On the Reducibility of Linear Differential Equations with Quasiperiodic Coefficients

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The system \( \dot{x} = (A + \varepsilon Q(t))x \) in \( \mathbb{R}^d \) is considered, where \( A \) is a constant matrix and \( Q \) a quasiperiodic analytic matrix with \( r \) basic frequencies. The eigenvalues of \( A \) are arbitrary including the purely imaginary case. Suppose that the set formed by the eigenvalues of \( A \) and the basic frequencies of \( Q \) satisfies a nonresonant condition. Then there is a positive measure cantorian set \( \mathcal{E} \) such that for \( \varepsilon \in \mathcal{E} \) the system is reducible to constant coefficients by means of a quasiperiodic change of variables, provided a nondegeneracy condition holds. This condition prevents locking at resonance.

0. Introduction

We say that a function \( f \) is a quasiperiodic function of time with basic frequencies \( \omega_1, ..., \omega_r \) if \( f(t) = F(\theta_1, ..., \theta_r) \), where \( F \) is \( 2\pi \) periodic in all its arguments and \( \theta_j = \omega_j t \) for \( j = 1, ..., r \). Furthermore, \( f \) will be called analytic quasiperiodic in a strip of width \( \rho \) if \( F \) is analytical for \( |\text{Im} \theta_j| < \rho \) for \( j = 1, ..., r \). In this case we denote by \( \|f\|_\rho \) the norm \( \sup\{|F(\theta_1, ..., \theta_r)|/|\text{Im} \theta_j| < \rho, 1 \leq j \leq r\} \). Let us consider first the equation,

\[
\dot{x} = A(t)x,
\]

(0.1)

where \( A(t) \) is an \( n \times n \) matrix that depends on time in a quasiperiodic way with basic frequencies \( \omega = (\omega_1, ..., \omega_r)^T \). We say that a change of variables \( x = P(t)y \) is a Lyapunov–Perron (LP) transformation if \( P(t) \) is nonsingular.
and \( P(t) \), \( P^{-1}(t) \), and \( \dot{P}(t) \) are bounded for all \( t \in \mathbb{R} \). Moreover, if \( P, P^{-1} \), and \( \dot{P} \) are quasiperiodic we refer to \( x = P(t)y \) as a quasiperiodic LP transformation. If \( x = P(t)y \) is a LP transformation, then \( y \) satisfies the equation

\[
\dot{y} = B(t)y,
\]

(0.2)

where \( B = P^{-1}(AP - \dot{P}) \). We say that (0.1) is reducible if there is a quasiperiodic LP transformation that transforms (0.1) to (0.2), where \( B \) is a constant matrix. Obviously if \( Q \) is periodic the reducibility in all cases is given by the classical Floquet theory. In [2] this problem is studied for different conditions on \( A \) and \( Q \) and the ideas used in the present paper are very close to the ones found in [2]. Another source of inspiration has been the proof of KAM theorem given in [1].

It is also known [4] that if \( A \) is sufficiently smooth, its frequencies satisfy a suitable nonresonant condition and it has the so called “full spectrum” (see [4] for the definition), then the system (0.1) is reducible.

In this paper we shall drop the “full spectrum” hypothesis and we shall consider \( A(t) \) analytical and close to a nonresonant constant matrix. Our system will be

\[
\dot{x} = (A + \varepsilon Q(t))x,
\]

(0.3)

with \( x \) a \( d \)-dimensional vector. Let \( \lambda_j, j = 1, ..., d \) be the eigenvalues of \( A \) and \( \lambda^T = (\lambda_1, ..., \lambda_d) \). The greatest difficulties are found when the real parts of all \( \lambda_j \) are equal (perhaps zero) and the authors are not aware of any result in this case. We present a theorem which holds in this case asking for some nonresonant conditions for the vector \( v^T = (\lambda^T, \omega^T) \). This condition is satisfied by a set of big relative measure in the space of the parameter \( v \).

Under some nondegeneracy conditions we shall prove that if \( \varepsilon_0 \) is small enough, there exists a cantorian subset \( \mathcal{E} \) of \([0, \varepsilon_0]\) of positive measure such that, if \( \varepsilon \in \mathcal{E} \) then (0.3) is reducible. Moreover, our proof is constructive using an iterative scheme with quadratic convergence with respect to \( \varepsilon \). That is, after \( n \) steps the transformed equation looks like (0.3) with \( A_n(\varepsilon), \varepsilon^{2^n}, \) and \( Q_n(t, \varepsilon) \) (bounded by some \( M_n \)) instead of \( A, \varepsilon, \) and \( Q(t) \) for \( \varepsilon \) in some cantorian set \( \mathcal{E}_n \).

