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Artificial discontinuities of single-parametric Gröbner bases

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ABSTRACT

Artificial discontinuity is a kind of singularity at a parametric point in computing the Gröbner basis of a specialized parametric ideal w.r.t. a certain term order. When it occurs, though parameters change continuously at the point and the properties of the parametric ideal have no sudden changes, the Gröbner basis will still have a jump at the parametric point. This phenomenon can cause instabilities in computing approximate Gröbner bases.

In this paper, we study artificial discontinuities in single-parametric case by proposing a solid theoretical foundation for them. We provide a criterion to recognize artificial discontinuities by comparing the zero point numbers of specialized parametric ideals. Moreover, we prove that for a single-parametric polynomial ideal with some restrictions, its artificially discontinuous specializations (ADS) can be *locally repaired* to continuous specializations (CS) by the TSV (Term Substitution with Variables) strategy.

1. Introduction

"Artificial discontinuity" is a kind of singular phenomenon that may appear in computing Gröbner bases. It has been described in several publications such as Stetter (1997), Stetter (2004), Kondratyev (2004), Mourrain (2007) and Faugère and Liang (2007), where the terminologies may be different.

Let us see a toy example. Take two polynomials $\hat{f}_1 = 4x^2 + y^2 - 4$ and $\hat{f}_2 = 4\theta xy + 15y^2 - 12$ in $\mathbb{R}[\theta][x, y]$. Consider the staircases (i.e., normal sets) $N_{\theta=\tilde{A}}$ corresponding to the Gröbner bases of $\langle \hat{f}_1, \hat{f}_2\rangle_{\theta=\tilde{A}}$ w.r.t. the DRL order with $x \succ y$ for all points $\tilde{A}$ around 0. Note that $\{1, y, x, y^2\} = N_{\theta=\tilde{A}\neq 0} \neq$

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$N_{\theta=0} = \{1, y, x, xy\}$. However, this does not mean that the quotient rings $\mathbb{R}[x, y]/\langle \hat{f}_1, \hat{f}_2\rangle_{\theta=\tilde{A}}$ have no common monomial bases. In fact, $\{1, y, x, xy\}$ is one of them (Faugère and Liang, 2007). Hence, the singularity is not intrinsic but comes from the Gröbner basis method itself. In Kondratyev (2004), this kind of phenomenon is called artificial discontinuity. It can lead to instabilities in computing approximate Gröbner bases. To understand artificial discontinuity more deeply and at last solve the instabilities numerically, we need to study it parametrically and theoretically.

The aim of this paper is to set up a theory for artificial discontinuities. Since there were no formal definitions or criteria but only some descriptions on one classical example (two ellipses) in the past two decades, we first give formal definitions of the continuous specialization (CS), the discontinuous specialization (DS) and the artificially discontinuous specialization (ADS). Then we provide their criteria. Theorem 3.1 tells us that a DS can be identified from CS by checking the specialized leading coefficients of the polynomials in the Gröbner bases of a parametric ideal $\hat{I}$. Theorem 3.4 and Corollary 3.5 show that an ADS can be found by comparing the numbers of zeros of the specialized systems. At last, we focus our study on how to repair ADS. In the paper (Faugère and Liang, 2007), we have provided a strategy called TSV (Term Substitution with Variables), whose main points are also listed at the end of Section 2. In this paper, the TSV strategy is applied to deal with ADS in parametric cases. We prove that if a specialization Γ_A of a parametric ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ saturated w.r.t. $(\theta - A)$ is an ADS, where A is a point in an infinite T_1-topological field $\mathbb{K}$ without isolated points, then it can be locally repaired to a CS by the TSV strategy (Theorem 3.8). For the example mentioned in the second paragraph, we only need to add a binomial $z - xy$ to the system and compute the minimal strong Gröbner basis (cf. Adams and Loustaunau (1994)) w.r.t. the DRL order with $x \succ y \succ z$. Then we can get a common monomial basis $\{1, z, y, x\}$ of $\mathbb{R}[x, y, z]/\langle \hat{f}_1, \hat{f}_2, z-xy\rangle_{\theta=\tilde{A}}$ for all $\tilde{A}$ in a neighborhood of 0. That is to say that the artificial discontinuity at 0 has been locally repaired. The relationship between an ideal and its extended ideals can be found in the paper (Faugère and Liang, 2007).

