70) who determined the primitive ovoids of the quadric $O_8^+(q)$. Transitive ovoids of the classical polar space arising from the Hermitian surface $\mathscr{H}(3, q^2)$ of $PG(3, q^2)$ with even q are investigated in this paper. There are known two such ovoids up to projectivity, namely the classical ovoid and the Singer-type ovoid. Both are *linearly* transitive in the sense that the subgroup of $PGU(4, q^2)$ preserving the ovoid is still transitive on it. Furthermore, the full collineation group preserving either of them is a subgroup of $P\Gamma U(3, q^2)$. Our main result states that for q even the only linearly transitive ovoids are the classical ovoids and the Singer-type ovoid. It remains open the problem of finding other (i.e. non-linearly) transitive ovoids, although we prove that the full collineation group of any transitive ovoid is a subgroup of $P\Gamma U(3, q^2)$.

Keywords: Ovoid; Hermitian surface; Hermitian curve; Singer cycle

1. Introduction

A non-degenerate Hermitian variety, defined as the set of all self-conjugate points of a non-degenerate unitary polarity of $PG(r, q^2)$, provides an important example of

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finite classical polar spaces. The combinatorial properties of non-degenerate Hermitian varieties have been studied together with those of quadrics and linear complexes in the general theory of polar spaces, see [32]. In this setting important objects are the ovoids. An *ovoid* $\mathcal{O}$ of a non-degenerate Hermitian variety $\mathcal{H}(r,q^2)$ with $r\geqslant 3$ is a set of points on $\mathcal{H}(r,q^2)$ which has exactly one common point with every generator of $\mathcal{H}(r,q^2)$. Here "generator" means "projective subspace of maximum dimension lying on $\mathcal{H}(r,q^2)$". In even dimension r, Thas [30] proved that $\mathcal{H}(r,q^2)$ has no ovoid. In odd dimension r, the existence problem is still open for $r>3$, apart from some special cases settled with negative answer by Blokhuis and Moorhouse [3] and Moorhouse [25]. On the other hand, so many projectively non-equivalent ovoids of $\mathcal{H}(3,q^2)$ are known to exist that their classification seems to be possible only under some extra condition.

In this paper we investigate the transitive ovoids of $\mathcal{H}(3,q^2)$ for even q. An ovoid $\mathcal{O}$ is said to be *transitive* if the full collineation group $\mathrm{P}\Gamma\mathrm{U}(4,q^2)$ of $\mathcal{H}(3,q^2)$ has a subgroup G that preserves $\mathcal{O}$ and acts transitively on $\mathcal{O}$. If G happens to be a subgroup of the projective unitary group $\mathrm{PGU}(4,q^2)$ then $\mathcal{O}$ is said to be *linearly* transitive. The generators of $\mathcal{H}(3,q^2)$ are the lines lying on $\mathcal{H}(3,q^2)$, and every ovoid of $\mathcal{H}(3,q^2)$ has size q^3+1. Any non-tangent plane of $\mathcal{H}(3,q^2)$ cuts out on $\mathcal{H}(3,q^2)$ a Hermitian curve $\mathcal{H}(2,q^2)$ which is a linearly transitive ovoid (called the *classical ovoid*) of $\mathcal{H}(3,q^2)$. In Section 3, we construct another linearly transitive ovoid $\mathcal{O}$ of $\mathcal{H}(3,q^2)$ for every even $q>2$. Our construction depends on the existence of a spread of $\mathcal{H}(2,q^2)$ left invariant under the Singer group of $\mathcal{H}(2,q^2)$, see [2]. We will use the term of *Singer-type ovoid* to denote any ovoid of $\mathcal{H}(3,q^2)$ projectively equivalent to $\mathcal{O}$. The subgroup of $\mathrm{PGU}(4,q^2)$ preserving the Singer-type ovoid $\mathcal{O}$ has order $3(q^3+1)$. More precisely, it is the semidirect product of a cyclic normal subgroup of order q^3+1 by a subgroup of order 3. Also, the non-linear collineations arising from the automorphisms of $\mathrm{GF}(q^2)$ preserves $\mathcal{O}$, and the subgroup of $\mathrm{P}\Gamma\mathrm{U}(4,q^2)$ preserving the Singer-type ovoid has order $3d(q^3+1)$ where $q=2^d$. Our main result is the following theorem.

Transitive ovoids of other classical polar spaces have been investigated in [8], [14], and [23].

Theorem 1.1. *The classical ovoid and the Singer-type ovoid are the only linearly transitive ovoids of $\mathcal{H}(3,q^2)$ with q even.*

We remark that Theorem 1.1 for the smallest case, $q=2$, follows from the classification of ovoids of $\mathcal{H}(3,4)$ given in [5].

Theorem 1.2 is Theorem 1.1 rephrased in terms of 1-spreads of the elliptic quadric $Q^-(5,q)$ of $\mathrm{PG}(5,q)$. This depends on the fact that the incidence structure consisting of all points and lines of $\mathcal{H}(3,q^2)$ is isomorphic to the dual of the incidence structure consisting of all points and lines of $Q^-(5,q)$. In fact, under such an isomorphism, ovoids of $\mathcal{H}(3,q^2)$ and 1-spreads of $Q^-(5,q)$ of $\mathrm{PG}(5,q)$ are equivalent objects. For some more information on this isomorphism, see [16, Chapter 19].

Theorem 1.2. *For even q, there exist exactly two projectively non-equivalent linearly transitive 1-spreads of the elliptic quadric $Q^-(5,q)$ of $\mathrm{PG}(5,q)$.*

Theorem 1.2 completes previous work on transitive 1-spreads of $Q^-(5,q)$ done by Dye [11], Thas [29] and Hamilton and Penttila [15]. In fact, one of the linear transitive 1-spread of $Q^-(5,q)$ corresponds to the classical ovoid of $\mathcal{H}(3,q^2)$ via the natural embedding of the unitary group $U_3(q^2)$ in the orthogonal group $\mathrm{O}_6^-(q)$, see [11]. The other 1-spread derives from a 2-spread of $Q(6,q)$, see [30], and has a transitive cyclic automorphism group, see [15]. According to Theorem 1.2, this 1-spread corresponds to the Singer-type ovoid.