1. Main Results

We define the average of \( Q(t) \) as

\[
\bar{Q} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t) \, dt.
\]
For the existence of the limit see [3]. We consider first Eq. (0.3) after averaging with respect to \( t \) and some rearrangement

\[
\dot{x} = (\tilde{A} + \varepsilon \tilde{Q}(t))x,
\]

where \( \tilde{Q}(t) = Q(t) - \bar{Q} \), \( \tilde{A} = A + \varepsilon \bar{Q} \). Next we do the change of variables \( x = (I + \varepsilon \tilde{P})y \) to obtain

\[
\dot{y} = [(I + \varepsilon \tilde{P})^{-1}(\tilde{A} + \varepsilon(\tilde{A}P - \tilde{P} + \tilde{Q})) + \varepsilon^2(I + \varepsilon \tilde{P})^{-1}\tilde{Q} \tilde{P}]y, \tag{1.1}
\]

where \( I \) denotes the identity matrix in \( \mathbb{R}^d \). We would like to have

\[
(I + \varepsilon \tilde{P})^{-1}((\tilde{A} + \varepsilon(\tilde{A}P - \tilde{P} + \tilde{Q})) = \tilde{A}
\]

and this implies

\[
\dot{\tilde{P}} = \tilde{A}P - P\tilde{A} + \tilde{Q}. \tag{1.2}
\]

Suppose now that we have a quasiperiodic solution of (1.2) with the same frequencies which appear in \( Q \). Then, (1.1) becomes

\[
\dot{y} = [(\tilde{A} + \varepsilon^2(I + \varepsilon \tilde{P})^{-1}\tilde{Q} \tilde{P}]y.
\]

Now we average again and restart the process. Obviously, if we can do this until the \( n \)th step, we shall obtain an equation like

\[
\dot{x}_n = (A_n + \varepsilon^n \bar{Q}_n)x_n,
\]

where \( \|\bar{Q}_n\| \) can be very large. We are going to see that, under suitable conditions, this method converges.

**Theorem.** Consider the equation \( \dot{x} = (A + \varepsilon Q(t))x, \quad \varepsilon \in (0, \varepsilon_0) \) and \( x \in \mathbb{R}^d \), where \( A \) is a constant matrix with different eigenvalues \( \lambda_1, \ldots, \lambda_d \) and \( Q(t) \) is a quasiperiodic matrix with basic frequencies \( \omega_1, \ldots, \omega_r \). Suppose that

1. \( Q \) is analytic on a strip of with \( p_0 > 0 \).
2. The vector \( v \), where \( v^T = (\lambda_1, \ldots, \lambda_d, \sqrt{-1}\omega_1, \ldots, \sqrt{-1}\omega_r) \) satisfies the nonresonance conditions

\[
|m| \geq \frac{c_v}{|m|^\gamma},
\]

for all \( m \in \{m_1 \in \mathbb{Z}^d, |m_1| = 0 \text{ or } |m_1| = 2\} \times \{m_2 \in \mathbb{Z}^r, |m_2| \neq 0\} \), where \( c_v \) is a positive number, \( \gamma = r + d + \beta, \beta > -1 \) and \( |m| = \sum_{j=1}^d |m_j| \).

3. Let \( \bar{Q} \) be the average of \( Q \) with respect to \( t \) and let \( \lambda_j^0(\varepsilon) \) be an eigenvalue of \( \bar{A} = A + \varepsilon \bar{Q} \) for \( j = 1, \ldots, d \). We require

\[
\left| \frac{d}{d\varepsilon} (\lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon)) \right|_{\varepsilon = 0} > 2\delta > 0, \quad \forall 1 \leq i < j \leq d.
\]
Then there exists a cantorian set $\delta \subset (0, \varepsilon_0)$ with positive Lebesgue measure such that the system $\dot{x} = (A + \varepsilon Q)x$ is reducible. If $\varepsilon_0$ is small enough the relative measure of $\delta$ in $(0, \varepsilon_0)$ is close to 1. Furthermore the quasiperiodic change of variables that transforms the system to $\dot{y} = By$ (B being a constant matrix) has the same basic frequencies as $Q$.

Remark 1. The nonresonance condition for $v$ is satisfied for most of the values of $v$. More concretely, if $v$ belongs to a ball of radius $R$ then we have that the condition is satisfied for all $v$ except by a set of relative Lebesgue measure less than $4c_\gamma (d+r)^{-\gamma/2}(\zeta(2+\beta)/R)$, where $\zeta$ denotes the Riemann zeta function. The third condition is a nondegeneracy condition, not allowing to be locked at resonance. This condition can be replaced by a higher order nondegeneracy condition but it is not so simple to state in the hypothesis.

Remark 2. We can suppose that $A = \text{diag}(\lambda_1, \ldots, \lambda_d)$. Let $\|Q\|_\rho$ be the matricial norm associated to the vector norm defined by $\|(f_1, \ldots, f_d)^T\|_\rho = \max_{1 \leq k \leq d} \|f_k\|_\rho$, where $\|f_k\|_\rho$ is the norm defined in the Introduction. Introducing a new time $\tau = st$, where

$$s = \max \left\{ \frac{\pi^2/3 + 1}{\rho_0}, \|Q\|_{\rho_0} \right\}$$

we can suppose $\rho_0 \geq \pi^2/3 + 1$, $\|Q\|_{\rho_0} \leq 1$. These bounds will be used in the proof of the theorem. The scaling can change the constant $c_v$ and, therefore the admissible set of $\varepsilon$ is scaled by the same factor.

2. LEMMAS

We need some lemmas.