There are also some other attempts to deal with artificial discontinuities, such as the extended Gröbner basis method (Stetter, 1997, 2004; Kondratyev, 2004), the border basis method (Auzinger and Stetter, 1988; Möller, 1993; Mourrain, 1999; Stetter, 2004; Kehrein et al., 2005; Kehrein and Kreuzer, 2005, 2006; Chen and Meng, 2007; Abbott et al., 2008) and the generalized normal form method (Trébuchet, 2004; Mourrain and Trébuchet, 2005). G. Reid, J. Tang, J. Yu and L. Zhi have also provided a method based on the partial differential equation theory (Reid et al., 2003, 2005). In this paper, we want to contribute to the study of artificial discontinuities by proposing a solid theoretical foundation for them.

The rest of this paper is structured as follows. In Section 2, we give some necessary preliminaries. Section 3 is devoted to proving our main theoretical results. Finally, we make the conclusion and state the future work in Section 4.

2. Preliminaries

Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_s\}$ be two finite sets of variables, $\Theta = \{\theta_1, \ldots, \theta_k\}$ be the set of parameters. Let $\mathbb{K}$ be an infinite T_1-topological field, i.e., topological field (cf. Warner (1989)) with T_1 separation axiom (cf. Munkres (2004) and Singer and Thorpe (2009)), without isolated points. Then each nonempty open set of $\mathbb{K}$ contains infinitely many elements and each rational function in Θ is continuous where it is well-defined in $\mathbb{K}^k$. Let I and $\hat{I}$ be an ideal of $\mathbb{K}[X]$ and $\mathbb{K}[\Theta][X]$ respectively, T^X be the set of terms in X, $\preceq$ be a term order on T^X. Denote a term set $\{1, t_1, \ldots, t_s\} \subset T^X$ by M and the substitution set $\{y_1 - t_1, \ldots, y_s - t_s\} \subset \mathbb{K}[X \cup Y]$ by E. Let $E = 0$ stand for $\{y_1 = t_1, \ldots, y_s = t_s\}$ and $X \succ Y$ stand for the condition that $x \succ y$ for every $x \in X$ and every $y \in Y$.

For a parametric polynomial $\hat{f} \in \mathbb{K}[\Theta][X]$, let $\mathrm{lt}(\hat{f})$, $\mathrm{lc}(\hat{f})$, $\mathrm{lm}(\hat{f})$, $\mathrm{T}(\hat{f})$ and $\mathrm{C}(\hat{f})$ be the leading term (with coefficient 1), the leading coefficient, the leading monomial, the term set and the coefficient set of $\hat{f}$, respectively. For any $S \subset T^X$, denote the term set $\{xs : x \in X, s \in S\} \setminus S$ by ∂S and denote the Dickson basis of S by $\mathrm{B}(S)$.

We call $\mathbb{K}[X \cup Y]$ an extended ring of $\mathbb{K}[X]$ w.r.t. Y, $I^E = \langle I \cup E\rangle$ an extended ideal of I w.r.t. E in $\mathbb{K}[X \cup Y]$, and $T^{X\cup Y}$ an extended term set of T^X w.r.t. Y. Specify a term order $\preceq$ on T^X. If another term order $\preceq^e$ on $T^{X\cup Y}$ coincides with $\preceq$ on T^X, then $\preceq^e$ is called an extended order of $\preceq$ on $T^{X\cup Y}$.

In this paragraph, some notions come from the paper (Weispfenning, 1992). Let A be a point in $\mathbb{K}^k$. A specialization $\Gamma_A : \mathbb{K}[\Theta] \to \mathbb{K}$ denotes the only ring homomorphism from $\mathbb{K}[\Theta]$ to $\mathbb{K}$ such that $\Gamma_A(\theta_i) = a_i$ for $i = 1, \ldots, k$. We denote the canonical extension $\bar{\Gamma}_A : \mathbb{K}[\Theta][X] \to \mathbb{K}[X]$ by Γ_A as well. Let $\hat{f}_A$ be $\Gamma_A(\hat{f})$, and $\hat{F}_A$ be the set of images of $\hat{F} \subset \mathbb{K}[\Theta][X]$ under the specialization Γ_A. Let $\mathrm{Lt}(\hat{F})$ denote the ideal generated by $\{\mathrm{lt}(\hat{f}) : \hat{f} \in \hat{F}\}$ in the corresponding ring.