It remains open the problem of finding all (that is non linearly) transitive ovoids of $\mathcal{H}(3,q^2)$. However, we are able to prove the following result.

Theorem 1.3. *For any transitive ovoid $\mathcal{O}$ of $\mathcal{H}(3,q^2)$, the collineation group preserving $\mathcal{O}$ fixes a point outside $\mathcal{H}(3,q^2)$ and hence it is isomorphic to a subgroup of $\mathrm{P}\Gamma\mathrm{U}(3,q^2)$.*

The concept of an ovoid of a polar space is generalized to that of a *partial ovoid*, see [24,26,27,32], which is defined to be any point set meeting every generator at most in one point. A partial ovoid is called *complete* (or *maximal*), if it is not properly contained in another partial ovoid. A natural generalization of the above problem is to find and classify transitive complete partial ovoids of $\mathcal{H}(3,q^2)$. In the last section, we give two examples for even q both arising from null polarities of $\mathrm{PG}(3,q^2)$ commuting with the unitary polarity associated with $\mathcal{H}(3,q^2)$. Actually, we show that if q is even then any transitive ovoid of the 3-dimensional symplectic space over $\mathrm{GF}(q)$ can be lifted to a transitive complete partial ovoid of $\mathcal{H}(3,q^2)$. In this way, two examples are obtained, namely one from the non-degenerate quadric and the other from the Suzuki–Tits ovoid.

2. Notation, terminology and preliminary results

Hermitian varieties and their combinatorial properties have been the subject of numerous papers. For previous results playing a role in the proof of Theorems 1.1 and 1.2, see [17, 2.1.5]; [16, Chapter 19]; [18, Chapters 22, 23]; [20, Chapter 2]. The proof of Theorem 1.1 also requires some new results on the Singer groups of a Hermitian curve, stated in the three lemmas below.

Let $\pi = \mathrm{PG}(2,q^2)$ denote the Desarguesian projective plane over the finite field $\mathrm{GF}(q^2)$ of order q^2, where q is any prime power. A (non-degenerate) *Hermitian curve* $\mathcal{H}(2,q^2)$ in π is defined as the set of all absolute points of a (non-degenerate) unitary polarity of π. The number of points of $\mathcal{H}(2,q^2)$ is equal to q^3+1. If P is a point in π, then the polar line $P^\perp$ of P with respect to $\mathcal{H}(2,q^2)$ meets $\mathcal{H}(2,q^2)$ in either 1 or $q+1$ points, according as P lies on $\mathcal{H}(2,q^2)$ or does not. Similarly, if ℓ is a line in π, then the pole $\ell^\perp$ of ℓ with respect to $\mathcal{H}(2,q^2)$ lies on $\mathcal{H}(2,q^2)$ or does not, according as ℓ meets $\mathscr{H}(2,q^2)$ in 1 or $q+1$ points. Lines of the first type are absolute lines and are called *tangents*, those of second type are non-absolute lines and are called *chords* or *secants* of $\mathscr{H}(2,q^2)$. There is just one tangent at every point $P \in \mathscr{H}(2,q^2)$, while the remaining q^2 lines through P are chords. If $P \notin \mathscr{H}(2,q^2)$, then through P there are $q+1$ tangents (meeting $\mathscr{H}(2,q^2)$ in the points of $P^\perp \cap \mathscr{H}(2,q^2)$) and q^2-q chords.

The Singer collineation group $\mathscr{S}$ of $\mathscr{H}(2,q^2)$, q even, is defined to be any one of the pairwise conjugate cyclic subgroups of order q^2-q+1 in the linear collineation group $\mathrm{PGU}(3,q^2)$ preserving $\mathscr{H}(2,q^2)$. In fact, $\mathscr{S}$ is a subgroup of a Singer group of $\mathrm{PG}(2,q^2)$ where the term of Singer group of $\mathrm{PG}(2,q^2)$ stands for any of the pairwise conjugate cyclic linear collineation groups of $\mathrm{PG}(2,q^2)$ which act on points (and lines) in π as a sharply transitive permutation group. The normaliser $N(\mathscr{S})$ of $\mathscr{S}$ in $\mathrm{PGU}(3,q^2)$ is $\mathscr{S} \rtimes C_3$, the semidirect product of $\mathscr{S}$ by a cyclic group of order 3, and it is a maximal subgroup of $\mathrm{PGU}(3,q^2)$, see [19]. If N_Γ is the normaliser of $N(\mathscr{S})$ in $\mathrm{P}\Gamma\mathrm{U}(3,q^2)$, then $N_\Gamma/N(\mathscr{S})$ is isomorphic to the Galois group of $\mathrm{GF}(q^2)/\mathrm{GF}(2)$; in particular the normaliser of $N(\mathscr{S})$ in $\mathrm{P}\gamma\mathrm{U}(3,q^2)$ has order $2|N(\mathscr{S})|$.

Although $\mathscr{S}$ is defined with respect to a given Hermitian curve $\mathscr{H}(2,q^2)$, some other non-degenerate Hermitian curves in π are also preserved by $\mathscr{S}$. More precisely, the following properties hold, see [7]:

(i) The number of S-invariant non-degenerate Hermitian curves in π is equal to q^2+q+1.
(ii) Any two distinct $\mathscr{S}$-invariant non-degenerate Hermitian curves have exactly $q+1$ common points; such points form a unique orbit under $\mathscr{S}$.
(iii) Any two distinct $\mathscr{S}$-invariant non-degenerate Hermitian curves have exactly $q+1$ common tangent lines; such lines form a unique orbit under $\mathscr{S}$.