**Lemma 1.** Let $N_r^m = \# \{k \in \mathbb{Z}^r/|k| = \sum_{i=1}^r |k_i| = m\}$. Then

$$N_r^m \leq \frac{2^r}{(r-1)!} \left( m + \frac{r}{2} \right)^{r-1}, \quad \forall r, m \geq 1.$$  

**Proof.** As $k_r$ ranges from $-r$ to $r$ we have the recurrence relation

$$N_r^m = 2 \sum_{k=1}^{m-1} N_{r-1}^{k} + N_{r-1}^{m-1} + 2$$

and $N_1^m = 2$ for all $m$. This satisfies the relation given on the statement. Suppose that this relation holds for all $m$ and some $r$. Then
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\[ N_r^{m+1} = 2 \sum_{k=1}^{m-1} N_r^k + N_r^m + 2 \]

\[ < 2 \int_{r}^{m+1} \frac{2^r}{(r-1)!} \left( x + \frac{r}{2} \right)^{r-1} dx + \frac{2^r}{(r-1)!} \left( m + \frac{r}{2} \right)^{r-1} + 2 \]

\[ = \frac{2^{r+1}}{r!} \left[ \left( m + \frac{r}{2} \right)^r - \left( 1 + \frac{r}{2} \right)^r + \frac{r}{2} \left( m + \frac{r}{2} \right)^{r-1} + \frac{r!}{2^r} \right]. \]

But

\[ \left( m + \frac{r}{2} \right)^r + \frac{r}{2} \left( m + \frac{r}{2} \right)^{r-1} < \left( m + \frac{r}{2} + 1 \right)^r \]

\[ = \left( m + \frac{r}{2} \right)^r + \frac{r}{2} \left( m + \frac{r}{2} \right)^{r-1} + \sum_{j=2}^{r} \frac{1}{2^j} \left( r \right) \left( m + \frac{r}{2} \right)^{r-j}, \]

and, using that \( r!/2^r < (1 + r/2)^r \), the result follows. \[ \blacksquare \]

Remark. A simpler (and worse) bound like \( N_r^m \leq 2rm^{-1} \) can also be obtained by induction. There is numerical evidence that the factor \( \frac{1}{2} \), which multiplies \( r \) on the statement of the lemma can be replaced by 0.1872183, slightly larger than \((2e)^{-1}\). The bound of the lemma is also true for \( m = 0 \).

**Lemma 2.** Let

\[ p = \sum_{k \in \mathbb{Z}^r} p^k e^{(k, \omega) \sqrt{-1} t} \]

be an analytic Fourier series satisfying \(|p^k| \leq A_1 |k|^\gamma e^{-\rho_1 |k|} \) for \( k \neq 0 \) with \( \gamma > 0 \). If \( \rho_2 \in (0, \rho_1) \) then, for \( k \neq 0 \), we have \(|p^k| \leq A_2 e^{-\rho_2 |k|} \), where \( A_2 = A_1 (\gamma / (\rho_1 - \rho_2))^{\gamma} \).

Proof. We know \(|p^k| \leq A_1 |k|^\gamma e^{-\rho_1 - \rho_2 |k|} \) for \( k \neq 0 \). Using that the maximum of \( g(x) = x^\gamma e^{-\rho_1 - \rho_2 x} \) is reached when \( x = \gamma / (\rho_1 - \rho_2) \) the proof is completed. \[ \blacksquare \]

**Lemma 3.** We consider \( \dot{P} = AP - PA + Q \), where \( A = \text{diag}(\lambda_1, \ldots, \lambda_d) \) and \( Q \) is a quasiperiodic matrix with basic frequencies \( \omega = (\omega_1, \ldots, \omega_r)^T \) and without constant term. Let \( q_{ij} \) be the elements of \( Q \),

\[ q_{ij} = \sum_{k \in \mathbb{Z}^r \setminus \{0\}} q^k_{ij} e^{(k, \omega) \sqrt{-1} t}. \]

We suppose also \(|q^k_{ij}| \leq M e^{-\rho_1 |k|} \) and \(|\lambda_i - \lambda_j - (k, \omega) \sqrt{-1} | > c/|k|^{\gamma} \) for all \( i, j \in \{1, \ldots, d\} \) and all \( k \in \mathbb{Z}^r \setminus \{0\} \). Then there exists a unique solution \( P \) of \( \dot{P} = AP - PA + Q \) with the same frequencies as \( Q \) and which satisfies \(|p^k_{ij}| \leq N e^{-\rho_2 |k|} \) with \( \rho_2 \in (0, \rho_1) \) and \( N = (M/c)(\gamma / (\rho_1 - \rho_2))^{\gamma} \).
Proof. We look for

\[ p_{ij} = \sum_{k \in \mathbb{Z} \setminus \{0\}} p_{ij}^k e^{(k, \omega) \sqrt{-1} t}, \]

and this means that we have to solve the linear system

\[ \dot{p}_{ij} = \alpha_{ij} p_{ij} + q_{ij}, \quad \alpha_{ij} = \lambda_i - \lambda_j, \quad 1 \leq i, j \leq d. \]

It is easy to obtain the coefficients \( p_{ij}^k \),

\[ p_{ij}^k = \frac{q_{ij}^k}{(k, \omega) \sqrt{-1} - \alpha_{ij}}. \]