In what follows, we give formal definitions of various specializations: the continuous specializations (CS), the discontinuous specializations (DS), the artificially discontinuous specializations (ADS) and the intrinsically discontinuous specializations (IDS). Although we focus our attention on parametric ideals with a single parameter in this paper, i.e., $\Theta = \{\theta\}$, we would like to define these concepts in a more general form.

Definition 2.1 (*CS/DS*)**.** Specify an ideal $\hat{I} \subset \mathbb{K}[\Theta][X]$, a term order $\preceq$ on T^X and a specialization Γ_A. If there exists a finite set $\hat{F} \subset \hat{I}$ such that

- $0 \notin \mathrm{lc}(\hat{F})_A$;
- $\mathrm{Lt}(\hat{F}) = \mathrm{Lt}(\hat{I})$;
- $\hat{F}_A$ is a minimal Gröbner basis of $\hat{I}_A$ w.r.t. $\preceq$,

then we say that Γ_A is a continuous specialization (CS) of $\hat{I}$ w.r.t. $\preceq$. Otherwise, Γ_A is called a discontinuous specialization (DS) of $\hat{I}$ w.r.t. $\preceq$.

Definition 2.2 (*ADS/IDS*)**.** Specify a term order $\preceq$ on T^X and a discontinuous specialization Γ_A of $\hat{I} \subset \mathbb{K}[\Theta][X]$ w.r.t. $\preceq$. If there exists a term set $M \subset \mathrm{T}^X$ and a neighborhood V of A such that

- $1 \in M$;
- $\forall \tilde{A} \in V$, M forms a basis of $\mathbb{K}[X]/\hat{I}_{\tilde{A}}$,

then we say that Γ_A is an artificially discontinuous specialization (ADS) of $\hat{I}$ w.r.t. $\preceq$. Otherwise, Γ_A is called an intrinsically discontinuous specialization (IDS) of $\hat{I}$ w.r.t. $\preceq$.

The rest contents of this section are from the paper (Faugère and Liang, 2007).

Definition 2.3 (*Direct Product*)**.** Specify an ordered set $(\mathrm{T}^X, \preceq)$, and two tuples $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N) \in (\mathrm{T}^X)^N$, where $\preceq$ is a term order on T^X, N is a positive integer and $(\mathrm{T}^X)^N$ is the Cartesian product. Construct a new ordered set $((\mathrm{T}^X)^N, \preceq')$, where $\preceq'$ is defined by $a \prec' b$ iff

- $a_i \preceq b_i$ for every $1 \le i \le N$;
- there exists an integer $1 \le j \le N$ such that $a_j \prec b_j$.

Then we call $((\mathrm{T}^X)^N, \preceq')$ the Nth direct product of $(\mathrm{T}^X, \preceq)$. (cf. p. 163, Becker et al. (1993))

Remark 2.4. "Direct Product" here is modified slightly, taking place "quasi-order" by "term order".

Given a term order $\preceq$ and a term sequence $m_1 \prec m_2 \prec \cdots$ (not necessarily finite) in T^X, if $\{m_1, m_2, \ldots\}$ forms a basis of $\mathbb{K}[X]/I$, then $m = (m_1, m_2, \ldots)$ is called a *basis tuple* of $\mathbb{K}[X]/I$ in T^X. Denote by MT^X the set of basis tuples of $\mathbb{K}[X]/I$ in T^X.

Proposition 2.5. MT^X *has a unique element* m' *such that* $m' \preceq' m$ *for every* $m \in \mathrm{MT}^X$*. More precisely, the components of* m' *form a staircase corresponding to the Gröbner basis* G *of* I *w.r.t.* $\preceq$*.*

Definition 2.6 (*TSV Strategy*)**.** Specify a polynomial ideal $I \subset \mathbb{K}[X]$, a substitution set E and a term order $\preceq$ on T^X. Compute the Gröbner basis G' and the corresponding normal set (staircase) N' of $I^E = \langle G \cup E \rangle$ w.r.t. some extended order $\preceq^e$ of $\preceq$ on $\mathrm{T}^{X \cup Y}$. Then substitute $E = 0$ to N', and a new term set $N \subset \mathrm{T}^X$ is obtained. This strategy to compute G', N or other objects is called TSV strategy.