Moreover, every point orbit under $\mathscr{S}$ has the interesting combinatorial property of being a (q^2-q+1)-arc, see [4,6,12,13,22]. There are q^2+q+1 of such point orbits, $q+1$ of them lying on $\mathscr{H}(2,q^2)$, and q^2 being disjoint from $\mathscr{H}(2,q^2)$. Since $q^2 \equiv 1 \pmod 3$, one of the latter orbits must also be preserved by $N(\mathscr{S})$. Actually, there exits only one. Geometrically, such an orbit is characterised by the fact that every tangent to $\mathscr{H}(2,q^2)$ meets it in exactly one point, that is a 1-secant of the orbit. On the other hand, let Ω be any point orbit under $\mathscr{S}$. From previous results due to Thas [31] we know that if q is even then the set of all 1-secants of Ω, the so-called algebraic envelope of Ω viewed as a q^2-q+1, consists of all tangents of a non-degenerate Hermitian curve, say $\mathscr{H}(\Omega)$. Also, Ω is disjoint from $\mathscr{H}(\Omega)$ and it is left invariant by the normaliser of $\mathscr{S}$ in the linear subgroup (isomorphic to $\mathrm{PGU}(3,q^2)$) preserving $\mathscr{H}(\Omega)$. For proofs and more information, the reader is referred to [2,7]. Summing up we have statement (a) in the following lemma.

Lemma 2.1. *Let $q>2$ be even. Let Ω be a point orbit of a Singer group $\mathscr{S}$ of order q^2-q+1 of a non-degenerate Hermitian curve $\mathscr{H}$ such that $\mathscr{H} \cap \Omega = \emptyset$.*

(a) *The algebraic envelope of* Ω, *regarded as a* (q^2-q+1)-*arc, consists of all tangents of a non-degenerate Hermitian curve* $\mathscr{H}$. *A necessary and sufficient condition for* $\mathscr{H}=\mathscr{H}(\Omega)$ *is that* Ω *be preserved by the normaliser* $N(\mathscr{S})$ *of* $\mathscr{S}$ *in* $\mathrm{PGU}(3,q^2)$. *If* $\mathscr{H}\neq\mathscr{H}(\Omega)$ *then they have exactly* q^2-q+1 *common tangents.*

(b) *If* $\mathscr{H}(\Omega)\neq\mathscr{H}$ *and* $\Omega\cap\mathscr{H}=\emptyset$, *then the number of those tangents to* $\mathscr{H}$ *which are secants to* Ω *is* $\frac{1}{2}q(q^2-q+1)$.

Proof. We prove (b). As tangents of Ω are tangents of $\mathscr{H}(\Omega)$ and vice versa, (ii*) yields that Ω and $\mathscr{H}$ have exactly $q+1$ common tangents. For any point $A\in\Omega$, let $i(A)$ denote the number of common tangents of Ω and $\mathscr{H}$. Then $\sum i(A)=q+1$ with A ranging over Ω. But $\mathscr{S}$ is sharply transitive on Ω, and so $i(A)$ does not depend on the choice of A. Hence, $i(A)=1$. Since $A\notin\mathscr{H}$, through A there are exactly $q+1$ tangents to $\mathscr{H}(2,q^2)$. As $i(A)=1$, it turns out that q of these tangents to $\mathscr{H}$ are secants to Ω. If A ranges over Ω we obtain $q(q^2-q+1)$ not necessarily distinct tangents to $\mathscr{H}$ which are secants to Ω. Actually, each one is obtained just twice, as Ω is an arc. □

By a *spread of* $\mathscr{H}(2,q^2)$ we mean any collection of chords partitioning the point set of $\mathscr{H}(2,q^2)$. In other words, a spread comprises q^2-q+1 chords any two of them meet in a point outside $\mathscr{H}(2,q^2)$. A spread is said to be *linear* if it consists of all chords through a point P together with $P^{\perp}$. Spreads of a Hermitian curve seem to be rare, as no non-linear spread is known yet for odd q, and only one non-linear spread has been found so far for q even and bigger than 2. The latter one, called cyclic spread, consists of all chords in the unique chord orbit of $\mathscr{S}$ which is also preserved by $N(\mathscr{S})$, see [2]. The following lemma which will occur in Section 3 shows that no other linearly transitive spread of $\mathscr{H}(2,q^2)$ exists.

Lemma 2.2. *For even* $q>2$, *the cyclic spread is the only linear spread of* $\mathscr{H}(2,q^2)$ *consisting of all chords in a single orbit of a subgroup of* $\mathrm{PGU}(3,q^2)$.

Proof. Let $\mathbf{f}$ be a spread of a Hermitian curve $\mathscr{H}$ such that the subgroup G of $\mathrm{PGU}(3,q^2)$ which preserves $\mathbf{f}$ acts transitively on $\mathbf{f}$. Then q^2-q+1 divides the order of G. This together with the classification of all maximal subgroups of $\mathrm{PGU}(3,q^2)$ given in [19] implies that G is conjugate to a subgroup of $N(\mathscr{S})$. Without loss of generality, we may assume G to be a subgroup of $N(\mathscr{S})$. Then, either $G=\mathscr{S}$ or $G=N(\mathscr{S})$ or $G=H\rtimes\langle\phi\rangle$ where H is the subgroup of $\mathscr{S}$ of index 3 and ϕ has order 3. In the first two cases, $\mathbf{f}$ is the cyclic spread. We have to show that the latter case cannot actually occur. To do this we take the pole F_1 of a chord in $\mathbf{f}$ and define $\mathbf{F}_1$ to be the point orbit of F_1 under H. Let $\mathbf{F}_2=\phi(\mathbf{F}_1)$ and $\mathbf{F}_3=\phi(\mathbf{F}_2)$. Then $\mathbf{F}=\mathbf{F}_1\cup\mathbf{F}_2\cup\mathbf{F}_3$ must have the property of containing exactly one point from every tangent to $\mathscr{H}$. In fact, if a tangent to $\mathscr{H}$ contained two distinct points A,B from $\mathbf{F}$, the tangency point $U\in\mathscr{H}$ would be a common point of $A^{\perp}$ and $B^{\perp}$, and hence $\mathbf{f}$ would not be a spread of $\mathscr{H}$. For $i=1,2,3$, $\mathbf{F}_i$ is contained in a cyclic (q^2-q+1)-