From the hypothesis one has the bound

\[ |p_{ij}^k| \leq |q_{ij}^k| \frac{|k|^\gamma}{c} \leq \frac{M}{c} |k|^\gamma e^{-\rho_1 |k|}, \]

and using Lemma 2 we obtain

\[ |p_{ij}^k| \leq \frac{M}{c} \left( \frac{\gamma}{(\rho_1 - \rho_2) e} \right)^\gamma e^{-\rho_2 |k|} = Ne^{-\rho_2 |k|}. \]

Remark. The worst situation is found when \( \lambda_i - \lambda_j \) are on the imaginary axis. If they are off of it the given bounds of \( |p_{ij}^k| \) are very high compared with the actual values. Therefore it is enough to restrict to the case when \( \text{Re}(\lambda_i - \lambda_j) = 0 \) for all \( i, j \in \{1, ..., d\} \), both for the initial matrix \( A \) and for all the matrices \( A_n \) found in the iterative process.

Lemma 4. Let

\[ q(t) = \sum_{k \in \mathbb{Z}'} q^k e^{(k, \omega) \sqrt{-1} t} \]

be such that \( |q^k| \leq Me^{-\rho_1 |k|} \). Then, for \( r \geq 2 \), one has

\[ \|q\|_{\rho_2} < M \left( \frac{2}{\rho_1 - \rho_2} \right)^r \exp \left( \frac{(\rho_1 - \rho_2) r}{2} \right) \left( 1 + \frac{\rho_1 - \rho_2}{\sqrt{2\pi(r-1)}} \right). \]

Proof. Let \( t \) be a complex number verifying \( |\text{Im} \theta_j| \leq \rho_2 \), where \( \theta_j = \omega_j t, 1 \leq j \leq r \). Then

\[ |q(t)| \leq \sum_{k \in \mathbb{Z}'} |q^k| |e^{(k, \omega t) \sqrt{-1}}| \leq M \sum_{k \in \mathbb{Z}'} e^{-\rho_1 |k|} e^{\rho_2 |k|} \]

\[ \leq M \sum_{k \in \mathbb{Z}'} e^{-(\rho_1 - \rho_2) |k|}. \]
Let us define $\delta = \rho_1 - \rho_2$. This implies that
\[
\|q\|_{\rho_2} \leq M \sum_{k \in \mathbb{Z}} e^{-\delta |k|},
\]
and using Lemma 1
\[
\|q\|_{\rho_2} \leq M \frac{2^r}{(r-1)!} \sum_{m=0}^{\infty} \left( m + \frac{r}{2} \right)^{r-1} e^{-\delta m}.
\]
As the function $x \mapsto (x + r/2)^{r-1} e^{-\delta x}$ has at most one maximum on $[0, \infty)$, the sum is bounded by the maximum plus the integral. Hence
\[
\|q\|_{\rho_2} < M \frac{2^r}{(r-1)!} \left[ \left( \frac{r-1}{\delta e} \right)^{r-1} e^{\delta r/2} + \left( \frac{r}{\delta} \right)^r \frac{1}{\delta} e^{\delta r/2} \right] = M \frac{2^r}{(r-1)!} \left( \delta \left( x + \frac{r}{2} \right) \right)^{r-1} e^{-\delta (x + r/2)} d\left( \delta \left( x + \frac{r}{2} \right) \right)
\]
\[
= M \frac{2^r}{(r-1)!} e^{\delta r/2} \left[ \frac{(r-1)^{r-1}}{e^{(r-1)!}} \delta + (r-1)! \right] = M \left( \frac{2}{\delta} \right)^r e^{\delta r/2} \left[ 1 + \frac{\delta}{\sqrt{2\pi}} \right].
\]

Remark. In the statement one should replace the last factor of the bound by $(1 + \delta/e)$ if $r = 1$.

**Lemma 5.** Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $K_n \leq a K_{n-1}^2$. Then
\[
K_n \leq \frac{1}{a} \left[ \left( \frac{5}{3} \right)^b a K_0 \right]^{2^n}.
\]

**Proof.** It is easy to see that
\[
K_n \leq a^1 + 2^2 + \ldots + 2^n \left[ 2 \left( \prod_{i=0}^{n-1} (n-i) \right)^b \right] K_0^{2^n}.
\]
To bound the expression in brackets we take logarithms,
\[
\ln \left( \prod_{i=0}^{n-1} (n-i)^2 \right) = \sum_{i=0}^{n-2} 2^i \ln(n-i) \leq 2^n \sum_{i=0}^{\infty} \frac{\ln(i+2)}{2^{i+2}} = c 2^n.
\]
Then

\[ c = \sum_{i=0}^{\infty} \frac{\ln(i+2)}{2^{i+2}} < \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \sum_{i \geq j} \frac{\ln((j+2)((j+3)/(j+2))^{i-j})}{2^{i+2}} \]

\[ = \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \frac{\ln(j+2)}{2^{j+1}} + \ln \left( \frac{j+3}{j+2} \right) \frac{1}{2^{j+3}} \sum_{k \geq 1} \frac{k}{2^{k-1}} \]

\[ = \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \frac{1}{2^{j+1}} \ln(j+3). \]

Hence,

\[ \exp c < \prod_{i=0}^{j-1} (i+2)^{2^{-i}} (j+3)^{2^{-j+1}}, \]

and taking \( j = 3 \) one obtains \( \exp c < 5/3 \) because \( 2^4 3^2 4 < (5/3)^{16} \). Finally \( K_n < a^{2n-1} (\exp c)^n K_0^n \) and the result follows.  