Theorem 2.7. *Specify a finite term set* $M \subset \mathrm{T}^X$ *(containing* 1*) and an extended graded order* $\preceq^e$ *of a graded order* $\preceq$ *on* $\mathrm{T}^{X \cup Y}$*. If there exists a subset* K *of* M *such that* $K \cup \{1\}$ *forms a basis of* $\mathbb{K}[X]/I$*, then the set* N *(or* N'*) computed by using the TSV strategy w.r.t.* $\preceq^e$ *is a monomial basis of* $\mathbb{K}[X]/I$ *(or* $\mathbb{K}[X \cup Y]/I^E$*); more precisely,* N *is a subset of* M*.*

3. Main results

In this part, we start to set up the theory for the artificial discontinuity of an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$. First, we give the criteria for a CS and an ADS in the sense of Definition 2.1 and Definition 2.2. Then we prove Theorem 3.8 which shows that an ADS can be locally repaired to a CS in the Gröbner basis framework by the TSV strategy.

3.1. Criteria for continuous specializations

To study the local behavior of the Gröbner basis of a specialized parametric ideal $\hat{I}$, one should know what a given specialization Γ_A is — a CS, a DS or an ADS. Since the latter two are based on the definition of the former, we first give Theorem 3.1 to identify a CS. In fact, Theorem 3.1 includes two criteria, by which a CS can be identified both in $\mathbb{K}[\theta][X]$ and in $\mathbb{K}[X]$. The criteria for a DS can be easily deduced from that for a CS.

Theorem 3.1. *Specify a point $A \in \mathbb{K}$, a term order $\preceq$ on T^X and an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ saturated w.r.t. $(\theta - A)$. Let $\hat{G}$ be a Gröbner basis of $\hat{I}$ w.r.t. $\preceq$. Then the following three conditions are equivalent.*

(1) *Γ_A is a CS of $\hat{I}$ w.r.t. $\preceq$.*
(2) *There exists a subset $\hat{H} \subset \hat{G}$ such that $\mathrm{lt}(\hat{H}) = \mathrm{B}(\mathrm{Lt}(\hat{G}))$, $0 \notin \mathrm{lc}(\hat{H})_A$ and $\hat{H}_A$ is a minimal Gröbner basis of $\hat{I}_A$ w.r.t. $\preceq$.*
(3) *There exists a neighborhood V of A such that $\mathbb{K}[X]/\hat{I}_{\tilde{A}}$ share the same staircase M corresponding to the Gröbner bases of $\hat{I}_{\tilde{A}}$ w.r.t. $\preceq$ for all $\tilde{A} \in V$.*

Proof. (1) ⇒ (2): For each $\hat{f} \in \hat{F}$ (cf. Definition 2.1), there exists a finite set $\hat{H}_{\hat{f}} \subset \hat{G}$ such that all the elements of $\hat{H}_{\hat{f}}$ have the same leading term $\mathrm{lt}(\hat{f})$ and $\sum_{\hat{h} \in \hat{H}_{\hat{f}}} (\gamma_{\hat{h}} \mathrm{lc}(\hat{h})) = \mathrm{lc}(\hat{f})$, where $\gamma_{\hat{h}} \in \mathbb{K}[\theta]$. Since $\mathrm{lc}(\hat{f})_A \neq 0$, there must exist an $\hat{h}_{\mathrm{lt}(\hat{f})} \in \hat{H}_{\hat{f}}$ such that $\hat{h}_{\mathrm{lt}(\hat{f})A} \neq 0$. Pick $\hat{H} = \{\hat{h}_{\mathrm{lt}(\hat{f})} : \hat{f} \in \hat{F}\}$.

(2) ⇒ (3): Since $\mathbb{K}$ is a topological field, polynomials in $\mathbb{K}[\theta]$ are continuous functions in $\mathbb{K}$. Note that $\mathbb{K}$ as a topological space satisfies the T_1 separation axiom. Thus, $\mathbb{K} \setminus \{0\}$ is open and nonempty. Hence, from $0 \notin \mathrm{lc}(\hat{H})_A$, we know that there exists a neighborhood V of A such that $0 \notin \mathrm{lc}(\hat{H})_{\tilde{A}}$ for every $\tilde{A} \in V$.