arc K_i whose algebraic envelope consists of all tangents of a non-degenerate Hermitian curve, say $\mathscr{H}_i$. Since none of the three arcs K_i is preserved by $N(\mathscr{S})$, we have that $\mathscr{H}_i \neq \mathscr{H}$. By Lemma 2.1 $\frac{1}{2}q(q^2-q+1)$ tangents to $\mathscr{H}$ are secants to K_i. Since $\frac{3}{2}q(q^2-q+1) > q^3+1$ by our assumption $q>2$, these three sets of tangents are not pairwise disjoint. So, there is a tangent r to $\mathscr{H}$ which is secant to at least two of the arcs K_i, say K_1 and K_2. Let $r \cap K_1 = \{P_{11}, P_{12}\}$ and $r \cap K_2 = \{P_{21}, P_{22}\}$. On the other hand K_i partitions into $K_{i1} \cup K_{i2} \cup K_{i3}$ where K_{ij} are orbits under H. We choose indices such that $\phi(K_{1j}) = K_{2j}$, $\phi(K_{2j}) = K_{3j}$ and $\phi(K_{3j}) = K_{1j}$. Clearly $\mathbf{F}_i = K_{i1}$, hence every tangent to $\mathscr{H}$ meets $K_{i1} \cup K_{i2} \cup K_{i3}$ in exactly one point. The points P_{11} and P_{12} are in two different members of the partition of K_1. In fact, since $\mathscr{S}$ is transitive on K_1, there is an element $s \in \mathscr{S}$ that takes P_{11} to a point of K_{11}. If P_{12} was in the same member, then s would take P_{12} to a point in K_{11}, and hence s would take r to a tangent to $\mathscr{H}$ containing two distinct points from K_{11}, a contradiction. Similarly, the points P_{21} and P_{22} are in two different members of the partition of K_2. Replacing G by its conjugate under s, we may assume that $P_{11} \in K_{11}$. Then, either K_{12} or K_{13} contains P_{12}. On the other hand, neither P_{21} nor P_{22} is in K_{21}, otherwise we would have $|r \cap (K_{11} \cup K_{21})| = 2$. Hence one of these points is in K_{22} and the other one is in K_{23}. Then $|r \cap (K_{1j} \cup K_{2j})| = 2$ follows for $j=2$ or for $j=3$. Again, we have an element $s' \in \mathscr{S}$ taking the point P_{12} to a point on K_{11}. But then s' takes r to a tangent r' to $\mathscr{H}$ meeting both K_{11} and K_{21}. Since $K_1 = \mathbf{F}_1$ and $K_2 = \mathbf{F}_2$, the tangent r' meets $\mathbf{F}$ in two distinct points. But this contradicts the definition of a spread, and hence the case $G = H \rtimes \langle \phi \rangle$ does not occur. □

It might happen a priori that some of the chord orbits under $N(\mathscr{S})$ is a 3-spread, as it consists of $3(q^2-q+1)$ chords which cover every point of $\mathscr{H}(2,q^2)$ three times. The following lemma which will be used in Section 3 shows however that such a 3-spread does not exist.

Lemma 2.3. *No* 3*-spread of* $\mathscr{H}$ *consists of all chords in a single orbit under the normalizer of* $\mathscr{S}$ *in* $\mathrm{PGU}(3,q^2)$.

Proof. Let K_1, K_2, K_3 be three distinct point orbits under $\mathscr{S}$, each disjoint from $\mathscr{H}$, such that $K = K_1 \cup K_2 \cup K_3$ is a point orbit under $\mathrm{N}(\mathscr{S})$. If every tangent to $\mathscr{H}$ meets K in exactly three points, then the polar lines of the points in K form a 3-spread. Since every 3-spread of $\mathscr{H}$ consisting of all chords in a single orbit under $N(\mathscr{S})$ arises in this way, we may assume that K has the property of containing exactly three points from every tangent to $\mathscr{H}$. As we have already shown in the proof of Lemma 2.2, the second statement in Lemma 2.1 implies the existence of a tangent r to $\mathscr{H}$ which is secant to both K_1 and K_2. But then $|r \cap K| > 3$, a contradiction. □

3. A non-classical transitive ovoid of $\mathscr{H}(3,q^2)$ with even q

We keep our notation and terminology introduced in Section 2. The following construction due to Payne and Thas [26] provides non-classical ovoids of $\mathscr{H}(3,q^2)$.

Given a classical ovoid $\mathcal{O}$ of $\mathcal{H}(3,q^2)$, choose two distinct points A, B on it. Then the line ℓ through A, B meets $\mathcal{O}$ in $q+1$ points. Replace these points with those in the intersection of the polar line of ℓ with $\mathcal{H}(3,q^2)$. The resulting set contains no two conjugate points, and has the same size of $\mathcal{O}$, and hence it is an ovoid. Sometimes, this procedure can be repeated several times. We limit ourselves to describe the simplest situation. Given a Hermitian curve $\mathcal{H}(2,q^2)$ cut out on $\mathcal{H}(3,q^2)$ by a non-tangent plane π, choose a set $\mathbf{f}$ of chords $\mathcal{H}(2,q^2)$ such that any two intersect in an external point of $\mathcal{H}(2,q^2)$. Remove all points of $\mathcal{H}(2,q^2)$ covered by these chords, and add the common points of their polar lines with $\mathcal{H}(3,q^2)$ to the remaining points of $\mathcal{H}(2,q^2)$. The resulting point set has size q^3+1 and turns out to be an ovoid of $\mathcal{H}(3,q^2)$. It may be noted that the maximum size of $\mathbf{f}$ is q^2-q+1, and if equality holds then $\mathbf{f}$ is a spread of $\mathcal{H}(2,q^2)$. Now, we investigate closer the case where $\mathbf{f}=\{f_1,\dots,f_{q^2-q+1}\}$ is the cyclic spread of $\mathcal{H}(2,q^2)$ arising from the Singer group $\mathcal{S}$. Embed $\mathcal{H}(2,q^2)$ in a non-degenerate Hermitian surface $\mathcal{H}(3,q^2)$ of $\mathrm{PG}(3,q^2)$ as a plane section cut out by a non-tangent plane π. For $i=1,\dots,q^2-q+1$, let e_i be the polar line of f_i with respect to the unitary polarity associated to $\mathcal{H}(3,q^2)$. Such lines e_i are chords of $\mathcal{H}(3,q^2)$ through the pole P of π. No three of them are coplanar, since no three lines in $\mathbf{f}$ are concurrent. Hence the resulting ovoid $\mathcal{O}$ is non-classical. We will use the term of *Singer-type* ovoid to indicate any ovoid of $\mathcal{H}(3,q^2)$ that is equivalent to $\mathcal{O}$ up to a collineation in $\mathrm{PGU}(4,q^2)$.