**Remark.** One can improve the bound on \( \exp c \) but not by more than three per thousand.

**Lemma 6.** Consider the expression \( a_n = ((n+1)^c/2)^{2^{-n}} \) for \( n \in \mathbb{N} \cup \{0\} \). If \( c \geq 3 \) the maximum is obtained for \( n = 1 \) and therefore \( a_n < 2^{(c-1)/2} \).

**Proof.** Let \( g(x) = \ln \left( \frac{(x+1)^{c/2}}{2^{x-1}} \right) = c^{2^{-x}} (\ln(x+1) - \ln \alpha) \), where \( \alpha = 2^{1/c} \) and \( x \geq 0 \). Computing the derivative and equating to zero one should have \( h(x) = \ln 2 (\ln(x+1) - \ln \alpha) - 1/(x+1) = 0 \) to obtain a maximum. The function \( h \) is monotonically increasing, as \( \ln \alpha \leq \frac{1}{c} \ln 2 \) one has \( h(1) < 0, h(2) > 0 \) for all \( c \geq 3 \). To see that the maximum over the integers is attained at \( n = 1 \) we compare the values for \( n = 1 \) and \( n = 2 \). One obtains \( (2^{1/2})^{1/2} \) and \( (3^{1/2})^{1/4} \) and the first one is larger than the second if \( c > (\ln 2/\ln (4/3)) \approx 2.41 \).

**Lemma 7.** Let \( M \) be a diagonal matrix with different eigenvalues \( \mu_j, j = 1, \ldots, d, \) and \( \alpha = \min_{i,j: i \neq j} |\mu_i - \mu_j| \). Let \( N \) be a matrix such that \( (d+1) \|N\| < \alpha \) (here \( \| \cdot \| \) is the sup norm). Let \( v_j, j = 1, \ldots, d \) be the eigenvalues of \( M + N \), \( B \) a suitable matrix such that \( B^{-1}(M+N)B = D = \text{diag}(v_j) \) with condition number \( C(B) \). Then

1. \( \beta = \min_{i,j: i \neq j} |v_i - v_j| \geq \alpha - 2 \|N\| \).
2. \( C(B) \leq (\alpha + (d-3) \|N\|)/(\alpha (d+1) \|N\|) \). In particular, if \( \|N\| < \alpha/(3d-1) \) then \( C(B) < 2 \).

**Proof.** From Gerschgorin Lemma it follows \( |\mu_i - v_j| < \|N\| \) and hence 1 holds. Let \( N = (n_{ij}), B = (b_{ij}) \). The matrix \( B \) is made of eigenvectors of
We choose a matrix $B$ such that $b_{ij} = 1$, $j = 1, \ldots, d$. To determine $b_{kj}$, $k = 1, \ldots, d$, $k \neq j$ we have to solve a $(d - 1)$-dimensional linear system, where the diagonal entries of the matrix are $\mu_k - v_j + n_{kk}$, $k \neq j$, and the out of diagonal entries are $n_{km}$, $k \neq j$, $m \neq j$. The independent term has entries $-n_{kj}$, $k \neq j$. Let $b_{kj}$ such that $|b_{kj}| = \max_{k \neq j} |b_{kj}|$. From

$$n_{s_1}b_{1j} + \cdots + n_{s_{j-1}}b_{j-1 j} + n_{s_j}b_{j+1 j} + \cdots + n_{s_d}b_{dj} + (\mu_j - v_j)b_{sj} = -n_{sj}$$

one has

$$|b_{sj}| \leq \frac{|n_{sj}|}{|\mu_j - v_j| - \|N\|} \leq \frac{\|N\|}{\alpha - 2 \|N\|}.$$

Therefore, $B = I + B'$ with

$$\|B\| \|B^{-1}\| \leq (1 + \|B\|)/(1 - \|B\|)$$

and (2) follows.

**Lemma 8.** Let $\omega \in \mathbb{R}$, $s = 1, \ldots, d$ such that

$$|\lambda_s - \lambda_j - \sqrt{-1}(k, \omega)| \geq \frac{c}{|k|^{\gamma_1}}.$$

for all $s, j \in \{1, \ldots, d\}$ and all $k \in \mathbb{Z} \setminus \{0\}$, where $c > 0$, $\gamma_1 > 0$. Define a resonant subset $\mathcal{R}_\mu$ as

$$\mathcal{R}_\mu = \{ \varphi \in \sqrt{-1}\mathbb{R}, |\varphi| < \mu / \exists s, j \in \{1, \ldots, d\} \land \exists k' \in \mathbb{Z} \setminus \{0\} \text{ such that } |\varphi + \lambda_s - \lambda_j - \sqrt{-1}(k', \omega)| < \frac{c/2}{|k'|^{\gamma_2}} \}.$$

Let $\psi(\mu) = m(\mathcal{R}_\mu)/2\mu$, where $m$ denotes the Lebesgue measure. If $\gamma_2 = \gamma_1 + r + 1$ then $\lim \inf_{\mu \to 0} \psi(\mu) = 0$.