(3) ⇒ (1): Pick a minimal strong Gröbner basis $\hat{G}$ of $\hat{I}$ w.r.t. $\preceq$. Since $\prod_{\hat{g} \in \hat{G}} \mathrm{lc}(\hat{g})$ is nonzero and V contains infinite number of points in $\mathbb{K}$, there exists a point $A^* \in V$ such that $\mathrm{lc}(\hat{g})_{A^*} \neq 0$ for every $\hat{g} \in \hat{G}$. Then $\mathrm{Lt}(\hat{G}) = \langle \partial M \rangle = \langle \mathrm{B}(\partial M) \rangle$ and every $t \in \mathrm{B}(\partial M) \subset \mathrm{lt}(\hat{G})$ corresponds to a unique $\hat{g}_t \in \hat{G}$ with $\mathrm{lt}(\hat{g}_t) = t$. Sort $\mathrm{B}(\partial M)$ by increasing order of $\preceq$. Pick $t_1 = \min_{\preceq}(\mathrm{B}(\partial M))$. Then $\mathrm{T}(\hat{g}_{t_1}) \setminus \{t_1\} \subset M$. Since $\hat{I}$ is saturated w.r.t. $(\theta - A)$ and M is $\mathbb{K}$-linearly independent in $\mathbb{K}[X]/\hat{I}_A$, we have $\mathrm{lc}(\hat{g}_{t_1})_A \neq 0$. Supposing that $\mathrm{lc}(\hat{g}_{t_j})_A \neq 0$ for all $1 \leq j \leq k-1$, we prove $\mathrm{lc}(\hat{g}_{t_k})_A \neq 0$. Otherwise, reduce $\hat{g}_{t_k} \prod_{1 \leq j \leq k-1} \mathrm{lc}(\hat{g}_{t_j})^{r_j}$ by every $\hat{g}_{t_j}$ to a polynomial $\hat{g}_{t_k\mathrm{red}}$ such that $\mathrm{T}(\hat{g}_{t_k\mathrm{red}}) \setminus \{t_k\} \subset M$, where r_j is the times that $\hat{g}_{t_j}$ is used. Hence, $\hat{g}_{t_k\mathrm{red}A} = 0$ and $\hat{g}_{t_k\mathrm{red}}$ can be written as $\hat{h}(\theta - A)^q$ where $q \in \mathbb{Z}^+$, $\hat{h} \in \hat{I}$ with $\hat{h}_A \neq 0$. However, this leads to a contradiction that $\mathrm{lm}(\hat{g}_{t_k}) \nmid \mathrm{lm}(\hat{h})$. □

3.2. Criteria for artificially discontinuous specializations

One of the main results in this paper is Theorem 3.4 which gives an algebraic criterion for an ADS by comparing the number of zeros of specified ideals. In order to prove this criterion, we first give two lemmas.

Lemma 3.2. *Specify a point $A \in \mathbb{K}$ and an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ saturated w.r.t. $(\theta - A)$. If a term set $M \subset \mathrm{T}^X$ is $\mathbb{K}$-linearly independent in $\mathbb{K}[X]/\hat{I}_A$, then it is $\mathbb{K}(\theta)$-linearly independent in $\mathbb{K}(\theta)[X]/\langle \hat{I} \rangle_{\mathbb{K}(\theta)[X]}$.*

Proof. If M is $\mathbb{K}(\theta)$-linearly dependent in $\mathbb{K}(\theta)[X]/\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$, then there exists a polynomial $\hat{f} \in \langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$ such that $\mathrm{T}(\hat{f}) \subset M$. Note that $\hat{f}$ can be written as $\sum_{i=1}^{m}\hat{p}_i\hat{f}_i$, where $\hat{p}_i \in \mathbb{K}(\theta)[X]$ and $\hat{f}_i$'s ($i = 1,\ldots,m$) are generators of $\hat{I}$. Thus, there exists a polynomial $d \in \mathbb{K}[\theta]$ such that $d\hat{f} \in \hat{I}$. If $(d\hat{f})_A \neq 0$, then M is $\mathbb{K}$-linearly dependent in $\mathbb{K}[X]/\hat{I}_A$. If $(d\hat{f})_A = 0$, then $d\hat{f}$ can be written as $(\theta - A)^r\hat{f}^*$ with $\hat{f}^*_A \neq 0$. Since $\hat{I}$ is saturated w.r.t. $(\theta - A)$, we know that $\hat{f}^*$ belongs to $\hat{I}$. Therefore, M is also $\mathbb{K}$-linearly dependent in $\mathbb{K}[X]/\hat{I}_A$. □

Lemma 3.3. *Specify a point $A \in \mathbb{K}$ and an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ with $\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$ zero-dimensional. Pick any monomial basis M of $\mathbb{K}(\theta)[X]/\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$. Then there exists a neighborhood V_M of A such that M is a basis of $\hat{I}_{\tilde{A}}$ for every $\tilde{A} \in V_M \setminus \{A\}$.*