Theorem 3.1. *The Singer-type ovoid $\mathcal{O}$ is transitive. More precisely, the subgroup G of $\mathrm{PGU}(4,q^2)$ preserving $\mathcal{O}$ contains a cyclic transitive subgroup C_{q^3+1} of order q^3+1 and a subgroup C_3 of order 3 such that $G\cong C_{q^3+1}\rtimes C$. For $q>2$ G is the full linear collineation group preserving $\mathcal{O}$. For $q=2$ the full linear collineation group is larger as it has order 162. Furthermore, $H/G\cong \mathrm{Gal}(\mathrm{GF}(q^2)/\mathrm{GF}(2))$ where H is the subgroup of $\mathrm{P\Gamma U}(4,q^2)$ preserving $\mathcal{O}$.*

Proof. Let M denote the subgroup of $\mathrm{PGU}(4,q^2)$ which fixes P and preserves π. Since $P\notin\pi$, M is isomorphic to $\mathrm{GU}(3,q^2)$, and hence it contains a metacyclic group $K=C_{q^3+1}\rtimes C_3$ of order $3(q^3+1)$ for $q>2$ (and a subgroup K of order 162 for $q=2$) such that $R=Z(M)\cap K$ has order $q+1$. Geometrically, R is a homology group with centre P and axis π, while the linear collineation group induced by K on π is the normaliser $N(\mathcal{S})$ of the Singer group $\mathcal{S}$ of the Hermitian curve $\mathcal{H}(\pi)$ cut out on $\mathcal{H}(3,q^2)$ by π. By construction the group K preserves $\mathcal{O}$, that is $K\leqslant G$. Since $\mathcal{S}$ acts transitively on the cyclic spread of π, as R does on the common points of $\mathcal{H}(3,q^2)$ with any chord of $\mathcal{H}(3,q^2)$ through P, it turns out that $\mathcal{O}$ is a transitive ovoid of $\mathcal{H}(3,q^2)$. Since G preserves π, the factor group G/R is a subgroup of $\mathrm{PGU}(3,q^2)$ containing K/R. But the latter one is a maximal subgroup of $\mathrm{PGU}(3,q^2)$, and this implies $K=G$. To prove the final assertion in Theorem 3.1, we note that $H/G\cong(H/R)/(G/R)$. Then the assertion follows from the fact already quoted in Section 2 that $\bar{N}/N(\mathcal{S})=\mathrm{Gal}(\mathrm{GF}(q^2)/\mathrm{GF}(2))$ where $\bar{N}$ denotes the normaliser in $\mathrm{P}\Gamma\mathrm{U}(3,q^2)$ of the normaliser $N(\mathcal{S})$ of the Singer group $\mathcal{S}$ in $\mathrm{PGU}(3,q^2)$. □

4. The proof of Theorem 1.3

By Brouwer and Wilbrink [5] we may assume that $q>2$. The proof proceeds in several steps. We will need some more notation:

- $q=2^t$ with $t>1$, $q'=2^{t'}$ with $t'|t$;
- $Q^-(5,q)$ the elliptic quadric of $\mathrm{PG}(5,q)$ associated to $\mathcal{H}(3,q^2)$ under the isomorphism $\mathrm{P}\Gamma\mathrm{U}(4,q^2)\cong\mathrm{P}\Gamma\mathrm{O}^-(6,q)$.
- K is some subgroup of $\mathrm{P}\Gamma\mathrm{U}(4,q^2)$ preserving $\mathcal{H}(3,q^2)$;
- $K^{(\infty)}$ last term of the commutator series for K;
- G a subgroup of $\mathrm{P}\Gamma\mathrm{U}(4,q^2)$ acting transitively on an ovoid $\mathcal{O}$.

The essential tool for proving Theorem 1.3 is the following deep classification theorem, see [21, Theorem 5.7].

Theorem 4.1. *If q is even and K has no subgroup of index 2, then one of the following holds*:

(i) $K^{(\infty)}=\mathrm{PSU}(4,q'^2)$, *with* $\mathrm{GF}(q'^2)\subseteq\mathrm{GF}(q^2)$;
(ii) *K fixes either a point of $\mathcal{H}(3,q^2)$, or a generator of $\mathcal{H}(3,q^2)$, or a point outside $\mathcal{H}(3,q^2)$, or a chord of $\mathcal{H}(3,q^2)$, or a self-conjugate simplex with vertices external to $\mathcal{H}(3,q^2)$*;
(iii) *K viewed as a subgroup of $\mathrm{P}\Gamma\mathrm{L}(6,q)$ fixes either a point outside of $Q^-(5,q)$, or a line disjoint from $Q^-(5,q)$, or a plane meeting $Q^-(5,q)$ in a conic*;

Since the size of $\mathcal{O}$ is an odd number, there is a subgroup K in G satisfying both conditions
(I) K acts transitively on $\mathcal{O}$;
(II) K contains no subgroup of index 2.
By virtue of (II), Theorem 4.1 applies. Our aim is to prove that (I) leaves us with only one possibility.