**Proof.** Take $\mu_n = c/n^{\gamma_1}$. For any $k'$ with $|k'| \geq n$ and any couple $s, j$ the measure of the resonant interval of $\varphi$ is bounded by $c/|k'|^{\gamma_2}$. Adding for all the values of $k'$ with $|k'| = n'$ and all $s, j$ and using the remark following Lemma 1, we have

$$m(\mathcal{R}_{n_1}) \leq c2r d^2 \sum_{n' \geq n} \frac{1}{(n')^{\gamma_2 - r + 1}} < 2crd^2(n - 1)^{-(\gamma_2 - r)}.$$
Furthermore the resonant intervals associated to $n'<n$ are disjoint with $\mathcal{R}_{\mu_n}$ if $n$ is large enough. Hence, for $n$ large enough, $\psi(\mu_n) < r d^2n^{11}(n-1)^{-(r+1)}$ which goes to zero if $n$ goes to infinity. 

3. **Proof of Theorem**

First we are going to do the proof without worrying about resonances, and then we shall take out the values of $\epsilon$ for which the proof fails.

We suppose that we have applied the method exposed in Section 1 until step $n$, and we are going to see that we can apply it again to obtain the $n+1$ step. In this way we shall obtain bounds for the quasiperiodic part at the $n$th step and for the transformation at this step, and this allows us to prove the convergence. Now suppose that we are at the $n$th step. This means that we have

$$\dot{x}_n = (A_n + e^{2\pi i n}Q_n)x_n,$$

where $A_n$ is a diagonal matrix with eigenvalues $\lambda_1^n, \ldots, \lambda_n^n$ satisfying

$$|\lambda_i^n - \lambda_j^n - (k, \omega)| \sqrt{-1} > \frac{c_n}{|k|^{\gamma}}, \quad \forall i, j,$$

with $\gamma = \gamma_n + r + 1$ and $c_n$ is taken as $c_n = c_0/(n+1)^2$. We have $Q_n = (q_{nij})$ with

$$q_{nij} = \sum_{k \neq 0} q_{nij}^k e^{(k, \omega)\sqrt{-1}t},$$

and $|q_{nij}| \leq M_n e^{-\rho_n|k|}$, where $M_n = \|Q_n\|_{\rho_n}$. Moreover, $\{\rho_n\}$ is a sequence defined by $\rho_n = \rho_{n-1} - 2/n^2$ with $\rho_0 = \pi^2/3 + 1$, and $\tilde{\rho}_n = \rho_n + 1/n^2$.

We note that the limit value $\lim_{n \to \infty} \rho_n$ is equal to 1. Finally we suppose that $Q_n$ has already been averaged: $\tilde{Q}_n = 0$. Now we need to solve $\dot{P}_n = A_n P_n - P_n A_n + Q_n$ and we use Lemma 3 to obtain a unique $P_n = (p_{nij})$ whose elements verify

$$p_{nij} = \sum_{k \in \mathbb{Z} \setminus \{0\}} p_{nij}^k e^{(k, \omega)\sqrt{-1}t},$$

and

$$|p_{nij}^k| \leq \frac{M_n}{c_n} \left(\frac{\gamma}{(1/(n+1)^2)e}\right)^\gamma e^{-\tilde{\rho}_{n+1}|k|}.$$
Now we can apply Lemma 4 to bound \( \|P_n\|_{\rho_{n+1}} \):

\[
\|P_n\|_{\rho_{n+1}} \leq d \max_{i,j} \|p_{nij}\|_{\rho_{n+1}}.
\]

Therefore

\[
\|P_n\|_{\rho_{n+1}} \leq d E M_n (n + 1)^{2(y+1)}
\]

\[
\times \left[ (2(n + 1)^2)^r e^r (2(n + 1)^2)^r \left(1 + \frac{1/(n + 1)^2}{\sqrt{2\pi(r-1)}}\right) \right].
\]

We can bound the previous expression by

\[
\|P_n\|_{\rho_{n+1}} \leq L M_n (n + 1)^{2(y + r + 1)},
\]

(3.1)

where

\[
L = d E 2^r e^r \left(1 + \frac{1}{\sqrt{2\pi(r-1)}}\right).
\]

Of course, if \( r = 1 \) we replace \( \sqrt{2\pi(r-1)} \) by \( e \).