Proof. Pick a Gröbner basis $\hat{G}$ of $\hat{I}$ w.r.t. a term order $\preceq$ on T^X. The reduced Gröbner basis $\hat{\mathcal{G}}$ of $\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$ w.r.t. $\preceq$ can be deduced from $\hat{G}$. Denote $\Pi_{\hat{g}\in\hat{G}}\mathrm{lc}(\hat{g})$ by w. Since w has finitely many zeros in $\mathbb{K}$ and $\mathbb{K}$ is an infinite T_1-topological field without isolated points, there exists a neighborhood V_0 of A such that $V_0 \setminus \{A\} \neq \emptyset$ and $w_{\tilde{A}} \neq 0$ for every $\tilde{A} \in V_0 \setminus \{A\}$. Then $\hat{\mathcal{G}}_{\tilde{A}}$ is a Gröbner basis of $\hat{I}_{\tilde{A}}$. Denote the corresponding staircase by M_s (finite, since $\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$ is zero-dimensional). There exists a matrix P whose entries are all rational functions of θ such that $M = PM_s$ and the determinant $\det(P) \neq 0$. Since $\mathbb{K}$ is an infinite T_1-topological field without isolated points, there exists a neighborhood V_1 of A such that $\det(P)_{\tilde{A}} \neq 0$ for every $\tilde{A} \in V_1 \setminus \{A\}$. Take $V_M = V_0 \cap V_1$. Then V_M is a neighborhood of A, and M is a basis of $\hat{I}_{\tilde{A}}$ for every $\tilde{A} \in V_M \setminus \{A\}$. □

Theorem 3.4. *Specify a point $A \in \mathbb{K}$, a term order $\preceq$ on T^X and an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ saturated w.r.t. $(\theta - A)$. If $\hat{I}_A$ is zero-dimensional and Γ_A is a discontinuous specialization of $\hat{I}$ w.r.t. $\preceq$, then the following three conditions are equivalent.*

(1) *Γ_A is an ADS of $\hat{I}$ w.r.t. $\preceq$.*
(2) *For each monomial basis M of $\mathbb{K}[X]/\hat{I}_A$, there exists a neighborhood V_M of A such that for every $\tilde{A}$ in V_M, M forms a basis of $\mathbb{K}[X]/\hat{I}_{\tilde{A}}$.*
(3) *There exists a neighborhood V of A such that for all $\tilde{A} \in V$, $\hat{I}_{\tilde{A}}$ have the same number of zeros in $\bar{\mathbb{K}}^n$ counting multiplicities.*

Proof. (2) ⇒ (1) and (1) ⇒ (3): Obvious.

(3) ⇒ (2): From the condition, we know that $\dim(\mathbb{K}[X]/\hat{I}_A) = \dim(\mathbb{K}[X]/\hat{I}_{\tilde{A}})$. Pick a Gröbner basis $\hat{G}$ of $\hat{I}$ w.r.t. $\preceq$. Then there exists a neighborhood $U \subset V$ of A such that $0 \notin \mathrm{lc}(\hat{G})_{\tilde{A}}$ for every $\tilde{A} \in U \setminus \{A\}$. Thus, we have the equality that $\dim(\mathbb{K}[X]/\hat{I}_{\tilde{A}}) = \dim(\mathbb{K}(\theta)[X]/\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]})$. As a result, $\dim(\mathbb{K}[X]/\hat{I}_A) = \dim(\mathbb{K}(\theta)[X]/\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]})$. Then, by Lemma 3.2, any monomial basis M of the quotient ring $\mathbb{K}[X]/\hat{I}_A$ forms a basis of $\mathbb{K}(\theta)[X]/\langle\hat{I}\rangle_{\mathbb{K}(\theta)[X]}$. Moreover, by Lemma 3.3, we know that there exists a neighborhood $V_M \subset U$ of A such that M is a basis of $\mathbb{K}[X]/\hat{I}_{\tilde{A}}$ for all $\tilde{A} \in V_M \setminus \{A\}$. Therefore, M is a monomial basis for every $\tilde{A} \in V_M$, and thus Γ_A is an ADS of $\hat{I}$ w.r.t. $\preceq$. □

Corollary 3.5. *Specify a point $A \in \mathbb{K}$, a term order $\preceq$ on $\mathrm{T}^{\{x,y\}}$ and a parametric ideal $\hat{I} = \langle\hat{f},\hat{h}\rangle \subset \mathbb{K}[\theta][x,y]$ such that $\hat{I}$ is saturated w.r.t. $(\theta - A)$ and Γ_A is a DS of $\hat{I}$ w.r.t. $\preceq$. If there exists a neighborhood V of A such that for every $\tilde{A} \in V$,*