Proposition 4.2. *K fixes a point outside $\mathcal{H}(3,q^2)$, and hence K is isomorphic to a subgroup of* $\mathrm{P}\Gamma\mathrm{U}(3,q^2)$.

Proposition 4.2 follows from the following six lemmas.

Lemma 4.3. *K fixes no point on $\mathcal{H}(3,q^2)$.*

Proof. Let $P \in (\mathscr{H}(3,q^2))$ be a fixed point of K. By (I) $P \notin \mathcal{O}$. Then the tangent plane π to $\mathscr{H}(3,q^2)$ at P meets $\mathcal{O}$ in $q+1$ points. In particular $\pi \cap \mathcal{O}$ is a proper non-empty subset of $\mathcal{O}$. But, as π is preserved by K, this is inconsistent with (I). □

Lemma 4.4. *K preserves neither a generator nor a chord of $\mathscr{H}(3,q^2)$.*

Proof. Since every generator meets $\mathcal{O}$ in a unique point, (I) implies that K preserves no generator. To prove the second assertion, assume K to preserve a chord ℓ of $\mathscr{H}(3,q^2)$. Then K also preserves the chord $\ell^{\perp}$ corresponding to ℓ under the unitary polarity associated to $\mathscr{H}(3,q^2)$. Note that every common point of ℓ with $\mathscr{H}(3,q^2)$ is conjugate to every point of $\ell^{\perp}$ with $\mathscr{H}(3,q^2)$. Thus, every line joining a point in $\ell \cap \mathscr{H}(3,q^2)$ to a point in $\ell^{\perp} \mathscr{H}(3,q^2)$ is a generator of $\mathscr{H}(3,q^2)$, and hence meets $\mathcal{O}$ in a unique point. Choose one of such points, say A, and count the points in the orbit A^K of A under K. Clearly A^K is covered by generators meeting both ℓ and $\ell^{\perp}$; more precisely, every generator covers just one point in A^K. As the total number of such generators is $(q+1)^2$, this shows that A^G has size at most $(q+1)^2$. But this contradicts (I) again. □

Lemma 4.5. *K preserves no self-conjugate simplex with vertices external to $\mathscr{H}(3,q^2)$.*

Proof. The subgroup of $\mathrm{P}\Gamma\mathrm{U}(3,q^2)$ preserving a self-conjugate simplex has order $24d(q+1)^3$ with $q=2^d$. By (I) q^2-q+1 divides $24d(q+1)^2$. Let u be an odd prime divisor of q^2-q+1. Since $gcd(q+1,q^2-q+1)$ is either 1 or 3, but q^2-q+1 is not divisible by 9, it follows that q^2-q+1 is divisible by 9. Hence $2^{2d}-2^d+1 \leqslant 3d$. But, this is only possible for $d=1$ contradicting our assumption $q>2$. □

Lemma 4.6. *Case* (iii) *cannot actually occur.*

Proof. Let $P \in \mathrm{PG}(5,q)$ be a fixed point of K outside $Q^-(5,q)$. The (four dimensional) polar space π of P with respect to the orthogonal polarity associated to $Q^-(5,q)$ meets $Q^-(5,q)$ in a parabolic quadric $Q(4,q)$ left invariant by K. Since $Q(4,q)$ has q^3+q^2+q+1 points, a line ℓ in the spread Σ of size q^3+1 arising from $\mathcal{O}$ meets $Q(4,q)$ more than one point. But then ℓ lies in π, and transitivity of K on Σ yields that every line in Σ is contained in π. But this is impossible Σ being a spread of $Q^-(5,q)$, and hence the first assertion is proved. Let ℓ be a K-invariant line disjoint from $Q^-(5,q)$. Then the (three dimensional) polar space π of ℓ meets $Q^-(5,q)$ in a hyperbolic quadric $Q^+(3,q)$ preserved by K. Since $Q^+(3,q)$ has size $(q+1)^2$, at least one but not more than $(q+1)^2$ lines in the spread Σ meet $Q^+(3,q)$. Since the set of such lines is K-invariant and the latter number is smaller than q^3+1, K cannot act transitively on Σ. Finally, consider a K-invariant plane meeting $Q^-(5,q)$ in a conic C. As q is even, C has a nucleus N. Clearly, N is a fixed point of K. Since $N \notin C$ implies $N \notin Q^-(5,q)$, it turns out that K has a fixed point outside $Q^-(5,q)$, but this is impossible as already showed. □

To rule out case (i), the following result will be needed, see [1,10,33].

Lemma 4.7. $\mathrm{PSU}(4,q^2)$ *has two classes of involutions. One consists of all transvections* (*that is elations with centre on* $\mathscr{H}(3,q^2)$ *and axis the tangent plane to* $\mathscr{H}(3,q^2)$ *at the centre*) *while every involution in the other class fixes each point of a generator but no point outside it.*

According to Lemma 4.7 we distinguish three types of linear involutions in $K^{(\infty)}$, namely

(i) elation whose centre lies on $\mathcal{O}$;
(ii) elation whose centre lies on $\mathscr{H}(3,q^2)\backslash\mathcal{O}$;
(iii) involution whose fixed points are those of a generator of $\mathscr{H}(3,q^2)$.

Then the number of fixed points of an involution of $g\in K^{(\infty)}$ on $\mathcal{O}$ is easily computed:

(i) If g is either of type (i) or type (iii) then g has exactly one fixed point on $\mathcal{O}$;
(ii) If g is of type (ii), then g has exactly $q+1$ fixed points on $\mathcal{O}$.

Lemma 4.8. *If* $K^{(\infty)}$ *has even order, then either* $\mathcal{O}$ *is classical, or* q^2-q+1 *divides the order of* $K^{(\infty)}$.