Now, remembering that \( M_n = \|Q\|_{\rho_n} \) we obtain the bound that we were looking for,

\[
\|P_n\|_{\rho_{n+1}} \leq L(n + 1)^{2(y + r + 1)} \|Q_n\|_{\rho_n},
\]

(3.2)

If we change variables through \( y_{n+1} = (I + \epsilon^{2n} P_n) x_n \) we obtain

\[
y_{n+1}' = (A_n + \epsilon^{2n+1} (I + \epsilon^{2n} P_n)^{-1} Q_n P_n) y_{n+1}.
\]

We suppose now that \( \|\epsilon^{2n} P_n\| \leq \frac{1}{3} \) (we shall see after that it can be achieved by selecting \( \epsilon \) small enough). Let \( Q^*_{n+1} = (I + \epsilon^{2n} P_n)^{-1} Q_n P_n \). We can now bound the new quasiperiodic part,

\[
\|Q^*_{n+1}\|_{\rho_{n+1}} \leq \frac{1}{1 - \|\epsilon^{2n} P_n\|_{\rho_{n+1}}} \|Q_n\|_{\rho_{n+1}} \|P_n\|_{\rho_{n+1}},
\]

and using (3.2) we obtain

\[
\|Q^*_{n+1}\|_{\rho_{n+1}} \leq 2 L (n + 1)^{2(y + r + 1)} \|Q_n\|_{\rho_n}^2.
\]

At this point we introduce the matrices \( \bar{Q}^*_{n+1} \) (see Section 1) and \( \bar{A}_{n+1} = A_n + \epsilon^{2n+1} \bar{Q}^*_{n+1} \) (we note that, in general, \( \bar{A}_{n+1} \) has no diagonal form). We still have

\[
\|\bar{Q}^*_{n+1}\|_{\rho_{n+1}} \leq 2 L (n + 1)^{2(y + r + 1)} \|Q_n\|_{\rho_n}^2.
\]
Now we have the following equation

\[ \dot{y}_{n+1} = (\tilde{A}_{n+1} + e^{2\pi i} Q_{n+1}^*) y_{n+1}. \]

Let \( B_{n+1} \) be a matrix such that \( B_{n+1}^{-1} \tilde{A}_{n+1} B_{n+1} = A_{n+1} \) is diagonal. We choose the diagonal of \( B_{n+1} \) equal to the identity as in Lemma 7. Making \( x_{n+1} = B_{n+1} y_{n+1} \), one obtains

\[ \dot{x}_{n+1} = (A_{n+1} + e^{2\pi i} Q_{n+1}) x_{n+1}, \]

where \( Q_{n+1} = B_{n+1}^{-1} \tilde{Q}_{n+1}^* B_{n+1} \). As \( \tilde{Q}_{n+1} = 0 \) we only need to control the size of \( \|Q_{n+1}\|_{\rho_{n+1}} \). We define the condition number \( C(B) = \|B^{-1}\| / \|B\| \) for all nonsingular constant matrices \( B \), and we shall see later that \( C(B_n) \leq 2, \forall n \).

Now we can bound \( \|Q_{n+1}\|_{\rho_{n+1}} \),

\[ \|Q_{n+1}\|_{\rho_{n+1}} = \|B_{n+1}^{-1} \tilde{Q}_{n+1}^* B_{n+1}\|_{\rho_{n+1}} \leq 4L(n+1)^{2(\gamma + r + 1)} \|Q_n\|_{\rho_n}^2. \]

If we suppose that the same inequality holds for \( \|Q_n\|_{\rho_n}, ... , \|Q_1\|_{\rho_1} \) and we use Lemma 5 together with \( \|Q_0\|_{\rho_0} = 1 \) one obtains

\[ \|Q_{n+1}\|_{\rho_{n+1}} \leq \frac{1}{4L} \left[ \left( \frac{5}{3} \right)^b 4L \right]^{2n+1}, \]

where \( b = 2(\gamma + r + 1) \).

At this point we are in a situation to prove the convergence. The quasiperiodic part at the \( n \)th step is \( e^{2\pi i} Q_n \) whose norm on the strip \( |\text{Im} \ z| \leq \rho_n \) is bounded by

\[ \frac{1}{4L} \left[ \varepsilon \left( \frac{5}{3} \right)^b 4L \right]^{2n}. \]

This converges to 0 if the bracket is less than 1, that is, if \( \varepsilon < K^{-1} \), where \( K = (5/3)^b 4L \).

We had left without proof the fact \( \|e^{2\pi i} P_n\|_{\rho_{n+1}} \leq \frac{1}{2} \). Recall (3.1) and then

\[ \|e^{2\pi i} P_n\|_{\rho_{n+1}} \leq \frac{(n+1)^{2(\gamma + r + 1)}}{4} (\varepsilon K)^{n}. \]

To have \( \|e^{2\pi i} P_n\|_{\rho_{n+1}} < \frac{1}{2} \) it is enough to take

\[ \varepsilon < \left( K \max_{n \in \mathbb{N} \cup \{0\}} \left\{ \left( \frac{(n+1)^2}{2} \right)^{-n} \right\} \right)^{-1}, \]

where \( c = 2(\gamma + r + 1) > 2(2r + d) \). Using Lemma 6 it is enough to take \( \varepsilon < (K^2)^{-1} = \varepsilon_1 \).
To end this part we need to prove that the condition $C(B_n) \leq 2$, $\forall n$ holds if $\varepsilon$ is sufficiently small. Let $\alpha = \min_{i \neq j} |\lambda_i^0 - \lambda_j^0|$. The successive steps change the minimum distance between eigenvalues (see Lemma 7) at most by

$$2 \sum_{n \geq 0} \varepsilon^{2n+1} \|Q_{n+1}^*\|_{\rho_{n+1}} \leq \frac{1}{4L} \sum_{n \geq 0} (\varepsilon K)^{2n+1} \leq \frac{1}{4L} \frac{(\varepsilon K)^2}{1 - (\varepsilon K)^2}.$$  