- *$\hat{f}_{\tilde{A}}$ and $\hat{h}_{\tilde{A}}$ have no common factors;*
- *$\deg(\hat{f}_{\tilde{A}})$, $\deg(\hat{h}_{\tilde{A}})$ and the number of zeros of $\hat{I}_{\tilde{A}}$ at infinity in the projective plane $\mathbb{P}\bar{\mathbb{K}}^2$ are constant,*

then Γ_A is an ADS of $\hat{I}$ w.r.t. $\preceq$.

Proof. Since $\hat{f}_{\tilde{A}}$ and $\hat{h}_{\tilde{A}}$ have no common factors, we know by Bézout's theorem that $\hat{I}_{\tilde{A}}$ have $\deg(\hat{f}_{\tilde{A}})\deg(\hat{h}_{\tilde{A}})$ zeros in $\mathbb{P}\bar{\mathbb{K}}^2$ counting multiplicities and including the ones at infinity. Note that $\deg(\hat{f}_{\tilde{A}})$, $\deg(\hat{h}_{\tilde{A}})$ and the number of zeros of $\hat{I}_{\tilde{A}}$ at infinity are constant for all $\tilde{A} \in V$. Hence, $\hat{I}_{\tilde{A}}$ have the same number of zeros in $\bar{\mathbb{K}}^2$ for all $\tilde{A} \in V$. By Theorem 3.4(3), we know that Γ_A is an ADS of $\hat{I}$ w.r.t. $\preceq$. □

Remark 3.6. The condition that "$\hat{I}$ is saturated w.r.t. $(\theta - A)$" is used to prevent the dependence of terms in a parametric polynomial from changing when $\theta = A$. The conclusions of Lemma 3.2, Theorem 3.4 and Corollary 3.5 are not always true without this assumption. Consider a parametric polynomial ideal $\hat{I} = \langle \theta x^3 + x^2, y^2, \theta xy\rangle \subset \mathbb{R}[\theta][x, y]$. No matter A equals 0 or not, $\hat{I}_A$ has four zeros in $\mathbb{C}^2$ counting multiplicities. Note that $\mathbb{R}[x, y]/\hat{I}_0$ has a unique monomial basis $M = \{1, x, y, xy\}$. But for $A \neq 0$, M is not a basis of $\mathbb{R}[x, y]/\hat{I}_A$.

Generally, to find an ADS of a parametric polynomial ideal $\hat{I}$ with a single parameter, one should first find a DS Γ_A. By Theorem 3.1(2), a Gröbner basis w.r.t. a block ordering needs to be computed. We have done this by the function fgb_gbasis($\hat{F}$, 0, X, $[\theta]$) in FGb package (Faugère, 1999, 2006) in Maple. Then one should check whether Γ_A is an ADS or not by Theorem 3.4(3) and Corollary 3.5.

Example 3.7. Consider the ideals $\hat{I} = \langle \hat{f}_1, \ldots, \hat{f}_n\rangle$ below. They are all saturated w.r.t. $(\theta - A)$. The corresponding Γ_A are all ADS of $\hat{I}$ w.r.t. the DRL order with $x \succ y \succ z$.

(1) $\hat{f}_1 = \theta x + y, \hat{f}_2 = x^2; A = 0.$
(2) $\hat{f}_1 = \theta x^3 + x^2 y, \hat{f}_2 = x^k + y^k, k \geq 2; A = 0.$
(3) $\hat{f}_1 = x^2 - y^2 - y, \hat{f}_2 = \theta xy - y^2 + x + 4; A = 0.$
(4) $\hat{f}_1 = x^2 - \theta xy - y^2, \hat{f}_2 = \theta x^2 - xy; A = \pm 1.$
(5) $\hat{f}_1 = x^3 + (2 - \theta)x^2 y + 3xy^2 + (4 - \theta)y^3, \hat{f}_2 = (4 + \theta)x^3 + 3x^2 y + (2 + \theta)xy^2 + y^3;$ $A = 0, -10/3, -1 \pm \sqrt{6}.$
(6) $\hat{f}_1 = \theta x^3 + \theta x^2 y - \theta y^3 - \theta xz^2 + (1 + \theta)z^3, \hat{f}_2 = (\theta - 1)y^3 + (1 + \theta)x^2 z + (1 + \theta)y^2 z - z^3,$ $\hat{f}_3 = (1 - \theta)x^3 + (1 + \theta)xy^2 + (\theta - 1)y^3 + (\theta - 1)x^2 z - y^2 z; A = 0.$