Proof. If $K^{(\infty)}$ contains an involution of type (i) or (iii), then $K^{(\infty)}\trianglelefteq K$ together with the lemma of Gleason, see for instance [9, 4.3.15], yields that $K^{(\infty)}$ is transitive on $\mathcal{O}$. In particular, q^3+1 divides the order of $K^{(\infty)}$, and this proves the assertion. Assume now that every involution in $K^{(\infty)}$ is of type (ii). Let Δ be the set of all points which are the centre of an involution $K^{(\infty)}$. We begin by proving that Δ is a partial ovoid. Let $B_1,B_2\in\Delta$ be any two distinct points. Let Φ_1 denote the elementary abelian subgroup comprising all involutions of $K^{(\infty)}$ with centre B_1 together with the identity. Choose an involution $\varphi_2\in K^{(\infty)}$ with centre B_2, and assume on the contrary that B_1 and B_2 are two perpendicular points. Then the line ℓ through them is a generator, and hence φ_2 also fixes B_1. This implies that φ_2 normalises Φ_1. As Φ_1 contains an odd number of involutions, it follows that φ_2 commutes with an involution of Φ_1, say φ_1. Since φ_1 and φ_2 are two involutory elations with different centres, no point outside ℓ is fixed by $\varphi_1\varphi_2$. It turns out that $\varphi_1\varphi_2$ is an involution of type (iii), rather than of type (ii). This proves that Δ is a partial ovoid. In particular, every involution in $K^{(\infty)}$ has exactly one fixed point in Δ, and Δ has odd size. By $K^{(\infty)}\trianglelefteq G$, we also have that $K^{(\infty)}$ is transitive on Δ. Now, count the pairs (A,B) with $A\in\mathcal{O}$ and $B\in\Delta$ two perpendicular points. (I) yields that each $A\in\mathcal{O}$ is perpendicular to a constant number, say n, of points in $B\in\Delta$. On the other hand, each point $B\in\Delta$ is perpendicular to exactly $q+1$ points $A\in\mathcal{O}$. Hence Δ has size $n(q^2-q+1)$. Since $K^{(\infty)}$ is transitive on Δ, this yields that q^2-q+1 divides the order of $K^{(\infty)}$. $\square$

Lemma 4.9. *Case* (i) *cannot actually occur.*

Proof. If (i) holds then $q^2 - q + 1$ divides $|\mathrm{PSU}(4, q'^2)|$ by Lemma 4.8. On the other hand, from $|\mathrm{PSU}(4, q'^2)| = q'^6(q'^4 - 1)(q' + 1)(q'^2 - 1)(q'^2 - q' + 1)$ together with $q = q'^b$, $b > 0$ we deduce that $|\mathrm{PSU}(4, q'^2)|$ divides $q^6(q^4 - 1)(q + 1)(q^2 - 1)(q'^2 - q' + 1)$. Since $q^2 - q + 1$ is either prime to $q + 1$, or their q.c.d. is 3, while $q^2 - q + 1$ is always prime to $(q^2 + 1)(q - 1)$, this yields that $q^2 - q + 1$ divides $27(q'^2 - q' + 1)$, a contradiction. □

From Proposition 4.2 we deduce Theorem 1.3. In fact, let G be the full collineation group preserving a transitive ovoid $\mathcal{O}$. If G has no subgroup of index 2, then Proposition 4.2 coincides with Theorem 1.3. Otherwise, there are a chain of subgroups $K = K_0 < K_1 < \cdots < K_m = G$ such that $[K_i : K_{i-1}] = 2$ for $i = 1, \ldots, m$. In particular, $K = K_0 \trianglelefteq \cdots \trianglelefteq K_m = G$. By Proposition 4.2, K has a unique fixed point P outside $\mathscr{H}(3, q^2)$. By $K_0 \trianglelefteq K_1$, this implies that P is the unique fixed point of K_1. This argument repeated for $K_1, \ldots, K_{m-1}$ completes the proof of Theorem 1.3.

5. The proof of Theorem 1.1

To prove Theorem 1.1 we assume G to be a subgroup of $\mathrm{PGU}(4, q^2)$, $q > 2$ acting transitively on $\mathcal{O}$. According to Theorem 1.3, let $A \notin \mathscr{H}(3, q^2)$ be the unique fixed point of G. Then $\mathscr{H}(3, q^2)$. Then G preserves the polar plane α under the polarity associated to $\mathscr{H}(3, q^2)$. Let M be the normal subgroup of G that fixes α pointwise. Then M is a group of homologies with centre A and axis α, and hence the order of M is a divisor of $q + 1$, say $(q + 1)/m$. The factor group $\bar{G} = G/M$ can be viewed as a linear collineation group of α which preserves the non-degenerate Hermitian curve $\mathscr{H}(\alpha)$ cut out on $\mathscr{H}(3, q^2)$ by α. Since the order of G is a multiple of $q^3 + 1$, say $c(q^3 + 1)$, the order of $\bar{G}$ can be written in the form $mc(q^2 - q + 1)$. According to the classification of all maximal subgroups of the projective unitary group $\mathrm{PGU}(3, q^2)$, see [19], there are four possibilities for $\bar{G}$, namely

(I) $m = 3, c = 1$ and $\bar{G} \cong C_3 \ltimes Z_{q^2-q+1}$;
(II) $m = 1, c = 1$ and $\bar{G} \cong C_{q^2-q+1}$;
(III) $m = 1, c = 1$ and $\bar{G} \cong C_3 \ltimes C_{(q^2-q+1)/3}$;
(IV) $m = 1, c = 3$ and $\bar{G} \cong C_3 \ltimes C_{q^2-q+1}$.

Finally, Theorem 1.1 follows from the following three lemmas.