We ask that this value be less than $\alpha/2$. Then $|\lambda_i^n - \lambda_j^n| > \alpha/2$ and the condition (2) of Lemma 7 to have $C(B_n) \leq 2$ is written as

$$\frac{1}{8L} (\varepsilon K)^{2n+1} \leq \frac{\alpha/2}{3d - 1}$$

that holds for all $n$ if it holds for $n = 0$. Hence it is enough to impose the condition

$$\varepsilon \left[ K \max \left\{ \left( \frac{4\alpha L}{3d - 1} \right)^{-1/2} \left( \frac{2L\alpha}{1 + 2L\alpha} \right)^{-1/2} \right\} \right]^{-1} = \varepsilon_2$$

to guarantee $C(B_n) \leq 2$ for all the transformations. Hence $\|\varepsilon^2 Q_n\|_{\rho = 1}$ goes to zero if $\varepsilon < \min(\varepsilon_1, \varepsilon_2) = \varepsilon_3$. To see that the composition of all the transformations $B_{n+1}(I + \varepsilon^2 P_n)$ is convergent we first bound the transformation at step $n$,

$$\|B_{n+1}(I + \varepsilon^2 P_n)\|_{\rho_{n+1}} \leq \left[ 1 + \frac{(d - 1) \varepsilon^{2n+1}}{\alpha/2} \frac{2}{2n+1} \|Q_{n+1}^*\|_{\rho_{n+1}} \right] \left[ 1 + \frac{\varepsilon^{2n}}{4L} \|P_n\|_{\rho_{n+1}} \right]$$

$$= (1 + a_n)(1 + b_n).$$

It is clear that $a_n$ and $b_n$ go to zero when $n$ goes to infinity and that the series

$$\sum_{n = 0}^{\infty} a_n, \sum_{n = 0}^{\infty} b_n$$

are convergent if $\varepsilon < \varepsilon_3$. Then the full procedure works for $\varepsilon < \min(\varepsilon_0, \varepsilon_3) = \varepsilon_4$ provided the nonresonance condition

$$|\lambda_i^n - \lambda_j^n - (k, \omega) \sqrt{-1}| > c_n/k^7$$

holds for all $i, j \in \{1, \ldots, d\}$, for all $k \in \mathbf{Z} \setminus \{0\}$ and for all $n \in \mathbf{N} \cup \{0\}$. To end the proof we are going to take into account the resonances. Let $\phi_i^n(\varepsilon)$ be the function that gives the values of $\lambda_i^n - \lambda_j^n$ at step $n$,

$$\phi_i^n(\varepsilon) = \lambda_i^n(\varepsilon) - \lambda_j^n(\varepsilon) + \varepsilon^2 d_{i,2}^n + \varepsilon^3 d_{i,3}^n + \cdots.$$
At every step the eigenvalues and the diagonalizing matrix, $B_{n+1}$, depend algebraically, and therefore analytically, on $\varepsilon$. Hence, as

$$\left| \frac{d}{d\varepsilon} \left( \lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon) \right) \right|_{\varepsilon = 0} > 2\delta$$

one has $(d/d\varepsilon) |\varphi^0_{ij}(\varepsilon)| > \delta$ if $\varepsilon$ is small enough, $\varepsilon < \varepsilon_s$. On the other side $|d\varphi^0_{ij}/d\varepsilon|$ is bounded by some $\delta$ for all $i, j, n$ in some interval $\varepsilon \in (0, \varepsilon) \subset (0, \varepsilon_4) \cap (0, \varepsilon_s)$. Here we use, for simplicity, the remark following Lemma 3 and consider all the $\varphi^0_{ij}$ as purely imaginary. If we take some $\mu_m$ (see Lemma 8), with $\gamma_1 = \gamma_1$, $\gamma_2 = \gamma$, $c = c = c_0 = c_0/2$, such that $\mu_m/\delta < \varepsilon$ when $\varepsilon$ ranges on $(0, \varepsilon)$ then $\varphi^0_{ij}$ ranges on $(-\mu_m, \mu_m)$.

To obtain the cantorian set $\mathcal{E}_0$ where the nonresonance conditions hold for $n = 0$ one should delete an infinity of intervals in the range of $\varepsilon$ with a measure at most $\psi(\mu_m) 2\mu_m(1/\delta) d^2$. The relative measure of $\mathcal{E}_0$ in $(0, \mu_m/\delta)$ is at least $1 - \psi(\mu_m) 2\delta^2/\delta$. In a similar way we obtain the set $\mathcal{E}_n \subset \mathcal{E}_{n-1}$, where the nonresonant condition holds up to $n$. Its relative measure in $(0, \mu_m/\delta)$ is at least

$$1 - \psi(\mu_m) \left( 2\delta^2/\delta \right) \sum_{j=0}^n \frac{1}{(j+1)^2} > 1 - \psi(\mu_m) \frac{\pi^2}{3} \delta^2$$

which goes to 1 if $n$ goes to infinity. The limit set

$$\mathcal{E}_\infty = \bigcap_{n \geq 0} \mathcal{E}_n$$

is the cantorian that we were looking for.

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