3.3. Local repairs of artificially discontinuous specializations

The other main result is Theorem 3.8 which tells us how an ADS can be transformed to a CS. We call this process *local repair*. Theorem 3.8 shows that though an ADS can cause a singularity in Gröbner basis computation, we can still overcome it in the Gröbner basis framework. The tool to realize this transformation is just the TSV strategy first introduced in Faugère and Liang (2007).

Theorem 3.8. *Specify a point $A \in \mathbb{K}$, a graded order $\preceq$ on T^X and an ideal $\hat{I} \subset \mathbb{K}[\theta][X]$ saturated w.r.t. $(\theta - A)$. If Γ_A is an ADS of $\hat{I}$ w.r.t. $\preceq$, then it can be locally repaired to a CS of some extended ideal of $\hat{I}$ w.r.t. some graded extended order of $\preceq$ by using the TSV strategy.*

Proof. Let M and V be the corresponding finite term set in T^X and the neighborhood of A in Definition 2.2 respectively. Denote $M \setminus \{1\}$ by $\{t_1, \ldots, t_s\}$. Pick $E = \{y_1 - t_1, \ldots, y_s - t_s\}$ and a graded extended order $\preceq^e$ of $\preceq$ with $Y \prec X$. Then by Theorem 2.7, the term set $M' = \{1\} \cup Y$ constitutes a monomial basis of $\mathbb{K}[X \cup Y]/\hat{I}^E_{\tilde{A}}$ for every $\tilde{A} \in V$. Since M' is the minimal element in the monomial basis set of $\mathbb{K}[X \cup Y]/\hat{I}^E_{\tilde{A}}$ w.r.t. $\preceq^{e'}$ for every $\tilde{A} \in V$, all these $\mathbb{K}[X \cup Y]/\hat{I}^E_{\tilde{A}}$ share the same staircase M' corresponding to the Gröbner bases of $\hat{I}^E_{\tilde{A}}$ w.r.t. $\preceq^e$ by Proposition 2.5. It is easy to see that $\hat{I}^E$ is saturated w.r.t. $(\theta - A)$. Therefore, Γ_A becomes a CS of $\hat{I}^e$ w.r.t. $\preceq^e$ by Theorem 3.1(3). □

Remark 3.9. The s mentioned in the proof of Theorem 3.8 is an upper bound on the number of binomials needed to locally repair an ADS; in practice, an ADS can be repaired with much less effort (see the following examples).

Example 3.10. Consider the examples of ADS in Example 3.7. We show how we can repair them by using the TSV strategy with the DRL order with $x \succ y \succ z \succ u_1 \succ u_2 \succ \cdots$ below. (In fact, we have shown this process by a toy example in the introduction of this paper.)

System	E
(1), (2)	$u_1 - x$
(3), (4), (5)	$u_1 - xy$
(6)	$u_1 - yz, u_2 - xz, u_3 - xy, u_4 - u_2u_3, u_5 - zu_1u_3, u_6 - xu_5$

When the TSV strategy is used to repair an ADS at one point, it may cause new ADS at other points in the extended system. For instance, when $\Gamma_{\pm 1}$ have been repaired to CS in Example 3.7(4), Γ_0 becomes an ADS of $\langle \hat{f}_1, \hat{f}_2, u_1 - xy \rangle$ w.r.t. the DRL order with $x \succ y \succ u_1$. That is why we say that $\Gamma_{\pm 1}$ have been locally repaired. In numerical computation, only the points in a small neighborhood of the coefficient vectors are concerned. Therefore, those new ADS would do no harm to the computations if the precision is high enough.

4. Conclusion and future work

In this paper, a theory has been set up for artificial discontinuities of single-parametric polynomial ideals with some restrictions. We first gave formal definitions of the continuous specialization (CS), the discontinuous specialization (DS) and the artificial discontinuous specialization (ADS). Then provided their criteria. At last, we show that an ADS can be locally repaired to a CS by the TSV strategy in an extended ring w.r.t. an extended order.

For the future work, we want to generalize the results in this paper to the cases with multiple parameters and design effective and optimal algorithms based on the TSV strategy to locally repair any ADS.

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