Lemma 5.1. *If either* (I) *or* (II) *occurs, then* $\mathcal{O}$ *is the Singer-type ovoid.*

Proof. According to Theorem 1.1, both cases (I) and (II) occur for the Singer-type ovoid. In proving Lemma 5.1 we will limit ourselves to consider (II), as (I) implies

(II). So, $M \cong C_{q+1}$, and $\bar{G} \cong C_{q^2-q+1}$. Our idea is to rebuild $\mathcal{O}$ from its collineation group. Take a line ℓ joining A with a point B of $\mathcal{O}$. The orbit of B under M is contained in ℓ, and hence ℓ is a secant of $\mathcal{O}$. More precisely, such an orbit contains all common points of ℓ with $\mathcal{H}(3,q^2)$, because M has order $q+1$. Furthermore, the orbit of ℓ under G consists of all lines through A which meet α in the points of an orbit under $\bar{G}$. Since $\mathcal{O}$ has size q^3+1, it turns out that the latter orbit Ω has size q^2-q+1, and hence $\bar{G}$ is a Singer group of order q^2-q+1 acting regularly on Ω. Hence, $\bar{G} = \mathcal{S}$, up to a collineation of $\mathrm{PGU}(3,q^2)$ which can be lifted to a collineation $g \in \mathrm{PGU}(4,q^2)$. For any two distinct points $U, V \in \Omega$, the corresponding polar lines u, v with respect to the unitary polarity associated to $\mathcal{H}(2,q^2)$ meet in a point outside $\mathcal{H}(2,q^2)$. By Lemma 2.1 this property is consistent with just one orbit under the Singer group $\mathcal{S}$. Therefore $\mathcal{O}$ is the Singer-type ovoid. □

Lemma 5.2. *Case* (III) *does not occur.*

Proof. We define Ω as in the proof of Lemma 4.4 and so Ω turns out to be an orbit under $\bar{G}$. Then the polar lines of the points in Ω form a spread consisting of all chords of a single orbit under $\bar{G}$. By Lemma 2.2, $\bar{G}$ must be cyclic, and hence case (III) is impossible. □

Lemma 5.3. *Case* (IV) *does not occur.*

Proof. If $\mathcal{O}$ is a counterexample, then $\mathcal{O}$ has exactly $3(q^2-q+1)$ secants through A, each of them containing exactly $(q+1)/3$ points from $\mathcal{O}$. These secants meet the plane α in a set K consisting of $3(q^2-q+1)$ points, and $\bar{G}$ acts transitively on K. Hence, the polar lines of the points on K with respect to the polarity associated to $\mathcal{H}(2,q^2)$ form a 3-spread of $\mathcal{H}(2,q^2)$ consisting of all chords in a single orbit under $\bar{G}$. But $\bar{G}$ is conjugate to the normaliser $N(\mathcal{S})$ of $\mathcal{S}$ in $\mathrm{PGU}(3,q^2)$ which is impossible by Lemma 2.3. □

6. Null polarities commuting with a unitary polarity

Let $\mathcal{U}$ be a unitary polarity of $\mathrm{PG}(3,q^2)$ associated with the Hermitian surface $\mathcal{H}(3,q^2)$. There exist $q^2(q^3+1)$ null polarities commuting with $\mathcal{U}$, and the product of any of them with $\mathcal{U}$ is an involutory linear collineation v preserving $\mathcal{H}(3,q^2)$. As Segre pointed out [28, pp. 128, 132], v fixes q^3+q^2+q+1 points on $\mathcal{H}(3,q^2)$ but no point outside $\mathcal{H}(3,q^2)$, and leaves the same number of lines of $\mathcal{H}(3,q^2)$ invariant such that each fixed point is incident with $q+1$ invariant lines and each invariant line is incident with $q+1$ fixed points. This symmetric configuration extends to a three-dimensional projective space $\Sigma \cong \mathrm{PG}(3,q)$ by adding the $q^2(q^2+1)$ v-invariant lines which are not generators of in $\mathcal{H}(3,q^2)$. In this context, Σ is naturally equipped with the symplectic polarity $\mathcal{A}$ whose absolute lines are the lines of the above symmetric configuration. Also, the projective symplectic group $\mathrm{PSp}(4,q)$ associated with $\mathscr{A}$ turns out to be a subgroup of the projective unitary group $\mathrm{PGU}(4,q^2)$ associated with $\mathscr{H}(3,q^2)$.

Theorem 6.1. *Assume q is even. Let $\mathcal{O}$ be an ovoid of Σ whose tangent lines are the absolute lines of $\mathscr{S}$. Then $\mathcal{O}$ is a complete partial ovoid of $\mathscr{H}(3,q^2)$.*

Proof. Since the generators of $\mathscr{H}(3,q^2)$ through the points of Σ are absolute lines of $\mathscr{A}$, no generator of $\mathscr{H}(3,q^2)$ meets $\mathcal{O}$ in two distinct points. This shows that $\mathcal{O}$ is a partial ovoid of $\mathscr{H}(3,q^2)$. To prove the completeness, assume on the contrary the existence of a point $P\in\mathscr{H}(3,q^2)$ which can be added to $\mathcal{O}$ to obtain a larger ovoid $\mathcal{O}'$ of $\mathscr{H}(3,q^2)$. Then no tangent line to $\mathcal{O}$ in $\mathrm{PG}(3,q)$ passes through P. This implies $P\notin\Sigma$, as the tangent lines to an ovoid of $\mathrm{PG}(3,q)$ cover all points in $\mathrm{PG}(3,q)$. In other words, v takes P to another point P'. By a result due to Segre [28, p. 132], the line through P and P' is a generator g of $\mathscr{H}(3,q^2)$. Since v is involutory, g is a v-invariant generator of $\mathscr{H}(3,q^2)$. But every v-invariant generator is an absolute line of $\mathscr{A}$. It follows that the generator g is line of Σ, more precisely a tangent line to $\mathcal{O}$. But then P together with the tangency point of g on $\mathcal{O}$ would provide two distinct common points of g with $\mathcal{O}'$, contradicting the assumption. □

Remark 6.2. Theorem 6.1 applies to a non-degenerate quadric for every even q, but also to the Suzuki–Tits ovoid provided that $q\geqslant 8$ is an odd power of 2. In both cases, the ovoid is preserved by a subgroup of $\mathrm{PSp}(4,q)$ which acts on it 2-transitively, see [16, Chapter 16]. Hence, there are at least two projectively non-equivalent 2-transitive complete partial ovoids of $\mathscr{H}(3,q^2)$.

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